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MATHEMATICS

Polynomial Criteria for the ν-Sufficiency of Jets in Classes of Finitely Smooth Mappings

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Many problems of nonlinear analysis can be reduced to studying the structure of the solution set of the underdetermined nonlinear equations

$$f(x) = 0, \tag{1}$$

where $f: \mathbb{R}^n \to \mathbb{R}^m$ for $n \geq m$ and $f(0) = 0$. Unfortunately, solutions of nonlinear equations are often hard to study even in small neighborhoods of zero, and the equations must be simplified somehow. One of the most popular simplification methods is the so-called truncation of Eq. (1). Formally, for a C^k-smooth mapping f, its r-truncation (where $r \leq k$) is defined as the interval $f^{(r)}(x)$ of its Taylor expansion at zero in which only terms of degrees at most r are retained. The passage to truncated equations is similar to the analysis of stability in the first approximation, studying bifurcations by passing to linearized equations, etc.

Simple examples show that the solution sets of Eq. (1) and the truncated equation

$$f^{(r)}(x) = 0 \tag{2}$$

may be topologically different. In this case, the question of under what conditions the solution sets of Eqs. (1) and (2) are locally topologically equivalent naturally arises. This question is closely related to the sufficiency problem for jets of mappings. Roughly speaking, the sufficiency of a jet is a condition under which all mappings with the same truncations have the same topological structure.

THE ν-SUFFICIENCY OF JETS OF MAPPINGS

In this section, we recall the basic definitions and results of the sufficiency theory of jets. In what follows, $\mathscr{E}_{[k]}(n, m)$ denotes the set of germs of C^k-smooth mappings $f: (\mathbb{R}^n, 0) \to (\mathbb{R}^m, 0)$. For $r \leq k$, $j^r f(0)$ denotes the r-jet of $f \in \mathscr{E}_{[k]}(n, m)$ at the point $0 \in \mathbb{R}^n$, which can be identified with the polynomial $f^{(r)}$; by $J^r(n, m)$ we denote the set of all r-jets in $\mathscr{E}_{[k]}(n, m)$. Mappings $f, g \in \mathscr{E}_{[k]}(n, m)$ are said to be C^0-equivalent if there exists a local homeomorphism h: $(\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ such that $f = g \circ h$. Mappings $f, g \in \mathscr{E}_{[k]}(n, m)$ are said to be ν-equivalent ($s\nu$-equivalent) if the germ at zero of the set $f^{-1}(0)$ is homeomorphic to that of the set $g^{-1}(0)$ (respectively, there exists a local homeomorphism $h: (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ such that $h(f^{-1}(0)) = g^{-1}(0)$). For $r \leq k$, an r-jet $w \in J^r(n, m)$ is C^0-sufficient (ν-sufficient, $s\nu$-sufficient) in $\mathscr{E}_{[k]}(n, m)$ if any two mappings f, $g \in \mathscr{E}_{[k]}(n, m)$ for which $j^r f(0) = j^r g(0) = w$ are C^0-equivalent (respectively, are ν-equivalent, are $s\nu$-equivalent).

Obviously, the C^0-sufficiency of jets implies $s\nu$-sufficiency, which in turn implies ν-sufficiency. In fact [1], ν-sufficiency is equivalent to $s\nu$-sufficiency.

For functions (at $m = 1$), a criterion for C^0-sufficiency, known as the Kuiper–Kuo criterion, was obtained in [2–4]. According to this criterion, a jet $j^r f(0)$ of a mapping $f \in \mathscr{E}_{[r]}(n, 1)$ is C^0-sufficient in $\mathscr{E}_{[r]}(n, 1)$ if and only if there exist numbers $C, \varepsilon > 0$ for which

$$|\operatorname{grad} f(x)| \geq C|x|^{r-1} \tag{3}$$

at $|x| < \varepsilon$. A jet $j^r f(0)$ of a mapping $f \in \mathscr{E}_{[r+1]}(n, 1)$ is C^0-sufficient in $\mathscr{E}_{[r+1]}(n, 1)$ if and only if there exist numbers $C, \delta, \varepsilon > 0$ for which

$$|\operatorname{grad} f(x)| \geq C|x|^{r-\delta} \tag{4}$$

at $|x| < \varepsilon$.

In [5], it was proved that condition (3) is equivalent to the following Thom condition: there exist numbers $K, \varepsilon > 0$ for which

$$\sum_{i<j} \left| x_i \frac{\partial f}{\partial x_j} - x_j \frac{\partial f}{\partial x_i} \right|^2 + |f(x)|^2 \geq K|x|^{2r} \tag{5}$$

at $|x| < \varepsilon$.

The verification of both the Kuiper-Kuo conditions (3) and (4) and the Thom conditions (5) reduces

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to estimating the growth rate of a polynomial in a neighborhood of its roots, which is equivalent to calculating the so-called Łojasiewicz local exponent of a polynomial. Recall that, according to Łojasiewicz's theorem [6, 7], for any polynomial p: $\mathbb{R}^n \to \mathbb{R}$ with $p(0) = 0$, there exist constants $C, \kappa > 0$ such that $|p(x)| \geq C|x|^\kappa$ in some neighborhood of zero. The minimum number κ for which this inequality holds is called Łojasiewicz's local exponent of p and denoted by $\mathcal{L}_0(p)$. If the zero root of the polynomial p is isolated, then κ exists and is rational [6–8]. Moreover, in this case, we have $\mathcal{L}_0(p) \leq (d-1)^n + 1$ [9], where d denotes the degree of the polynomial p. The problem of calculating Łojasiewicz's exponent has an extensive literature; see, e.g., [9–11] and the references therein.

In the general case of $n \geq m$, where m is arbitrary, a criterion for v-sufficiency (sv-sufficiency) was obtained by Kuo [12]. According to this criterion, for $n \geq m$, the jet $j^r f(0)$ of a mapping $f = (f_1, f_2, \ldots, f_m) \in \mathcal{E}_{[r]}(n, m)$ is v-sufficient (sv-sufficient) in $\mathcal{E}_{[r]}(n, m)$ if and only if there exist numbers $C, \varepsilon, \sigma > 0$ such that

$$\mathcal{D}(\mathrm{grad} f_1^{(r)}(x), \mathrm{grad} f_2^{(r)}(x), \ldots, \mathrm{grad} f_m^{(r)}(x)) \geq C|x|^{r-1} \quad (6)$$

for $x \in \mathcal{H}_r(f^{(r)}; \sigma) \cap \{|x| < \varepsilon\}$. The jet $j^r f(0)$ of a mapping $f = (f_1, f_2, \ldots, f_m) \in \mathcal{E}_{[r+1]}(n, m)$ is v-sufficient (sv-sufficient) in $\mathcal{E}_{[r+1]}(n, m)$ if and only if, for any polynomial $g = (g_1, g_2, \ldots, g_m)$ of degree $r + 1$ with $j^r g(0) = j^r f(0)$, there exist numbers $C, \delta, \varepsilon, \sigma > 0$ (depending on g) such that

$$\mathcal{D}(\mathrm{grad} f_1^{(r)}(x), \mathrm{grad} f_2^{(r)}(x), \ldots, \mathrm{grad} f_m^{(r)}(x)) \geq C|x|^{r-\delta} \quad (7)$$

for $x \in \mathcal{H}_{r+1}(g; \sigma) \cap \{|x| < \varepsilon\}$.

In this statement of Kuo's criterion, $\mathcal{H}_s(f; \sigma)$ denotes the so-called horn neighborhood $\{x \in \mathbb{R}^n : |f(x)| < \sigma|x|^s\}$ of the set $f^{-1}(0)$, and the quantity $\mathcal{D}(v_1, v_2, \ldots, v_m)$ for a set of vectors $v_1, v_2, \ldots, v_m$ is defined as the minimum distance over $i = 1, 2, \ldots, m$ from a vector v_i to the linear span V_i of the vectors v_j with $j \neq i$.

The verification of Kuo's conditions (6) and (7) is substantially more complicated than that of the Kuiper–Kuo conditions (3), (4) or of Thom's condition (5). One of the reasons for this is that the function $\mathcal{D}(v_1, v_2, \ldots, v_m)$ has a rather complicated expression, which is very hard to deal with in practice. A more serious problem is that the values of the function $\mathcal{D}$ under conditions (6) and (7) must be estimated in horn neighborhoods of the sets $(f^{(r)})^{-1}(0)$ and $g^{-1}(0)$, which are generally unknown a priori! Finally, in the case of v-sufficiency in $\mathcal{E}_{[r+1]}(n, m)$, condition (7) must be verified in infinitely many horn neighborhoods determined by all possible polynomials g of degree $r + 1$ and satisfying the relation $j^r g(0) = j^r f(0)$.

The purpose of this paper is to formulate analogues of Kuo's conditions (6) and (7) which do not require verifying any inequalities in horn neighborhoods of the set $f^{-1}(0)$, which is not known a priori. Another purpose is to replace the function $\mathcal{D}$ in conditions (6) and (7) by something easier to compute in applications.

QUALIFIED REGULARITY AND TRANSVERSALITY

We use the following notions and notation. By $\langle \cdot, \cdot \rangle$ we denote Euclidean inner product in $\mathbb{R}^n$, and $|\cdot|$ denotes the corresponding Euclidean norm. For a smooth mapping $f = (f_1, f_2, \ldots, f_m)$: $\mathbb{R}^n \to \mathbb{R}^m$, $df(x) = \frac{\partial f_i(x)}{\partial x_j}$ denotes its derivative and $(df)^*(x)$, the matrix adjoint to $df(x)$, which is composed of the m column vectors $\mathrm{grad} f_i(x)$ with $i = 1, 2, \ldots, m$.

To any mapping f: $\mathbb{R}^n \to \mathbb{R}^m$ and any integer $p \geq 1$ we assign the two auxiliary functions of $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$ defined by

$$\mathcal{R}_p(f; x, y) = |f(x)|^p |y|^p + |(df)^*(x)y|^p |x|^p,$$

$$\mathcal{T}_p(f; x, y) = \mathcal{R}_p(f; x, y) - |\langle (df)^*(x)y, x\rangle|^p.$$

Note that both functions $\mathcal{R}_p(f; x, y)$ and $\mathcal{T}_p(f; x, y)$ are nonnegative and homogeneous with respect to y. They are polynomials in x and y if f is a polynomial and p is even.

The positivity of the function $\mathcal{R}_p(f; x, y)$ for $y \neq 0$ and small $x \neq 0$ means that $|(df)^*(x)y| > 0$ for $y \neq 0$ and any small nonzero solutions x of Eq. (1), i.e., the derivative of the mapping $f(x)$ is regular on small nonzero solutions of Eq. (1). Thus, the inequality $\mathcal{R}(f; x, y) > 0$ for $x, y \neq 0$ can be interpreted as a regularity condition for small nonzero solutions of Eq. (1), and the inequality

$$\mathcal{R}_p(f; x, y) \geq C|x|^{pq}|y|^p, \quad (8)$$

which holds with some constants $C, q > 0$ for small x and all y, can be called the qualified regularity condition for small nonzero solutions of Eq. (1).

Similarly, the positivity of the function $\mathcal{T}_p(f; x, y)$ for $y \neq 0$ and small $x \neq 0$ means that $|(df)^*(x)y| \cdot |x| > |\langle (df)^* (x)y\rangle, x|$ for $y \neq 0$ and any small nonzero solutions x of Eq. (1). This inequality is an algebraic expression of the fact that the solution set of Eq. (1) is transverse to all sufficiently small spheres $|x| = \varepsilon$. Thus, the inequality

$$\mathcal{T}_p(f; x, y) \geq C|x|^{pq}|y|^p, \quad (9)$$

which holds with some constants $C, q > 0$ for small x and any y, can be called the condition for the qualified transversality of the solution set of Eq. (1) to small spheres.

According to the following lemma, for polynomial mappings f, the functions $\mathcal{R}_p(f; x, y)$ and $\mathcal{T}_p(f; x, y)$ are comparable in a natural sense for small x.

Lemma 1. *If a mapping f: $\mathbb{R}^n \to \mathbb{R}^m$ with $f(0) = 0$ is a polynomial, then, for every positive integer p, there exists a $\mu_p > 0$ such that*

$$2^{1-p}\mathcal{R}_1^p(f; x, y) \le \mathcal{R}_p(f; x, y) \le 2\mathcal{R}_1^p(f; x, y),$$

$$\mu_p\mathcal{R}_p(f; x, y) \le \mathcal{T}_p(f; x, y) \le \mathcal{R}_p(f; x, y)$$

for small x and all y.

If the mapping f is polynomial, then the set of small nonzero solutions of Eq. (1) is regular if and only if it is transverse to small spheres [13]. According to Lemma 1, for a polynomial mapping f, the set of small nonzero solutions of Eq. (1) is qualifiedly regular (with some parameter q) if and only if it is qualifiedly transverse (with the same parameter q) to small spheres. Moreover, conditions (8) and (9) with the same $q > 0$ but different p are equivalent to each other.

MAIN RESULTS

Theorem 1. *The jet $j^rf(0)$ of a mapping $f \in \mathcal{E}_{[r]}(n, m)$, where $n \ge m$, is v-sufficient (sv-sufficient) in $\mathcal{E}_{[r]}(n, m)$ if and only if, for any positive integer p, there exists a number $q > 0$ such that*

$$\mathcal{K}(f^{(r)}; x, y) \ge q|x|^{pr}|y|^p \qquad (10)$$

for small x and all y, where $\mathcal{K}$ is any of the functions $\mathcal{R}_p$ and $\mathcal{T}_p$.

The jet $j^rf(0)$ of a mapping $f \in \mathcal{E}_{[r+1]}(n, m)$, where $n \ge m$, is v-sufficient (sv-sufficient) in $\mathcal{E}_{[r+1]}(n, m)$ if and only if, for any positive integer p,

$$\frac{\mathcal{K}(f^{(r)}; x, y)}{|x|^{pr+p}|y|^p} \to \infty \qquad (11)$$

as $x \to 0$, $x \ne 0$ uniformly with respect to $y \ne 0$, where $\mathcal{K}$ is any of the functions $\mathcal{R}_p$ and $\mathcal{T}_p$.

For $p = 2$, Theorem 1 was proved in [14; 15, Chapter 8].

A standard argument [7, 8] shows that (11) is equivalent to a condition similar to (7), namely, that there exist numbers $q, \delta > 0$ such that $\mathcal{K}(f^{(r)}; x, y) \ge q|x|^{pr+p-\delta}|y|^p$ for small x and any y. This inequality can be verified by using the technique of estimating Łojasiewicz' exponents mentioned above.

Each of the functions $\mathcal{K}(f^{(r)}; x, y)$ in Theorem 1 is a y-homogeneous polynomial in x and y. This makes it possible to simplify the statement of Theorem 1 in the case of functions (for $m = 1$). We set

$$\mathcal{R}_p^*(f^{(r)}; x) = (f^{(r)}(x))^p + |\operatorname{grad} f^{(r)}(x)|^p|x|^p,$$

$$\mathcal{T}_p^*(f^{(r)}; x) = \mathcal{R}_p^*(f^{(r)}; x) - |\langle \operatorname{grad} f^{(r)}(x), x\rangle|^p.$$

Theorem 2. *The jet $j^rf(0)$ of a mapping $f \in \mathcal{E}_{[r]}(n, 1)$ is v-sufficient (sv-sufficient) in $\mathcal{E}_{[r]}(n, 1)$ if and only if, for any positive integer p, there exists a number $q > 0$ such that*

$$\mathcal{K}^*(f^{(r)}; x) \ge q|x|^{pr} \qquad (12)$$

for small x, where $\mathcal{K}^*$ is any of the functions $\mathcal{R}_p^*$ and $\mathcal{T}_p^*$.

The jet $j^rf(0)$ of a mapping $f \in \mathcal{E}_{[r+1]}(n, 1)$ is v-sufficient (sv-sufficient) in $\mathcal{E}_{[r+1]}(n, 1)$ if and only if, for any positive integer p,

$$\frac{\mathcal{K}^*(f^{(r)}; x)}{|x|^{pr+p}} \to \infty \qquad (13)$$

as $x \to 0$, $x \ne 0$, where $\mathcal{K}^*$ is any of the functions $\mathcal{R}_p^*$ and $\mathcal{T}_p^*$.

For any analytic germ h: $\mathbb{R}^n \to \mathbb{R}^1$ satisfying the condition $h(0) = 0$, any $0 < \theta < 1$, and any sufficiently small x, the Bochnak–Łojasiewicz inequality $|\operatorname{grad} h(x)| \cdot |x| \ge \theta|h(x)|$ is valid [4, Lemma 2]. Therefore, for the polynomial $f^{(r)}(x)$ in Theorem 2, there exists a number $\gamma > 0$ such that

$$\gamma\mathcal{R}_1^*(f^{(r)}; x) \le |\operatorname{grad} f^{(r)}(x)| \cdot |x| \le \mathcal{R}_1^*(f^{(r)}; x)$$

for small x. This inequality means that conditions (12) and (13) with $\mathcal{K}^* = \mathcal{R}_1^*$ are equivalent to the Kuiper–Kuo conditions (3) and (4). Therefore, conditions (10) and (11) in Theorem 1 can be interpreted as natural generalizations of the Kuiper–Kuo conditions (3) and (4), respectively.

A direct verification shows that

$$\mathcal{T}_2^*(f^{(r)}; x) = \sum_{i<j}\left|x_i\frac{\partial f^{(r)}}{\partial x_j} - x_j\frac{\partial f^{(r)}}{\partial x_i}\right|^2 + |f^{(r)}(x)|^2,$$

hence, condition (12) with $\mathcal{K}^* = \mathcal{T}_2^*$ is nothing but Thom's condition (5) for the mapping $f^{(r)}$. Therefore, conditions (10) and (11) in Theorem 1 can be interpreted as a natural generalization of Thom's condition (5) to the case of mappings ($m > 1$).

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REFERENCES

1. D. J. A. Trotman and L. C. Wilson, Proc. London Math. Soc. (3) **78** (2), 334–368 (1999).
2. T. C. Kuo, Topology **8**, 167–171 (1969).
3. N. H. Kuiper, Ann. Math. Stud, No. 69, 199–218 (1972).
4. J. Bochnak and S. Łojasiewicz, Lect. Notes Math. **192**, 254–261 (1971).
5. K. Bekka and S. Koike, Sūrikaisekikenkyūsho Kōkyūroku, No. 952, 41–49 (1996).
6. S. Łojasiewicz, Stud. Math. **18**, 87–136 (1959).
7. S. Łojasiewicz, *Introduction to Complex Analytic Geometry* (Birkhäuser, Basel, 1991).

8. E. A. Gorin, Russ. Math. Surveys **16** (1), 93–119 (1961) [Usp. Mat. Nauk **16** (1), 91–118 (1961)].
9. J. Gwoździewicz, Comment. Math. Helv. **74** (3), 364–375 (1999).
10. E. Garcia Barroso and A. Płoski, C. R. Math. Acad. Sci. Paris **336** (7), 585–588 (2003).
11. O. M. Abderrahmane, Kodai Math. J **28** (1), 106–110 (2005).
12. T. C. Kuo, Topology **11**, 115–131 (1972).
13. J. Milnor, *Singular Points of Complex Hypersurfaces* (Princeton Univ. Press, Princeton, 1968; Mir, Moscow, 1971).
14. V. S. Kozyakin, Autom. Remote Control **45**, 1275–1280 (1984) [Avtom. Telemekh., No. 10, 38–43 (1984)].
15. N. A. Bobylev, V. G. Boltyanskii, S. Yu. Vsekhsvyatskii, et al., *Matematical Systems Theory* (Nauka, Moscow, 1986) [in Russian].