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Construction of a System of Linear Differential Equations from a Scalar Equation

I. V. V'yugin[a] and R. R. Gontsov[a]

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Abstract—As is well known, given a Fuchsian differential equation, one can construct a Fuchsian system with the same singular points and monodromy. In the present paper, this fact is extended to the case of linear differential equations with irregular singularities.

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1. INTRODUCTION

The classical Riemann–Hilbert problem, i.e., the question of existence of a system

$$\frac{dy}{dz} = B(z)y, \qquad y(z) \in \mathbb{C}^p, \tag{1}$$

of p linear differential equations with given *Fuchsian* singular points $a_1, \dots, a_n \in \overline{\mathbb{C}}$ and monodromy

$$\chi\colon\ \pi_1(\overline{\mathbb{C}} \setminus \{a_1, \dots, a_n\}, z_0) \to \mathrm{GL}(p, \mathbb{C}), \tag{2}$$

has a negative solution in the general case. Recall that a singularity a_i of system (1) is said to be Fuchsian if the matrix differential 1-form $B(z)\,dz$ has simple poles at this point. The monodromy of a system is a representation of the fundamental group of a punctured Riemann sphere in the space of nonsingular complex matrices of dimension p. Under this representation, a loop γ is mapped to a matrix G_γ such that $Y(z) = \widetilde{Y}(z)G_\gamma$, where $Y(z)$ is a fundamental matrix of the system in the neighborhood of the point z_0 and $\widetilde{Y}(z)$ is its analytic continuation along γ.

The first counterexample to the Riemann–Hilbert problem was produced by A.A. Bolibruch in the case of $p = 3$ and $n = 4$ (see [1, Ch. 2] for more details). He also obtained various sufficient conditions for the positive solution of the problem. Here we focus on one of them that served as the basis for the present study.

($\triangle$) *If representation* (2) *is the monodromy of a linear differential equation*

$$\frac{d^p u}{dz^p} + b_1(z)\frac{d^{p-1}u}{dz^{p-1}} + \ldots + b_p(z)u = 0 \tag{3}$$

of order p all of whose singularities $a_1, \dots, a_n$ are Fuchsian, then the Riemann–Hilbert problem has a positive solution (see [4, Addendum 1]). A singular point a_i of the scalar equation (3) is Fuchsian if the coefficient $b_j(z)$ has a pole of order at most j at this point, $j = 1, \dots, p$.

Note that the Fuchsian singularities of both the system and the scalar equation are *regular* singular points; i.e., all solutions near these points have at most power growth. Therefore, the Riemann–Hilbert problem, as well as the above-mentioned sufficient condition for its solvability, can be reformulated in terms of meromorphic equivalence of systems of linear differential equations.

[a] Institute for Information Transmission Problems (Kharkevich Institute), Russian Academy of Sciences, Bol'shoi Karetnyi per. 19, Moscow, 127994 Russia.

E-mail addresses: vyugin@gmail.com (I.V. V'yugin), rgontsov@inbox.ru (R.R. Gontsov).

A linear transformation

$$\widetilde{y} = \Gamma(z)y \tag{4}$$

(either local in a neighborhood O_i of a singular point a_i, or global) is said to be *meromorphically invertible* if the matrix $\Gamma(z)$ is meromorphic (either in O_i or in $\overline{\mathbb{C}}$, respectively) and $\det\Gamma(z) \not\equiv 0$. Such a transformation makes system (1) into the system

$$\frac{d\widetilde{y}}{dz} = \widetilde{B}(z)\widetilde{y}, \qquad \widetilde{B}(z) = \frac{d\Gamma}{dz}\Gamma^{-1} + \Gamma B(z)\Gamma^{-1}, \tag{5}$$

which is said to be *meromorphically equivalent* to the original system (1).

According to Plemelj's theorem, there always exists a system (1) with given *regular* singular points $a_1,\dots,a_n$ and monodromy (2). Therefore, the negative solution to the Riemann–Hilbert problem implies that some systems with regular singular points cannot be (globally) meromorphically equivalent to Fuchsian systems with the same singularities.[1] At the same time, in view of the sufficient condition ($\triangle$), a system of special form

$$\frac{dy}{dz} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & 0 \\ 0 & \dots\dots & & 0 & 1 \\ -b_p & \dots\dots\dots & & & -b_1 \end{pmatrix} y, \tag{6}$$

obtained from the *Fuchsian* equation (3) by the standard substitution

$$y^1 = u, \qquad y^2 = \frac{du}{dz}, \qquad \dots, \qquad y^p = \frac{d^{p-1}u}{dz^{p-1}} \tag{7}$$

(hence, all singular points of this system are regular), is globally meromorphically equivalent to a Fuchsian system with the same singularities.

In this paper, we consider a *generalized* Riemann–Hilbert problem for linear systems with *irregular* (i.e., nonregular) singular points and an analog of the sufficient condition ($\triangle$) for this problem. The basic concepts and facts related to irregular singularities of linear differential equations are presented in the next section.

Prior to formulating the generalized Riemann–Hilbert problem (posed in [5]), we recall the definition of the *minimal Poincaré rank* of system (1) at a singular point.

If the Laurent series of the matrix $B(z)$ of coefficients of system (1) has the form

$$B(z) = \frac{B_{-r-1}}{(z-a)^{r+1}} + \dots + \frac{B_{-1}}{z-a} + B_0 + \dots, \qquad B_{-r-1} \neq 0,$$

in a neighborhood of a singular point $a = a_i$, then the number r is called the *Poincaré rank* of the system at this point.

It is easy to notice that (local) meromorphic transformations (4) can either increase or decrease the Poincaré rank. The *minimal* Poincaré rank of system (1) at a singular point a_i is the least of the Poincaré ranks of systems (5) that are meromorphically equivalent to system (1) in a neighborhood O_i of the point a_i.

For example, the minimal Poincaré rank of a regular singular point is zero, while the minimal Poincaré rank of an irregular singularity is positive.

[1]Here we use the simple fact that two linear systems with identical sets of regular singular points are (globally) meromorphically equivalent if and only if they have the same monodromy.

The generalized Riemann–Hilbert problem for systems with irregular singular points can be formulated as follows.

For every $i = 1, \dots, n$, *consider a local system*

$$\frac{dy}{dz} = B_i(z)y, \qquad B_i(z) = \frac{B^i_{-r_i-1}}{(z-a_i)^{r_i+1}} + \ldots + \frac{B^i_{-1}}{z-a_i} + B^i_0 + \ldots, \tag{8}$$

in a neighborhood O_i *of a singular point* a_i *of minimal Poincaré rank* r_i. *Does there exist a global system* (1) *with singularities* $a_1, \dots, a_n$ *of Poincaré ranks* $r_1, \dots, r_n$ *that has a given monodromy* (2) *and is meromorphically equivalent to systems* (8) *in the respective neighborhoods* O_i?

Naturally, this problem may have a positive solution only if the monodromy groups of the local systems (8) coincide with the corresponding restrictions $\chi|_{\pi_1(O_i\setminus\{a_i\})}$ of representation (2).

Representation (2) together with local systems (8) will be referred to as *generalized monodromy data*.

The classical Riemann–Hilbert problem is a particular case of the generalized problem (with all $r_i = 0$).[2] In Section 3, we consider some types of counterexamples to the latter with positive minimal Poincaré ranks r_i, i.e., with irregular singularities of systems (8).

The main result of this study is the following generalization of the sufficient condition ($\triangle$). This result is proved in Section 4.

Theorem 1. *Suppose that system* (6) *corresponds to equation* (3) *with more than one singularity and each of these singularities is formally unramified. Then such a system* (6) *is meromorphically equivalent to a system* (1) *with the same singular points whose Poincaré ranks are minimal.*

In other words, the generalized Riemann–Hilbert problem has a positive solution for generalized monodromy data corresponding to a scalar equation (3) whose singularities are formally unramified.

2. A METHOD FOR SOLVING THE GENERALIZED RIEMANN–HILBERT PROBLEM

First, recall the basic concepts and facts from the local theory of linear differential equations and systems with irregular singular points.

It is well known (see, for example, [2]) that in a neighborhood of an irregular singularity $a = a_i$ of Poincaré rank $r = r_i$, system (8) has a formal fundamental matrix $\widehat{Y}(z)$ of the form

$$\widehat{Y}(z) = \widehat{F}(z)(z-a)^{\widehat{E}} e^{Q(z)}, \tag{9}$$

where

(a) $\widehat{F}(z)$ is a formal (matrix) Laurent series about $z = a$ with finite principal part and $\det \widehat{F}(z) \not\equiv 0$;

(b) $Q(z) = \operatorname{diag}(Q^1, \dots, Q^N)$, where the diagonal matrices $Q^j(z)$ are polynomials P^j in $(z-a)^{-1/s}$ without free terms of degree at most rs, and each block $Q^j(z)$ is closed with respect to the analytic continuation around the singular point $z = a$ (i.e., the matrices $Q^j(a + ze^{2\pi i})$ and $Q^j(a+z)$ differ only by some permutation of the diagonal elements);

(c) $\widehat{E} = (1/2\pi i)\ln \widehat{G}$, $\widehat{G} = \operatorname{diag}(\widehat{G}^1, \dots, \widehat{G}^N)$, is the *formal monodromy matrix* (of block diagonal form corresponding to the form of the matrix Q) defined by

$$\widehat{Y}(a + ze^{2\pi i}) = \widehat{Y}(a+z)\widehat{G};$$

the eigenvalues ρ of the matrix $\widehat{E}$ satisfy the condition $0 \le \operatorname{Re}\rho < 1$.

[2] In this case, systems (8) can be omitted because they are uniquely defined by the monodromy representation (2).

A neighborhood of a singular point a can be covered by a finite set $\{S_\alpha\}$ of sectors with vertices at this point so that in each sector S_α there exists a fundamental matrix $Y_\alpha(z)$ of system (8) for which the formal fundamental matrix (9) is asymptotic. The latter means that the formal matrix Laurent series $\widehat{F}(z) = \sum_{j=-m}^{\infty} F_j(z-a)^j$ is asymptotic in the sector S_α for the matrix function $Y_\alpha(z)e^{-Q(z)}(z-a)^{-\widehat{E}}$; i.e.,

$$Y_\alpha(z)e^{-Q(z)}(z-a)^{-\widehat{E}} - \sum_{j=-m}^{l} F_j(z-a)^j = o(|z-a|^l) \qquad \text{as} \quad z \to a, \quad z \in S_\alpha,$$

for all $l \geq -m$.

It is also known that each diagonal element $q(z)$ of the matrix $Q(z)$ has the form

$$q(z) = -\frac{\lambda}{r}(z-a)^{-r} + o(|z-a|^{-r}), \qquad z \to a,$$

where λ is an eigenvalue of the matrix $B^i_{-r_i-1}$ (for different $q(z)$, the corresponding eigenvalues of the matrix $B^i_{-r_i-1}$ are different).

Definition 1. The *Katz index* K_i of a singular point $a = a_i$ is the number $(\deg P)/s$, where $P = \operatorname{diag}(P^1, \dots, P^N)$ is regarded as a polynomial in $(z-a)^{-1/s}$.

Since the matrix $Q(z)$ is a meromorphic invariant of system (8), it follows from the properties of this matrix that the Katz index K_i is not greater than the minimal Poincaré rank r_i of the singularity.

Definition 2. An irregular singularity of system (8) is said to be *formally unramified* if the diagonal elements of the matrix $Q(z)$ in decomposition (9) are linear combinations of integer powers of $z-a$, i.e., if $s = 1$. Otherwise a singularity is said to be *formally ramified.* (The Fuchsian singularities can also be naturally regarded as unramified ones.)

In the case of a formally unramified singularity, each block $Q^j(z)$ of the matrix $Q(z)$ in decomposition (9) is scalar and the matrix $\widehat{E}$ is a Jordan matrix.

The Katz index of a singular point of the scalar equation (3) is defined as the corresponding Katz index of system (6); however, it can also be calculated directly from the coefficients of the equation. Namely (see [12] or [10, § 9.E]),

$$K_i = \max\left(0, \max_{j=1,\dots,p} \frac{-j - \operatorname{ord}_{a_i} b_j}{j}\right), \qquad a_i \neq \infty. \tag{10}$$

In particular, the Katz index of a Fuchsian singularity is zero. If equation (3) also has a singularity at infinity, then the Katz index K_∞ of the singularity at infinity is equal to the Katz index of the singularity $t = 0$ of the equation obtained by passing to the coordinate $t = 1/z$ in equation (3). Calculations lead to the formula

$$K_\infty = \max\left(0, \max_{j=1,\dots,p} \frac{j - \operatorname{ord}_\infty b_j}{j}\right). \tag{11}$$

According to (10), $K_i \geq -1 - \operatorname{ord}_{a_i} b_j/j$; i.e., the order of the pole of the function $b_j(z)$ at the point a_i is not greater than $j(\lceil K_i \rceil + 1)$, where $\lceil \cdot \rceil$ denotes the ceiling function. Therefore, the transformation (4) with

$$\Gamma(z) = (z-a_i)^{(\lceil K_i \rceil + 1)D}, \qquad D = \operatorname{diag}(0, 1, \dots, p-1),$$

reduces the Poincaré rank of the singularity a_i of system (6) to the number $\lceil K_i \rceil$ (and the Poincaré rank cannot be reduced further). Since any system (8) is meromorphically equivalent to a system of the form (6) (see [9, Lemma II.1.3]), the minimal Poincaré rank r_i of a singular point of system (8) is related to the Katz index K_i of this singularity as follows: $r_i = \lceil K_i \rceil$.

A method for solving the generalized Riemann–Hilbert problem is expounded in [5]. It is analogous to the method for solving the classical problem (see [4, Lecture 8]). Recall it briefly.

First, over the punctured Riemann sphere $\overline{\mathbb{C}} \setminus \{a_1, \dots, a_n\}$, one constructs a holomorphic vector bundle $\widetilde{F}$ of rank p with holomorphic connection $\widetilde{\nabla}$ that has a given monodromy (2). The bundle is defined by a set $\{U_\alpha\}$ of small disks that cover $\overline{\mathbb{C}} \setminus \{a_1, \dots, a_n\}$ and by a set $\{g_{\alpha\beta}\}$ of constant matrices that define a gluing cocycle. The connection $\widetilde{\nabla}$ is defined by the set $\{\omega_\alpha\}$ of matrix differential 1-forms $\omega_\alpha \equiv 0$.

Then one extends the pair $(\widetilde{F}, \widetilde{\nabla})$ to a bundle F^0 with meromorphic connection ∇^0 that are defined over the whole Riemann sphere and called a *canonical* extension of the pair $(\widetilde{F}, \widetilde{\nabla})$. To this end, one adds neighborhoods $O_1, \dots, O_n$ of the points $a_1, \dots, a_n$, respectively, to the covering $\{U_\alpha\}$. In a nonempty intersection $O_i \cap U_\alpha$, the gluing function $g_{i\alpha}(z) = Y_i(z)$ is defined by the fundamental matrix $Y_i(z)$ of the corresponding system (8), and the analytic continuations of this function to other nonempty intersections $O_i \cap U_\beta$ define the gluing functions $g_{i\beta}(z)$, so that the set $\{g_{\alpha\beta}, g_{i\alpha}(z)\}$ defines a cocycle corresponding to the covering $\{U_\alpha, O_i\}$ of the Riemann sphere. The meromorphic connection ∇^0 is defined by the set $\{\omega_\alpha, \omega_i\}$ of matrix differential 1-forms, where $\omega_i = B_i(z)dz = (dY_i)Y_i^{-1}$ are 1-forms of the coefficients of systems (8).

Next, one can construct a family $\mathcal{F}$ of extensions of the pair $(\widetilde{F}, \widetilde{\nabla})$ by replacing the matrices $g_{i\alpha}(z)$ in the construction of the pair (F^0, ∇^0) with the matrices

$$\widetilde{g}_{i\alpha}(z) = \Gamma_i(z) g_{i\alpha}(z) \tag{12}$$

and the forms ω_i with the forms

$$\widetilde{\omega}_i = (d\Gamma_i)\Gamma_i^{-1} + \Gamma_i \omega_i \Gamma_i^{-1}, \tag{13}$$

where $\widetilde{y} = \Gamma_i(z)y$ are all possible meromorphic transformations of system (8) that do not increase its Poincaré rank r_i, $i = 1, \dots, n$.

If the bundle F is holomorphically trivial for some pair $(F, \nabla) \in \mathcal{F}$, then the corresponding connection ∇ defines a global system (1) that solves the generalized Riemann–Hilbert problem. Therefore, *the generalized Riemann–Hilbert problem for systems with irregular singular points is solvable if and only if one of the bundles of the family $\mathcal{F}$ constructed by the generalized monodromy data* (2), (8) *is holomorphically trivial* (see [5]).

The degree $\deg F$ of a bundle F with a meromorphic connection ∇ that has singularities $a_1, \dots, a_n$ can be calculated by the formula $\deg F = \sum_{i=1}^n \operatorname{tr} \operatorname{res}_{a_i} \nabla$, where $\operatorname{res}_{a_i} \nabla$ is the residue of the matrix differential 1-form that defines the connection in a neighborhood of the point a_i. According to the Birkhoff–Grothendieck theorem, any holomorphic vector bundle F of rank p over a Riemann sphere is equivalent to the direct sum of line bundles, $F \cong \mathcal{O}(k_1) \oplus \ldots \oplus \mathcal{O}(k_p)$; i.e., such a bundle can be described in the coordinate form as

$$U_0 = \mathbb{C}, \qquad U_\infty = \overline{\mathbb{C}} \setminus \{0\}, \qquad g_{0\infty}(z) = z^K,$$

where $K = \operatorname{diag}(k_1, \dots, k_p)$. The set of integers $k_1 \le \ldots \le k_p$ is called the *splitting type* of the bundle F. The splitting type uniquely defines the bundle. The sum $\sum_{j=1}^p k_j$ coincides with the degree of the bundle F.

Now, in the family $\mathcal{F}$, we distinguish a subset $\mathcal{E} \subset \mathcal{F}$ constructed by means of meromorphic transformations with matrices $\Gamma_i(z)$ from (12) and (13) of special form. To this end, we need the following definition of an admissible matrix.

Definition 3. Consider system (8) with an (irregular) singular point $a = a_i$ and a formal fundamental matrix $\widehat{Y}(z)$ of the form (9). A diagonal integer matrix $\Lambda_i = \operatorname{diag}(\Lambda_i^1, \dots, \Lambda_i^N)$ split into blocks in the same way as the matrix $Q(z)$ is called an *admissible matrix* for this system if

- the diagonal elements of the block Λ_i^j form a nonincreasing sequence for any unramified block $Q^j(z)$;
- Λ_i^j is a scalar matrix for any ramified block $Q^j(z)$.

We represent the matrix $\widehat{Y}(z)$ as follows:

$$\widehat{Y}(z) = \widehat{F}(z)(z-a)^{-\Lambda_i}(z-a)^{\Lambda_i}(z-a)^{\widehat{E}_i}e^{Q(z)}. \tag{14}$$

By virtue of an analog of the Sauvage lemma (see [11, Lemma 11.2]) for formal matrix series, there exists a matrix $\Gamma_i'(z)$ holomorphically invertible in O_i such that

$$\Gamma_i'(z)\widehat{F}(z)(z-a)^{-\Lambda_i} = (z-a)^K \widehat{F}_0(z), \tag{15}$$

where K is a diagonal integer matrix and $\widehat{F}_0(z)$ is an invertible formal (matrix) Taylor series around $z=a$.

Now, we define the required meromorphic transformation for each irregular singular point $a=a_i$ by the matrix $\Gamma^{\Lambda_i}(z) = (z-a)^{-K}\Gamma_i'(z)$, which depends on an admissible matrix Λ_i (since $\Gamma_i'(z)$ depends on Λ_i). It follows from (14) and (15) that the transformation $y' = \Gamma^{\Lambda_i}(z)y$ maps system (8) to a system with the formal fundamental matrix

$$\widehat{Y}'(z) = \widehat{F}_0(z)(z-a)^{\Lambda_i}(z-a)^{\widehat{E}_i}e^{Q(z)}.$$

As shown in [5], such a transformation does not increase the Poincaré rank r_i of system (8). The family $\mathcal{E}$ of extensions $(F^\Lambda, \nabla^\Lambda)$ of the pair (F, ∇) to the whole Riemann sphere is obtained by means of all possible sets $\Lambda = \{\Lambda_1, \dots, \Lambda_n\}$ of admissible matrices for the singularities $a_1, \dots, a_n$. Thus, the family $\mathcal{E}$ is a subset of the family $\mathcal{F}$. The degree of the bundle F^Λ can be calculated by the formula $\deg F^\Lambda = \sum_{i=1}^n \operatorname{tr}(\Lambda_i + \widehat{E}_i)$.

Note that the holomorphic triviality of one of the bundles in the family $\mathcal{E}$ implies the positive solvability of the Riemann–Hilbert problem (since $\mathcal{E} \subset \mathcal{F}$); however, the absence of the holomorphically trivial bundles in the family $\mathcal{E}$ does not imply the negative solution of the problem.

3. COUNTEREXAMPLES TO THE GENERALIZED RIEMANN–HILBERT PROBLEM

The generalized monodromy data (2), (8) are said to be *reducible* if both representation (2) and the local systems (8) are reducible. The reducibility of the latter means that they are meromorphically equivalent to systems with coefficients matrices of the same block upper triangular form. Otherwise the generalized monodromy data are said to be *irreducible*.

Among the sufficient conditions for the positive solution of the generalized Riemann–Hilbert problem that were obtained in [5], there was the following: if the singularity of one of the local systems (8) is formally *unramified* and the generalized monodromy data (2), (8) are *irreducible*, then the problem is solvable.

If the singularities of all local systems (8) are formally ramified, then the generalized Riemann–Hilbert problem may have a negative solution even in the case of irreducible monodromy data.

Example 1 (van der Put and Saito [7])**.** Consider a linear system of two equations,

$$\frac{dy}{dz} = B(z)y, \qquad B(z) = \begin{pmatrix} 0 & f(z) \\ 1 & 0 \end{pmatrix}, \tag{16}$$

where $f(z) = f_3z^3 + f_1z + f_0$ is a third-degree polynomial. Below we will show that the only singularity $z=\infty$ of this system is formally ramified; therefore, the system is irreducible.

System (16), together with the trivial monodromy representation, defines *irreducible* generalized monodromy data for which the generalized Riemann–Hilbert problem is unsolvable.

First of all, note that system (16) is obtained from the scalar equation

$$\frac{d^2u}{dz^2} - f(z)u = 0$$

by the substitution $y^1 = \frac{du}{dz}$ and $y^2 = u$; therefore, the Katz index K_∞ of the singular point $z = \infty$ of this system coincides with the corresponding Katz index of the equation, and, according to (11),

$$K_\infty = \frac{2 - \mathrm{ord}_\infty f(z)}{2} = \frac{5}{2}.$$

Thus, the singularity $z = \infty$ of system (16) is formally ramified and its minimal Poincaré rank is $r_\infty = \left\lceil \frac{5}{2} \right\rceil = 3$.

Suppose that the generalized Riemann–Hilbert problem has a positive solution, i.e., there exists a meromorphic transformation (in a neighborhood of infinity) $\widetilde{y} = \Gamma(z)y$ that maps system (16) to a system with the coefficient matrix

$$\widetilde{B}(z) = \frac{d\Gamma}{dz}\Gamma^{-1} + \Gamma B(z)\Gamma^{-1} = B_0 + B_1 z + B_2 z^2. \tag{17}$$

Then the entries of the matrix $\Gamma(z)$ are polynomials because they satisfy the system of linear differential equations $\frac{d\Gamma}{dz} = \widetilde{B}\Gamma - \Gamma B$ whose coefficients (the entries of the matrices $B(z)$ and $\widetilde{B}(z)$) are holomorphic in $\mathbb{C}$. It follows from the Liouville formula that the determinant of the matrix $\Gamma(z)$ satisfies the linear differential equation $\frac{d}{dz}\det\Gamma = \mathrm{tr}(\widetilde{B} - B)\det\Gamma$ and therefore does not vanish and is constant. We may assume that $\det\Gamma(z) \equiv 1$.

Introducing the notation $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)^* = \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$, we write the matrices Γ and Γ^{-1} in the form

$$\Gamma(z) = \Gamma_0 + \Gamma_1 z + \ldots + \Gamma_s z^s, \qquad \Gamma^{-1}(z) = \Gamma_0^* + \Gamma_1^* z + \ldots + \Gamma_s^* z^s, \qquad \Gamma_s \neq 0.$$

Calculating the coefficients of z^{2s+3} and z^{2s+1} in the matrix polynomial $\frac{d\Gamma}{dz}\Gamma^{-1} + \Gamma B(z)\Gamma^{-1}$ (which vanish in view of (17)) leads to the following conclusions:

$$\Gamma_s \begin{pmatrix} 0 & f_3 \\ 0 & 0 \end{pmatrix} \Gamma_s^* = 0 \quad \Rightarrow \quad \Gamma_s = \begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix}, \qquad \Gamma_{s-1} \begin{pmatrix} 0 & f_3 \\ 0 & 0 \end{pmatrix} \Gamma_{s-1}^* = 0 \quad \Rightarrow \quad \Gamma_{s-1} = \begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix}.$$

If $s = 1$, then the matrix $\Gamma(z) = \Gamma_0 + \Gamma_1 z$ has the form $\left(\begin{smallmatrix} * & * \\ 0 & 0 \end{smallmatrix}\right)$, which contradicts the fact that it is nondegenerate.

If $s > 1$, then the coefficient of z^{2s} in the matrix polynomial $\frac{d\Gamma}{dz}\Gamma^{-1} + \Gamma B(z)\Gamma^{-1}$ also vanishes; this implies that $\Gamma_s \left(\begin{smallmatrix} 0 & f_0 \\ 1 & 0 \end{smallmatrix}\right) \Gamma_s^* = 0$, i.e., $\Gamma_s = 0$. This fact contradicts the assumption $\Gamma_s \neq 0$.

Example 1 also shows that if all singularities of the scalar equation (3) are formally ramified, then the corresponding system (6) may not be meromorphically equivalent to system (1) with the same singular points whose Poincaré ranks are minimal.

Below, we give a counterexample to the generalized Riemann–Hilbert problem. This counterexample is based on the following proposition (see [3, Proposition 2.2.4, proof of Theorem 2.3.3, Corollary 2.3.2]).

Proposition 1. *For any set of points $a_1, \ldots, a_n \in \overline{\mathbb{C}}$ and any even number γ, $0 < \gamma \leq n - 2$, one can construct a three-dimensional representation $\chi^* \colon \pi_1(\overline{\mathbb{C}} \setminus \{a_1, \ldots, a_n\}) \to \mathrm{GL}(3, \mathbb{C})$ possessing the following properties*:

(a) *there exists a system* (1) *with singularities $a_1, \ldots, a_n$ and monodromy χ^*; the singularity a_1 is regular and its Poincaré rank is $\gamma/2$, while the other singularities are Fuchsian*;

(b) *there does not exist a similar system with smaller Poincaré rank at the point a_1.*

Example 2. Let $a_1, \dots, a_6 \in \overline{\mathbb{C}}$, $\gamma = 4$, and χ^* be an appropriate representation from Proposition 1. Denote by $G_1, \dots, G_6 \in \mathrm{GL}(3, \mathbb{C})$ generators of this representation and by $E_1, \dots, E_6$ the normalized logarithms of these generators (i.e., $E_k = \frac{1}{2\pi i} \ln G_k$).

Consider the following generalized monodromy data:

($*$) the representation χ^* and

($**$) the local systems $\frac{dy}{dz} = B_k(z)y$, $k = 1, \dots, 6$, where

$$B_1(z) = \frac{E_1}{z - a_1} - \frac{I}{(z - a_1)^2}, \qquad B_k(z) = \frac{E_k}{z - a_k}, \quad k = 2, \dots, 6.$$

Let us show that the generalized Riemann–Hilbert problem for the monodromy data ($*$), ($**$) has a negative solution.

Suppose that there exists a (global) system

$$\frac{dy}{dz} = B(z)y \tag{18}$$

that has singular points $a_1, \dots, a_6$ of Poincaré ranks $r_1 = 1$, $r_2 = \ldots = r_6 = 0$, respectively, and the monodromy χ^* and is meromorphically equivalent to the system $\frac{dy}{dz} = B_1(z)y$ in a neighborhood of the (irregular) singularity a_1. Then the fundamental matrix of this system can be written as

$$Y(z) = \Gamma_1(z)(z - a_1)^{E_1} e^{\frac{1}{z - a_1}},$$

where the matrix $\Gamma_1(z)$ is meromorphically invertible at the point a_1. Hence, the system

$$\frac{d\widetilde{y}}{dz} = \left(B(z) + \frac{I}{(z - a_1)^2} \right) \widetilde{y},$$

obtained from (18) by the substitution $\widetilde{y} = e^{-\frac{1}{z - a_1}} y$, has the same singularities and monodromy, but the point a_1 is a regular singularity of Poincaré rank 1 for the latter system. This fact contradicts one of the properties of the representation χ^* (property (b) in Proposition 1).

4. PROOF OF THEOREM 1

The proof of Theorem 1 consists of two parts. In the first part, using equation (3), we construct a pair consisting of a holomorphic vector bundle and a meromorphic connection and prove its stability (Lemmas 1 and 2). The second part is a modification of Theorem 2 from [5] (Lemma 3).

Recall that the family $\mathcal{F}$ consists of bundles with connections.

Lemma 1. *Among the elements of the family $\mathcal{F}$ constructed from equation* (3), *there exists a pair (F, ∇) such that*

$$F \cong \mathcal{O} \oplus \mathcal{O}(R + n - 2) \oplus \ldots \oplus \mathcal{O}((p - 1)(R + n - 2)), \qquad R = \sum_{i=1}^{n} r_i. \tag{19}$$

Proof. Let $a_1 = \infty, a_2, \dots, a_n \in \mathbb{C} \backslash \{0\}$ be the singular points of equation (3). This assumption does not restrict the generality because one singularity can always be transferred to infinity and another can be removed from zero by an appropriate linear fractional change of the independent variable. These transformations do not change the Katz indices of the equations.

Let us examine the singular points $a_1, \dots, a_n$ of system (6) constructed from equation (3) by the standard substitution (7). Recall that the minimal Poincaré rank r_i of the singularity a_i is related to its Katz index K_i by the formula $r_i = \lceil K_i \rceil$.

As shown in Section 2, the gauge transformation

$$\widetilde{y} = (z - a_i)^{(r_i+1)D} y, \qquad D = \mathrm{diag}(0, 1, \ldots, p-1), \quad i = 2, \ldots, n,$$

reduces system (6) to a system with Poincaré rank r_i at the point a_i (for all points except $a_1 = \infty$). At the same time, in view of (11),

$$\mathrm{ord}_\infty\, b_j(z) \geq -j(r_1 - 1), \qquad j = 1, \ldots, p; \tag{20}$$

therefore, the gauge transformation $\widetilde{y} = z^{-(r_1-1)D} y$ reduces system (6) to a system with Poincaré rank r_1 at the point $a_1 = \infty$.

Now, we consider a holomorphic vector bundle F of rank p over $\overline{\mathbb{C}}$ defined by the coordinate description $U_0 = \mathbb{C}$, $U_\infty = \overline{\mathbb{C}} \setminus \{0\}$, $g_{0\infty}(z) = z^{(R+n-2)D}$. Let us show that one can introduce a meromorphic connection ∇ in F such that the matrix differential 1-forms ω_0 and ω_∞ of the connection (defined in the neighborhoods U_0 and U_∞, respectively) possess the following properties:

(a) the systems $dy = \omega_0 y$ and $dy = \omega_\infty y$ are meromorphically equivalent to system (6) in U_0 and U_∞, respectively;

(b) the Poincaré ranks of the singularities ($a_2, \ldots, a_n$ and $a_1, \ldots, a_n$, respectively) of these systems are minimal.

Let us apply gauge transformations of the form $\widetilde{y} = \Gamma_0(z) y$ and $y' = \Gamma_\infty(z) y$ with

$$\Gamma_0(z) = \prod_{i=2}^{n} (z - a_i)^{(r_i+1)D}, \qquad \Gamma_\infty(z) = \prod_{i=2}^{n} \Big(\frac{z - a_i}{z}\Big)^{(r_i+1)D} z^{-(r_1-1)D}$$

to system (6) (in U_0 and U_∞, respectively). Then, according to the construction, the systems

$$d\widetilde{y} = \omega_0 \widetilde{y}, \qquad dy' = \omega_\infty y' \tag{21}$$

possess properties (a) and (b), and the matrix differential 1-forms ω_0 and ω_∞ define a meromorphic connection ∇ in the bundle F. The latter follows from the fact that systems (21) are related in $U_0 \cap U_\infty$ by the gauge transformation $\widetilde{y} = g_{0\infty}(z) y'$,

$$g_{0\infty}(z) = \Gamma_0(z) \Gamma_\infty^{-1}(z) = z^{(R+n-2)D};$$

hence, the 1-forms ω_0 and ω_∞ satisfy the required gluing condition

$$\omega_0 = (dg_{0\infty}) g_{0\infty}^{-1} + g_{0\infty} \omega_\infty g_{0\infty}^{-1}.$$

The lemma is proved. $\square$

The following lemma completes the first part of the proof of Theorem 1.

Lemma 2. *The pair (F, ∇) constructed in Lemma 1 is stable.*

Proof. We have to prove that for every subbundle $F^1 \subset F$ that is *stabilized* by the connection ∇, the *slope* $\mu(F) = \deg F / \mathrm{rk}\, F$ of the bundle F is greater than the slope $\mu(F^1)$ of F^1. Recall that the existence of a subbundle stabilized by the connection implies that the pair (F, ∇) admits a coordinate description in which the matrices that define the cocycle of the bundle F have the same block upper triangular form as the matrix differential 1-forms that define the connection ∇.

Note at once that the holomorphic type of the bundle F is known (see (19)); therefore, we can calculate its slope

$$\mu(F) = \frac{(R + n - 2)(p - 1)}{2}.$$

If the bundle F has a subbundle F^1 of rank l that is stabilized by the connection ∇, then the monodromy of equation (3) is reducible. Consider a basis $u_1(z), \dots, u_p(z)$ of the solution space of equation (3) such that the first l of its elements form a basis of a monodromy-invariant subspace X^l of solutions. Note that the fundamental matrix $Y(z)$ of system (6) has the form

$$Y(z) = \begin{pmatrix} u_1(z) & u_2(z) & \dots & u_p(z) \\ \frac{du_1(z)}{dz} & \frac{du_2(z)}{dz} & \dots & \frac{du_p(z)}{dz} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{d^{p-1}u_1(z)}{dz^{p-1}} & \frac{d^{p-1}u_2(z)}{dz^{p-1}} & \dots & \frac{d^{p-1}u_p(z)}{dz^{p-1}} \end{pmatrix}.$$

Now, consider a linear differential equation

$$\frac{1}{W(u_1, \dots, u_l)} \det \begin{pmatrix} u_1 & \dots & u_l & u \\ \frac{du_1}{dz} & \dots & \frac{du_l}{dz} & \frac{du}{dz} \\ \vdots & \ddots & \vdots & \vdots \\ \frac{d^l u_1}{dz^l} & \dots & \frac{d^l u_l}{dz^l} & \frac{d^l u}{dz^l} \end{pmatrix} = 0 \tag{22}$$

of order l for the unknown $u(z)$, where

$$W(u_1, \dots, u_l) = \det \begin{pmatrix} u_1 & \dots & u_l \\ \frac{du_1}{dz} & \dots & \frac{du_l}{dz} \\ \vdots & \ddots & \vdots \\ \frac{d^{l-1}u_1}{dz^{l-1}} & \dots & \frac{d^{l-1}u_l}{dz^{l-1}} \end{pmatrix}$$

is the Wronskian of the functions $u_1(z), \dots, u_l(z)$. Since these functions are linearly independent and generate a monodromy-invariant subspace, it follows that $W(u_1, \dots, u_l) \not\equiv 0$ and the coefficients of equation (22) are single-valued functions on the Riemann sphere. Below we will need the following important proposition.

Proposition 2. *The coefficients of equation* (22) *are meromorphic functions on the Riemann sphere.*

Proof. Outside the singular points $a_1, \dots, a_n$ of the original equation (3), equation (22) may have singularities only at the zeros of the Wronskian $W(u_1, \dots, u_l)$. The latter are poles for the coefficients of equation (22); therefore, to prove Proposition 2, it suffices to establish that

(a) the set $\{b_\alpha\}$ of zeros of the Wronskian $W(u_1, \dots, u_l)$ is finite, and

(b) the coefficients of equation (22) have poles at the points $a_1, \dots, a_n$.

It suffices to prove that the zero set of the Wronskian $W(u_1, \dots, u_l)$ has no limit points. Clearly, the point $z_0 \in \overline{\mathbb{C}} \setminus \{a_1, \dots, a_n\}$ cannot be a limit point because the function $W(u_1(z), \dots, u_l(z)) \not\equiv 0$ is holomorphic in a neighborhood of this point. Now, we show that the points $a_1, \dots, a_n$ cannot be limit points for the set $\{b_\alpha\}$.

Due to formula (9) and the reducibility of system (6), a fundamental system $(\widehat{u}_1(z), \dots, \widehat{u}_p(z))$ of formal solutions of equation (3) in a neighborhood $D(a)$ of the point $a = a_i$ has the form

$$(\widehat{u}_1(z), \dots, \widehat{u}_p(z)) = \big(\widehat{f}_1(z), \dots, \widehat{f}_p(z)\big)(z-a)^{\widehat{E}} e^{Q(z)}, \tag{23}$$

$$\widehat{E} = \begin{pmatrix} \widehat{E}^1 & * \\ 0 & \widehat{E}^2 \end{pmatrix}, \qquad Q(z) = \begin{pmatrix} Q^1(z) & 0 \\ 0 & Q^2(z) \end{pmatrix},$$

where $\widehat{f}_j(z)$ are formal Laurent series about $z = a$ with finite principal part and the blocks $\widehat{E}^1$ and $Q^1(z)$ are of dimension $l \times l$.

As mentioned in Section 2, the neighborhood $D(a)$ can be covered by a finite set of sectors with vertices at the point a so that in each sector the formal fundamental system (23) is asymptotic for some real system. If the point a is a limit point of the set $\{b_\alpha\}$, then one of the sectors (denote it by S_a) contains an infinite number of points b_α that approximate to a. Take a fundamental system $(v_1,\dots,v_p)$ of solutions of equation (3) for which (23) is asymptotic in S_a and in which the first l elements form a basis of the space X^l; i.e., $(v_1,\dots,v_l)=(u_1,\dots,u_l)C$, $C\in \mathrm{GL}(l,\mathbb{C})$. The latter is possible since system (6) is reducible. Then the points b_α are zeros of the Wronskian $W(v_1,\dots,v_l)$.

By construction, the vector function $(v_1(z),\dots,v_l(z))e^{-Q^1(z)}(z-a)^{-\widehat{E}^1}$ has an asymptotic representation $(\widehat{f}_1(z),\dots,\widehat{f}_l(z))$ in the sector S_a. Therefore, using the basic properties of asymptotic series (addition, multiplication, and differentiation; see [6, Theorems 8.2, 8.3, 8.8]) and the properties of the determinant of a matrix, we find that the formal series $W(\widehat{f}_1(z),\dots,\widehat{f}_l(z))=\sum_{j=-m}^{\infty} f_j(z-a)^j$ is asymptotic for the function

$$f(z)=W(v_1(z),\dots,v_l(z))e^{-\operatorname{tr}Q^1(z)}(z-a)^{-\operatorname{tr}\widehat{E}^1}$$

in the sector S_a. However, since this function vanishes at the points b_α, it follows that this series is a zero series. Indeed,

$$(z-a)^m f(z)-f_{-m}=o(1),\qquad z\to a,\quad z\in S_a;$$

therefore, approaching the point a along the set $\{b_\alpha\}\cap S_a$, we obtain $f_{-m}=0$. Then, we successively prove that all f_j, $j\ge -m$, vanish.

Since $W(\widehat{f}_1(z),\dots,\widehat{f}_l(z))\equiv 0$, the series $\widehat{f}_1(z),\dots,\widehat{f}_p(z)$ are linearly dependent. Consequently, $W(\widehat{u}_1(z),\dots,\widehat{u}_p(z))\equiv W(\widehat{f}_1(z),\dots,\widehat{f}_p(z))\equiv 0$, which is impossible for the formal fundamental system $(\widehat{u}_1(z),\dots,\widehat{u}_p(z))$. Thus, we have shown that the set of zeros of the Wronskian $W(u_1,\dots,u_l)$ is finite. It remains to verify that the points $a_1,\dots,a_n$ cannot be essential singular points for the coefficients of equation (22).

Denote by $V_j(u_1,\dots,u_l)$ the determinant of the matrix obtained by eliminating the $(j+1)$th row of the $(l+1)\times l$ matrix

$$\begin{pmatrix} u_1 & \dots & u_l\\ \frac{du_1}{dz} & \dots & \frac{du_l}{dz}\\ \vdots & \ddots & \vdots\\ \frac{d^lu_1}{dz^l} & \dots & \frac{d^lu_l}{dz^l}\end{pmatrix}.$$

Then the coefficient of the derivative $\frac{d^ju}{dz^j}$ in equation (22) is $\frac{V_j(u_1,\dots,u_l)}{W(u_1,\dots,u_l)}$, $j=0,1,\dots,l$. It follows from the above analysis that the functions

$$V_j(u_1(z),\dots,u_l(z))e^{-\operatorname{tr}Q^1(z)}(z-a)^{-\operatorname{tr}\widehat{E}^1}$$

have the asymptotic representation $\mathrm{const}\cdot V_j(\widehat{f}_1(z),\dots,\widehat{f}_l(z))$ in each of the sectors covering the neighborhood $D(a)$ of the point $a=a_i$ (the constant depends only on the sector). Therefore, each ratio $\frac{V_j(u_1,\dots,u_l)}{W(u_1,\dots,u_l)}$ has the same asymptotic representation (obtained by dividing the series $V_j(\widehat{f}_1(z),\dots,\widehat{f}_l(z))$ and $W(\widehat{f}_1(z),\dots,\widehat{f}_l(z))$, see [6, Theorem 8.5]) in all sectors.

Thus, the coefficients of equation (22), which are holomorphic in the annular neighborhood $D(a)\setminus\{a\}$, can be asymptotically represented by Laurent series in this neighborhood; i.e., the point a is a pole for these coefficients (see [6, Theorem 8.6]). $\square$

Now we return to the proof of Lemma 2. We have established that the singularities of equation (22) with meromorphic coefficients are the singular points $a_1,\dots,a_n$ of the original equation (3) and the additional false singularities $b_1,\dots,b_h\in\overline{\mathbb{C}}\setminus\{a_1,\dots,a_n\}$, the zeros of the Wronskian $W(u_1(z),\dots,u_l(z))$. The latter are Fuchsian because the solutions are holomorphic at these points.

It is also clear that the Katz indices of the singularities a_i of equation (22) are not greater than the corresponding Katz indices of equation (3).

Thus, the coefficients of equation (22) expressed in the standard form

$$\frac{d^l u}{dz^l} + q_1(z)\frac{d^{l-1}u}{dz^{l-1}} + \ldots + q_l(z)u = 0 \tag{24}$$

have the corresponding orders of poles; i.e., the functions

$$q_j(z)\prod_{i=2}^{n}(z-a_i)^{j(r_i+1)}\prod_{k=1}^{h}(z-b_k)^j \qquad \text{and} \qquad q_j(z)z^{-j(r_1-1)}$$

are holomorphic in $\mathbb{C}$ and at ∞, respectively ($j = 1, \ldots, l$).

Using equation (24), we can obtain expressions for the derivatives of order $m \geq l$ of any of the functions $u_1(z), \ldots, u_l(z)$ in terms of the derivatives of order less than m. One can easily verify that these expressions have the form

$$\frac{d^m u}{dz^m} = \widetilde{q}_1^{\,m}(z)\frac{d^{m-1}u}{dz^{m-1}} + \ldots + \widetilde{q}_m^{\,m}(z)u, \qquad m \geq l,$$

where the functions

$$\widetilde{q}_j^{\,m}(z)\prod_{i=2}^{n}(z-a_i)^{j(r_i+1)}\prod_{k=1}^{h}(z-b_k)^j \qquad \text{and} \qquad \widetilde{q}_j^{\,m}(z)z^{-j(r_1-1)}$$

are also holomorphic in $\mathbb{C}$ and at ∞, respectively ($j = 1, \ldots, m$).

Let us apply the gauge transformation $\widetilde{y} = \Gamma(z)y$, where

$$\Gamma(z) = \begin{pmatrix} 1 & & & & & & \\ 0 & 1 & & & & & \\ \vdots & \ddots & \ddots & & & & \\ 0 & \ldots & 0 & 1 & & & \\ -\widetilde{q}_l^{\,l}(z) & \ldots\ldots & & -\widetilde{q}_1^{\,l}(z) & 1 & & \\ \vdots & \ldots\ldots\ldots\ldots & & & \ddots & \ddots & \\ -\widetilde{q}_{p-1}^{\,p-1}(z) & \ldots\ldots\ldots\ldots\ldots & & & & -\widetilde{q}_1^{\,p-1}(z) & 1 \end{pmatrix}$$

is a lower triangular matrix, to the fundamental matrix $Y(z)$ of system (6). This transformation subtracts a linear combination of the previous rows (with coefficients $\widetilde{q}_1^{\,m-1}, \ldots, \widetilde{q}_{m-1}^{\,m-1}$) from each row of the matrix $Y(z)$ with number $m > l$ and leaves the first l rows unchanged. The entries of the transformed matrix $\Gamma(z)Y(z)$ that occupy the intersection of the last $p-l$ rows and first l columns are zero.

By construction, the matrices $\widetilde{\Gamma}_0(z) = \Gamma_0(z)\Gamma(z)\Gamma_0^{-1}(z)$ and $\widetilde{\Gamma}_\infty(z) = \Gamma_\infty(z)\Gamma(z)\Gamma_\infty^{-1}(z)$ are holomorphically invertible in the domains $U_0 \setminus \{b_1, \ldots, b_h\}$ and $U_\infty \setminus \{b_1, \ldots, b_h\}$, respectively; therefore, the transformations $\widetilde{x} = \widetilde{\Gamma}_0(z)\widetilde{y}$ and $x' = \widetilde{\Gamma}_\infty(z)y'$ applied to system (21) map the matrix differential 1-forms ω_0 and ω_∞, which define the meromorphic connection ∇ with singularities $a_1, \ldots, a_n$ in the bundle F, to the matrix differential 1-forms

$$\widetilde{\omega}_0 = \widetilde{\Gamma}_0\omega_0\widetilde{\Gamma}_0^{-1} + (d\widetilde{\Gamma}_0)\widetilde{\Gamma}_0^{-1}, \qquad \widetilde{\omega}_\infty = \widetilde{\Gamma}_\infty\omega_\infty\widetilde{\Gamma}_\infty^{-1} + (d\widetilde{\Gamma}_\infty)\widetilde{\Gamma}_\infty^{-1}$$

of block upper triangular form, which define a meromorphic connection $\widetilde{\nabla}$ with singularities $a_1, \ldots, a_n$, $b_1, \ldots, b_h$ in the same bundle F (although $\widetilde{\Gamma}_0(z)$ and $\widetilde{\Gamma}_\infty(z)$ have singularities in U_0 and U_∞, respectively):

$$\widetilde{g}_{0\infty} = \widetilde{\Gamma}_0 g_{0\infty}\widetilde{\Gamma}_\infty^{-1} = (\Gamma_0\Gamma\Gamma_0^{-1})\Gamma_0\Gamma_\infty^{-1}(\Gamma_\infty\Gamma^{-1}\Gamma_\infty^{-1}) = \Gamma_0\Gamma_\infty^{-1} = g_{0\infty}.$$

Thus, the bundle F has a subbundle $\widetilde{F}^1 \subset F$ that is stabilized by the connection $\widetilde{\nabla}$, and

$$\widetilde{F}^1 \cong \mathcal{O} \oplus \mathcal{O}(R+n-2) \oplus \ldots \oplus \mathcal{O}((l-1)(R+n-2)).$$

Hence,

$$\mu(\widetilde{F}^1) = (R+n-2)\frac{l-1}{2} < (R+n-2)\frac{p-1}{2} = \mu(F). \tag{25}$$

On the other hand, denoting by $\widetilde{\nabla}^1$ and ∇^1 the restrictions of the connections $\widetilde{\nabla}$ and ∇ to the subbundles $\widetilde{F}^1$ and F^1, respectively, we obtain

$$\deg \widetilde{F}^1 = \sum_{i=1}^{n} \operatorname{tr}\operatorname{res}_{a_i} \widetilde{\nabla}^1 + \sum_{k=1}^{h} \operatorname{tr}\operatorname{res}_{b_k} \widetilde{\nabla}^1 = \sum_{i=1}^{n} \operatorname{tr}\operatorname{res}_{a_i} \nabla^1 + \sum_{k=1}^{h} \operatorname{res}_{b_k} \frac{dW(u_1,\ldots,u_l)}{W(u_1,\ldots,u_l)} = \deg F^1 + H,$$

where

$$H = \sum_{k=1}^{h} \operatorname{res}_{b_k} \frac{dW(u_1,\ldots,u_l)}{W(u_1,\ldots,u_l)} = \sum_{k=1}^{h} \operatorname{ord}_{b_k} W(u_1,\ldots,u_l) \geq 0$$

(the equalities $\operatorname{tr}\operatorname{res}_{a_i}\widetilde{\nabla}^1 = \operatorname{tr}\operatorname{res}_{a_i}\nabla^1$ follow from the holomorphic equivalence at a_i of the corresponding matrix differential 1-forms that define the connections $\widetilde{\nabla}^1$ and ∇^1).

Thus, $\mu(F^1) \leq \mu(\widetilde{F}^1) < \mu(F)$, which proves the stability of the pair (F, ∇). $\square$

The first part of the proof of Theorem 1 is complete.

Due to (25), the inequality $\mu(F) > \mu(F^1)$ is oversatisfied:

$$\mu(F) - \mu(F^1) > (R+n-2)\frac{p-l}{2}. \tag{26}$$

This allows us to apply the following lemma, which completes the proof of Theorem 1.

Lemma 3. *Suppose that the singular point of each system* (8) *is formally unramified. If the family $\mathcal{F}$ constructed by the generalized monodromy data* (2), (8) *contains a stable pair* (F, ∇) *for which the inequalities for the slopes hold in the strengthened form* (26), *then the corresponding generalized Riemann–Hilbert problem has a positive solution.*

This lemma complements one of the main results of [5]: *If the singularity of at least one of the systems* (8) *is formally unramified and the subfamily $\mathcal{E} \subset \mathcal{F}$ contains a stable pair, then the corresponding generalized Riemann–Hilbert problem has a positive solution.*

Proof. According to [5], it suffices to show that the existence of a stable pair $(F, \nabla) \in \mathcal{F}$ satisfying the conditions of the lemma implies the existence of a stable pair $(F^\Lambda, \nabla^\Lambda) \in \mathcal{E}$. To find a set $\Lambda = \{\Lambda_1, \ldots, \Lambda_n\}$ of admissible matrices generating a stable pair $(F^\Lambda, \nabla^\Lambda)$, we will use the following procedure, which is composed of three transformations and is applied to each system (8). This procedure is thoroughly described in [8].

1. Conjugating by a constant nondegenerate matrix, we reduce the leading coefficient $B^i_{-r_i-1}$ of expansion (8) to the block diagonal form $\operatorname{diag}(B^1, \ldots, B^m)$, where each block B^j of dimension p_j $(p_1 + \ldots + p_m = p)$ has a unique eigenvalue λ_j and $\lambda_j \neq \lambda_k$ for $j \neq k$. According to Sibuya's splitting lemma (see, for example, [6, Theorem 11.1]), system (8) can be formally reduced by a holomorphic transformation $\widehat{y} = \widehat{\Gamma}(z)y$ to a system with the matrix of coefficients (which are formal in general) $\widehat{B}_i(z) = \operatorname{diag}(\widehat{B}^1_i(z), \ldots, \widehat{B}^m_i(z))$ of the same block diagonal form.

The formal fundamental matrices $\widehat{Y}^j(z)$ of the subsystems corresponding to the blocks $\widehat{B}^j_i(z)$ have the form (9):

$$\widehat{Y}^j(z) = \widehat{F}^j(z)(z-a_i)^{\widehat{E}^j} e^{Q^j(z)},$$

where $\widehat{F}^j(z)$ is a formal (matrix) Laurent series about $z = a_i$ with finite principal part and $\det \widehat{F}^j(z) \not\equiv 0$, $\widehat{E}^j$ is an upper triangular matrix, and $Q^j(z)$ is a diagonal matrix whose entries are polynomials in $1/(z-a_i)$. Each block $\widehat{F}^j(z)$ can be represented as $\widehat{F}^j(z) = \widehat{V}^j(z)M^j(z)$, where $\widehat{V}^j(z)$ is an invertible formal (matrix) Taylor series about $z = a_i$ and $M^j(z)$ is an *upper triangular* matrix meromorphic at the point a_i (see, for example, [8]). Thus, the holomorphic transformation

$$y^{\mathbf{1}} = \widehat{V}^{-1}(z)\widehat{\Gamma}(z)y, \qquad \widehat{V}(z) = \operatorname{diag}\big(\widehat{V}^1(z), \dots, \widehat{V}^m(z)\big),$$

reduces formally system (8) to a system with the coefficient matrix $B_i^{\mathbf{1}}(z) = \operatorname{diag}(B_i^1(z), \dots, B_i^m(z))$ of block diagonal form, with each block $B_i^j(z)$ being an upper triangular matrix:

$$B_i^j(z) = \frac{dM^j}{dz}(M^j)^{-1} + M^j\left(\frac{\widehat{E}^j}{z-a_i} + (z-a_i)^{\widehat{E}^j}\frac{dQ^j}{dz}(z-a_i)^{-\widehat{E}^j}\right)(M^j)^{-1}.$$

2. The transformation

$$y' = e^{q^j(z)}y, \qquad q^j(z) = \frac{\lambda_j/r_i}{(z-a_i)^{r_i}},$$

applied to each subsystem $\frac{dy}{dz} = B_i^j(z)y$ of the system obtained above reduces the former to the system

$$\frac{dy'}{dz} = \left(B_i^j(z) - \frac{\lambda_j}{(z-a_i)^{r_i+1}}I\right)y', \qquad j = 1, \dots, m.$$

Since the leading coefficient of the Laurent series of the matrix $B_i^j(z)$ is an upper triangular matrix with a unique eigenvalue λ_j, it follows that the leading coefficient of the Laurent series of the matrix $B_i^j(z) - \frac{\lambda_j}{(z-a_i)^{r_i+1}}I$ is an upper triangular nilpotent matrix. Thus, the transformation

$$y^{\mathbf{2}} = e^{Q(z)}y^{\mathbf{1}}, \qquad Q(z) = \operatorname{diag}\big(q^1(z)I^{p_1}, \dots, q^m(z)I^{p_m}\big),$$

where I^{p_j} is the identity matrix of dimension p_j, takes the system

$$\frac{dy^{\mathbf{1}}}{dz} = B_i^{\mathbf{1}}(z)y^{\mathbf{1}}$$

to a system with the block diagonal coefficient matrix $B_i^{\mathbf{2}}(z) = B_i^{\mathbf{1}}(z) + \frac{dQ}{dz}$.

3. Each diagonal block $B_i^j(z) - \frac{\lambda_j}{(z-a_i)^{r_i+1}}I$ of the matrix $B_i^{\mathbf{2}}(z)$ is an upper triangular matrix such that the leading coefficient of its Laurent series is a nilpotent matrix. Hence, the transformation

$$y^{\mathbf{3}} = (z-a_i)^{-D}y^{\mathbf{2}}, \qquad D = \operatorname{diag}(D^1, \dots, D^m), \qquad D^j = \operatorname{diag}(0, 1, \dots, p_j - 1),$$

takes the system

$$\frac{dy^{\mathbf{2}}}{dz} = B_i^{\mathbf{2}}(z)y^{\mathbf{2}}$$

to a system with the coefficient matrix

$$B_i^{\mathbf{3}}(z) = (z-a_i)^{-D}B_i^{\mathbf{2}}(z)(z-a_i)^D - \frac{D}{z-a_i}.$$

The Poincaré rank of the latter at the point a_i is less than the Poincaré rank of the original system (8) at least by one. Moreover, the form of the above-described transformations implies the relation

$$\operatorname{tr}\operatorname{res}_{a_i} B_i^{\mathbf{3}}(z) = \operatorname{tr}\operatorname{res}_{a_i} B_i(z) - \operatorname{tr} D = \operatorname{tr}\operatorname{res}_{a_i}\nabla - \sum_{j=1}^{m}\frac{p_j(p_j-1)}{2}. \tag{27}$$

Applying at most r_i steps of the above-described procedure, we obtain a system

$$\frac{dy}{dz} = B_i^{\mathrm{F}}(z)y$$

with a Fuchsian singularity a_i. Such a system has a fundamental matrix of the form

$$Y_i^{\mathrm{F}}(z) = U_i(z)(z-a_i)^{\Lambda_i}(z-a_i)^{\widehat{E}_i},$$

where $U_i(z)$ is a matrix that is holomorphically invertible at the point a_i and Λ_i is an admissible matrix. Therefore, $\operatorname{tr}(\Lambda_i + \widehat{E}_i) = \operatorname{tr}\operatorname{res}_{a_i} B_i^{\mathrm{F}}(z)$ and, according to (27),

$$\operatorname{tr}\operatorname{res}_{a_i}\nabla = \sum h + \operatorname{tr}\operatorname{res}_{a_i} B_i^{\mathrm{F}}(z). \tag{28}$$

By $\sum h$ we denoted the sum of quantities of the form $h = \sum_{j=1}^{m} p_j(p_j-1)/2$ over all steps of the procedure (at different steps, the numbers m of blocks and their dimensions p_j are generally different).

The set Λ of the obtained admissible matrices $\Lambda_1,\dots,\Lambda_n$ for the singularities $a_1,\dots,a_n$ defines an element $(F^\Lambda,\nabla^\Lambda)$ of the subfamily $\mathcal{E}\subset\mathcal{F}$. Let us show that the pair $(F^\Lambda,\nabla^\Lambda)$ is stable.

First of all, note that

$$\operatorname{tr}\operatorname{res}_{a_i}\nabla^\Lambda = \operatorname{tr}(\Lambda_i + \widehat{E}_i) = \operatorname{tr}\operatorname{res}_{a_i} B_i^{\mathrm{F}}(z). \tag{29}$$

Let $E^1\subset F^\Lambda$ be a rank l subbundle stabilized by the connection ∇^Λ, and $F^1\subset F$ be the corresponding subbundle stabilized by the connection ∇. Denote by ∇_1^Λ and ∇^1 the restrictions of the connections ∇^Λ and ∇ to these subbundles. In a neighborhood of the point a_i, system (8) is holomorphically equivalent to the system

$$\frac{dy}{dz} = \begin{pmatrix} C_i(z) & * \\ 0 & * \end{pmatrix} y$$

with the coefficient matrix of block upper triangular form, where $C_i(z)$ is an $l\times l$ block. Consider a step of the procedure applied to the subsystem defined by this block. Assume that this subsystem, just as system (8), is split into m blocks of dimensions l_j, but some l_j may vanish: $0\le l_j\le p_j$, $l_1+\dots+l_m = l$. Then, according to (27), using the notation introduced above, we obtain the relations

$$\operatorname{tr}\operatorname{res}_{a_i}\nabla^1 = h^1 + \operatorname{tr}\operatorname{res}_{a_i} C_i^{\mathbf{3}}(z), \qquad h^1 = \sum_{j=1}^{m}\frac{l_j(l_j-1)}{2},$$

$$\operatorname{tr}\operatorname{res}_{a_i}\nabla = h + \operatorname{tr}\operatorname{res}_{a_i} B_i^{\mathbf{3}}(z), \qquad h = \sum_{j=1}^{m}\frac{p_j(p_j-1)}{2}.$$

The following proposition plays a key role in the proof of the stability of the pair $(F^\Lambda,\nabla^\Lambda)$.

Proposition 3. *The numbers h^1 and h satisfy the inequality*

$$\frac{h}{p} - \frac{h^1}{l} \le \frac{p-l}{2}. \tag{30}$$

Proof. Taking into account the relations $\sum_{j=1}^{m} p_j = p$ and $\sum_{j=1}^{m} l_j = l$, we rewrite inequality (30) as

$$\frac{\sum_{j<k} p_j p_k}{\sum_{j=1}^{m} p_j} \ge \frac{\sum_{j<k} l_j l_k}{\sum_{j=1}^{m} l_j}.$$

To prove this inequality, it suffices to verify that the function

$$F(x_1,\dots,x_m)=\frac{\sum_{j<k}x_jx_k}{\sum_{j=1}^m x_j}$$

increases in each variable on the set $\mathbb{R}^m_{\geq 0}\setminus\{0\}$. Let us fix some values of the variables $x_1,\dots,x_{m-1}$: $x_1=c_1\geq 0$, …, $x_{m-1}=c_{m-1}\geq 0$, and consider the function $f(x)=F(c_1,\dots,c_{m-1},x)$ of a variable $x\in\mathbb{R}_{\geq 0}$:

$$f(x)=\frac{ax+b}{x+a},\qquad a=\sum_{j=1}^{m-1}c_j,\qquad b=\sum_{1\leq j<k\leq m-1}c_jc_k.$$

Its derivative

$$\frac{df}{dx}=\frac{a^2-b}{(x+a)^2}=\frac{\sum_{j=1}^{m-1}c_j^2+b}{(x+a)^2}$$

is nonnegative; this proves that the function $F(x_1,\dots,x_m)$ increases in the variable x_m. For the other variables, the proof is similar. $\square$

In at most r_i steps we obtain a system with a Fuchsian singular point a_i; so, according to the construction and equalities (28) and (29), we have

$$\operatorname{tr}\operatorname{res}_{a_i}\nabla^1=\sum h^1+\operatorname{tr}\operatorname{res}_{a_i}\nabla_1^\Lambda,\qquad \operatorname{tr}\operatorname{res}_{a_i}\nabla=\sum h+\operatorname{tr}\operatorname{res}_{a_i}\nabla^\Lambda,$$

where

$$\frac{\sum h}{p}-\frac{\sum h^1}{l}\leq\frac{p-l}{2}r_i$$

in view of Proposition 3. Hence,

$$\frac{1}{p}\operatorname{tr}\operatorname{res}_{a_i}\nabla-\frac{1}{l}\operatorname{tr}\operatorname{res}_{a_i}\nabla^1=\frac{1}{p}\operatorname{tr}\operatorname{res}_{a_i}\nabla^\Lambda-\frac{1}{l}\operatorname{tr}\operatorname{res}_{a_i}\nabla_1^\Lambda+\Delta_i,\qquad \Delta_i\leq\frac{p-l}{2}r_i.$$

Summing over all singular points, we obtain the relation

$$\mu(F)-\mu(F^1)=\mu(F^\Lambda)-\mu(E^1)+\Delta,\qquad \Delta\leq\frac{p-l}{2}R.$$

Since $\mu(F)-\mu(F^1)>(R+n-2)(p-l)/2$, it follows that the quantity

$$\mu(F^\Lambda)-\mu(E^1)=\mu(F)-\mu(F^1)-\Delta>\frac{p-l}{2}(n-2)$$

is positive for $n>1$, which proves the stability of the pair $(F^\Lambda,\nabla^\Lambda)$. $\square$

Note that applying the arguments from the proof of Lemma 1 and the sufficient condition ($\triangle$) (see [4, Addendum 1]), we can augment Theorem 1 with the following statement.

Theorem 2. *System* (6) *corresponding to equation* (3) *one of whose singularities is Fuchsian is meromorphically equivalent to system* (1) *with the same singular points whose Poincaré ranks are minimal.*

In conclusion, we conjecture that in the hypotheses of Theorem 1 it suffices to require that at least *one* singularity of equation (3), rather than *all* of them, should be formally unramified.

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