On oscillations in resonant equations with complex nonlinearities

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Abstract

In the paper the analysis is presented of forced periodic oscillations in systems described by the second order ODE with resonant linear part and complex nonlinearities: with hysteresis and with delay. For such equations we give conditions of the existence of at least one periodic solution and conditions of the existence of unbounded sequences of such solutions. Analogous results are formulated for forced periodic oscillations in resonant control systems.

1 The statement of the problem

Consider the equation

$$x'' + x = f(t, x) \tag{1}$$

with a continuous bounded function $f(t,x) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, 2π-periodic in t. Consider the problem of existence of 2π-periodic solutions for this equation. The linear part of this equation is resonant; equation (1) with various right-hand sides may have or may have not 2π-periodic solutions. For $f(t,x) \equiv b(t)$ the answer is given by the Fredholm alternative lemma: the 2π-periodic solutions exist iff

$$\bar{b} \stackrel{\text{def}}{=} \int_0^{2\pi} b(t) e^{it}\, dt = 0, \tag{2}$$

if this condition is valid then there exist infinitely many such solutions. For $f(t,x)$ depending on x even particular answers are much more cumbersome. We consider cases where the nonlinearities have the form "time-independent nonlinearity" + "forcing term $b(t)$". The value $|\bar{b}|$ defined by (2) plays the main role in the following statements, the value $|\bar{b}|/\sqrt{\pi}$ is equal to the norm in L^2

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of the orthogonal projection of the function $b(t)$ onto two dimensional subspace with the basis $\cos t,\ \sin t$.

The first theorems on 2π-periodic solutions of nonlinear equation (1) where obtained in (Lazer and Leach, 1969). Later, various authors generalized the Landesman – Leach results in different directions. They used various topological methods, potential methods, lower and upper solutions, etc., and proved the existence of at least one periodic solution and the non-existence of such solutions. Generally speaking, the results obtained can be formulated in the following way. Under suitable (and rather strong) conditions for the nonlinearity $f(t,x)$, two numbers k and K are calculated, $0 \le k \le K$. If $|\bar{b}| < k$, then at least one 2π-periodic solution exists. If $|\bar{b}| > K$, then 2π-periodic solutions do not exist at all or they may exist, but their common topological index equals zero. Results of this type were obtained for more general than (1) types of equations, in particular, for equations with hysteresis and delays. In (Krasnosel'skii, 1996) a class of nonlinearities was presented such that $k = K$, and in a natural sense the results are sharp.

Recently, in (Krasnosel'skii and Mawhin, to appear), more general results were obtained. For arbitrary bounded nonlinearity $f(t,x) = f(x) + b(t)$, again, two numbers $0 \le k \le K$ are presented. If $|\bar{b}| < k$, then, again, at least one 2π-periodic solution exists. The most interesting case is $k < |\bar{b}| < K$. For this case there exists an infinite sequence of 2π-periodic solutions, norms (in any reasonable sense) of these solutions tend to infinity.

In this paper we consider the equations

$$x'' + x = f(x(t), x(t-h)) + b(t) \tag{3}$$

with the delay h and the equations

$$x'' + x = G(x) + b(t) \tag{4}$$

where $G(x)$ is the special type of hysteresis nonlinearity considered in Section 3 (for more details, properties and general theory, see (Krasnosel'skiĭ and Pokrovskiĭ, 1984)). The results for equations (3) and (4) are generalized in Section 5 for some equations arising in control theory.

Definition 1. *We say that some equation* ***has correct boundaries*** $\boldsymbol{k \le K}$ ***for the forcing term*** $\boldsymbol{b(t)}$ *if the following statements are valid:*

- *The inequality $|\bar{b}| < k$ guarantees an existence of at least one 2π-periodic solution of the equation and an* a priori *estimate $\|x\|_C \le c$ for all such solutions;*
- *The inequality $k < |\bar{b}| < K$ guarantees the existence of an infinite sequence x_n of 2π-periodic solutions of the equation: $\|x_n\|_C \to \infty$ as $n \to \infty$;*
- *The inequality $K < |\bar{b}|$ guarantees an* a priori *estimate $\|x\|_C \le c$ for all 2π-periodic solutions of the equation.*

Generally speaking, if the inequality $K < |\bar{b}|$ holds, then the equation may have not 2π-periodic solutions.

The paper is organized in the following way: in the next section we present a result about second order ODE with delay, in Section 3 we give a minimal description of hysteresis nonlinearity named hysteron and in Section 4 we present a result about second order ODE with the hysteron. Remarks are in Section 6, Section 7 contains all the proofs.

2 Equations with delay

For $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ continuous and bounded, set

$$\psi_s(\xi) = \int_0^{2\pi} \sin t\, f(\xi \sin t, \xi \sin(t-h))\, dt, \qquad \psi_c(\xi) = \int_0^{2\pi} \cos t\, f(\xi \sin t, \xi \sin(t-h))\, dt,$$

$$\Psi(\xi) = \sqrt{[\psi_c]^2 + [\psi_s]^2} \tag{5}$$

and

$$k = \liminf_{\xi \to +\infty} \Psi(\xi), \qquad K = \limsup_{\xi \to +\infty} \Psi(\xi). \tag{6}$$

We say that the function $f(x, y)$ satisfies a proper Lipschitz condition in x if for any δ and Δ a function[1] $\zeta(r)$ exists such that

$$\lim_{r \to +\infty} \zeta(r) = 0 \tag{7}$$

and

$$|f(x_1, y) - f(x_2, y)| \le \zeta(r)\, |x_1 - x_2|, \qquad \delta\, r \le |x_1|, |x_2|, |y| \le \Delta\, r. \tag{8}$$

Similarly, we say that the function $f(x, y)$ satisfies a proper Lipschitz condition in y if for any δ and Δ a function $\zeta(r)$ exists such that (7) and

$$|f(x, y_1) - f(x, y_2)| \le \zeta(r)\, |y_1 - y_2|, \qquad \delta\, r \le |y_1|, |y_2|, |x| \le \Delta\, r. \tag{9}$$

We say that the function $f(x, y)$ has proper behavior at infinity, if it can be represented as a sum of two functions, one satisfying a proper Lipschitz condition in x and another one satisfying a proper Lipschitz condition in y.

Theorem 1. *Suppose the function $f(x, y)$ has proper behavior at infinity. Equation* (3) *has correct boundaries* (6) *for the forcing term $b(t)$.*

The typical example of the function $f(x, y)$ having proper behavior at infinity is the function $f_1(x) + f_2(y)$ with arbitrary bounded and continuous f_1 and f_2.

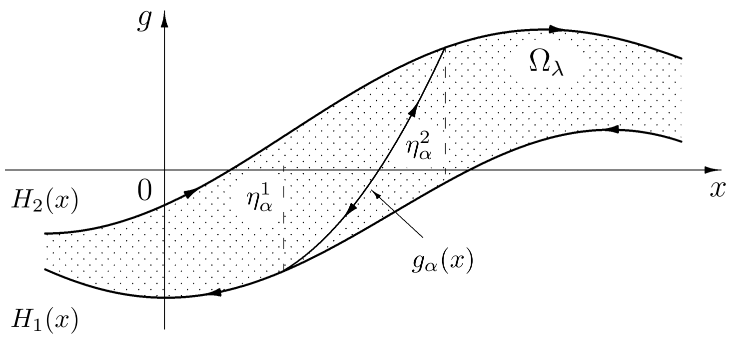

Fig. 1. Hysteron

3 Hysteresis nonlinearity

Only a very simple modification of the nonlinearity "hysteron" is described below. See (Krasnosel'skiĭ and Pokrovskiĭ, 1984) for the general definition. Consider in the plane $\{x, g\}$ the graphs of two continuous functions $H_1(x)$, $H_2(x)$ satisfying the inequality $H_1(x) < H_2(x)$, $x \in \mathbb{R}$. Suppose that the set $\Omega = \{\{x, g\} : x \in \mathbb{R},\ H_1(x) \le g \le H_2(x)\}$ in the plane $\{x, g\}$ is sliced into the disjoint union of continuous family of graphs of continuous functions $g_\alpha(x)$, where α is a parameter. Each function $g_\alpha(x)$ is defined on its interval $[\eta_\alpha^1, \eta_\alpha^2]$, $\eta_\alpha^1 < \eta_\alpha^2$ and $g_\alpha(\eta_\alpha^1) = H_1(\eta_\alpha^1)$, $g_\alpha(\eta_\alpha^2) = H_2(\eta_\alpha^2)$, that is, the endpoints of the graphs of the functions $g_\alpha(x)$ belong to the union of the graphs $H_1(x)$, $H_2(x)$ (see Fig. 1). The figure graphs one of the functions $g_\alpha(x)$.

The hysteron is the transducer with internal states μ from the segment $[0, 1]$ and the input–output operators which are described as follows. The variable output $\mathcal{H}(\mu_0)x(t) \equiv \mathcal{H}(\mu_0, t_0)x(t)$ $(t \ge t_0)$ is defined by the formula

$$\mathcal{H}(\mu_0)x(t) = \begin{cases} g_\alpha(x(t)), & \text{if} \quad \eta_\alpha^1 \le x(t) \le \eta_\alpha^2, \\ H_1(x(t)), & \text{if} \quad x(t) \le \eta_\alpha^1, \\ H_2(x(t)), & \text{if} \quad \eta_\alpha^2 \le x(t) \end{cases}$$

for the monotone inputs $x(t)$, $t \ge t_0$. The value of α is defined by the initial state μ_0 to satisfy $g_\alpha(x(t_0)) = \mu_0 H_1(x(t_0)) + (1 - \mu_0) H_2(x(t_0))$ and the corresponding variable internal state is defined by

$$\Xi(\mu_0)x(t) = \frac{\mathcal{H}(\mu_0)x(t) - H_1(x(t))}{H_2(x(t)) - H_1(x(t))}.$$

[1] It may depend on δ and Δ.

For the piecewise monotone continuous inputs, the output is constructed by the semigroup identity. The input–output operators can then be extended to the totality of all continuous inputs by continuity (see (Krasnosel'skiĭ and Pokrovskiĭ, 1984)). The operators $\mathcal{H}(\mu_0)x(t)$, $\Xi(\mu_0)x(t)$ are defined for each continuous input and for each initial state. They are continuous as operators in the spaces of continuous functions with the uniform metric.

4 Equations with hysteron

Suppose that both functions $H_i(x)$ are bounded. Set

$$R(t,\xi) = \begin{cases} H_1(\xi \sin t), & \cos t > 0, \\ H_2(\xi \sin t), & \cos t < 0; \end{cases}$$

$$\Phi(\xi) = \sqrt{\left[\int_0^{2\pi} \sin t \, R(t,\xi)\, dt\right]^2 + \left[\int_0^{2\pi} \cos t \, R(t,\xi)\, dt\right]^2}$$

and

$$k = \liminf_{\xi \to +\infty} \Phi(\xi), \qquad K = \limsup_{\xi \to +\infty} \Phi(\xi). \tag{10}$$

Theorem 2. *Let, for any* α*,*

$$\eta_\alpha^2 - \eta_\alpha^1 \le \theta(\max\{|\eta_\alpha^2|, |\eta_\alpha^1|\}) \tag{11}$$

where the function $\theta(u): \ (0,\infty) \to (0,\infty)$ *is sublinear at infinity:*

$$\lim_{u\to\infty} \frac{\theta(u)}{u} = 0. \tag{12}$$

Then the system

$$x'' + x = \mathcal{H}(\mu_0)x + b(t), \qquad \Xi(\mu_0)x(t)\Big|_{t\,=\,2\pi} = \mu_0 \tag{13}$$

has correct boundaries (10) *for the forcing term* $b(t)$*.*

The second equation in (13) means that the 2π-periodic function $x(t)$ is a solution of equation with hysteresis if the corresponding time-depending state of hysteresis nonlinearity is also 2π-periodic (and this function satisfies the first equation (13)).

5 Control theory equations

In this section we consider equations arising in control theory:

$$L\left(\frac{d}{dt}\right) x = M\left(\frac{d}{dt}\right) \big(\, f(x(t), x(t-h)) + b(t) \, \big) \tag{14}$$

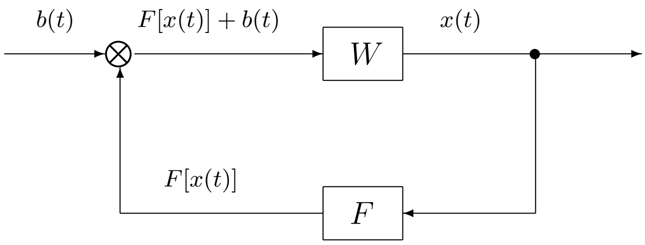

Fig. 2. Control system

and

$$\begin{cases} L\left(\dfrac{d}{dt}\right) x = M\left(\dfrac{d}{dt}\right)\left(\,\mathcal{H}(\mu_0)x + b(t)\,\right), \\[2ex] \Xi(\mu_0)x(t)\Big|_{t\,=\,2\pi} = \mu_0. \end{cases} \tag{15}$$

Here $L(p)$ and $M(p)$ are real coprime polynomials, $l = \deg L > m = \deg M$. Again, $\mathcal{H}(\mu_0)x$ is a hysteron of the type considered in Section 3.

In Fig. 2, one can see a block diagram of such systems. Nonlinearity is denoted as F and W is a linear element with rational transfer function $W(p) = M(p)/L(p)$.

Theorem 3. *Let $L(i) = 0$ and let $L(ni) \neq 0$ for $n = 0, 2, 3, 4, \ldots$. Let the function $f(x, y)$ be continuous and bounded and let it have proper behavior at infinity. Then system* (14) *has correct boundaries* (6) *for the forcing term $b(t)$.*

Theorem 4. *Let $L(i) = 0$ and let $L(ni) \neq 0$ for $n = 0, 2, 3, 4, \ldots$. Let, for any α, inequality* (11) *be valid with sublinear $\theta(u)$. Then system* (15) *has correct boundaries* (10) *for the forcing term $b(t)$.*

6 Remarks

6.1 Index at infinity

For any differential equation presented above we consider (see the proofs) some equivalent operator equation. This equation has the form $x = Ax$, where A is a completely continuous nonlinear operator. For the vector field $x - Ax$, one can calculate its index at infinity (see (Krasnosel'skiĭ and Zabreĭko, 1984)).

If $|\bar{b}| < k$, then the index equals ± 1; if $k < |\bar{b}| < K$, then the index is undefined; if $K < |\bar{b}|$, then the index is equal to 0.

6.2 Bifurcation at infinity

Let, in a Banach space E, the equation $B(x, \lambda) = 0$ be given with some operator $B(x, \lambda)$ depending on a parameter $\lambda \in \Lambda = [a, b]$.

A value λ_0 of the parameter is called a *bifurcation point at infinity* or (the same) an *asymptotic bifurcation point* if, for every $\varepsilon > 0$, there exists a $\lambda_\varepsilon \in (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon) \bigcap \Lambda$ such that, for $\lambda = \lambda_\varepsilon$, the equation $B(x, \lambda) = 0$ has at least one solution x_ε satisfying $\|x_\varepsilon\| > \varepsilon^{-1}$.

Let us formulate an application example of Theorem 1 for some equation with a parameter.

Consider the equation

$$x'' + x = f(x(t), x(t-h)) + \lambda \sin t \tag{16}$$

with a real parameter λ.

Let us consider function (5) and numbers (6). Suppose $k \neq K$.

Theorem 5. *The set* $[-K/\pi, -k/\pi] \bigcup [k/\pi, K/\pi]$ *is the set of asymptotic bifurcation points for equation* (16).

This set is a union of 2 intervals if $k > 0$ and the interval $[-K/\pi, K/\pi]$ if $k = 0$.

6.3

After all, a natural question arises: How to calculate the numbers k and K, used in the hypotheses of Theorems 1 – 4 or, the same, how to estimate the behaviour of the integrals

$$\int_0^{2\pi} \sin t\, f(\xi \sin t, \xi \sin(t-h))\, dt \quad \text{and} \quad \int_0^{2\pi} \cos t\, f(\xi \sin t, \xi \sin(t-h))\, dt \tag{17}$$

or

$$\int_0^{2\pi} \sin t\, R(\xi, t)\, dt \quad \text{and} \quad \int_0^{2\pi} \cos t\, R(\xi, t)\, dt \tag{18}$$

for $\xi \to \infty$?

The general approach is the following: for various functions $f(x, y)$, at least one or both such integrals tend to zero or even are equal to zero. Therefore one can split the function $f(x, y)$ into the sum of functions such that for some of these functions integrals (17) tend to zero and for others the integrals can be computed in an obvious form.

More information on the calculation of the integrals (17) for the simplest case $f(x, y) = f(x)$ without delays can be found in (Krasnosel'skii and Mawhin, to appear).

Integrals (18) can be rewritten in the form

$$\int_0^{2\pi} \sin t\, R(\xi, t)\, dt = \int_{\pi/2}^{\pi/2} \sin t(H_1(\xi \sin t) + H_2(\xi \sin t))\, dt$$

and

$$\int_0^{2\pi} \cos t\, R(\xi, t)\, dt = H_1(\xi) - H_2(\xi) - H_1(-\xi) + H_2(-\xi).$$

It is easy to see that the even parts of the functions $H_j(x)$ do not play any role, the answers are defined by the odd parts.

6.4 Arbitrary period

Of course, instead of the left-hand side $x'' + x$ it is possible to consider left-hand sides of more general type $x'' + n^2 x$ with integer $n > 1$. Formulations for this case are almost the same. It is possible to rewrite all theorems for T-periodic problem with arbitrary period T.

7 Proofs

7.1 General scheme

The proofs of all theorems have common schemes. In the beginning we reduce the periodic problem for initial differential equation to an operator equation in an appropriate Banach space. This operator equation has the form $x = Ax + Fx$ with linear completely continuous A and nonlinear completely continuous bounded F. The asymptotically linear vector field $\Phi x = x - Ax - Fx$ is degenerate: the value 1 is an eigenvalue for the linear operator A. For the case $\overline{b} < k$, we calculate the index at infinity for this field, the index is well defined and equal to ± 1. For the case $\overline{b} > K$, the index is also well defined and equal to 0. This proves the corresponding parts of the theorems.

The most difficult case is $k < \overline{b} < K$. For this case we calculate the rotation of the vector field Φx on the boundary of some infinite-dimensional cylinder in L^2; this cylinder has a 2-dimensional component and a bounded 2-codimensional part. Using the rotation product formula the rotation calculation can be reduced to the rotation calculation of some planar vector field on the boundary of annulus. The last calculation can be done in obvious form, the rotation of Φx equals ± 1. Therefore, in such cylinder at least one solution exists. As a last step of the proof, we see that our cylinders may not intersect arbitrary large balls $\{\|x\| \leq r\}$, consequently these solutions may have arbitrary large norms. The main part of the proof is the reduction of the rotation of the infinite-dimensional vector field to the rotation of a 2-dimensional one. This 2-dimensional vector field has the form $\Delta(|z|)z/|z| + z_0$, where z is the point of the plane, considered as a complex number, $\Delta(|z|)$ is a bounded complex function, z_0 is some vector. Denote $K' = \limsup |\Delta(\xi)|$, and $k' = \liminf |\Delta(\xi)|$, then if $|z_0| < k'$ the index at infinity of the field $\Delta(|z|)z/|z| + z_0$ is equal to ± 1, if $k < |z_0| < K$ then the equation $\Delta(|z|)z/|z| + z_0 = 0$ has an unbounded set of solutions.

The idea of the proof was already used for functional nonlinearity $f(x)$ in (Krasnosel'skii and Mawhin, to appear). The main part is the reduction of infinite dimensional vector fields to some planar ones. This reduction follows from Lemmas 1 and 2 below. The final study of the planar vector fields is common for both Theorems 1 and 2.

We give the complete proof for Theorem 1 only. For equations with hysteresis we give only the proof of the main lemma and give some explanations for the equivalent operator equation construction. Other parts of the proof are very close to the proof of Theorem 1.

7.2 Main lemma for nonlinearities with delay

Denote

$$t_h = \begin{cases} t - h, & t \geq h, \\ t - h + 2\pi, & t < h. \end{cases}$$

Lemma 1. *Let the function $g(t)$ be Lipschitz. Let the function $e(t) \in C^1$ satisfy the condition* $\operatorname{mes}\{t \in [0, 2\pi] : e(t)e'(t) = 0\} = 0$. *Let the function $f(x, y)$ have proper behavior at infinity. Then the following relation is valid for any $c > 0$:*

$$\lim_{\xi \to \infty} \sup_{\|z\|_{C^1} \leq c} \left| \int_0^{2\pi} g(t) \left(f(\xi e(t) + z(t), \xi e(t_h) + z(t_h)) - f(\xi e(t), \xi e(t_h)) \right) dt \right| = 0. \tag{19}$$

Proof. Without loss of generality we prove this lemma for the case where the function $f(x, y)$ satisfy a proper Lipschitz condition in y only.

Let us choose an $\varepsilon > 0$ and let us show that the supremum in (19) is less than ε for sufficiently large $|\xi|$:

$$\sup_{\|h\|_{C^1} \leq c} \left| \int_0^{2\pi} g(t) \left(f(\xi e(t) + z(t), \xi e(t_h) + z(t_h)) - f(\xi e(t), \xi e(t_h)) \right) dt \right| < \varepsilon. \tag{20}$$

To this end, let us split the interval $[0, 2\pi]$ into a finite number of subintervals $[a_i, b_i]$ and $[b_i, a_{i+1}]$ as follows. The intervals (b_i, a_{i+1}) contain the set $\{t \in [0, 2\pi] : e(t)e'(t) = 0\}$, the union of these intervals can have any arbitrarily small measure, they can be chosen such that

$$t \in \bigcup [a_i, b_i] \Rightarrow t_h \in \bigcup [a_i, b_i], \qquad \sup |f(x, y)| \int\limits_{\bigcup [b_i, a_{i+1}]} |g(t)|\, dt < \varepsilon/2. \tag{21}$$

Suppose that the points a_i and b_i are fixed till the end of the proof of the lemma. For any $[a_i, b_i]$, the estimates

$$\inf_{t \in [a_i, b_i]} \min\{|e(t)|,\ |e'(t)|,\ |e(t_h)|\} \geq \delta > 0 \tag{22}$$

hold. This means that the function $e(t)$ is strictly monotone on every $[a_i, b_i]$, and, for sufficiently large $|\xi|$ ($|\xi| > 2c\delta^{-1}$), the function $\xi e(t) + z(t)$ is also strictly monotone, and $|\xi e'(t) + z'(t)| > 1/2\,|\xi|\delta$. Consider the integrals

$$\mathcal{J}_i = \int_{a_i}^{b_i} g(t) f(\xi e(t) + z(t), \xi e(t_h) + z(t_h))\, dt.$$

Fix any one of them, and perform in this integral, for any ξ, the change of variables $t = t(\tau) = t(\xi, \tau)$ defined by the formula $\xi e(\tau) = \xi e(t) + z(t)$:

$$\mathcal{J}_i = \int_{t^{-1}(\xi, a_i)}^{t^{-1}(\xi, b_i)} g(t(\xi, \tau))\, f(\xi e(\tau), \xi e(t(\xi, \tau)_h) + z(t(\xi, \tau)_h))\, t'_\tau(\xi, \tau)\, d\tau.$$

The function $t(\xi, \tau)$ is one-to-one, $t(\xi, \tau) \to \tau$ and $t'_\tau(\xi, \tau) \to 1$ uniformly in τ as $|\xi| \to \infty$. Now

$$t^{-1}(\xi, a_i) \to a_i, \quad t^{-1}(\xi, b_i) \to b_i,$$

and $g(t(\xi, \tau)) \to g(\tau)$ due to the continuity of $g(\cdot)$. One can see that $|t(\xi, \tau)_h - \tau_h| \le \mathit{const}\,\xi^{-1}$ hence $|\xi e(t(\xi, \tau)_h) + z(t(\xi, \tau)_h) - \xi e(\tau_h)| \le \mathit{const}$. Consequently from the Lipschitz condition (9) it follows that

$$\mathcal{J}_i - \int_{a_i}^{b_i} g(\tau) f(\xi e(\tau), \xi e(\tau_h))\, d\tau \to 0$$

for every i. This, together with (21), proves (20) and the lemma. ■

7.3 Main lemma for hysteron

Let

$$r(t, \xi; e) = \begin{cases} H_1(\xi e(t)), & e'(t) > 0, \\ H_2(\xi e(t)), & e'(t) < 0; \end{cases}$$

Lemma 2. *Let the function $g(t)$ be Lipschitz. Let the function $e(t) \in C^1$ satisfy the condition* $\mathrm{mes}\{t \in [0, 2\pi] : \ e'(t) = 0\} = 0$. *Then the following relation is valid for any $c > 0$:*

$$\lim_{\xi \to \infty} \sup_{\mu \in [0,1], \|z\|_{C^1} \le c} \left| \int_0^{2\pi} g(t) \left(\mathcal{H}(\mu)(\xi e(t) + z(t)) - r(t, \xi; e) \right) dt \right| = 0. \quad (23)$$

Proof. Again, let us choose an $\varepsilon > 0$ and let us show that the supremum in (23) is less than ε for sufficiently large $|\xi|$:

$$\sup_{\mu \in [0,1], \|z\|_{C^1} \le c} \left| \int_0^{2\pi} g(t) \left(\mathcal{H}(\mu)(\xi e(t) + z(t)) - r(t, \xi; e) \right) dt \right| \le \varepsilon.$$

Let us split the interval $[0, 2\pi]$ into interval in the same way as in the proof of Lemma 1, the intervals (b_i, a_{i+1}) have an arbitrarily small common measure. Consider the integrals

$$\mathcal{J}_i = \int_{a_i}^{b_i} g(t) \left(\mathcal{H}(\mu)(\xi e(t) + z(t)) - r(t, \xi; e) \right) dt.$$

On any interval (a_i, b_i), the function $x(t) = \xi e(t) + z(t)$ is monotone for sufficiently large ξ. At the beginning of the interval the state of hysteron can be different from 0 and 1, after some time the state becomes 0 if $x(t)$ decreases and it becomes 1 if $x(t)$ increases. Let us estimate how long the state can be in the interior $(0, 1)$ of the interval $[0, 1]$. Since we have the uniform estimate (22), for sufficiently large ξ the velocity $x'(t)$ is arbitrary large of the order ξ. This means that $x(t)$ always reaches the end η_α^j $(j = 1, 2)$ after the time

$$\frac{\eta_\alpha^2 - \eta_\alpha^1}{\xi} \leq \frac{\theta(c_1\xi)}{\xi} \to 0.$$

Therefore

$$\mathcal{J}_i = o(\xi) + \int_{a_i+\sigma_0}^{b_i} g(t) \left(\mathcal{H}(\mu)(\xi e(t) + z(t)) - r(t, \xi; e) \right) dt$$

where for $t \geq a_i + \sigma_0$ one has

$$\mathcal{H}(\mu)(\xi e(t) + z(t)) = H_j(\xi e(t) + z(t))$$

with the corresponding $j = 1, 2$. The rest of the proof, i.e. the equality

$$\lim_{\xi\to\infty} \sup_{\|z\|_{C^1}\leq c} \left| \int_{a_i+\sigma_0}^{b_i} g(t) \left(H_j(\xi e(t) + z(t)) - H_j(\xi e(t)) \right) dt \right| = 0$$

can be done with a change of variables similar to one in the proof of Lemma 1.
■

7.4 Equivalent operator equations

Consider the space $L^2 = L^2(0, 2\pi)$ of square integrable functions with the usual norm. Denote by Π_0 the 1-dimensional subspace of constant functions and by Π_n, $n = 1, 2, \ldots$ the 2-dimensional subspaces spanned by the functions $\sin t$ and $\cos t$. Define for any $u(t) \in L^2$ the linear self-adjoint operator

$$Au(t) = \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{1}{2 - n^2} P_n u$$

where by $P_n u$ we denote the orthogonal projector onto Π_n. This operator is completely continuous in L^2, it is also completely continuous as the operator from L^2 to C^1, it maps any function $u(t)$ into the 2π-periodic solution $x = Au$

of the equation $x'' + 2x = u(t)$. If $u(t)$ is continuous, then this solution is the classical one, if $u(t) \in L^2$, then $x(t) \in W^{1,2}$.

This means that 2π-periodic problem for the equation (3) is equivalent to the equation

$$x = A\left(x + f(x(t), x(t_h)) + b(t) \right) \tag{24}$$

and this problem for the equation (13) is equivalent to the system

$$x = A\left(x + \mathcal{H}(\mu_0)x + b(t) \right), \qquad \Xi(\mu_0)x(t)\Big|_{t\,=\,2\pi} = \mu_0. \tag{25}$$

Analogously we can rewrite the periodic problem for equation (14) and for system (15), we do not write the exact formulas.

We prefer to consider the operator equations in the space L^2 but the operators $\mathcal{H}$ and Ξ are defined for continuous functions $x(t)$ only. Instead of (24) and (25) we consider the systems

$$y = Ay + f(Ay(t), Ay(t_h)) + b(t) \tag{26}$$

and

$$y = Ay + \mathcal{H}(\mu_0)Ay + b(t), \qquad \Xi(\mu_0)Ay(t)\Big|_{t\,=\,2\pi} = \mu_0. \tag{27}$$

Equation (26) is considered in L^2, any its solution $y \in L^2$ generates the solution $x = Ay \in C$ of equation (24). Analogously, any solution $\{y, \mu_0\}$, $y \in L^2$ of the equation (27) generates the solution $\{x = Ay, \mu_0\}$ of equation (25), $x \in C$.

The second equation in systems (25) and (27) is defined only for $\mu \in [0, 1]$. Let us formally continue the operator $\Xi(\mu)x$ for $\mu \notin [0, 1]$ as follows:

$$\Xi(\mu)x = \Xi(0)x, \text{ if } \mu < 0 \quad \text{and} \quad \Xi(\mu)x = \Xi(1)x, \text{ if } \mu > 0.$$

Now the operator $\Xi(\mu)x(t)$ can be considered as the operator defined for $\mu \in \mathbb{R}$ and $x \in C$.

Equations (26) and (27) have the form $x = Bx$ where x is an element of some Banach space E. We have $E = L^2$ and $By = Ay + f(Ay(t), Ay(t_h)) + b(t)$ for equation (26) and $E = L^2 \times \mathbb{R}$ and the corresponding B for system (27). For both theorems the operator B is completely continuous in the corresponding space E.

The proof of Theorems 1 and 2 is different for 3 different items of Definition 1. We give the proofs only for Theorem 1. Theorem 2 has additional difficulty: we need to control the state μ for the hysteresis. From the proofs it is clear that this difficulty does not give any troubles.

7.5 Index at infinity calculation, the case $|\bar{b}| < k$

The proof of this part is very close to the proof of its analog from (Krasnosel'skii and Mawhin, to appear).

In the proof we use the notation

$$Pu(t) = \frac{1}{\pi} \int_0^{2\pi} \cos(t - s)u(s)\, ds$$

for the orthogonal projector onto the plane Π_1 and the notation $Qx = x - Px$.

Let $k > |\bar{b}|$. Consider the homotopy

$$\Theta(\lambda, x) = x - Ax - \lambda f(Ax, Ax(t_h)) - (1-\lambda)Pf(Px, Px(t_h)) - b, \qquad \lambda \in [0,1]. \tag{28}$$

Now we have to do two things, namely to prove an a priori estimate for all possible zeros of the homotopy $\Theta(\lambda, x)$, and to study the vector field $\Theta(0, x)$.

Suppose that $x(t) = \xi \sin(t + \varphi) + z(t)$ where $z(t) = Qx(t)$ and $\Theta(\lambda, x) = 0$. Then $Q\Theta(\lambda, x) = 0$ and $P\Theta(\lambda, x) = 0$. The equality $Q\Theta(\lambda, x) = 0$ implies the estimate

$$\|Az\|_{C^1} \le c < \infty.$$

The equality $P\Theta(\lambda, x) = 0$ has the form

$$\lambda Pf(Ax, Ax(t_h)) - (1-\lambda)Pf(Px, Px(t_h)) - Pb = 0. \tag{29}$$

If $\xi \to +\infty$, then according to Lemma 1, $Pf(Ax, Ax(t_h)) - Pf(Px, Px(t_h)) \to 0$, therefore (29) implies

$$\lim_{\xi\to\infty} Pf(Px, Px(t_h)) = -Pb(t) \tag{30}$$

Since [2]

$$\liminf_{\xi\to\infty} \|Pf(Px, Px(t_h))\|_{L^2} \ge \liminf_{\xi\to\infty} \Psi(\xi)/\sqrt{\pi} = k/\sqrt{\pi}$$

and $\sqrt{\pi}\,\|Pb\|_{L^2} = |\bar{b}|$ the condition $|\bar{b}| < k$ contradicts to (29). This proves the required a priori estimate.

Now consider the vector field $\Theta(0, x) = x - Ax - Pf(Px, Px(t_h)) - b$. This vector field in L^2 has two independent components: in Π_1 and in QL^2. In QL^2 this field is asymptotically linear and non-degenerate, its index at infinity is ± 1. On the plane Π_1 this vector field depends on Px only. Consider this planar vector field on circles $\{\xi = \rho\}$ of fixed large radius ρ. Since

$$Pf(Px, Px(t_h)) = Pf(\xi\sin(t+\varphi), \xi\sin(t_h+\varphi)) = \frac{1}{\sqrt{\pi}}\left(\ \psi_s(\xi)\sin(t+\varphi) + \psi_c(\xi)\cos(t+\varphi)\ \right)$$

the image $P\Theta(0, Px)\{\xi = \rho\}$ of the circle $\{\xi = \rho\}$ is one-to-one passed circle with the center $|\bar{b}|/\sqrt{\pi}$ and the radius $\psi(\xi)$. The origin lies inside this circle; this means that the rotation of the vector field $P\Theta(0, Px)$ on the circle $\{\xi = \rho\}$ is equal to 1. The rotation product formula completes the proof: the index of infinity of the vector field $\Theta(0, x)$ is ± 1 as well as the index of $\Theta(1, x)$.

7.6 Infinite sequences of solutions, the case $k < \bar{b} < K$

For this case there exist unbounded sequences ξ_n and ξ^n with $\xi_{n+1} > \xi^n > \xi_n$ such that

$$\Psi(\xi_n) + \varepsilon < |\bar{b}| < \Psi(\xi^n) - \varepsilon, \qquad \Psi(\xi_n) < \Psi(\xi) < \Psi(\xi^n), \quad \xi_n < \xi < \xi^n \tag{31}$$

[2] We write "$\ge$" instead of "$=$" in the next formula because the value ξ is not arbitrary.

for some fixed $\varepsilon > 0$. Without loss of generality, suppose that any ξ_n is sufficiently large so that the supremum in formula (19) is small enough for $\xi \geq \xi_n$.

Below we prove that the rotation γ of the vector field $x - Ax - f(Ax, Ax(t_h)) - b$ on the boundary of the set $\Omega_n = \{\|Qx\| \leq R_1 + 1\} \times \{\xi \sin(t + \varphi) : \ \xi \in [\xi_n, \xi^n]\} \subset L^2$ is defined and that $|\gamma| = 1$. This equality proves the remaining part of the theorem: any Ω_n contains its own solution of equation (26), and the sets Ω_n are disjoint. The constant R_1 will be chosen below, and it does not depend on n.

Let us fix some n and let us calculate $|\gamma|$ for this number n. Consider again the homotopy (28)

For $\lambda = 0$ this homotopy is our vector field $x - Ax - f(Ax, Ax(t_h)) - b$, for $\lambda = 1$ it is equal to $\Theta(x, 1) = x - Ax - Pf(Px, Px(t_h)) - b$.

Let us prove that the homotopy is nonzero on $\partial\Omega_n$. If it is not the case, then $\Theta(x, \lambda) = 0$ for some $\lambda \in [0, 1]$ and $x(t) = \xi \sin(t + \varphi) + z(t)$. Therefore, $Q\Theta(x, \lambda) = 0$ and $P\Theta(x, \lambda) = 0$. The first equality again implies the estimates

$$\|z\|_{L^2} \leq R_1, \qquad \|Az\|_{C^1} \leq c \tag{32}$$

where the constants c and R_1 are independent from λ and ξ. With this definition of the constant R_1, we see that $Q\Theta(x, \lambda)$ is nonzero if $\|Qx\|_{L^2} = R_1 + 1$.

Now consider the remaining part of the set $\partial\Omega_n$, which is made of the sets $\{\|Px\|_{L^2} = \xi_n, \ \|Qx\|_{L^2} \leq R_1 + 1\}$ and $\{\|Px\|_{L^2} = \xi^n, \ \|Qx\|_{L^2} \leq R_1 + 1\}$. The equality $P\Theta(x, \lambda) = 0$ can be rewritten as

$$P[f(Px, Px(t_h)) + b] = \lambda P[f(Px, Px(t_h)) - f(Ax, Ax(t_h))]$$

But the last equality is impossible for large n: the left-hand side is uniformly nonzero due to (31), and the right-hand side is arbitrarily small for large ξ.

Now consider the vector field $\Theta(x, 1) = x - Ax - Pf(Px, Px(t_h)) - b$. The rotation $\gamma(\Theta(x, 1), \partial\Omega_n)$ can be calculated with the use of rotation product formula. This formula for our case takes the form

$$\gamma(\Theta(x, 1), \partial\Omega_n) = (-1)^\beta \gamma(Pf(Px, Px(t_h)) - Pb, \partial Z_n),$$

where β is an integer. Here $Z_n = \{\xi_n \leq \|Px\|_{L^2} \leq \xi^n\} \subset \Pi_1$ is an annulus in the plane Π_1, and the value of $\gamma(Pf(Px, Px(t_h)) - Pb, \partial Z_n)$ can be calculated directly. The map $Pf(Px, Px(t_h)) - Pb$ is one-to-one on the boundary ∂Z_n, and the image of this boundary is again the annulus. Simple computation shows that the origin lies in this second annulus, this means that the rotation γ is equal to 1.

7.7 A priori estimate, the case $\bar{b} > K$

This is the most simple part of the proof.

Consider equation (26) and suppose that the set of its solution is unbounded. Then these solutions have the form $\xi_n \sin(t + \varphi) + z_n(t)$ with $\xi_n \to \infty$.

Again $\|Az_n\|_{C^1} \leq c$ and we can apply Lemma 1. Since $Py = PAy$ and

$$Pf(Ay(t), Ay(t_h)) + Pb(t) = 0, \tag{33}$$

we have for $\xi_n \to \infty$:

$$\|Pf(Ay(t), Ay(t_h)) - Pf(Py, Py(t_h))\|_{L^2} \to 0$$

therefore for sufficiently large ξ_n

$$\limsup_{\xi_n \to \infty} \|Pf(Ay(t), Ay(t_h))\|_{L^2} \geq K\sqrt{\pi} < \sqrt{\pi}|\bar{b}| = \|Pb(t)\|.$$

This contradicts to $K < |\bar{b}|$. ■

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