Learning Volatility of Discrete Time Series Using Prediction with Expert Advice

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The talk in outline

- Arbitrage strategies based on difference between macro and micro volatilities of a time series
- Prediction with expert advice - Hannan's algorithm in case of unbounded gains
- Merging the arbitrage strategies using Hannan's algorithm
- Results of experiments

Consider a discrete time series of stock prices

$$S_0 = S(0), S_1 = S(T/(KM)), S_2 = S(2T/(KM)) \dots, S_{KM} = S(T).$$

The macro volatility is represented by the sum

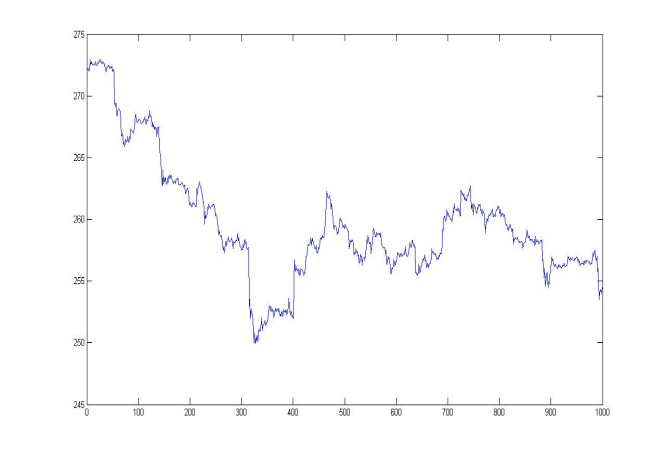
$$\sum_{i=0}^{K-1} (S_{(i+1)T} - S_{iT})^2,$$

and the micro volatility is represented by the sum

$$\sum_{i=0}^{KT-1} (\Delta S_i)^2,$$

where $\Delta S_i = S_{i+1} - S_i$, $i = 1, \dots KT$
In this paper for simplicity we consider the case $K = 1$.

Stock price



Evolution of the price of a stock

The difference between macro and micro volatility sums

$$(S_T - S_0)^2 - \sum_{t=0}^{T-1} (\Delta S_t)^2 =$$

$$(\sum_{t=0}^{T-1} \Delta S_t)^2 - \sum_{t=0}^{T-1} (\Delta S_t)^2 =$$

$$\sum_{t=0}^{T-1} 2(S_t - S_0)\Delta S_t$$

Vovk, V.: A game-theoretic explanation of the $\sqrt{dt}$ effect, (2003), http://www.probabilityandfinance.com

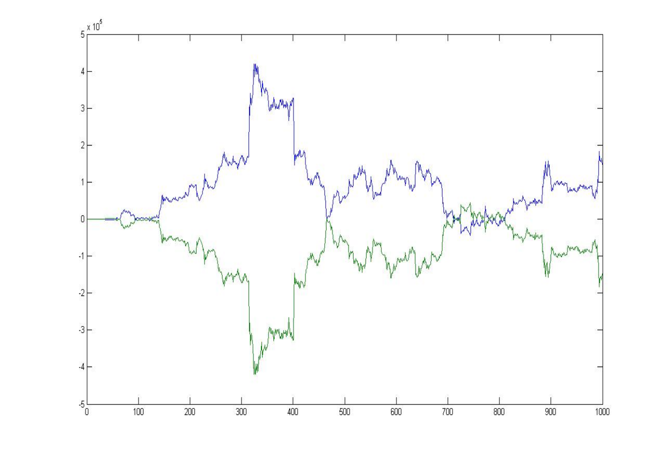
We consider two strategies

$$s_t^1 = 2(S_t - S_0)\Delta S_t,$$
$$s_t^2 = -2(S_t - S_0)\Delta S_t.$$

These strategies earn the incomes on steps $t = 1, \dots T$

$$s_{1:T}^1 = \sum_{t=0}^{T-1} s_t^1 = (S_T - S_0)^2 - \sum_{t=0}^{T-1} (\Delta S_t)^2,$$
$$s_{1:T}^2 = \sum_{t=0}^{T-1} s_t^2 = \sum_{t=0}^{T-1} (\Delta S_t)^2 - (S_T - S_0)^2.$$

Zero-sum game



Gains and losses of two experts.
Green line - micro volatility expert; Blue line - macro volatility expert.

We use methods of prediction with expert advice to merge these two strategies online

It is a peculiarity of our strategies that

- s_t^1 and s_t^2 can not be represented as values of a specific gain or loss function. Only general gains can be used.
- the absolute value of one-step gains or losses s_t^i, $i = 1, 2$, can be unrestrictedly large.

We use the Hannan's Follow the Perturbed Leader FPL algorithm (1957); rediscovered by Kalai and Vempala (2005).

Prediction with expert advice

s_t^i - gain of expert $i = 1, \ldots N$ at step t, $s_t^i \in (-\infty, +\infty)$

s_t - Learner gain at step t, $s_t \in (-\infty, +\infty)$

$s_{1:T}^i = \sum_{t=1}^{T} s_t^i$ - cumulative gain of expert i on steps $\leq T$

$s_{1:T} = \sum_{t=1}^{T} s_t^i$ - cumulative gain of Learner on steps $\leq T$

Learner's goal

$$s_{1:T} \geq \max_i s_{1:T}^i - \text{regret}$$

Following the Perturbed Leader FPL algorithm:

Output prediction of an expert i which maximizes

$$s^i_{1:t-1} + \frac{1}{\varepsilon}\xi^i,$$

where ξ^i, $i = 1, \dots N$, $t = 1, 2, \dots$, is a sequence of i.i.d random variables distributed according to the exponential distribution with the density $p(x) = \exp\{-x\}$, and ε is *a learning rate*.

Why randomization? An example.

In the deterministic framework, Learner can perform much worse than each expert:

let the current losses of two experts on steps $t = 0, 1, \dots$ be $s^1_{0,1,2,3,4,5,6,\dots} = \frac{1}{2}, -1, 1, -1, 1, -1, 1, \dots$ and $s^2_{0,1,2,3,4,5,6\dots} = 0, 1, -1, 1, -1, 1, -1, \dots$.

"Follow Leader" algorithm always chooses the wrong prediction.

Case: one-step gains of experts are bounded $0 \leq s_t^i \leq 1$.

In case, where algorithms suffer gains, Kalai and Vempala, Hutter and Poland results can be reformulated such that the expected cumulative gains of the FPL algorithm with variable learning rate $\epsilon_t = O(1/\sqrt{t})$ has the lower bound

$$E(s_{1:T}) \geq \max_{i=1,\ldots,N} s^i_{1:T} - O(\sqrt{T \log N}),$$

where N is the number of experts.

In that follows we allow gains at any step to be unbounded

$$s_t^i \in (-\infty, +\infty).$$

We use only general gains - the notion of a gain function is not used

Loss is a negative gain.

Volume of a game at step t

$$v_t = \sum_{j=1}^{t} \max_i |s_j^i|.$$

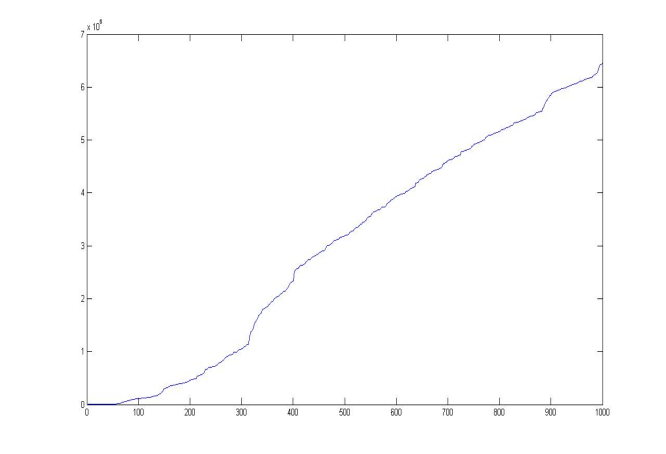
Scaled fluctuation of a game at step t.

$$\mathrm{fluc}(t) = \frac{\Delta v_t}{v_t} = \frac{\max_i |s_t^i|}{v_t},$$

where $\Delta v_t = v_t - v_{t-1}$.

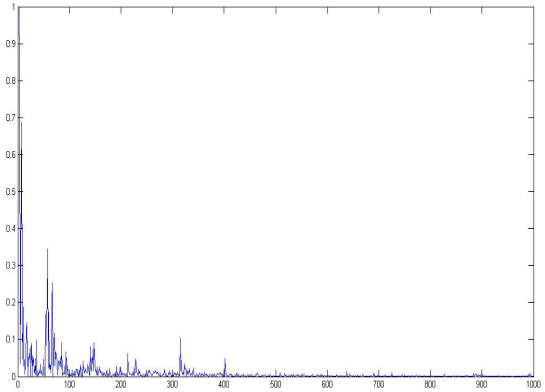
By definition $v_{t-1} \le v_t$ for all t and $0 \le \mathrm{fluc}(t) \le 1$ for all t.

Volume of the game



Evolution of the volume of the zero-sum game

Fluctuation of the game



Fluctuations of the zero-sum game

Asymptotic performance of probabilistic algorithms

A probabilistic algorithm is called asymptotically consistent in the mean if

$$\liminf_{T\to\infty} \frac{1}{T} E(s_{1:T} - \max_{i=1,\dots N} s^i_{1:T}) \ge 0.$$

A probabilistic algorithm is called asymptotically consistent in the mean in the modified sense if

$$\liminf_{T\to\infty} \frac{1}{v_T} E(s_{1:T} - \max_{i=1,\dots N} s^i_{1:T}) \ge 0.$$

Theorem

For any probabilistic algorithm of choosing an expert and for any ε such that $0 < \varepsilon < 1$, two experts exist such that

$$v_t \to \infty \text{ as } t \to \infty,$$
$$\mathrm{fluc}(t) \geq 1 - \varepsilon,$$
$$\frac{1}{v_t} E(\max_{i=1,2} s^i_{1:t} - s_{1:t}) \geq \frac{1}{2}(1 - \varepsilon)$$

for all t, where $s_{1:t}$ is the cumulative gain of this algorithm.

A probabilistic algorithm can be asymptotically consistent only in games where

$$\mathrm{fluc}(t) = \frac{\Delta v_t}{v_t} \to 0 \text{ as } t \to \infty.$$

What is a suitable sufficient condition for a game?

We consider games, such that $\mathrm{fluc}(t) \le \gamma(t)$ for all t, where $\gamma(t)$ is a computable non-increasing real function such that $0 \le \gamma(t) \le 1$ for all t and $\gamma(t) \to 0$ as $t \to \infty$.
We consider non-degenerate games, i.e., such that $v_t \to \infty$ as $t \to \infty$.

Learning rate of the FPL algorithm

We define a learning rate

$$\varepsilon_t = \frac{1}{\mu_t v_{t-1}}, \text{ where}$$

$$\mu_t = \sqrt{\frac{6}{1+\ln N}}(\gamma(t))^{1/2}.$$

It holds $\mu_t \leq \mu_{t-1}$ and $v_t \geq v_{t-1}$ for all t.

FPL algorithm.
FOR $t = 1, \dots T$
Define

$$I_t = \mathrm{argmax}_{i=1,2,\dots N} \left\{ s^i_{1:t-1} + \frac{1}{\varepsilon_t} \xi^i \right\},$$

where $\varepsilon_t = 1/(\mu_t v_{t-1})$ is the learning rate.

Receive one-step gains s^i_t for experts $i = 1, \dots, N$, and receive one-step gain $s_t = s^{I_t}_t$ of the master algorithm.
ENDFOR
Here, $\xi^1, \dots \xi^N$ are i.i.d random variables, distributed according to the density $p(x) = \exp\{-x\}$.

FPL algorithm: regret

Theorem

Let a game satisfies

$$\mathrm{fluc}(t) \le \gamma(t) \text{ for all } t,$$

where $\gamma(t)$ is a computable non-increasing real function such that $0 \le \gamma(t) \le 1$ for all t.
Then the expected cumulated gain of the FPL algorithm with the variable learning rate is bounded by

$$E(s_{1:T}) \ge \max_i s^i_{1:T} - 2\sqrt{6(1+\ln N)} \sum_{t=1}^{T} (\gamma(t))^{1/2} \Delta v_t.$$

FPL algorithm: asymptotic consistency in the mean

Theorem

Let a game satisfies

$$\mathrm{fluc}(t) \le \gamma(t) \text{ for all } t,$$

where $\gamma(t)$ is a computable non-increasing real function such that

- $0 \le \gamma(t) \le 1$ *for all* t,
- $\gamma(t) \to 0$, *and also*,
- $v_t \to \infty$ *as* $t \to \infty$.

Then the algorithm is asymptotically consistent in the mean

$$\liminf_{T\to\infty} \frac{1}{v_T} E(s_{1:T} - \max_{i=1,\dots N} s^i_{1:T}) \ge 0.$$

Details of the proof

We use an auxiliary **IFPL** algorithm. General scheme:

$$\mathrm{Loss}_{1:T}(\mathbf{FPL}) \geq \mathrm{Loss}_{1:T}(\mathbf{IFPL}) - \mathrm{regret}_1,$$
$$\mathrm{Loss}_{1:T}(\mathbf{IFPL}) \geq \max_i s^i_{1:T} - \mathrm{regret}_2,$$
$$\mathrm{Loss}_{1:T}(\mathbf{FPL}) \geq \max_i s^i_{1:T} - \mathrm{regret}_1 - \mathrm{regret}_2$$

IFPL algorithm.
FOR $t = 1, \ldots T$
Define the learning rate

$$\epsilon'_t = \frac{1}{\mu_t v_t}, \text{ where}$$

$$\mu_t = \sqrt{\frac{6}{1 + \ln N}} (\gamma(t))^{1/2}$$

Choose an expert with the minimal perturbed cumulated loss on steps $\leq t$

$$J_t = \mathrm{argmax}_{i=1,2,\ldots N} \left\{ s^i_{1:t} + \frac{1}{\epsilon'_t} \xi^i \right\}.$$

Receive the one step loss $\tilde{s}_t = s^{J_t}_t$ of the IFPL algorithm.
ENDFOR

Lemma

The cumulated expected losses of the FPL *and* IFPL *algorithms satisfy the inequality*

$$E(s_{1:T}) \geq E(\tilde{s}_{1:T}) - 6\sum_{t=1}^{T}(\gamma(t))^{1-\alpha_t}\Delta v_t$$

for all T, where $\alpha_t < 1$.

The optimal choice

$$\alpha_t = \frac{1}{2}\left(1 - \frac{\ln\frac{1+\ln N}{6}}{\ln\gamma(t)}\right)$$

Lemma 2

Lemma

The expected cumulative loss of the IFPL *algorithm is bounded*

$$E(\tilde{s}_{1:T}) \geq \max_i s^i_{1:T} - (1 + \ln N) \sum_{t=1}^{T} (\gamma(t))^{\alpha_t} \Delta v_t$$

for all T.

By definition $\mu_t = (\gamma(t))^{\alpha_t}$.

Corollary

Assume that $|s_t^i| \leq t^a$ *for all* t *and* $i = 1, \ldots N$, *and*

$$\liminf_{t \to \infty} \frac{v_t}{t^{a+\delta}} > 0,$$

where a *and* δ *are positive real numbers. Let also,* $\gamma(t) = t^{-\delta}$ *and* μ_t *is defined above. Then*

- *our algorithm is asymptotically consistent in the mean for any* $a > 0$ *and* $\delta > 0$;
- *the expected cumulated gain of this algorithm is bounded*

$$E(s_{1:T}) \geq \max_i s_{1:T}^i - O\left(\sqrt{\ln N} T^{1-\frac{1}{2}\delta+a}\right)$$

as $T \to \infty$.

Derandomized version of the algorithm

Learner's gain at step t

$$G_t = P\{I_t = 1\}s_t^1 + P\{I_t = 2\})s_t^2$$

Learner's cumulative gain at steps $\leq T$

$$G_{1:T} = \sum_{t=1}^{T} G_t = E(s_{1:T})$$

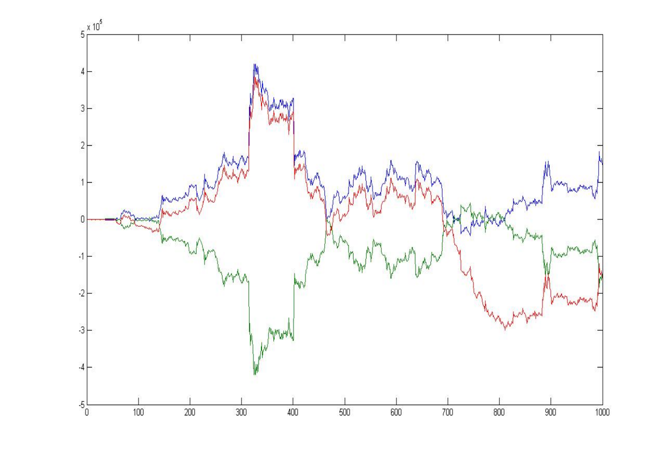
Lower bound

$$G_{1:T} \geq |s_{1:T}^1| - 8\sum_{t=1}^{T} (\gamma(t))^{1/2} |s_t^1|$$

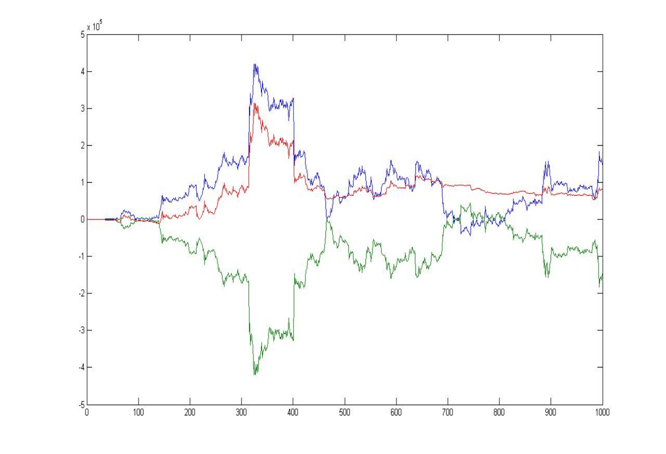
Corollary

The FPL algorithm is asymptotically riskless

$$\liminf_{T\to\infty} \frac{1}{v_T} G_{1:T} \geq 0.$$

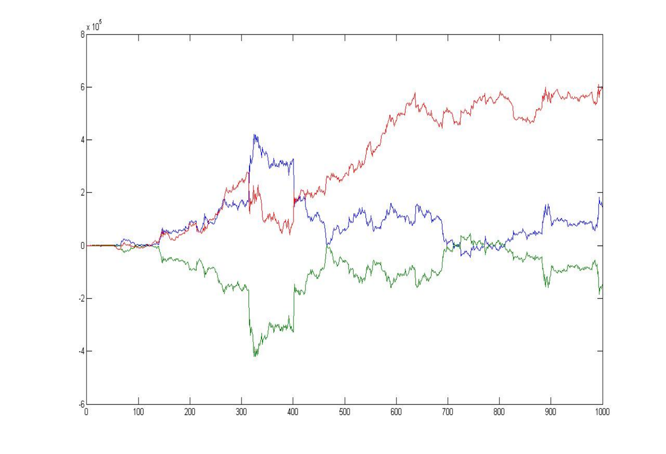


Gain of the FL (Follow Leader) algorithm (without randomization). Green and Blue lines - gains of experts; Red line - Learner gain



The FPL algorithm's expected gain.
Green and Blue lines - gains of experts; Red line - Learner gain

Merging strategies



Some runs of FPL are excellent

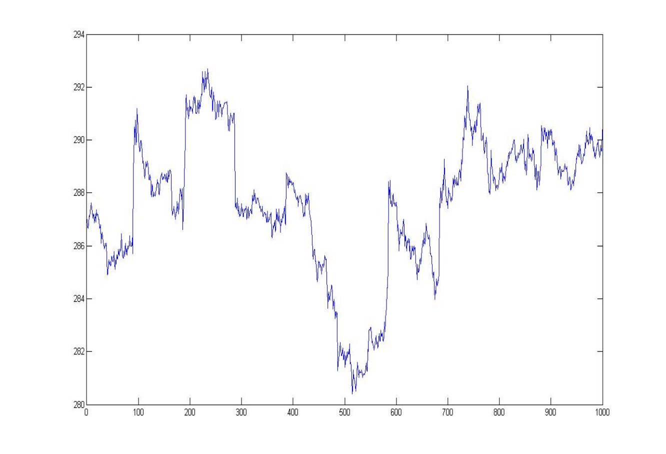
Conclusions

We have studied two different problems:

- How to use the fractional Brownian motion of prices to suffer gain with “minimal” risk on financial market;
- How to extend methods of the theory of prediction with expert advice for the case when experts one-step gains are unbounded.

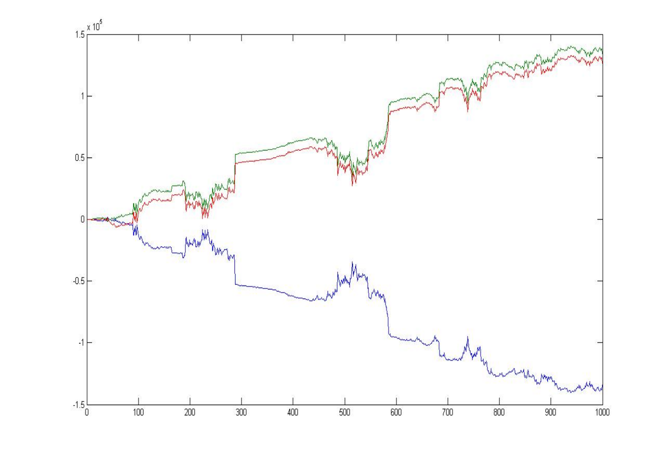
We use a solution of the second problem to solve the first one.

Stock price



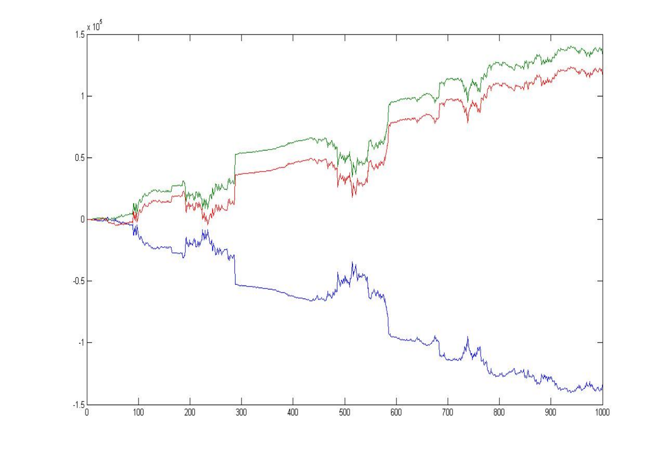
Evolution of the stock price (s=1000,1000)

Gain from high volatility. Constant leader



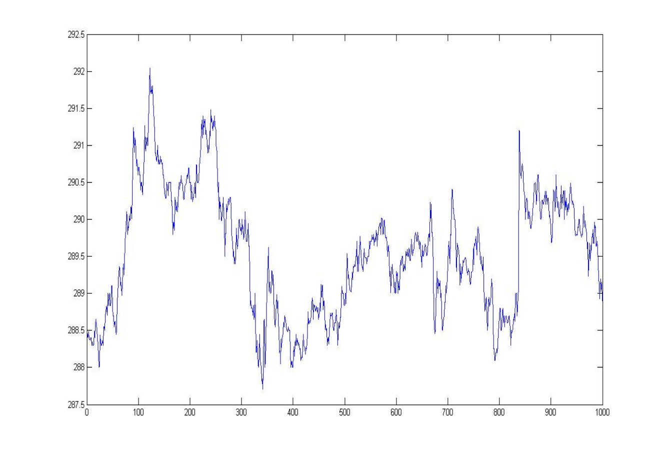
Gain of the FL algorithm (s=1000,1000)

Gain from high volatility. Constant leader



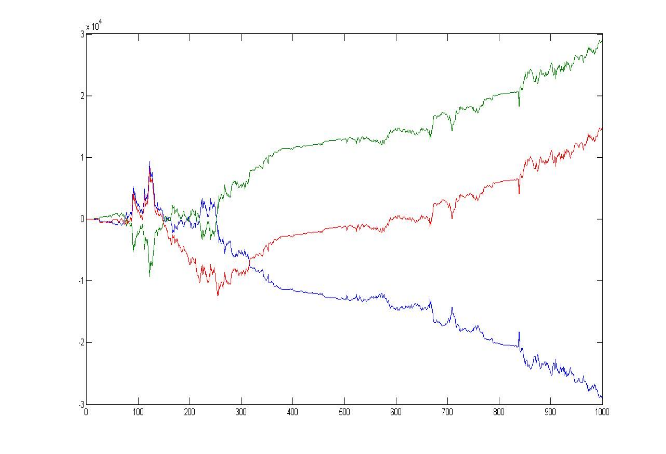
Expected gain of the FPL algorithm (s=1000,1000)

Stock price



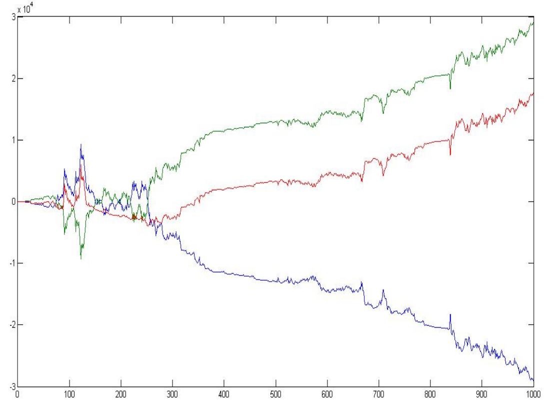
Evolution of the stock price (s1=4100,1000)

Changing the leader



Gain of the FL algorithm (s1=4100,1000)

Changing the leader



Expected gain of the FPL algorithm (s1=4100,1000)