# The index at infinity of some twice degenerate compact vector fields

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## 1 Introduction

Various problems concerning quasilinear equations can be reduced to the computation of some topological characteristics of compact vector fields. We can mention the problems of solvability, multiple solutions, different types of bifurcations, justification of approximate methods.

The usual approach to compute those topological characteristics (degree, rotation, index) goes as follows. Consider the dominating terms of the vector field. If these terms make up a nondegenerate field (in some sense), and if this field has an appropriate form, then we can calculate the required characteristics from the behavior of these dominating terms. If the field given by the dominating terms is degenerate, then we have to consider the "next order" terms. If these "next order" terms are nondegenerate on the set where the dominating terms are degenerate, then we can again find our characteristics, using now both the dominating and "next order" terms. And

*This paper was written during the visit of A.M.Krasnosel'skii to Louvain-La-Neuve, Belgium at 1994

if the "next order" terms are also degenerate, then then it is necessary to consider higher order terms.

Usually the first step of this programme is the most productive.

In the case of the index of an isolated singular point of a completely continuous vector field, this programme was done for asymptotically linear fields by Leray and Schauder [12] when the linear part is nondegenerate, and by [8], [9] and [4], chapter X, when it is not the case. For index calculation at infinity, the case of a nondegenerate linear part still follows from Leray-Schauder's result [12]. The case of fields with degenerate linear part was considered by various authors in seventies after the pioneering works of E.N. Landesman, A.C. Lazer and D.E. Leach [10, 11] (see [2],[4],[15],[16] for references). In the present paper we consider situations where both the linear and Landesman-Lazer terms ("next order" terms) are degenerate.

# 2 Definitions

Denote by $\Omega$ a domain in a finite dimensional space with a continuous finite measure $\mu$. We consider the space $L^2 = L^2(\mathbb{R};\Omega)$ of square integrable real functions $x(t): \ \Omega \to \mathbb{R}$ with a usual norm $\|\cdot\|$ and a inner product $(\cdot,\cdot)$ generated by the measure $\mu$.

Through all this paper we denote by $A: \ L^2 \to L^2$ a linear, compact and normal ($AA^* = A^*A$) operator (see [1] for the properties of such operators). The main example is a compact self-adjoint operator. Another example is the inverse of some linear differential operator $L(p) = p^l + a_1 p^{l-1} + \ldots + a_l \quad (p = d/dt, \ a_j$ independent of $t$) with periodic boundary conditions. All the results are new even for a self-adjoint operator $A$.

Let $g(t,x): \ \Omega \times \mathbb{R} \to \mathbb{R}$ be a bounded continuous function. Consider the vector field

$$\Theta x = x - A(x + g(\cdot, x)). \tag{1}$$

This vector field is considered on the spheres $S_\rho = \{x \in L^2 : \ \|x\| = \rho\}$ of sufficiently large radii $\rho$.

If$\Theta x \neq 0$ for $\|x\| \geq \rho_0$, then for all $\rho \geq \rho_0$ the rotation of $\Theta$ on $S_\rho$ (or the topological degree of $\Theta$ on the open ball $B(\rho)$) is defined (see [8]). This rotation does not depend on $\rho$, and its value is called the *index at infinity* of $\Theta$ and is denoted by *ind* $(\Theta)$.

If $1 \notin \sigma(A)$ (where $\sigma(A)$ is the spectrum of $A$), then *ind* $(\Theta) = (-1)^\beta$ [12],[8], where $\beta$ denotes the sum of the multiplicities of all real eigenvalues of $A$ which are greater than one. This notation will be used throughout the paper.

If $1 \in \sigma(A)$, then the linear part $I - A$ (dominating term) is degenerate, and it is necessary to use some sharp properties of the bounded nonlinearity $g(t,x)$. Let $E_0 = \mathrm{Ker}(I - A) = \{e(t) : \ Ae = e\}$ and let $P$ be the orthogonal projector on $E_0$, $E_1 = E_0^{\perp}$, $Q = I - P$ the orthogonal projector on $E_1$.

# 3 The Landesman-Lazer case

Let our nonlinearity $g(t,x)$ have the form

$$g(t,x) = f(t,x) + c\, sign\,(x) - b(t), \tag{2}$$

with

$$\lim_{|x|\to\infty} \sup_{t\in\Omega} |f(t,x)| = 0. \tag{3}$$

Since $g(t,x)$ is continuous, the function $f(t,x)$ has of course a jump at zero to compensate the jump at zero of $sign\,(x)$. The representation (2) means that

$$\lim_{x\to\pm\infty} g(t,x) = \pm c - b(t).$$

The last assumption is especially natural for $g(t,x) = g(x) - b_1(t)$. In this case it is equivalent to the existence of $\lim g(x)$ at $+\infty$ and at $-\infty$, with $c = \frac{1}{2}(g(+\infty) - g(-\infty))$, $b(t) = b_1(t) - \frac{1}{2}(g(+\infty) + g(-\infty))$.

The representation (2) is equivalent to the other representation

$$g(t,x) = f(t,x) + c\, s(x) - b(t)$$

with continuous terms, where $s(x) = sign\,(x),\ |x| > 1$ and $s(x) = x,\ |x| \le 1$. Further computations are simpler with the representation (2).

The important property of such type of nonlinearities $f(t,x) + c\, sign\,(x)$ is the following one: if

$$\mu\{t \in \Omega : \ e(t) = 0\} = 0 \tag{4}$$

then

$$\lim_{\xi\to+\infty} \sup_{\|h\|\le cst} \|f(t, \xi e(t) + h(t)) + c\, sign\,(\xi e(t) + h(t)) - c\, sign\,(e(t))\| = 0. \tag{5}$$

The following statement can be found in [7]. Similar results can be traced to [17],[14] and [4], Chapter VIII.

**Statement 1** *Let (4) be valid for all $e(t) \in U = \{e(t) \in E_0 : \ \|e\| = 1\}$ and*

$$\Psi e = P(c\,sign\,(e(t)) - b(t)) \neq 0, \quad e(t) \in U. \tag{6}$$

*Then ind $(\Theta) = (-1)^\beta \cdot \gamma$ where $\gamma$ is the rotation of the finite dimensional field $\Psi$ on the sphere $U$.*

This statement is applicable to the case when the dominating (linear) terms are degenerate, but the "next order" term $c\,sign(e(t))$ is nondegenerate in the sense of (6).

# 4 The problem

Condition (6) fails in two principal cases.

The first one is the case where $c = 0$ and $Pb(t) = 0$. Its study was started by Gaetano Villari [18] for periodic solutions of third order ordinary differential equations, by the second author [13] for the Neumann problem, by S. Fučik [3] for the Dirichlet problem, and was developed in a more general and systematic way in [5, 6].

The main type of results for this case is the following one.

Let $f(t, x)$ satisfy

$$f(t, x) \cdot sign\,(x) \geq \varphi(t, |x|), \quad |x| \geq u_0 \tag{7}$$

or

$$f(t, x) \cdot sign\,(x) \leq -\varphi(t, |x|), \quad |x| \geq u_0, \tag{8}$$

for some $u_0 > 0$. Here $\varphi(t, u) : \Omega \times \{u \geq u_0\} \to \mathbb{R}^+ \in G(u_0)$, where $G(u_0)$ is the class of nonnegative, Carathéodory and nonincreasing functions $\varphi(t, u)$ which are positive for $t \in \Omega_0 \subset \Omega$, with $\mu(\Omega_0) > 0$. If $\varphi(t, u)$ tends to 0 as $u \to \infty$ slowly enough, then the index at infinity of the vector field $x - A(x + f(\cdot, x) - b)$ is equal to $\pm 1$.

The possible restrictions on $\varphi(t, u)$ are defined by the behavior at zero of the distribution function $\chi_e$ of $e \in U$ defined by

$$\chi_e(\delta) = \mu\{t \in \Omega : \ |e(t)| \leq \delta\}. \tag{9}$$

The theorems in [5, 6] are based upon some results on integral functional inequalities, similar to some inequalities widely used in this paper.

The second case when (6) fails occurs when $c \neq 0$ and the following equality holds:

$$P(c\,sign\,(e_0(t)) - b(t)) = 0, \tag{10}$$

for some fixed $e_0 \in U$, and $\Psi e \neq 0$ for the other $e \in U$. Here again, if $f(t,x)$ satisfies (7) or (8) for an appropriate $\varphi(t,u)$, it is possible to compute $ind(\Theta)$. This is the case we shall consider in this paper.

# 5 The case where dim $E_0$=1

We first consider the case where dim $E_0 = 1$. Then $U$ contains only two elements respectively denoted by $e_0(t)$ and $-e_0(t)$. We suppose that (4) is valid for $e = e_0$ and that (10) holds.

To make proofs simpler, let us suppose that the operator $A$ acts continuously from $L^2$ to $L^\infty$.

**Theorem 1** *Let either $c > 0$ and (7) or $c < 0$ and (8) hold for some $u_0$. Let $\varphi(t,u) \in G(u_0)$ and assume that, for every $R > 0$ and $u_* \geq u_0$, one has*

$$\lim_{\delta \to 0} \frac{\chi_{e_0}(\delta)}{\int_\Omega |e_0(t)|\varphi(t, u_* + R\delta^{-1}|e_0(t)|)d\mu} = 0. \tag{11}$$

*Then $ind\,(\Theta) = (-1)^\beta sign\,(c)$.*

**Theorem 2** *Let either $c < 0$ and (7) or $c > 0$ and (8) hold for some $u_0$. Let $\varphi(t,u) \in G(u_0)$ and let condition (11) hold for every $R > 0$ and $u_* \geq u_0$. Then $ind\,(\Theta) = 0$.*

Condition (11) can be rewritten in particular cases in a rather simple form. It always holds if $|e_0(t)| > \delta_0 > 0$: in this case $\chi_{e_0}(\delta) \equiv 0$ for $0 \leq \delta \leq \delta_0$. This occurs in particular for the first (constant) eigenfunction of the periodic and the Neumann problems.

Let $\varphi(t,u) \equiv \varphi(u)$ for $u \geq u_0$. Condition (11) can be rewritten as

$$\lim_{\delta \to 0} \frac{\chi_{e_0}(\delta)}{\int_0^\infty \xi\varphi(u_* + R\delta^{-1}\xi)d\chi_{e_0}(\xi)} = 0. \tag{12}$$

For example, if $c_1\delta^\alpha \leq \chi_{e_0}(\delta) \leq c_2\delta^\alpha$ for small values of $\delta$, then (12) is equivalent to

$$\int^\infty u^{\alpha-1}\varphi(u)\,du = \infty.$$

A sufficient condition for (12) can also be given in the form

$$\lim_{\delta\to 0}\frac{\chi_{e_0}(\delta)}{\varphi(u_*+R_1\delta^{-1})}=0.$$

# 6 The case where dim $E_0 \geq 2$

In this section we study the vector field (1) in the case where dim $\mathrm{Ker}(I-A)>1$. This case differs from the case dim $\mathrm{Ker}(I-A)=1$ for the following reasons. When we estimate norms of the possible singular points $x$ of the homotopy (14), the projection $Px$ had the form $\xi e_0$ with the same $e_0$ as in (10). This situation was based upon the equality dim $E_0=1$. Now, when dim $E_0>1$, this projection may have the form $\xi e$ with $e\neq e_0$. This is the main reason why we do not get any extension of Theorem 2 for the case where dim $E_0>1$.

Again let $A$ act continuously from $L^2$ to $L^\infty$.

**Theorem 3** *Let either $c>0$ and (7) or $c<0$ and (8) hold for some $u_0$. Let $\varphi(t,u)\in G(u_0)$ and, for every $R>0$ and $u_*\geq u_0$, let us assume that*

$$\lim_{\delta\to 0}\sup_{e\in U}\frac{\chi_e(\delta)}{\int_\Omega|e(t)|\varphi(t,u_*+R\delta^{-1}|e(t)|)d\mu}=0. \tag{13}$$

*Then ind $(\Theta)=(-1)^\beta\gamma_0$, where $\gamma_0$ is the rotation of the finite dimensional vector field $\Psi_0 e=cP\mathrm{sign}\,(e(t))$ on $U$. In particular, ind $(\Theta)\neq 0$.*

The last conclusion follows from the fact that the field $\Psi_0 e$ is odd, so that its rotation $\gamma_0$ is also odd (see e.g. [8]).

As it was the case for condition (11), condition (13) can be reduced to rather simple forms in special situations.

# 7 Proofs of Theorems 1 and 2

Consider the homotopy

$$\Phi(x,\lambda)=x-A(x+f(\cdot,x)+c\,\mathrm{sign}\,(x)-\lambda b(\cdot)). \tag{14}$$

For $\lambda = 1$ the field $\Phi(x,\lambda)$ coincides with $\Theta x$. Homotopy (14) is considered either for $\lambda \in [\lambda_0, 1]$ ($\lambda_0 < 1$) or for $\lambda \in [1, \lambda_0]$ ($\lambda_0 > 1$). We choose $0 < \lambda_0 < 1$ to prove Theorem 1 and $\lambda_0 > 1$ to prove Theorem 2. The following lemma will be used in both proofs.

**Lemma 1** *Under the assumptions of Theorems* 1 *or* 2*, there exists some* $\rho > 0$ *such that the homotopy* $\Phi(x,\lambda)$ *is nonzero for* $\|x\| \geq \rho$*.*

To prove the lemma consider a zero $x(t) = \xi e_0(t) + h(t)$ of the homotopy (14) for some fixed $\lambda$. Since 1 is a regular value of the linear operator $A$ in $E_1$ we have, for some $r$,

$$\|h\|_{L^\infty} \leq r. \tag{15}$$

The estimate (15) follows from the equality

$$h = Ah + AQg(t,x) - \lambda AQb(t)$$

with

$$r = \|A\|_{\mathcal{L}(L^2,L^\infty)} \|(I-A)|_{E_1}^{-1}\| \left(\sup|g(t,x)| + \max\{1,\lambda_0\}\max|b(t)|\right)\sqrt{\mu(\Omega)}.$$

Consequently it is sufficient to prove an a priori estimate for the real number $\xi$ to prove Lemma 1.

The case $\xi \to -\infty$ is impossible due to the following relations:

$$P\Phi(x,\lambda) \equiv -Pf(t,x(t)) - cP sign\,(x(t)) + \lambda cP sign\,(e_0(t))$$

$$\to c(1+\lambda)P sign\,(e_0(t)) = c(1+\lambda)\int_\Omega |e_0(t)|\,d\mu \neq 0.$$

Let $\xi \to +\infty$. We shall prove Lemma 1 for the case (7) only, the other case (8) being absolutely analogous. It follows from the choice of $\lambda$ that $(1-\lambda)c \geq 0$. Since equation $P\Phi(x,\lambda) = 0$ is, according to (10), equivalent to

$$\int_\Omega e_0(t)f(t,x(t))\,d\mu + c\int_\Omega e_0(t)(sign\,(x(t)) - sign\,(e_0(t)))d\mu$$

$$+(1-\lambda)c\int_\Omega |e_0(t)|d\mu = 0,$$

the following inequality is true:

$$\int_\Omega e_0(t)f(t,x(t))\,d\mu + c\int_\Omega e_0(t)(sign\,(x(t)) - sign\,(e_0(t)))d\mu \leq 0. \tag{16}$$

And since $\textit{sign}\,(x(t)) \neq \textit{sign}\,(e_0(t))$ if and only if

$t \in T = \{t:\ \xi e_0(t)+h(t) \leq 0,\ e_0(t) > 0\} \bigcup \{t:\ \xi e_0(t)+h(t) \geq 0,\ e_0(t) < 0\}$

and, for $\xi > 0$,

$$T \subset \{t \in \Omega:\ \xi|e_0(t)| \leq |h(t)|\}$$
$$\subset \{t \in \Omega:\ \xi|e_0(t)| \leq r\} \subset \{t \in \Omega:\ \xi|e_0(t)| \leq r + u_0\} = \Omega^*$$

inequality (16) implies

$$\int_\Omega e_0(t) f(t, \xi e_0(t) + h(t))\, d\mu \leq c_1 \mu(\Omega^*). \tag{17}$$

Here and below in proofs, the $c_i$ are some positive constants.

Let us estimate the integral in the left-hand side of (17). If we set

$$G = \{t \in \Omega:\ \textit{sign}\,(e_0(t)) = \textit{sign}\,(x(t)),\ |x(t)| \geq u_0\},$$

(so that obviously $\Omega \backslash G \subset \Omega^*$), we have

$$\int_\Omega e_0(t) f(t, \xi e_0(t) + h(t))\, d\mu$$

$$\geq \int_G |e_0(t)| f(t, \xi e_0(t) + h(t))\, \textit{sign}\,(\xi e_0(t) + h(t)) d\mu - c_2 \mu(\Omega^*)$$

$$\geq \int_G |e_0(t)| \varphi(t, |\xi e_0(t) + h(t)|)\, d\mu - c_2 \mu(\Omega^*)$$

$$\geq \int_\Omega |e_0(t)| \varphi(t, u_0 + r + \xi |e_0(t)|)\, d\mu - c_3 \mu(\Omega^*).$$

The last inequality and (17) imply the estimate

$$\int_\Omega |e_0(t)| \varphi(t, u_0 + r + \xi |e_0(t)|) d\mu \leq c_4 \mu(\Omega^*),$$

and hence the estimate

$$\int_\Omega |e_0(t)| \varphi(t, u_0 + r + \xi |e_0(t)|) d\mu \leq c_4 \chi_{e_0}\left(\frac{u_0 + r}{\xi}\right).$$

Because of (11), this is impossible for $\xi \to +\infty$ and the proof of Lemma 1 is complete.

Now we can prove Theorems 1 and 2.

According to the Statement from Section 3, if $\lambda_0 \neq 1$ then $ind\,(\Phi(x,\lambda_0))$ is well-defined and $ind\,(\Phi(x,\lambda_0)) = (-1)^\beta\gamma$. It easy to calculate that for $c > 0$ and $\lambda_0 \in [0,1[$ we have $\gamma = 1$ and

$$ind\,(\Phi(x,\lambda_0)) = (-1)^\beta,$$

for $c < 0$ and $\lambda_0 < 1$, we have $\gamma = -1$ and

$$ind\,(\Phi(x,\lambda_0)) = (-1)^{\beta+1},$$

and for $\lambda_0 > 1$, we have

$$ind\,(\Phi(x,\lambda_0)) = \gamma = 0.$$

Both Theorems 1 and 2 follow then from the homotopy invariance of the Leray-Schauder degree.

# 8 Proof of Theorem 3

Consider again the homotopy (14) for $\lambda \in [0,1]$. If $\lambda = 1$, then $\Phi(x,\lambda) = \Theta x$, if $\lambda = 0$, then $\Phi(x,\lambda) = x - A(x + f(\cdot,x) + c\,sign\,(x))$. The last field satisfies all the conditions of the Statement from Section 3 and its index at infinity equals $(-1)^\beta\gamma_0$. Now Theorem 3 follows from the following lemma.

**Lemma 2** *For $\lambda \in [0,1]$ all possible zeros of the homotopy* (14) *are a priori bounded.*

The lemma will be proved for the case$c > 0$ and (7). Let again $x(t) = \xi e(t) + h(t)$, where $\xi > 0$, $e(t) \in U$ and $h(t) = Qx(t)$. For some $r > 0$ the estimate (15) is again true. This means that we need only to estimate the values of $\xi$. Since

$$0 = P\Phi(x,\lambda) \equiv -P(f(t,x) - c\,sign\,(x) + \lambda c\,sign\,(e_0)),$$

we have

$$\int_\Omega e(t)(f(t,x(t)) + c\,sign\,(x(t)) - \lambda c\,sign\,(e_0(t)))d\mu = 0$$

and

$$\int_\Omega e(t)f(t,x(t))d\mu + c\int_\Omega e(t)(\textit{sign}\,(x(t)) - \textit{sign}\,(e(t)))d\mu$$

$$+c\int_\Omega e(t)[\textit{sign}\,(e(t)) - \lambda \textit{sign}\,(e_0(t))]d\mu = 0. \tag{18}$$

We estimate the three terms in the left-hand side of (18) separately.

Analogously to the estimates in the proof of Lemma 1 we obtain

$$\int_\Omega e(t)f(t,x(t))d\mu \geq \int_\Omega |e(t)|\varphi(t, u_* + \xi|e(t)|)d\mu - c_5\mu(\Omega^*)$$

where $u_* = u_0 + r,\ \Omega^* = \{t \in \Omega:\ \xi|e(t)| \leq u_*\}$. To estimate the second term, we note that $\{t \in \Omega:\ \textit{sign}\,(x(t)) \neq \textit{sign}\,(e(t))\} \subset \Omega^*$ and

$$c\int_\Omega e(t)(\textit{sign}\,(x(t)) - \textit{sign}\,(e(t)))d\mu \geq -c\int_{\Omega^*}|e(t)|d\mu \geq -c_6\mu(\Omega^*).$$

The third term is always nonnegative: $c > 0$ and

$$\int_\Omega e(t)[\textit{sign}\,(e(t)) - \lambda \textit{sign}\,(e_0(t))]d\mu =$$

$$= \int_\Omega |e(t)|d\mu - \lambda\int_\Omega |e(t)|\textit{sign}\,(e_0(t)e(t))d\mu \geq (1-\lambda)\int_\Omega |e(t)|d\mu.$$

Now we obtain the estimate

$$\int_\Omega |e(t)|\varphi(t, u_* + \xi|e(t)|)d\mu \leq c_7\chi_e\left(\frac{u_*}{\xi}\right)$$

which contradicts (13) when $\xi \to +\infty$.

Lemma 2 and consequently Theorem 3 are proved.

# 9 Applications

Applications of Theorems 1 and 3 to the solvability of nonlinear boundary value problems are very clear: under the conditions of these theorems the equation $x = A(x + g(\cdot, x))$ has at least one solution in $L^\infty$. For example, if we consider the following simple periodic problem

$$x'(t) = \frac{|x|^{1/2} + |x|}{1+|x|}\text{sgn}\ x - b(t),\ x(0) = x(T),$$

where $b$ is continuous and T-periodic, the classical Landesman-Lazer condition implies its solvability for every $b$ such that

$$-1 < \overline{b} \equiv \frac{1}{T} \int_0^T b(t)\, dt < 1.$$

Now, with the notations of Section 5, we can take $e_0(t) = 1$
$sqrtT$ or $e_0(t) = -1/\sqrt{T}$. Consequently, Theorem 1 implies the existence of a solution if $\overline{b} = 1$ or $\overline{b} = -1$, so that now the solvability holds for each $\overline{b} \in [-1, 1]$. Of course, such a simple result would also follow from the conditions given in [13] for example, but Theorem 1 allows the obtention of similar results when the kernel is not made of constant functions and $f$ satisfies a sign condition of type (7) or (8) for a suitable $\varphi$. Notice that Theorem 1 and the invariance of the index for small perturbations also implies that given $\widetilde{b}(t) = b(t) - \overline{b}$, the existence holds for $\overline{b}$ belonging to some open interval containing $[-1, 1]$.

To illustrate Theorem 2, we can consider the simple periodic problem

$$x'(t) = \frac{|x| - |x|^{1/2}}{1 + |x|} \mathrm{sgn}\ x - b(t),\ x(0) = x(T),$$

where $b$ is continuous and T-periodic. Again, the classical Landesman-Lazer condition implies its solvability for every $b$ such that

$$-1 < \overline{b} \equiv \frac{1}{T} \int_0^T b(t)\, dt < 1.$$

Now, as

$$-1 < \frac{|x| - |x|^{1/2}}{1 + |x|} \mathrm{sgn}\ x < 1$$

for all $x \in \mathbb{R}$, the Landesman-Lazer condition is also necessary for the existence. The conclusion of the nullity of the index in Theorem 2 when $\overline{b} = \pm 1$ is consistent with that conclusion.

The applications to problems on asymptotic bifurcation points are also easy (see [8] for the definition and for the principle of changing index, which allows to study bifurcation at infinity by topological methods).

We formulate only one further example: a multiplicity statement for the solutions of an equation with a parameter near an asymptotic bifurcation point of special type. This type of bifurcation was described in [7] without the multiplicity result. We keep the terminology from the previous sections.

Consider the equation

$$x = A(x + f(\cdot, x) + c\, sign\,(x) - \lambda b(\cdot)). \tag{19}$$

This equation coincide with $\Phi(x, \lambda) = 0$ where $\Phi$ is the homotopy (14).

**Theorem 4** *Let* (10) *hold for some* $e_0 \in U$. *Let all the assumptions of Theorem* 3 *be valid. Then:*
(i) $\lambda = 1$ *is an asymptotic bifurcation point for* (19)*;*
(ii) *for all* $\lambda \in [0, 1]$ *at least one solution of* (19) *exists;*
(iii) *all the solutions* $x_\lambda$ *of all the equations* (19) *for* $\lambda \in [0, 1]$ *satisfy an a priori estimate* $\|x_\lambda\| \leq \rho < \infty$*;*
(iv) *for* $\lambda > 1$ *and* $\lambda$ *close to* 1 *equation* (19) *has at least two solutions.*

Proof. Conclusion (i) follows from the changing index principle and was given in [7]. Conclusion (ii) follows from the main result in [7] when $\lambda \in [0, 1[$, and from Theorem 3 when $\lambda = 1$. Conclusion (iii) is equivalent to Lemma 2. Conclusion (iv) follows from the next construction. For $\lambda = 1$, the index at infinity of the field $\Phi(x, \lambda)$ is odd. This means that at least one solution of (19) exists in the ball $\{\|x\| \leq r + 1\}$. Hence, for $\lambda$ greater then 1 and close to 1, we have at least one solution of (19) in the same ball. But the index at infinity of (14) for $\lambda > 1$ is equal to zero ([7]). This means that at least a second solution exists.

# 10 Remarks

A. It is possible to consider twice degenerate vector fields with non-normal linear part. The main additional assumption for that case is the sign-coincidence (see [6]) of the subspaces $\mathrm{Ker}(I - A)$ and $\mathrm{Ker}(I - A^*)$.

B. Continuous functions can be replaced by proper discontinuous ones. Functions of two variables may be assumed to satisfy Carathéodory conditions, functions of one variable may be assumed to be integrable or square integrable.

C. Instead of $c\, sign\,(x)$ the Landesman-Lazer term may have the form $c(t)\, sign\,(x)$ with positive (or negative) $c(t)$. This is equivalent to the assumptions

$$\lim_{x \to \pm\infty} g(t, x) = g_\pm(t)$$

for some $g_\pm : \Omega \to \mathbb{R}$.

D. We assumed that $A$ acts from $L^2$ to $L^\infty$. This assumption can be omitted, but in this case the estimates of $\mu\{t \in \Omega : \ \xi|e(t)| \leq |h(t)|\}$ in the proofs of Lemmas 1 and 2 are more difficult, and, moreover, the possible restrictions (of the type of (11)) on the behavior of $\varphi(t,x)$ at infinity are stronger than the ones in the Theorems 1–3. One can consult [6] in this direction.

E. In the proof of Lemma 1 we can use some weaker inequalities instead of the estimates

$$f(t,x) \geq \varphi(t,x), \quad x \geq u_0 \tag{20}$$

and

$$f(t,x) \leq -\varphi(t,-x), \quad -x \geq u_0, \tag{21}$$

equivalent to (7). More precisely, we used (20) for $t \in \{t \in \Omega : \ e_0(t) > 0\}$ and (21) for $t \in \{t \in \Omega : \ e_0(t) < 0\}$. In this way it is possible to generalize both Theorems 1 and 2. As an example we give here a variant of Theorem 1 for the case where $e_0(t) \geq 0$.

**Theorem 5** *Let $c > 0$ and, instead of (7), assume that the estimate (20) is valid for some $u_0$. Let $\varphi(t,u) \in G(u_0)$ and let equality (11) hold for every $R > 0$ and $u_* \geq u_0$. Then $ind\,(\Theta) = (-1)^\beta$.*

F. We considered nonlinear vector fields with a superposition operator as nonlinearity. It is possible to consider other types of nonlinearities, for example, nonlinearities with delays, derivatives or hysteresis. It is only necessary to suppose that this nonlinearity satisfies an analog of (5).

G. The dependence on the parameter in (19) is very particular. With the use of Theorems 1–3 it is possible to consider much more general cases, when $f(t,x)$ and $c$ also depend on the parameter and the term $\lambda b(t)$ is replaced by $b(t,\lambda)$.

H. The generalization of Theorems 1–3 for equations in $L^2(\Omega;\mathbb{R}^n)$ is an interesting open problem, as well as the extension of Theorem 2 to the case where $\dim E_0 > 1$ (at least for particular cases of linear and nonlinear parts).

I. In (13), the supremum with respect to all $e \in U$ can be replaced by the supremum with respect to $e \in U$ close to $e_0$, in the sense that $\|sign\,(e) -$

$sign(e_0)\|_{L^2} \leq \sigma$. This remark with Remark E makes it possible to generalize Theorem 3.

J. If $\dim E_0 > 1$ and (10) holds for $e_0(t)$ in some subset of $U$ containing more than one point, then Theorem 3 is also valid. This case is rather natural. For example, if $e_0$ is strictly positive, then (10) is valid for functions $e(t)$ close to $e_0$. In the proofs we did not use that (10) is valid only for one element $e_0 \in U$.

# References


[1] Dunford N., Schwartz J. *Linear Operators. I: General Theory*, Interscience Publishers, New York, 1985

[2] Fučik S. *Solvability of Nonlinear Equations and Boundary Value Problems*, Reidel, Dordrecht and Soc. Czech. Math. Phys., Prague, 1980

[3] Fučik S., Further remark on a theorem by E.M. Landesman and A.C. Lazer, *Comm. Math. Univ. Carolinae* **15** (1974), 259-271

[4] Gaines R.E., Mawhin J., *Coincidence Degree and Nonlinear Differential Equations,* Lecture Notes in Math. vol. 568, Springer, Berlin, 1977

[5] Krasnosel'skii A.M. *Asymptotics of Nonlinearities and Operator Equations*, Nauka, Moscow, 1992 (Russian)

[6] Krasnosel'skii A.M. On a method of analysis of resonance problems, *Nonlinear Analysis. Theory, Methods* & *Applications*, **16**, 4, (1991) 321-345

[7] Krasnosel'skii A.M. On bifurcation points of equations with Landesman–Lazer type nonlinearities, *Nonlinear Analysis. Theory, Methods* & *Applications*, **18** (1992) 1187-1199

[8] Krasnosel'skii M.A., Zabreiko P.P., *Geometric Methods of Nonlinear Analysis*, Springer, Berlin, 1984

[9] Laloux B., Mawhin J., Coincidence index and multiplicity, *Trans. Amer. Math. Soc.* **217** (1976) 143-162

[10] Landesman E.N., Lazer A.C. Nonlinear perturbations of linear elliptic boundary value problems at resonance, *J. Math. Mech.*, **19** (1970) 609-623

[11] Lazer A.C., Leach D.E., Bounded perturbations of forced harmonic oscillators at resonance, *Ann. Mat. Pura Appl.* (4) **82** (1969) 49-68

[12] Leray J. and Schauder J., Topologie et équations fonctionnelles, *Ann. Sci. Ecole Normale Sup.* (3) **51** (1934) 45-78

[13] Mawhin J., Problèmes aux limites du type de Neumann pour certaines équations différentielles ou aux dérivées partielles non linéaires, in *Equations différentielles et fonctionnelles non linéaires,* Hermann, Paris, 1973, 124-134

[14] Mawhin J., Topology and nonlinear boundary value problems, in *Dynamical Systems: An International Symposium,* vol. I, Academic Press, New York, 1976, 51-82

[15] Mawhin J., Landesman-Lazer's type problems for nonlinear equations, *Confer. Semin. Mat. Univ. Bari*, vol. 147, 1977

[16] Mawhin J., Willem M. *Critical Point Theory and Hamiltonian Systems*, Springer, New York, 1989

[17] Nirenberg L., An application of generalized degree to a class of nonlinear problems, in *Troisième Colloque du C.B.R.M. d'analyse fonctionnelle,* Vander, Louvain, 1971, 57-74

[18] Villari G., Soluzioni periodiche di una classe di equazioni differenziali del terz' ordine quasi lineari, *Ann. Mat. Pura Appl.* (4) 73 (1966), 103-110