On Some Higher Order BVP at Resonance

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1 Introduction

Let us consider a nonlinear boundary value problem of shooting type

$$x^{(n)} + \lambda x = f(t, x) + y(t) \tag{1}$$

$$x^{(i)}(a) = 0 \quad (0 \le i \le n-1,\ i \ne m), \quad x^{(p)}(b) = 0 \tag{2}$$

where $n \ge 2$, $-\infty < a < b < \infty$ and $m, p \quad (0 \le m < p \le n-1)$ are fixed integer numbers. Let the function $f(t, x) : [a, b] \times \mathbb{R} \to \mathbb{R}$ be locally bounded, satisfy Caratheodory condition and be sublinear:

$$\lim_{|x|\to\infty} \sup_{a \le t \le b} x^{-1}|f(t, x)| = 0. \tag{3}$$

For $\lambda = 0$ problem (1)–(2) was studied in [1] in the case where $f(t, s)$ satisfies some Landesman-Lazer type conditions. Point $\lambda = 0$ is an asymptotic bifurcation point (point of bifurcation at infinity) for (1)–(2) . Zero is a simple eigenvalue of the linear operator $Lx = x^{(n)}$ with boundary conditions (2).

In our paper problem (1)–(2) is considered for λ from a neighborhood of zero. Restrictions for nonlinearity are given which guarantee

i. *a priori* estimate and solvability for $\lambda \le 0$;

ii. multiplicity results for $\lambda > 0$.

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Under the restrictions any Landesman-Lazer type conditions are not valid. Similar multiplicity result was introduced in [2] for periodic problem and for two point boundary value problem for a quasilinear second order differential equation.

Main results are formulated in the next section and are proved in Sections 3–5. Theorem 2 being given in Section 6 concerns the problem (1)–(2) with nonlinearity, depending on λ. In Section 8 multiplicity results of special type are presented.

2 The main theorem

Let the function $f(t,x)$ satisfy the estimate

$$|f(t,x)| \le c(1+|x|^{\gamma}) \qquad (t \in [a,b], \;\; x \in \mathbb{R}) \tag{4}$$

with $\gamma \in [0,1)$ for some $c > 0$. Of course the function $f(t,x)$ is sublinear in this case.

Let some set Ω of positive measure exist such that

$$f(t,x)\mathbf{sign}x \ge \begin{cases} \Phi(|x|), & t \in \Omega, \;\; (|x| \ge u_0), \\ 0, & t \notin \Omega \end{cases} \tag{5}$$

where $u_0 > 0$ and $\Phi(u)$ $(u \ge u_0)$ is positive (strictly!) nonincreasing and continuous.

Theorem 1 *Let $y(t) \in L_1$ and*

$$\int_a^b y(t)(b-t)^{n-p-1} dt = 0. \tag{6}$$

Let, if $m > 0$, the following conditions be valid:

either, if $\gamma = 0$ and $\Omega = [a, a+a_0]$,

$$\int_{u_0}^{\infty} \Phi(u) u^{m^{-1}-1} du = \infty, \tag{7}$$

or, in all other cases,

$$2m\gamma < \sqrt{1+4m} - 1 \tag{8}$$

and

$$\lim_{u \to \infty} \Phi(u) u^{(1-\gamma m - \gamma^2 m)m^{-1}}. \tag{9}$$

Then there exists an $\varepsilon > 0$ such that:

i. for $0 < \lambda \le \varepsilon$ problem (1)–(2) has at least three solutions;

ii. for $-\varepsilon \le \lambda \le 0$ problem (1)–(2) has at least one solution, the set of all solutions of (1)–(2) is a priori *bounded.*

Note, that in the case $m = 0$ Fredholm condition (6) garantees the conclusions of Theorem 1 for our type of nonlinearities.

The dual to Theorem 1 preposition is valid. Under the assumptions of Theorem 1 the equation

$$x^{(n)} + \lambda x = -f(t,x) + y(t)$$

with boundary condition (2) has at least one solution for $0 \le \lambda \le \varepsilon$ and at least three solutions for $-\varepsilon \le \lambda < 0$.

Under the assumptions of Theorem 1 the point $\lambda_0 = 0$ is an asymptotic bifurcation point ([3]) (the point of bifurcation at infinity).

3 The case $\lambda \le 0$. Proof

Consider our problem for $\lambda \le 0$. Fix some sufficiently small $\lambda_0 > 0$. Since 0 is the simple isolated eigenvalue of the operator $x^{(n)}$ with boundary conditions (2) we see that equation $x^{(n)} - \lambda_0 x = z(t)$ has an unique solution $x(t) = Az(t)$ for every $z(t) \in L_1 = L_1([a,b];\mathbb{R})$. This solution $x(t)$ has absolutely continuous derivative of the $(n-1)$th order and $x^{(n)} \in L_1$. The operator A acts from L_1 to C^{n-1} (and in L_1) being completely continuous. The function $e(t) = (t-a)^m$ is the simple eigenfunction of A and $Ae = -\lambda_0^{-1} e$. Consider the family of operator equations

$$x = A[-(\lambda + \lambda_0)x + f(t,x) + y(t)] \tag{10}$$

for $\lambda \in [-\varepsilon, 0]$ ($\varepsilon > 0$ is sufficiently small). Every solution $x \in L_1$ of (10) belongs to C^{n-1} and $x(t)$ is a solution of (1)–(2) . On the contrary, every solution $x(t)$ of (1)–(2) is a solution of (10).

All the operators

$$B_\lambda x = A[-(\lambda + \lambda_0)x + f(t,x) + y(t)] \qquad (\lambda \in [-\varepsilon, 0])$$

are completely continuous in L_1 and for $\lambda = -\varepsilon$ the rotation of the field $x - B_\lambda x$ on spheres

$$S_\rho = \{x \in L_1,\ \|x\|_{L_1} = \rho\}$$

of sufficiently large radii ρ is equal to $(-1)^\beta$. The value β is the sum of multiplicities of all the real eigenvalues of $(\varepsilon - \lambda_0)A$ which are greater than 1. Therefore if we prove that all equations (10) with $\lambda \in [-\varepsilon, 0]$ have no any solutions outside of some ball

$$S_\rho = \{x \in L_1,\ \|x\|_{L_1} \le \rho\}$$

it will complete the proof of the conclusion of Theorem 1 for $\lambda \le 0$.

The function $g(t) = (b-t)^{n-p-1}$ is the eigenfunction of the operator A^* corresponding to the eigenvalue λ_0^{-1}. Every function $x(t) \in L_1$ has the form $x(t) = \xi e(t) + h(t)$ where

$$< h(t), g(t) >= \int_a^b h(t)\, g(t)\, dt = 0.$$

Lemma 1 *All solutions* $x = \xi e(t) + h(t) \quad (< h, g >= 0)$ *of* (10) *for* $\lambda \in [-\varepsilon, 0]$ *satisfy the inequalities*

$$\xi \int_a^b g(t)\, f[t, x(t)]\, dt \le 0 \tag{11}$$

and

$$\|h\|_{L_\infty} \le c(1 + |\xi|)^\gamma \tag{12}$$

for some $c > 0$.

Proof. Let $x = \xi + h$ be a solution of (10) for some $\lambda \in [-\varepsilon, 0]$. Then

$$\xi \int_a^b g(t)\, x(t)\, dt = \xi \int_a^b g(t)\, A[-(\lambda_0 + \lambda)x + f(t,x) + y(t)]\, dt$$

and (since $< g, Au >= -\lambda_0^{-1} < g, u > \quad (u \in L_1)$)

$$\xi^2 < g, e >= -\lambda_0^{-1} \left\{ -(\lambda_0 + \lambda)\xi^2 < g, e > +\xi \int_a^b g(t)\, f[t, x(t)]\, dt \right\}.$$

Therefore

$$\lambda \xi^2 < g, e >= \xi \int_a^b g(t)\, f[t, x(t)]\, dt$$

and by $\lambda \leq 0$ (11) is proved. Denote

$$Qx(t) = x(t) - \frac{e(t) < g, x >}{< e, g >}.$$

The operator Q is projector in the space L_1 on the invariant for A subspace $E_1 = \{x(t) \in L_1,\ < x, g >= 0\}$ of co-dimension 1. All the linear operators $Qx - AQ(-(\lambda_0 + \lambda)x)$ act in E_1 being continuously invertible uniformly in $\lambda \in [-\varepsilon, 0]$. Thus by

$$Qx = AQ[-(\lambda_0 + \lambda)x + f(t, x) + y(t)]$$

the following inequalities hold:

$$\|h\|_{L_\infty} \leq \|A\|_{L_1 \to L_\infty} \cdot \|\, \{Q - AQ[-(\lambda_0 + \lambda)]\}^{-1} \,\|_{L_1 \to L_1} \cdot \|f(t, x)\|_{L_1}.$$

Therefore (by (4)) a $c > 0$ exists such that (12) is valid. Lemma 1 is completely proved.

Under the assumptions of Theorem 1 the theorems on integral-functional inequalities can be used introduced in [4].

Theorem 2.1 from [4] garantees *a priori* estimate of all solutions $\{\xi; h(t)\}$ of inequalities (11) and (12) if

$$\chi(\delta) = \int_{\{|e(t)| \leq \delta\}} |g(t)|\, dt > 0 \qquad (\delta > 0) \tag{13}$$

and

$$\lim_{u \to \infty} \frac{\int_\Omega |g(t)|\, \Phi[u_0 + 2u|e(t)|]\, dt}{\chi\Big(u^{-1}\big[u_0 + c(1 + |u|)^\gamma\big]\Big) u^{\gamma^2}} = \infty \tag{14}$$

Condition (13) is obviously valid for our $e(t) = (t - a)^m$ if $m \geq 1$. In this case function (13) can be exactly estimated:

$$c_2 \delta^{1/m} \leq \chi(\delta) \leq c_1 \delta^{1/m} \quad (\delta \geq 0).$$

Using last estimates one can easily obtain that conditions (14) is also holds under the assumptions of Theorem 1.

If $m = 0$ then $\chi(\delta) = 0$ for $0 \leq \delta < 1$. In this case Theorem 2.2 from [4] garantees the *a priori* estimate of $\{\xi, h(t)\}$.

The conclusion of Theorem 1 for $\lambda \leq 0$ is proved.

4 The case $\lambda > 0$. General lemma

Definition 1 *Two vectors x_1 and x_2 of some space are called* **absolutely different (AD)**, *if $\mu x_1 + (1-\mu)x_2 = 0$ for some $\mu \in [0,1]$. Two vector fields are called AD on some set if they are AD at least in one point of the set.*

The following fact is the corollary of general degree theory: if two non-zero on the boundary of some domain Γ vector fields are not AD on Γ, then the rotations of these fields on Γ coincide.

Consider in a Banach space H a linear completely conyinuous (= compact + continuous) operator B having a simple eigenvalue 1. Let $Be = e$, $B^*g = g$, $E_0 = \{\xi e,\ \xi \in \mathbb{R}\}$, $E_1 = \{< g, e >= 0\}$, P and Q be projectors on E_0 and E_1 correspondently. The projectors P and Q commute with B.

Consider the following set in H:

$$G = G(c, \gamma, r, R) =$$

$$\{x \in H : \ \|Qx\| \le c(1 + \|Px\|)^\gamma;\ r \le \|x\| \le R;\ \gamma \in [0,1);\ Px = \xi e;\ \xi > 0;\ c, r, R > 0\}.$$

The boundary ∂G of G is the unification of three surfaces:

$$\Gamma_1 = \{r \le \|x\| \le R,\ \|Qx\| = c(1 + \|Px\|)^\gamma,\ Px = \xi e,\ \xi > 0\}$$

$$\Gamma_2 = \{\|x\| = r,\ \|Qx\| = c(1 + \|Px\|)^\gamma,\ Px = \xi e,\ \xi > 0\}$$

$$\Gamma_3 = \{\|x\| = R,\ \|Qx\| = c(1 + \|Px\|)^\gamma,\ Px = \xi e,\ \xi > 0\}.$$

Lemma 2 *Let some completely continuous vector field Φx (i.e. $\Phi x = x - Tx$ where T is a completely continuous operator) satisfy the following assumptions:*

i. On Γ_1 the fields $Q\Phi x$ and $Q(x - Bx)$ are not AD;

ii. On Γ_2 the fields $P\Phi x$ and e are not AD;

iii. On Γ_3 the fields $P\Phi x$ and $-e$ are not AD.

Then the rotation of the field Φx on ∂G is equal to $(-1)^\beta$ where β is the sum of multiplicities of real eigenvalues of B which are greater than 1.

Proof of this Lemma one can obtain by general methods of the degree theory (cf., e.g., [3]).

5 The case $\lambda > 0$. Proof

Let G, Γ_1, Γ_2, Γ_3 be the sets being defined in previous Section. The proof consists from 5 stages.

Stage 1. The fields $Q(x - B_\lambda x)$ and $Q(x + (\lambda_0 + \lambda)Ax)$ are not AD on Γ_1 for some large $c > 0$ and for sufficiently small λ.

Stage 2. The fields $P(x - B_0 x)$ and e are not AD on Γ_2. Let us prove this statement. Consider the family of the fields

$$\Phi_\mu x = \mu P(x - B_0 x) + (1-\mu)e \qquad (0 \le \mu \le 1).$$

Since

$$\Phi_\mu x = \left[(1-\mu) + \frac{\mu}{\lambda_0 < e, g >} \int_a^b g(s)\, f[s, x(s)]\, ds\right] e(t) \tag{15}$$

therefore $\Phi_\mu x \neq 0$ for sufficiently small μ for some $r > 0$ (because $f(t,x)$ is sublinear). For $\mu \geq \mu_0$ by (15) the equality $\Phi_\mu x = 0$ implies (11) hence for sufficiently large r field $\Phi_\mu x$ is non-zero for all $\mu \in [0,1]$.

Stage 3. The fields $P(x - B_\lambda x)$ and e are not AD on Γ_2 for some sufficiently small $\lambda > 0$. It follows from Stage 2 and complete continuity of $B_\lambda x$.

Stage 4. The fields $P(x - B_\lambda x)$ and $-e$ are not AD on Γ_3 for every $\lambda > 0$ for sufficiently large $R > 0$. It follows from the relations

$$P(x - B_\lambda x) = \left[-Px\frac{\lambda}{\lambda_0} + \frac{< g, f(t,x) >}{\lambda_0 < e, g >} e(t)\right], \quad Px = \xi e(t)$$

and

$$\xi\frac{\lambda}{\lambda_0} > 0$$

due to sublinearity of $f(t,x)$.

Stage 5. Due to Lemma 2 for every small $\lambda > 0$ problem (1)–(2) has at least one solution in G (sets G depend on λ). One can show that in the set $-G$ another solution of (1)–(2) lays. And the third solution of (1)–(2) lays in the ball $\{\|x\| \leq r\}$. $\square$

6 Nonlinearities, depending on a parameter

Consider for $|\mu_0 - \mu| \leq \varepsilon_0$ the function

$$d(x;\mu) = \begin{cases} d_+(\mu) & for \quad x \geq |d_+(\mu)|, \\ d_-(\mu) & for \quad x \leq |d_-(\mu)|, \\ x\mathbf{sign}\{d_+(\mu)\} & for \quad 0 \leq x \leq |d_+(\mu)|, \\ -x\mathbf{sign}\{d_-(\mu)\} & for \quad -|d_-(\mu)| \leq x \leq 0. \end{cases} \tag{16}$$

The graph of this function for $d_+(\mu) > 0$ and $d_-(\mu) < 0$ one can see in Fig. 1.

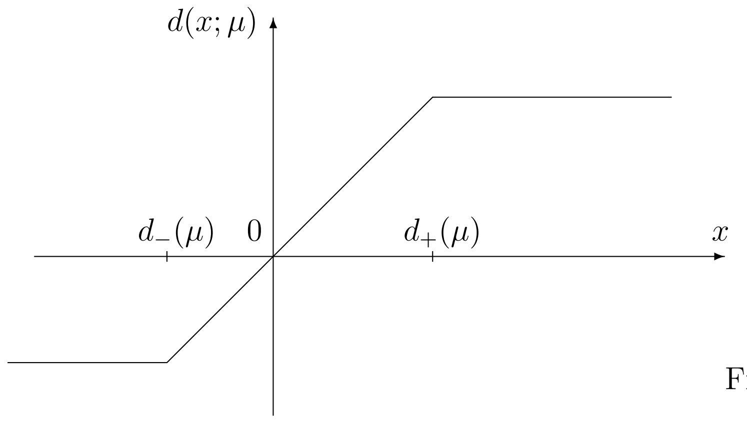

Fig.1

The functions $d_+(\mu)$ and $d_-(\mu)$ assumed to be continuous in μ. Consider the problem

$$\begin{gathered} x^{(n)} = d(x;\mu) + f(t,x;\mu) + y(t;\mu), \\ x^{(i)}(a) = 0 \quad (0 \le i \le n-1,\ i \ne m), \quad x^{(p)}(b) = 0 \end{gathered} \tag{17}$$

Let $f(t,x;\mu) \to 0$ for $|x| \to \infty$. In this case nonlinearity $g(t,x;\mu) = d(x;\mu) + f(t,x;\mu)$ satisfies Landesman-Lazer conditions

$$\lim_{x\to\pm\infty} g(t,x;\mu) = d_\pm(\mu)$$

for every μ.

Theorem 2 *Let $f(t,x;\mu)$ for every μ satisfy assumption* (5) *for $x \ge u_0$ where Ω, Φ are common for all values of μ. Let Φ satisfy assumptions of Theorem 1 with $\gamma = 0$. Let* (6) *be valid for every μ. Let*

i. $d_-(\mu_0) < 0$,
ii. $d_+(\mu_0) = 0$,
iii. $d_+(\mu) > 0$ for $\mu > \mu_0$,
iv. $d_+(\mu) < 0$ for $\mu < \mu_0$.

Then μ_0 is an asymptotic bifurcation point for (17)*, for $\mu \ge \mu_0$ at least one solution of* (17) *exists, for $\mu < \mu_0$ at least two solutions of* (17) *exist.*

Some dual to Theorem 2 results can be given.

7 Proof scheme of Theorem 2

For every $\mu \ge \mu_0$ the nonlineariry $g(t,x;\mu)$ satisfies all the conditions of Theorem 1 (uniformly in μ) for the case $\lambda = 0$. Therefore *a priori* estimate is valid for all such values of μ and the rotation of the corresponding vector field

$$\Psi_\mu x = x - A\left[-\mu_0 x + d(x;\mu) + f(t,x;\mu) + y(t,\mu)\right]$$

on spheres S_ρ $(\rho \ge \rho_0)$ is equal to $(-1)^\beta$. The existence of at least one solution of (17) for this case is proved.

The rotation of the field $\Psi_\mu x$ on the sphere S_{ρ_0} is also equal to $(-1)^\beta$ for $\mu < \mu_0$ and sufficiently small $\mu_0 - \mu$.

But the rotation of the field $\Psi_\mu x$ on spheres S_R of sufficiently large radii R is equal to 0 under the assumptions of Theorem 2. This fact is the corollary of the main theorem on rotation from [5], it can be checked by direct methods.

Due to principle of changing index [3] μ_0 is an asymptotic bifurcation points for (17). For $\mu < \mu_0$ we have at least two solutions of (17): one from the ball B_{ρ_0} and the other onelays outside B_{ρ_0} but inside B_R. $\square$

8 Bifurcation by forcing term

In this Section a new type of similar results are introduced.

Consider our problem when only forcing term depends on μ:

$$\begin{gathered} x^{(n)} = f(t,x) + y(t;\mu), \\ x^{(i)}(a) = 0 \quad (0 \le i \le n-1,\ i \ne m), \quad x^{(p)}(b) = 0. \end{gathered} \tag{18}$$

Let the function $f(t,x)$ satisfy all the assumptions of Theorem 1 with $\gamma = 0$. More then this let $f(t,x)$ tend to 0 for $|x| \to \infty$.

Consider the function

$$\Gamma(\mu) = \int_a^b y(t;\mu)\,(b-t)^{n-p-1} dt.$$

Theorem 3 *Let $\mu = \mu_0$ is an isolated zero of the function $\Gamma(\mu)$. Then μ_0 is an asymptotic bifurcation point of* (18)*; for $\mu = \mu_0$ problem* (18) *has at least one solution, the set of all solutions of* (18) *in this case is bounded. For $\mu \ne \mu_0$ and $|\mu - \mu_0|$ is sufficiently small problem* (18) *has at least two solutions.*

To prove Theorem 3 we consider the field

$$\Phi_\mu x = x - A\left[-\lambda_0 x + f(t,x) + y(t;\mu)\right].$$

For $\mu = \mu_0$ this field has been studied in the proof of Theorem 1. We obtained that for $\mu = \mu_0$ problem (18) has at least one solution, the set of all solutions of (18) is bounded, index at infinity of the field is equal to $(-1)^\beta$. Thus we have for sufficiently close to μ_0 values of μ a bounded brunch of solutions of (18). But for $\mu \ne \mu_0$ the index at infinity of Φ_μ is equal to 0 (see [5]). Therefore according to principle of changing index ([3]) μ_0 is an asymptotic bifurcation point of (18) and there are exist unbounded continua from the both sides of μ_0. $\square$

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