Krasnosel'skii A.M.

Asymptotic homogeneity of hysteresis operators

The present paper deals with the property of asymptotic homogeneity of nonlinear operators. This property is valid for various types of bounded nonlinear operators in functional spaces: for Landesman-Lazer type Nemytski operators, for some nonlinearities with delays and for wide class of hysteresis operators listed below. For asymptotically linear operators with degenerate linear part and asymptotically homogeneous sublinear part it is possible to calculate some global topological invariant: index at infinity. The theorem on the index calculation is a general approach to different problems and it allows (for concrete cases) to prove theorems on solvability of boundary value problems, operator and integral equations. This theorem can be also applied to study of the phenomenon of bifurcation at infinity. The examples are given of application of this theorem to some problems on forced periodic oscillations. Some results, formulated in the paper, were obtained in collaboration with P.-A. Bliman, M.A. Krasnosel'skii, M. Sorine, A.A. Vladimirov.

1. Definitions

Let E be some Banach space.

Definition 1. *An operator $L: E \to E$ is called* homogeneous *if*

$$L(\lambda x) \equiv Lx, \qquad \lambda > 0,\ x \in E. \tag{1}$$

The simplest example of homogeneous operator is a constant operator. If L is a homogeneous operator then LA with linear A is also homogeneous as well as FL with arbitrary F. The sum of homogeneous operators is also a homogeneous one.

In the functional spaces it is possible to present more concrete examples. Suppose E is a space (e.g. C^k, L^p, $W^{1,1}$) of scalar functions $x(t): \Omega \to \mathbb{R}$ defined on some set Ω. Then the typical examples are the following: $\operatorname{sign} x(t)$ is homogeneous in L^p and $\operatorname{sign} \dot{x}(t)$ is homogeneous in $W^{1,1}$. One can consider their linear combinations of the type $a(t) + b(t) \operatorname{sign} x(t)$ etc.

Let A be some linear operator acting from L^p to C^0. Then the operator $A(\operatorname{sign} x(t))$ is homogeneous in C.

Let $\hat{E} \supset E$ be a space with a weaker norm. Let E_0 be a finite dimensional subspace of E, denote by U the unite circle in E_0: $U = \{e: e \in E_0, \|e\| = 1\}$. Here $\|\cdot\|$ is an arbitrary norm in E_0.

Definition 2. *An operator B is call* asymptotically homogeneous *(or $E_0, \hat{E}, E$-*asymptotically homogeneous*) if it can be presented as $B = L + C$ where L is homogeneous in E and C is "small" in the following sense: for any $c > 0$*

$$\lim_{R\to\infty} \sup_{e \in U;\, h \in E,\ \|h\|_E < c} \|C(Re + h)\|_{\hat{E}} = 0. \tag{2}$$

If $f(x): \mathbb{R} \to \mathbb{R}$ satisfies

$$\lim_{|x|\to\infty} f(x) = 0 \tag{3}$$

and

$$mes\{t:\ t \in \Omega,\ e(t) = 0\} = 0, \qquad e(t) \in U, \tag{4}$$

then operator $Cx(t) = f[x(t)]$ is small in sense of (2) with $\hat{E} = E = L^1$ (or L^p with $p < \infty$).

This fact (and its modifications) can be used in the proofs of theorems on asymptotic homogeneity. The assumption (4) on E_0 will be often used in the paper. It was considered by a number of authors: S. Fučik, J.Mawhin, P. Hess and many others.

2. Examples

In this section we present various examples on asymptotic homogeneity of nonlinear operators. Any such example is a theorem (often rather difficult one). We formulate these examples for the simplest cases, one can easily generalize them.

Let $E = \hat{E} = L^1$.

Example 1. *Suppose the function $f(x): \mathbb{R} \to \mathbb{R}$ satisfies so-called Landesman–Lazer conditions:*

$$\lim_{x\to+\infty} f(x) = f^+; \qquad \lim_{x\to-\infty} f(x) = f^-. \tag{5}$$

Suppose E_0 satisfies (4). Then the operator $f[x(t)]$ is asymptotically homogeneous and

$$Lx(t) = \frac{1}{2}(f^+ + f^-) + \frac{1}{2}(f^+ - f^-)\,\mathrm{sign}\,x(t). \tag{6}$$

The various reformulations of this statement were used by various authors starting from the pioneering work [1].

Let $E = \hat{E} = L^1(0,T)$. Denote

$$S_h x(t) = \begin{cases} x(t+T-h), & 0 \le t < h; \\ x(t-h), & h \le t \le T. \end{cases} \tag{7}$$

Let $F(u,v) : \{u^2+v^2=1\} \to \mathbb{R}$ be a scalar continuous function.

Example 2. *Let*

$$\lim_{R\to\infty} f(Rx, Ry) = F\left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right). \tag{8}$$

Suppose E_0 satisfies (4). Then the operator $f[x(t), S_h x(t)]$ is asymptotically homogeneous and

$$Lx = F\left(\frac{x}{\sqrt{x^2+(S_h x)^2}}, \frac{S_h x}{\sqrt{x^2+(S_h x)^2}}\right). \tag{9}$$

This example can be used in the study of forced periodic oscillations for nonlinear equations with delay [2].

Now we describe briefly a hysteresis nonlinearity, named *hysteron.* Consider the graphs of two continuous functions $H_1(x)$ and $H_2(x)$ in the plane $\{x, g\}$ and suppose $H_1(x) < H_2(x) \quad (x \in \mathbb{R})$. Let the set $\Omega = \{\{x, g\} : x \in \mathbb{R}, H_1(x) \le g \le H_2(x)\}$ be included in the union of nonintersected graphs of a family of continuous functions $g_\alpha(x)$, where α is a parameter. Every function $g_\alpha(x)$ is defined on its own finite interval $[\eta^1_\alpha, \eta^2_\alpha]$ ($\eta^1_\alpha < \eta^2_\alpha$ for every α) and $g_\alpha(\eta^1_\alpha) = H_1(\eta^1_\alpha)$, $g_\alpha(\eta^2_\alpha) = H_2(\eta^2_\alpha)$. This means that the ends of the graphs of the functions $g_\alpha(x)$ lie on the graphs of the functions $H_1(x)$ and $H_2(x)$.

The output $\mathcal{H}(g_0)x(t)\,(t \ge 0)$ (it is also the state of the hysteron at time t) is defined for monotonous for $t \ge t_0$ inputs as

$$\mathcal{H}(g_0)u(t) = \begin{cases} g_\alpha(u(t)), & \eta^1_\alpha \le u(t) \le \eta^2_\alpha, \\ H_1(u(t)), & u(t) \le \eta^1_\alpha, \\ H_2(u(t)), & \eta^2_\alpha \le u(t); \end{cases} \tag{10}$$

the value of α is chosen such that $g_0 = g_\alpha(u(t_0))$. For piecewise monotonous inputs the output is constructed by the semigroup identity. Piecewise monotonous functions are dense in C^0, we define our operator onto C^0 by continuity. See [3] for the correctness of this procedure. The hysteron $\mathcal{H}(g_0)x(t)$ is defined for every continuous input and for every admissible initial state $g_0 \in [H_1(x(t_0)), H_2(x(t_0))]$; it is continuous as an operator from $\mathbb{R} \times C^0$ into C^0.

E x a m p l e 3. *Let $E = C^0$, $\hat{E} = L^1$ and let E_0 satisfy (4). Let the hysteron have saturation at infinity:*

$$\lim_{x\to\pm\infty} H_i(x) = g_\pm, \qquad i = 1,2. \tag{11}$$

Then the hysteron $\mathcal{H}(g_0)u(t)$ is asymptotically homogeneous and

$$Lx = \frac{1}{2}(g_+ + g_-) + \frac{1}{2}(g_+ - g_-)\,\mathrm{sign}\, x(t). \tag{12}$$

In the next example we use hysteresis model of friction, presented in [5]. Consider a stable square $n \times n$ matrix A and two n-dimensional vectors $\mathbf{b}$ and $\mathbf{c}$. We denote by $(\cdot,\cdot)$ the scalar product in $\mathbb{R}^n$. The model of friction has absolutely continuous scalar-valued inputs $u(t)$, $t \geq 0$ and variable state $\mathbf{x} \in \mathbb{R}^n$. The scalar output $F(\mathbf{x}_0)u(t)$ is defined for any initial state $\mathbf{x}_0$ by the following formulas. Let $\mathbf{x}(t)$ be the solution of

$$\dot{\mathbf{x}} = A\mathbf{x}(t)|\dot{u}| + \dot{u}\,\mathbf{b} \tag{13}$$

satisfying $\mathbf{x}(0) = \mathbf{x}_0$. Then

$$F(\mathbf{x}_0)u(t) = (\mathbf{c}, \mathbf{x}(t)). \tag{14}$$

E x a m p l e 4. *Let $E = W^{1,1}$, $\hat{E} = L^1$ and let E_0 satisfy (4). Then hysteresis model of friction is asymptotically homogeneous and*

$$Lu = -(\mathbf{c}, A^{-1}\mathbf{b})\,\mathrm{sign}\,\dot{u}(t). \tag{15}$$

A hysteron is called a *stop* if

$$H_1(x) \equiv -1, \;\; H_2(x) \equiv 1; \quad g_\alpha = x - \alpha, \quad \alpha - 1 \leq x \leq \alpha + 1, \quad \alpha \in \mathbb{R}. \tag{16}$$

We denote by $\mathcal{S}(g_0)$ the corresponding hysteresis operator, its variable state $g(t)$ (it coincides with the output of the stop) belongs to $[-1, 1]$.

E x a m p l e 5. *Let $E = W^{1,1}$, $\hat{E} = L^1$ and let E_0 satisfy (4). Then the stop is asymptotically homogeneous and*

$$Lu = \mathrm{sign}\,\dot{u}(t). \tag{17}$$

Note that in the last example we can not replace $W^{1,1}$ by C^0.

Last two examples are considered in details in [4]. Other examples are also possible. Nonideal relays, Preisach and Ishlinskii models are also asymptotically homogeneous.

3. Theorem on index at infinity

In this section we inrtoduce a theorem on computation of an important topological characteristics: the index at infinity. Suppose in E some asymptotically linear [6] completely continuous vector field Φx is given: $\Phi x = x - Ax - Bx$, where A is linear completely continuous, and B is bounded (or sublinear) and also completely continuous. We are interested in the computation of the index $\mathrm{ind}\,\Phi$ at infinity (see [6]) of the vector field Φx.

If 1 is a regular point of the operator A then $\mathrm{ind}\,\Phi = (-1)^\beta$ where β is the sum of the multiplicities of all real eigenvalues of A which are greater than 1.

Suppose that 1 is an eigenvalue of A and that the finite dimensional subspace $E_0 = Ker(I - A)$ consists only from eigenvectors: $E_0 = \{e \in E : \; e = Ae\}$. Let the operator B be asymptotically homogeneous: $Bx = Cx + Lx$ $(\hat{E} = E)$. Denote by P the projector on E_0 which commutes with A.

T h e o r e m 1. *Suppose the finite dimensional vector field PLe is non-zero on $U = \{e \in E_0, \; \|e\| = 1\}$ and the operator Lx is continuous for $x \in U \subset E$. Then $\mathrm{ind}\,\Phi$ is well-defined and*

$$\mathrm{ind}\,\Phi = (-1)^\beta \gamma(PL, U) \tag{18}$$

where $\gamma(PL, U)$ is the rotation of the vector field PL on the sphere U.

The rotation coincides with the degree of the map $PL/\|PL\|$ on U with respect to zero. If this rotation is different from zero, then the equation $x = Ax + Bx$ has at least one solution. The index at infinity can be also used in the study of bifurcations at infinity. In applications $B = AB_1$ where B_1 is $E_0, \hat{E}, E$-asymptotically homogeneous and the linear operator A acts from $\hat{E}$ to E being completely continuous.

4. Forced oscillations in a system with the stop

Consider the equation

$$x'' + x = S(x) + b(t), \qquad b(t + 2\pi) \equiv b(t). \tag{19}$$

Theorem 2, [4]. *Let*

$$\left| \int_0^{2\pi} b(t) e^{it} dt \right| < 4. \tag{20}$$

Then equation (19) has at least one 2π-periodic solution.

Definition 3. *Let us have an equation $x = F(x;\lambda)$ in a Banach space with a real parameter $\lambda \in \Lambda = (\lambda_1, \lambda_2)$. A value $\lambda_0 \in \Lambda$ of the parameter is called an* asymptotic bifurcation point *if, for every $\varepsilon > 0$, there exists a $\lambda = \lambda(\varepsilon) \in \Lambda \bigcap (\lambda - \varepsilon, \lambda + \varepsilon)$ such that the equation $x = F(x;\lambda)$ has at least one solution x_λ such that $\|x_\lambda\| > \varepsilon^{-1}$.*

M.A. Krasnosel'skii developed an important topological tool for analysis of such asymptotic bifurcation points called *the principle of changing index* [6].

From this principle and Theorem 1 one can obtain the following result.

Theorem 3, [4]. *Let the function*

$$\left| \int_0^{2\pi} b(t;\lambda) e^{it}\, dt \right| - 4 \tag{21}$$

take both positive and negative values in every neighborhood of λ_0. Then λ_0 is an asymptotic bifurcation point for 2π-periodic problem for

$$x'' + x = S(x) + b(t;\lambda), \qquad b(t + 2\pi;\lambda) \equiv b(t;\lambda). \tag{22}$$

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5. References

1 Landesman E.N., Lazer A.C.: Nonlinear perturbations of linear elliptic boundary problems at resonance, *J. Math. Mech*, **19** (1970) 609-623

2 Krasnosel'skii A.M.: On nonlinear resonance in systems with delays, *Avtomatika i Telemekhanika*, #2 (1995) 14-21 [Russian]

3 Krasnosel'skii M.A., Pokrovskii A.V.: *Systems with Hysteresis*, Springer, 1989

4 Bliman P.A., Krasnosel'skii A.M., Sorine M., Vladimirov A.A.: Nonlinear resonance in systems with hysteresis, *Nonlinear Analysis. Theory, Methods & Applications*, to appear

5 Bliman P.A., Sorine M.: A system-theoretic approach of systems with hysteresis. Application to friction modelling and compensation. *Proc. of the 2nd European Control Conf.*, Groningen, **4** (1993) 1844-1849

6 Krasnosel'skii M.A., Zabrejko P.P.: *Geometrical Methods of Nonlinear Analysis*, Springer, Berlin, 1984

Addresses: Prof. Dr. Krasnosel'skii Alexander M., Institute for Information Transmission Problems, Russian Academy of Sciences, 19 Bol'shoj Karetny per., 101447 Moscow, Russia, amk@ippi.msk.su