Bifurcation at infinity for equations in the spaces of vector-valued functions*

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Abstract

New result is suggested which allows to obtain new existence conditions of bifurcations at infinity for asymptotically linear equations in spaces of vector-valued functions. Under these conditions an index at infinity can be calculated. The case when bounded nonlinearity has discontinuous principal homogeneous part is considered. Applications are given to 2π-periodic problems for a system of two nonlinear first order ODE and to a vector two-point boundary value problem.

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Contents

1 Asymptotic bifurcation points and a changing index principle

Let in a Banach space E the equation $B(x,\lambda)=0$ be given with some operator $B(x,\lambda)$ depending on a parameter $\lambda\in\Lambda=[a,b]$.

Definition 1. *A value λ_0 of the parameter is called a* bifurcation point at infinity *or (the same) an* asymptotic bifurcation point *if for every $\varepsilon>0$ there exists a $\lambda_\varepsilon\in(\lambda_0-\varepsilon,\lambda_0+\varepsilon)\bigcap\Lambda$ such that for $\lambda=\lambda_\varepsilon$ the equation $B(x,\lambda)=0$ has at least one solution x_ε satisfying $\|x_\varepsilon\|>\varepsilon^{-1}$.*

The notion of the asymptotic bifurcation point was introduced by Mark Krasnosel'skii in the early 50s. He started to study the points by topological methods with the use of so-called *changing index principle* [8, 1]. This principle is applicable for equations $B(x,\lambda)=0$ of the type $x=T(x,\lambda)$ with completely continuous (compact and continuous) in the both variables operator $T(x,\lambda)$. An operator $\Phi x=x-Tx$ is called a vector field, if T is completely continuous then the vector field Φx is also called completely continuous.

Definition 2. *Let completely continuous vector field Φx be defined and nondegenerate for $\|x\|\geq r_0$. Then the rotation (see [8]) of the field on the boundary of every ball $B(r,0)=\{x\in E;\ \|x\|\leq r\}$ is defined and the value of this rotation is common for all $r>r_0$. This common value is called an* index at infinity *of the field Φx and is denoted as $\operatorname{ind}_\infty\Phi$.*

If the value of the index at infinity of the field $\Phi_\lambda x=x-T(x,\lambda)$ is not defined for some $\lambda=\lambda_0$ then this λ_0 is an asymptotic bifurcation point for the equation $x-T(x,\lambda)=0$.

Proposition 1 (Changing index principle). *Consider in the Banach space E some equation $x=T(x,\lambda)$ with the completely continuous with respect to the variables $x\in E,\ \lambda\in[a,b]$ operator $T(x,\lambda)$. Let for two different values of the parameter: λ_1 and λ_2 the indices at infinity of the field $\Phi_\lambda x=x-T(x,\lambda)$ be defined and let*

$$\operatorname{ind}_\infty\Phi_{\lambda_1}\neq\operatorname{ind}_\infty\Phi_{\lambda_2}. \tag{1}$$

Then at least one asymptotic bifurcation point for the equation $x=T(x,\lambda)$ exists on the interval $[\lambda_1,\lambda_2]$.

This statement and its various reformulations are called the *changing index principle.* The reformulations are mainly related with the problem how to find such two values of the parameter with the different indices.

The following variant of the changing index principle is the most widely used.

Let for every λ from some neighborhood of λ_0 the index at infinity is defined of the field Φ_λ. Let for $\lambda<\lambda_0$ this index is constant, denote it as $\operatorname{ind}_\infty\Phi_{\lambda_0-0}$. Let for $\lambda>\lambda_0$ this index is also constant, denote it as $\operatorname{ind}_\infty\Phi_{\lambda_0+0}$.

Proposition 2 ([8]). *Let among the three numbers*

$$\operatorname{ind}_\infty\Phi_{\lambda_0},\quad \operatorname{ind}_\infty\Phi_{\lambda_0-0},\quad \operatorname{ind}_\infty\Phi_{\lambda_0+0}$$

at least two numbers are defined and different. Then λ_0 is an asymptotic bifurcation point for the equation $x = T(x, \lambda)$.

If the index $\operatorname{ind}_\infty \Phi_{\lambda_0}$ is defined and differs from zero, then under assumptions of the last statement multiplicity results are valid for close to λ_0 values of the parameter. For example if $\operatorname{ind}_\infty \Phi_{\lambda_0} = 1$ and $\operatorname{ind}_\infty \Phi_{\lambda_0+0} = -1$ then, generally speaking, for close to λ_0 values $\lambda > \lambda_0$ the equation $x = T(x, \lambda)$ has three solutions: one of the index 1 which is situated in a bounded set for any $\lambda \in [\lambda_0, \lambda_0 + \varepsilon]$, and two branches of solutions of the index -1, tending to infinity for $\lambda \to \lambda_0$.

In applications the most natural situation is where nothing is known about the index $\operatorname{ind}_\infty \Phi_{\lambda_0}$, but other two numbers: $\operatorname{ind}_\infty \Phi_{\lambda_0-0}$ and $\operatorname{ind}_\infty \Phi_{\lambda_0+0}$ are known and different.

The most known theorems on asymptotic bifurcation points are formulated for asymptotically linear equations (the definition is given in the next section). In this case if some value of the parameter is an asymptotic bifurcation point then the kernel of the principal at infinity linear part $x - A_\lambda x$ of the field $x - T(x, \lambda)$ is nontrivial. If the parameter is included in the principal linear part as a multiplier ($A_\lambda x = \lambda A x$) then any eigenvalue μ of the linear operator A of odd multiplicity (e.g. a simple eigenvalue) generates an asymptotic bifurcation point $\lambda = \mu^{-1}$.

The paper is organized as following. In the next section we give the definition of asymptotically linear vector field and the theorem on the index calculation for the case of non-degenerate asymptotically linear completely continuous vector field. In Section 3 the notion is introduced of asymptotically homogeneous nonlinearities. For vector fields in abstract Banach spaces with degenerate linear part and continuous non-degenerate asymptotically homogeneous nonlinearity the theorem is given on the index at infinity calculation. Section 4 contains a new theorem on asymptotic homogeneity of superposition operators, which is proved in Section 5. The corresponding theorem on the index calculation for fields in spaces of vector-valued functions which have degenerate linear part and discontinuous non-degenerate asymptotically homogeneous nonlinearity can be found in Section 6. Such type nonlinearities appear naturally in applications. Examples and remarks are given in section 7-9.

2 Index at infinity of asymptotically linear non-degenerate vector field

A vector field $\Phi(x)$ is called *linear* if it can be represented as $\Phi x = x - Ax$, where A is a linear operator. A linear vector field is always zero at $x = 0$. Except this singular point a linear vector field either has no other singular points at all (if 1 does not belong to the spectrum of the operator A), or it degenerates on a non-trivial subspace (if 1 belongs to the spectrum of A).

If 1 is a regular value for a completely continuous linear operator A, then 0 is an isolated (and as a matter of fact the unique) singular point of the vector field $\Phi x = x - Ax$. Its index coincides with the index of the vector field Φx at infinity. The rotation of this vector field on the boundary of a given domain $\mathcal{D}$ either is equal to zero, if $0 \notin \mathcal{D}$, or it coincides with the index of zero, if $0 \in \mathcal{D}$.

Proposition 3. *The equality*

$$\operatorname{ind}_\infty \Phi = (-1)^\beta \tag{2}$$

holds where β denotes the sum of multiplicites of all real eigenvalues of A which are greater than 1.

A proof of this assertion see, for instance, in [8].

Definition 3. *A vector field $\Phi x = x - Tx$ and the operator T are called* assymptotically linear *if the operator T admits the representation $Tx = Ax + Fx$ where A is a linear operator and an operator F satisfies the condition*

$$\lim_{\|x\| \to \infty} \frac{\|Fx\|}{\|x\|} = 0.$$

The operator A is called the asymptotical derivative *of an asymptotically linear operator T or the* derivative of T at infinity*. A linear vector field $x - Ax$ is called the* main linear part *of the vector field $x - Tx$. The main linear part is said to be* non-degenerate *if 1 does not belong to the spectrum of the operator A, and is said to be* degenerate *otherwise.*

Asymptotical derivatives of completely continuous operators are always completely continuous [8].

The following theorem (by Leray-Schauder) holds by virtue of theorems on calculating of rotation of a vector field in terms of its main part.

Proposition 4 ([8, 1]). *Let a vector field $\Phi x = x - Tx$ be asymptotically linear with the non-degenerate main linear part $x - Ax$. Then the index of the vector field Φ at infinity is defined and*

$$\operatorname{ind}_\infty \Phi = (-1)^\beta,$$

where β denotes the sum of multiplicites of real eigenvalues of A which are greater than 1.

The main part of the present paper is devoted to the index at infinity calculation of asymptotically linear vector fields with a degenerate main linear part. In the pioneering papers [9, 10] the conditions were presented which allow to calculate the index at infinity of Hammerstein type vector fields $\Phi x = x - A(x + f(x))$ with bounded scalar nonlinearities $f(x) : \Omega \to I\!R^1$. These conditions have the form

$$\lim_{x \to +\infty} f(x) = f^+, \qquad \lim_{x \to -\infty} f(x) = f^-.$$

An extensive literature is devoted to investigate concrete boundary value problems with such nonlinearities, see [2] and references therein.

A generalization of the results mentioned above to vector-valued functions have been carried out in [3]. This paper contains some results concerning the index at infinity calculation of vector fields with a degenerate main linear part and with a non-degenerate next order term. These results use heavily the continuity of the next order (after linear) non-degenerate term of the vector field.

3 Asymptotically homogeneous vector fields

The results of this section on vector fields in Banach spaces have been announced in [4] and proved in [5].

Definition 4. *A nonlinear operator Q in the Banach space E is said to be* homogeneous *or more precisely* homogeneous of degree 0 *if*

$$Q(x) = Q(\lambda x), \qquad \lambda > 0, x \in E.$$

Any constant vector field is homogeneous by definition. Linear combinations of homogeneous vector fields are also homogeneous. Only functions of the form

$$q(x) = \begin{cases} q^-, & x < 0, \\ q^0, & x = 0, \\ q^+, & x > 0 \end{cases} \tag{3}$$

are homogeneous for the case of one-dimensional E.

A homogeneous nonlinearity is determined by its values on the unit sphere and at coordinate origin.

If A is a linear operator and Q is a homogeneous one then the operator QA is homogeneous; in fact if F is an arbitrary operator, then FQ is homogeneous.

In functional spaces (of, for instance, scalar-valued functions defined on a given set $\Omega \subset I\!\!R^m$) a superposition operator $x(t) \mapsto q(t, x(t))$ is homogeneous, if it is generated by a homogeneous function $q(t, x)$ which admits at each t a representation (3), that is

$$q(t,x) = \begin{cases} q^-(t), & x < 0, \\ q^0(t), & x = 0, \\ q^+(t), & x > 0. \end{cases} \tag{4}$$

In spaces of vector functions $\Omega \to I\!\!R^n$ examples of homogeneous nonlinearities can be given by the superposition operators $x(t) \mapsto f(t, x(t))$ generated by functions $f(t,x) = C(t)x/|x|$ where $C(t)$ is a $n \times n$ matrix and $|\cdot|$ is a given norm in $I\!\!R^n$. Functions $f(t,x)$ of the form $f(t, \operatorname{sign} x_1, ..., \operatorname{sign} x_n)$ also generate a homogeneous superposition operator.

If a homogeneous operator is not constant, then it must be discontinuous at zero. Moreover such operators can have others discontinuity points. A natural example of discontinuous homogeneous operator on the plane $\{x_1, x_2\}$ is given by the superposition operator $\{x_1, x_2\} \mapsto \{\operatorname{sign} x_1, 0\}$. This operator is discontinuous not only at zero but also on the whole straight line $x_1 = 0$.

The superposition operator $Qx(t) = q(t, x(t))$ generated by the function (4) is also discontinuous in functional spaces. If, for instance, $q(t,x) = q(x)$ and $q^- \neq q^+$ then discontinuity points of the operator Q are dense in the spaces L^p. The totality of these points is also dense in C outside of the sets of strictly positive or strictly negative functions. Nevertheless this operator has an amount of points of continuity and just these points are often enough for applications. For criteria of continuity of a superposition operator with discontinuous characteristics at a given point in spaces of integrable functions see in [7].

In the spaces L^∞ or C the operator Q can be discontinuous even on "very good" functions $x_0(t)$, which are equal to zero at a single point. For instance the operator $x(t) \mapsto \operatorname{sign} x(t)$, $t \in [0,1]$ is discontinuous at the function $x^*(t) = t - \dfrac{1}{2}$. Fortunately, superposition operators Q are often combined with linear integral operators A which possess some substantional improvability properties. For instance, the operator $x(t) \mapsto \operatorname{sign} x(t)$ is continuous at the function $x^*(t)$ as an operator from the space L^∞ to the space L^2 providing that the function $x(t)$ vanishes only at the set of zero Lebesgue measure, whereas a corresponding linear operator A is often continuous as an operator acting from the space L^2 back to the space L^∞. As a result the operator AQ is continuous in L^∞ at all points $x_0 = x_0(t)$ satisfying the condition

$$\operatorname{mes}\{t \in \Omega:\ x_0(t) = 0\} = 0.$$

Let in a Banach space E be chosen a finite dimensional subspace E_1 and a fixed projector P_1 on this subspace: $PE = E_1$, $P_1^2 = P_1$.

Definition 5 ([4]). *An operator F is said to be* asymptotically homogeneous *in the space E (with respect to the subspace E_1 and the projector P_1) if it can be represented as the sum $F = Q + B$ where the operator Q is homogeneous and the operator B satisfies the following condition of "vanishing at infinity": for each $c > 0$ the equality holds:*

$$\lim_{R\to+\infty} \sup_{e_1\in E_1,\ \|e_1\|=1,\ h\in E,\ \|h\|<c} \|P_1 B(Re_1 + h)\| = 0. \tag{5}$$

The main example of asymptotically homogeneous operator in a functional space is given by the superposition operator $f(t,x) = q(t,x) + \psi(t,x)$ where the function $q(t,x)$ is homogeneous and $\psi(t,x)$ satisfies the condition

$$\lim_{|x|\to\infty} \sup_{t\in\Omega} |\psi(t,x)| = 0. \tag{6}$$

The equality

$$\lim_{R\to+\infty} \sup_{e_1\in E_1,\ \|e_1\|=1,\ h\in E,\ \|h\|_{L^1}<c} \|\psi(Re_1 + h)\|_{L^1} = 0$$

which is stronger than (5) can be proved for such operators. The corresponding operator is asymptotically homogeneous with respect to an arbitrary projector on the subspace E_1.

If $f(t,x)$ satisfies Caratheodory condition and $q(t,x)$ is discontinuous in x at some points of S, then (6) is never valid.

Let us come back to the index calculation of the completely continuous asymptotically linear vector field $\Phi x = x - Ax - Fx$ with the degenerate linear part $x - Ax$. Let $E_1 = \operatorname{Ker}(I - A)$ and let P_1 be the projector on E_1 which commutes with A.

Theorem 1 ([5]). *Let the operator F be asymptotically homogeneous: $F = Q+B$, where Q is homogeneous and B satisfies condition (5) with the finite dimensional subspace E_1 and the projector P_1, to be defined by the linear operator A. Suppose* *that the finite dimensional vector field P_1Qe on the sphere $U = \{e \in E_1, \ \|e\| = 1\}$ is non-degenerated: $P_1Qe \neq 0, \ e \in U$ and that the operator $P_1Qx : E \to E_1$ is continuous at each point of U. Then the index* $\operatorname{ind}_\infty \Phi$ *is defined and*

$$\operatorname{ind}_\infty \Phi = (-1)^\beta \gamma(P_1Q, U),$$

where $\gamma(P_1Q, U)$ denotes the rotation of the vector field P_1Q on the sphere U in the finite dimensional subspace E_1.

In applications the subspace E_1 is often one- or two-dimensional and the rotation $\gamma(P_1Q, U)$ can be calculated in an explicit form.

4 Asymptotic homogeneity of superposition operator in spaces of vector functions

Let Ω be a closed bounded domain in a finite dimensional space, for instance $\Omega = [0, 1]$ or Ω is a square or a circle in a plane. We will consider operators, vector fields and equations in spaces E of functions $x(t) : \Omega \to I\!\!R^n$. Denote by $\langle \cdot, \cdot \rangle$ the scalar product in the space $I\!\!R^n$ and by $|\cdot|$ the corresponding norm.

Consider an arbitrary finite dimensional subspace $E_1 \subset E$ of continuous on Ω vector-valued functions and denote $U = \{e(t) : e(t) \in E_1, \|e\| = 1\}$. Suppose that each non-zero function $e(t) \in E_1$ satisfies the condition

$$\operatorname{mes}\{t \in \Omega : \ e(t) = 0\} = 0. \tag{7}$$

This condition was used in various publications (see again [2] and the references therein).

Let us fix a closed set $\Delta \subset S$ on the unit sphere $S = \{x \in I\!\!R^n : |x| = 1\} \subset I\!\!R^n$. Generally speaking in applications this set is "small": it has the co-dimension $n - 2$.

Let $u \in S$. Denote by $\rho(u, \Delta)$ the distance between a point u of the sphere and the set Δ. For each function $e(t) \in E_1$ introduce the notation

$$\chi(\delta, \Delta, e) = \operatorname{mes}\{t \in \Omega : \ \rho(\frac{e(t)}{|e(t)|}, \Delta) \leq \delta\}.$$

The main assumption in the theorem formulated below on asymptotic homogeneity of the superposition operator $x(t) \mapsto f(t, x(t))$ is the following: there exist a set Δ such that

1. The limit

$$\lim_{R \to +\infty} f(t, Ru) = q(t, u) \tag{8}$$

exists for each $u \in S, \ u \notin \Delta$. The limit function $q(t, u)$ satisfies the Caratheodory condition for $u \notin \Delta$: it is continuous in u and measurable in t. The limit in (8) is supposed to be uniform in $t \in \Omega$ and in u belonging to any given closed subset of S which is disjoint with Δ.

2. The equality

$$\chi(0, \Delta, e) = 0. \tag{9}$$

holds for each function $e(t) \in E_1$.

The assumption 1 can be reformulated as follows:

1*. The equality

$$\lim_{R\to+\infty} \sup_{t\in\Omega,\, u\in\Delta_*} |f(t,Ru) - q(t,u)| = 0 \tag{10}$$

holds for each $\Delta_* \in S$ such that $\overline{\Delta}_* \bigcap \Delta = \emptyset$.

Equality (7) together with the main assumption guarantee that the operator

$$Qx(t) \begin{cases} q(t, \dfrac{x(t)}{|x(t)|}), & x(t) \neq 0, \\ 0, & x(t) = 0 \end{cases} \tag{11}$$

is continuous as an operator in L^1 (and in others L^p for $p < \infty$) at every point of U (see [7]). The compactness of U guarantees the uniform continuity of this operator on U.

Let us suppose also that the functions $f(t,x)$ and $q(t,u)$ are both uniformly bounded.

Theorem 2. *The operator $x(t) \mapsto f(t,x(t))$ is asymptotically homogeneous in the space $E = L^2 = L^2(\Omega, \mathbb{R}^n)$ under the listed above assumptions.*

This theorem was proved in [3] in another terminology for the case $\Delta = \emptyset$. The closure G of the totality of discontinuity points of the function $q(t,u)$ may play the role of the set Δ. Theorem 2 can be generalized to the case when the set G varies in t. Note also that all what is said in this section is interesting only for vector-valued functions. For scalar functions the sphere S consists only from two points and the question about the continuity of the corresponding functions does not appear: $q(t,u)$ must be continuous at the both two points.

An example when the condition (9) does not hold is given in Section 9.

5 Proof of Theorem 2

Lemma 1. *The equality*

$$\lim_{\delta\to 0} \chi(\delta, \Delta, E_1) = 0 \tag{12}$$

holds where

$$\chi(\delta, \Delta, E_1) \stackrel{\text{def}}{=} \sup_{e(t)\in U} \chi(\delta, \Delta, e). \tag{13}$$

Let us suppose the contrary. Then there exists a number $\varepsilon > 0$ and a sequence of functions $e_n(t) \in U$ satisfying the inequalities

$$\chi(\frac{1}{n}, \Delta, e_n) > \varepsilon,$$

or, what is the same,

$$\text{mes}\,\{t \in \Omega :\ \rho(\frac{e_n(t)}{|e_n(t)|}, \Delta) \leq \frac{1}{n}\} > \varepsilon.$$

Without loss of generality we can suppose that the sequence $e_n(t)$ converges uniformly (E_1 is finite dimensional and hence all the norms in E_1 are equivalent) to a function $e^*(t) \in U$. The continuity of measure, condition (7) and again the finite dimensionality of the subspace E_1 imply together the equality

$$\lim_{\delta \to 0} \sup_{e(t) \in U} \operatorname{mes} \{t \in \Omega : \, |e(t)| \le \delta\} = 0. \tag{14}$$

Therefore

$$\operatorname{mes} \{t \in \Omega : \, \rho(\frac{e_n(t)}{|e_n(t)|}, \Delta) \le \frac{1}{n}, \; |e_n(t)|, |e^*(t)| > \delta_0\} > \frac{\varepsilon}{2} \tag{15}$$

for all sufficiently large n and some fixed δ_0. The inequality (15) contradicts (9) with $e = e^*$ because the value

$$\delta_n = \sup_{|e_n(t)|, |e^*(t)| > \delta_0} \left| \frac{e_*(t)}{|e_*(t)|} - \frac{e_n(t)}{|e_n(t)|} \right|$$

tends to zero as $n \to \infty$ and

$$\operatorname{mes} \{t \in \Omega : \, \rho(\frac{e_n(t)}{|e_n(t)|}, \Delta) \le \frac{1}{n}, \; |e_n(t)|, |e^*(t)| > \delta_0\} \le$$

$$\le \operatorname{mes} \{t \in \Omega : \, \rho(\frac{e_*(t)}{|e_*(t)|}, \Delta) \le \frac{1}{n} + \delta_n\} = \chi(\frac{1}{n} + \delta_n, \Delta, e^*) \to 0.$$

The lemma is proved. □

Let us complete now the proof of Theorem 2.

To this end we will prove the equality

$$\lim_{R \to +\infty} \sup_{e \in U, \; \|h\|_{L_1} \le c} \|f(t, Re + h) - q(t, \frac{e(t)}{|e(t)|})\|_{L^p} = 0, \tag{16}$$

where $p \in [1, \infty)$, which is stronger than (5) with $E = L^2$. This equality for $p > 1$ follows from the same equality at $p = 1$ by virtue of the uniform boundedness of the functions $f(t, x)$ and $q(t, u)$.

It remains to estimate the value

$$J = \int_\Omega |f(t, Re(t) + h(t)) - q(t, \frac{e(t)}{|e(t)|})| \, dt.$$

Let us chose an arbitrary $\varepsilon > 0$ and prove that the estimate $J < \varepsilon$ holds for all sufficiently large R.

By Chebyshev inequality

$$\operatorname{mes} \{t \in \Omega : \, |h(t)| > \mu\} \le \frac{\|h\|_{L^1}}{\mu}$$

and boundedness of the functions $f(t, x)$ and $q(t, x)$ the inequality

$$\int_{\{t:\,|h(t)|>\mu\}} |f(t, Re(t) + h(t)) - q(t, \frac{e(t)}{|e(t)|})|\, dt \le \frac{\varepsilon}{5}$$

is valid for some sufficiently large μ and all R.

By virtue of (14) and the boundedness of the functions $f(t, x)$ and $q(t, x)$, the analogous inequality

$$\int_{\{t:\,|e(t)|\le\delta\}} |f(t, Re(t) + h(t)) - q(t, \frac{e(t)}{|e(t)|})|\, dt \le \frac{\varepsilon}{5}$$

is valid for sufficiently small δ for all R. Let us surround the set Δ on the sphere S with a sufficiently small neighbourhood $N = \{u : \rho(u, \Delta) < \eta\}$. By Lemma 1 the point $e(t)/|e(t)|$ belongs to this neighbourhood N at t from the set $G(e, \eta)$ which has arbitrarily small measure uniformly with respect to all $e(t) \in U$. Let us fix now a neighbourhood N satisfying the inequality

$$\int_{G(e,\eta)} |f(t, Re(t) + h(t)) - q(t, \frac{e(t)}{|e(t)|})|\, dt < \frac{\varepsilon}{5}.$$

Below the values μ, δ and the set $G = G(e, \eta)$ are supposed to be fixed. Denote

$$\Omega^* \overset{\text{def}}{=} \{t \in \Omega : \, |h(t)| \le \mu, \, |e(t)| > \delta, \, t \notin G(e, \eta)\}.$$

The inequality $J < \varepsilon$ for large R will be proved if we show that

$$J_1 = \int_{\Omega^*} \left| q(t, \frac{Re(t) + h(t)}{|Re(t) + h(t)|}) - q(t, \frac{e(t)}{|e(t)|}) \right| dt$$

and

$$J_2 = \int_{\Omega^*} \left| f(t, Re(t) + h(t)) - q(t, \frac{Re(t) + h(t)}{|Re(t) + h(t)|}) \right| dt$$

satisfy the estimates

$$J_1, \; J_2 \le \frac{\varepsilon}{5}$$

for all sufficiently large R. To prove it, note that for large R and for $t \in \Omega^*$ the value

$$\left| \frac{e(t)}{|e(t)|} - \frac{Re(t) + h(t)}{|Re(t) + h(t)|} \right|$$

can be made arbitrarily small uniformly with respect to e, h and t. Therefore we can suppose without loss of generality that the both values

$$\frac{e(t)}{|e(t)|}, \qquad \frac{Re(t) + h(t)}{|Re(t) + h(t)|}$$

are uniformly separated from the set Δ for all sufficiently large R for $t \in \Omega^*$.

Hence J_2 tends to zero as $R \to \infty$ by the assumption (8) ($|Re(t) + h(t)| \to \infty$ uniformly), and J_1 tends to zero by virtue of the uniform continuity of superposition operator (11). $\square$

6 Theorem on index

In this section we will use again the space $L^2 = L^2(\Omega, I\!R^n)$ of integrable with the square functions $x(t): \ \Omega \to I\!R^n$ with the usual norm $\|\cdot\|$ generated by the scalar product in $\langle \cdot, \cdot \rangle$ in $I\!R^n$:

$$\|\cdot\| = \sqrt{(\cdot,\cdot)}, \quad (x, y) = \int_\Omega \langle x(t), y(t) \rangle \, dt.$$

Denote by $A: \ L^2 \to L^2$ a linear completely continuous operator. Let us suppose that a bounded function $f(t,x): \ \Omega \times I\!R^n \to I\!R^n$ satisfies the Caratheodory condition. Consider in L^2 the completely continuous vector field

$$\Phi x = x - A(x + f(t,x)). \tag{17}$$

This field is asymptotically linear and its asymptotic derivative is equal to $I - A$.

If $1 \notin \sigma(A)$ (where $\sigma(A)$ is the spectrum of the operator A), then $\operatorname{ind}_\infty \Phi = (-1)^\beta$, where β denotes the sum of multiplicities of all real eigenvalues of the operator A which are greater than 1.

If $1 \in \sigma(A)$ then the asymptotic derivative $I - A$ is degenerate and to compute the index one has to use some properties of the nonlinearity $f(t,x)$.

Denote $E_1 = \operatorname{Ker}(I - A)$ and suppose that $E_1 = \{e(t): \ Ae = e\}$ holds. The last assumption means that the eigenvalue 1 of A has not generalized eigenvectors. Denote by P_1 a projector on E_1 which commutes with A.

The projector P_1 can be constructed as follows. Denote by $e_1, \ , \ldots, e_m$ ($m = \dim E_1$) a basis in the finite dimensional space E_1 and denote by $g_1, \ , \ldots, g_m$ a basis in the finite dimensional space $E_1^* = \operatorname{Ker}(I - A^*) \subset L^2$, which satisfies the condition

$$\int_\Omega \langle e_i(t), g_j(t) \rangle \, dt = \delta_{ij},$$

where δ_{ij} is the Kronecker symbol. Then the projector P_1 can be defined as

$$P_1 x(\cdot) = \sum_{i=1}^m e_i(\cdot) \int_\Omega \langle g_i(t), x(t) \rangle \, dt.$$

Theorem 3. *Let a bounded nonlinearity $f(t,x)$ satisfy the conditions of Theorem 2 for some set Δ and function $q(t,u)$. Let the vector field $\Psi e = P_1 q(t, e(t)/|e(t)|)$ is non-degenerate on U. Then*

$$\operatorname{ind}_\infty \Phi = (-1)^\beta \gamma(\Psi, U).$$

Theorem 3 follows immediately from Theorems 1 and 2.

7 Example 1

Consider 2-dimensional system

$$\left\{ \begin{array}{l} x_1' = \quad x_2 + \arctan(x_1) + b_1(t,\lambda), \\ x_2' = -x_1 + \arctan(x_2) + b_2(t,\lambda) \end{array} \right. \tag{18}$$

or (what is the same) the equation

$$\mathbf{x}' = A\mathbf{x} + \mathbf{f}(\mathbf{x}) + \mathbf{b}(t, \lambda),$$

where

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathbf{x} = \{x_1, x_2\}, \quad \mathbf{f}(\mathbf{x}) = \{\arctan(x_1), \arctan(x_2)\},$$

$$\mathbf{b}(t, \lambda) = \{b_1(t, \lambda), b_2(t, \lambda)\}.$$

Functions $b_j(t, \lambda)$ are 2π-periodic in t and continuous with respect to both variables.

We study the existence of 2π-periodic solutions of this system and its asymptotic bifurcation points.

The problem is that the linear part $\mathbf{x}' - A\mathbf{x}$ is degenerate for 2π-periodic problem: the equation $\mathbf{x}' = A\mathbf{x}$ has two-dimensional subspace E_1 of 2π-periodic solutions. This subspace has the orthonormed basis $\mathbf{e}_1(t), \mathbf{e}_2(t)$ where

$$\mathbf{e}_1(t) = \frac{1}{\sqrt{2\pi}}\{\sin t, \cos t\}, \quad \mathbf{e}_2(t) = \frac{1}{\sqrt{2\pi}}\{\cos t, -\sin t\}.$$

Consider the function

$$\varphi(\lambda) = \left| \int_0^{2\pi} (b_2(t, \lambda) + ib_1(t, \lambda))e^{-it}\, dt \right| - 8.$$

Here $|\cdot|$ is the usual modulus of a complex number.

Theorem 3. *If $\varphi(\lambda) < 0$ then for this value of λ system* (18) *has at least one 2π-periodic solution.*

Theorem 4. *Let $\varphi(\lambda_0) = 0$ and let in any neighbourhood of the point λ_0 the function $\varphi(\lambda)$ take values of the both signs. Then λ_0 is an asymptotic bifurcation point for system* (18).

Consider the space L^2 of the vector functions $\mathbf{x}(t) : [0, 2\pi] \to I\!R^2$ with the usual scalar product denoted as $(\cdot, \cdot)$. Consider the completely continuous operator $\mathbf{y} = \mathbf{A}\mathbf{x}$ which put into correspondence to any $\mathbf{x} \in L^2$ the 2π-periodic solution $\mathbf{y}(t)$ of the linear equation $\mathbf{y}' - A\mathbf{y} + \mathbf{y} = \mathbf{x}$. In other words, the operator $\mathbf{A}$ is the inverse operator for differential operator $\mathbf{y} \mapsto \mathbf{y}' - A\mathbf{y} + \mathbf{y}$ with 2π-periodic boundary conditions. The operator $\mathbf{A}$ is completely continuous in L^2. This operator $\mathbf{A}$ exists since the spectrum of the differential operator is separated from zero. The value 1 belongs to the spectrum $\sigma(\mathbf{A})$ of the operator $\mathbf{A}$, the subspace E_1 corresponds to the eigenvalue 1. The operator equation

$$\mathbf{x} = \mathbf{A}(\mathbf{x} + \mathbf{f}(\mathbf{x}) + \mathbf{b}(t, \lambda))$$

is equivalent in a natural sense to 2π-periodic problem for system (18).

Then consider in L^2 the completely continuous vector field

$$\Phi_\lambda \mathbf{x} = \mathbf{x} - \mathbf{A}(\mathbf{x} + \mathbf{f}(\mathbf{x}) + \mathbf{b}(t, \lambda)).$$

We want to prove that if $\varphi(\lambda) \neq 0$ then the index at infinity of this vector field is defined and

- if $\varphi(\lambda) < 0$ then $\operatorname{ind}_\infty \Phi_\lambda = (-1)^{\cdots}$,
- if $\varphi(\lambda) > 0$ then $\operatorname{ind}_\infty \Phi_\lambda = 0$.

This proves both Theorems 3 and 4.

To calculate the index for the case considered we use Theorem 2. Put $\mathbf{q}(t, \mathbf{x}, \lambda) = \{\operatorname{sign} x_1, \operatorname{sign} x_2\} + \mathbf{b}(t, \lambda)$ and let Δ consists from 4 points: $u_1 = 0$, $u_2 = \pm 1$ and $u_1 = \pm 1$, $u_2 = 0$. Obviously, all the conditions of Theorem 2 are fulfilled and to calculate $\operatorname{ind}_\infty \Phi_\lambda$ it is only necessary to calculate the rotation $\gamma(\lambda)$ of the field $P_1\mathbf{q}(t, \mathbf{x}, \lambda)$ on U.

The operator P_1 has the form $P_1\mathbf{x} = (\mathbf{e}_1, \mathbf{x})\mathbf{e}_1(t) + (\mathbf{e}_2, \mathbf{x})\mathbf{e}_2(t)$. Let us parameterize the circle $U \in E_1$ as $U = \{\mathbf{e}_\psi(t) = \cos\psi\, \mathbf{e}_1(t) + \sin\psi\, \mathbf{e}_2(t)\}$ ($\psi \in [0, 2\pi]$) and calculate $\Psi_\lambda(\mathbf{e}_\psi) = P_1\mathbf{q}(t, \mathbf{e}_\psi(t), \lambda)$. After rather simple (but cumbersome) computations we get that

$$\Psi_\lambda(\mathbf{e}_\psi) = \frac{8}{\sqrt{2\pi}}\mathbf{e}_\psi(t) + P_1\mathbf{b}(t, \lambda).$$

It means that Ψ_λ is one-to-one mapping of the circle U to the circle U_λ with the radius $8/\sqrt{2\pi}$ and with the center in the point $P_1\mathbf{b}(t, \lambda)$. If $\varphi(\lambda) < 0$ then the origin is surrounded by U_λ and $\operatorname{ind}_\infty \Phi_\lambda = (-1)^{\cdots}$, if $\varphi(\lambda) > 0$ then the origin is not surrounded by U_λ and $\operatorname{ind}_\infty \Phi_\lambda = 0$. $\square$

More details on the rotation computation of planar vector fields see [6].

8 Example 2

Consider the two-point boundary value problem

$$\begin{cases} x_1'' - 4x_1 + 5x_2 = \arctan(x_1 + 2x_2) + b_1(t, \lambda), \\ x_2'' - 2x_1 + 3x_2 = \arctan(2x_1 - x_2) + b_2(t, \lambda), \\ x_1(0) = x_2(0) = x_1(\pi) = x_2(\pi) = 0 \end{cases} \tag{19}$$

or (what is the same) the equation

$$\mathbf{x}'' = A\mathbf{x} + \mathbf{f}(\mathbf{x}) + \mathbf{b}(t, \lambda), \qquad \mathbf{x}(0) = \mathbf{x}(\pi) = 0.$$

Here

$$A = \begin{pmatrix} 4 & -5 \\ 2 & -3 \end{pmatrix}, \quad \mathbf{x} = \{x_1, x_2\}, \quad \mathbf{f}(\mathbf{x}) = \{\arctan(x_1 + 2x_2), \arctan(2x_1 - x_2)\},$$

$$\mathbf{b}(t, \lambda) = \{b_1(t, \lambda), b_2(t, \lambda)\}.$$

Functions $b_j(t, \lambda)$ are continuous with respect to the both variables.

Theorem 5. *Let for some λ the function*

$$\varphi(\lambda) = \left| \int_0^\pi (b_1(t, \lambda) + b_2(t, \lambda)) \sin t\, dt \right| - 4 \tag{20}$$

be strictly negative. Then system (19) *has for this* λ *at least one solution.*

Theorem 6. *Let for some* $\lambda = \lambda_0$ *function* (20) *be equal zero. Let in any neighborhood of* λ_0 *this function takes values of the both signs. Then this* λ_0 *is an asymptotic bifurcation point for system* (19).

The differential operator $\mathbf{x}'' - \mathbf{A}\mathbf{x}$ has non-trivial one-dimensional kernel

$$E_1 = \{\alpha \mathbf{e}(t),\, \alpha \in I\!R^1\}, \qquad \mathbf{e}(t) = \frac{1}{\sqrt{\pi}} \sin t\{1,1\}.$$

Put $\mathbf{A} = (\mathbf{x}'' - A\mathbf{x} + \mathbf{x})^{-1}$ with the boundary conditions $\mathbf{x}(0) = \mathbf{x}(\pi) = 0$. Then system (19) is equivalent to the operator equation $\mathbf{x} = \mathbf{A}(\mathbf{x}+\mathbf{f}(\mathbf{x})+\mathbf{b}(t,\lambda))$. Consider the vector field $\Phi_\lambda \mathbf{x} = \mathbf{x} - \mathbf{A}(\mathbf{x} + \mathbf{f}(\mathbf{x}) + \mathbf{b}(t,\lambda))$.

Put

$$\mathbf{q}(t,\mathbf{x},\lambda) = \{\,\mathrm{sign}\,(x_1 + 2x_2),\ \mathrm{sign}\,(2x_1 - x_2)\} + \mathbf{b}(t,\lambda)$$

and

$$\Delta = S \setminus (S_1 \cup S_2)$$

where

$$S_1 = \left\{ \{x_1,x_2\} \in S,\ (x_1 - \frac{\sqrt{2}}{2})^2 + (x_2 - \frac{\sqrt{2}}{2})^2 < \varepsilon \right\}$$

and

$$S_2 = \left\{ \{x_1,x_2\} \in S,\ (x_1 + \frac{\sqrt{2}}{2})^2 + (x_2 + \frac{\sqrt{2}}{2})^2 < \varepsilon \right\}$$

for some $\varepsilon \in \left(0, \dfrac{\sqrt{3}-1}{\sqrt{2}}\right)$.

After easy computations we have

$$P_1\mathbf{x} = (\mathbf{e},\mathbf{x})\mathbf{e}, \quad \Psi_\lambda(\pm\mathbf{e}) = P_1\mathbf{q}(t,\pm\mathbf{e},\lambda)) = s^\lambda_\pm \mathbf{e},$$

$$s^\lambda_+ = \frac{1}{\sqrt{\pi}} \int_0^\pi \sin t \Big(\,\mathrm{sign}\,(3\sin t) + \,\mathrm{sign}\,(\sin t) + b_1(t,\lambda) + b_2(t,\lambda)\Big)\, dt$$

$$= \frac{1}{\sqrt{\pi}} \int_0^\pi \sin t \Big(2 + b_1(t,\lambda) + b_2(t,\lambda)\Big)\, dt = \frac{1}{\sqrt{\pi}} \left(\int_0^\pi \sin t \Big(b_1(t,\lambda) + b_2(t,\lambda)\Big)\, dt + 4 \right)$$

and

$$s^\lambda_- = \frac{1}{\sqrt{\pi}} \left(\int_0^\pi \sin t \Big(b_1(t,\lambda) + b_2(t,\lambda)\Big)\, dt - 4 \right).$$

Therefore

- if $\varphi(\lambda) > 0$ then $s^\lambda_+ \cdot s^\lambda_- > 0$ and $\mathrm{ind}_\infty\, \Phi_\lambda = 0$,
- if $\varphi(\lambda) < 0$ then $s^\lambda_+ \cdot s^\lambda_- < 0$ and $\mathrm{ind}_\infty\, \Phi_\lambda = (-1)^{\cdots}$.

This proves both theorems of this section. $\square$

9 Remarks

1. Without any changes theorems 3 – 6 can be rewritten for nonlinearities $\mathbf{f}(\mathbf{x}) + \theta(t, \mathbf{x}, \lambda)$ with arbitrary Caratheodorian $\theta(t, \mathbf{x}, \lambda)$ satisfying

$$\lim_{|\mathbf{x}|\to\infty} \sup_{t\in\Omega,\, \lambda\in\Lambda} |\theta(t, \mathbf{x}, \lambda)| = 0.$$

2. For the system

$$\begin{cases} x_1'' - 4x_1 + 5x_2 = \arctan(x_1 + 2x_2) + b_1(t, \lambda), \\ x_2'' - 2x_1 + 3x_2 = \arctan(x_1 - x_2) + b_2(t, \lambda), \\ x_1(0) = x_2(0) = x_1(\pi) = x_2(\pi) = 0 \end{cases}$$

Theorem 2 is inapplicable: $\mathbf{q}(t, \mathbf{x}, \lambda)$ contains the term $\operatorname{sign}(x_1 - x_2)$ which is discontinuous at the point $\mathbf{e}(t)$.

3. The function $\mathbf{b}(t, \lambda)$ can be only integrable in t, not continuous. But the continuity in λ is essential.

4. With the use of Theorem 2 solvability results given in [11] can be generalized.

5. Analogues of Theorem 2 and 3 can be formulated for the space L^p $(p \neq 2)$. Such analogues can be used to study nonlinear degenerate elliptic PDE.

6. We used the function $q(t, x)$ defined on $\Omega \times S$. One can suppose this function to be defined on $\Omega \times I\!\!R^n$: $q(t, x) = q(t, \dfrac{x}{|x|})$ or $q(t, x) = 0$ if $x = 0$.

7. It is possible to consider various continuous measures on Ω.

8. In the proofs of Theorems 3 – 6 we did not calculate the exponent in the formula $\operatorname{ind}_\infty \Phi_\lambda = (-1)^{\cdots}$ for $\varphi(\lambda) < 0$. This exponent depend on the spectrum $\sigma(\mathbf{A})$ and can be easely calculated.

9. Naturally, the closure of the set of discontinuity points for the function $q(t, u)$ may be chosen as the set Δ. But it can appear a situation when the essential part of the sphere S is not covered by the points $u = e(t)/|e(t)|$ for $e \in E_1$ and $t \in \Omega$. Such a situation is natural, for instance, for one-dimensional sets Ω and E_1 (and of course $n > 2$). In this case the set of points u is a one-dimensional submanifold of the sphere S, which is a manifold of the dimension $n - 1 > 1$. In this case there is no need to assume that condition (8) holds “almost entire” on the sphere S: it suffices to asume that it holds in a neighbourhood of the corresponding one-dimensional submanifold.

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