Differential Inequalities in Problems of Forced Nonlinear Oscillations *

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1 Main results

1.1. In the present paper we consider the nonlinear system of ordinary differential equations

$$\frac{dx}{dt} = f(t,x), \quad x \in R^N \tag{1}$$

with right-hand side $f(t,x)$ T-periodic in t:

$$f(t+T,x) \equiv f(t,x), \quad -\infty < t < +\infty,\ x \in R^N.$$

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We introduce new criteria for the existence of T-periodic solution of system (1). These criteria essentially use classical theorems on differential inequalities [6].

For simplicity we restrict our considerations to continuous functions $f(t,x)$ such that every initial condition

$$x(t_0) = x_0 \tag{2}$$

defines a unique solution

$$x(t) = p(t; t_0, x_0) \tag{3}$$

of system (1). The initial time t_0 is fixed, and plays an important role in our constructions.

We suppose that all the solutions (3) are defined for $t \in [t_0, +\infty)$, i.e. that the solutions (3) are non-locally continuable to the right. Therefore the translation operators $U(t, t_0)$ ([2]) are well-defined for $t \geq t_0$ by the equality

$$U(t, t_0)x = x(t; t_0, x), \quad x \in R^N. \tag{4}$$

The theorem on the continuity of solutions (3) with respect to initial data implies that the non-decreasing continuous function

$$\varphi(\rho) = \max\{|U(t, t_1)x| : \ 0 \leq t_1 \leq T, \ t_1 \leq t \leq t_1 + T, \ x \in R^N, \ |x| \leq \rho\} \tag{5}$$

is well-defined for $\rho > 0$. We denote by $|\cdot|$ a norm in R^N generated by some inner product $[\cdot, \cdot]$.

1.2. To study the system (1), we use some continuously differentiable functions $W(\cdot) : \ R^N \to R$. The derivatives of these functions along the trajectories of system (1) satisfy special estimates. These functions are similar to the usual guiding functions and their various modifications (see, for example, [2, 7, 4, 3]).

Definition 1. *We say that the function $W(x)$ is a t_0-*guiding function *if the following conditions hold for some $\rho_0 > 0$:*
(A)

$$\nabla W(x) \neq 0, \quad x \in R^N, \ |x| \geq \rho_0; \tag{6}$$

(B)

$$[\nabla W(x), f(t,x)] > a(t, W(x)), \quad -\infty < t < +\infty,\ x \in R^N,\ |x| \geq \rho_0; \quad (7)$$

(C) for every initial value ξ_0 the upper solution

$$\xi^*(t) = \xi^*(t, t_0, \xi_0), \quad t_0 \leq t \leq t_0 + T \quad (8)$$

of the Cauchy problem

$$\frac{d\xi}{dt} = a(t, \xi), \quad \xi(t_0) = \xi_0 \quad (9)$$

satisfies the inequality

$$\xi^*(t) \geq \xi_0, \quad t_0 \leq t \leq t_0 + T. \quad (10)$$

Definition 2. *We say that the function $W(t)$ is a* generalized t_0-guiding function *if conditions (A) and (C) are valid and, instead of (B), the following weaker estimate holds*

$$[\nabla W(x), f(t,x)] \geq a(t, W(x)), \quad -\infty < t < +\infty,\ x \in R^N,\ |x| \geq \rho_0. \quad (11)$$

The term "upper solution" is used in different senses; it means here the supremum of all the solutions of problem (9) (the uniqueness is not supposed).

If $a(t, \xi) \equiv 0$, then the t_0-guiding functions are the usual guiding functions and the generalized t_0-guiding functions are the generalized guiding functions in the sense of [3]. Analogs of t_0-guiding functions were used in [4].

Under the assumptions of our paper, the functions $a(t, \xi)$ may take values of different signs, and this makes possible to apply the guiding function method in new situations. Without loss of generality we suppose that the function $a(t, \xi)$ is T-periodic and continuous from the right in t, and continuous in ξ.

1.3. Condition (6) implies that the rotation $\gamma(\nabla W; \rho)$ of the vector field ∇W on the sphere $S_\rho = \{x \in R^N,\ |x| = \rho\}$ (see, for example, [1, 5]) is defined for any $\rho \geq \rho_0$. This rotation $\gamma(\nabla W; \rho)$ is an integer. It coincides with the Poincaré index of the field $\nabla W(x)$ on S_ρ and with the Brouwer-Hopf degree of the map $\nabla W(x)$ on the ball $B_\rho = \{x \in R^N,\ |x| \le \rho\}$ with respect to zero. As the value of $\gamma(\nabla W; \rho)$ does not depend on $\rho \ge \rho_0$, this common rotation is called the *index of the function* $W(x)$ *at infinity,* and denoted by $ind(W, \infty)$.

The main assumption we use is

$$ind(W, \infty) \ne 0. \tag{12}$$

This assumption (12) can be easily checked in a lot of specific cases. Let us give some examples.

Let $W(x)$ be a nondegenerate quadratic form: $W(x) = [Ax, x],\ x \in R^N$. Its gradient

$$\nabla W(x) = (A + A^*)x, \quad x \in R^N$$

satisfies (12), and we have

$$ind(W, \infty) = (-1)^\beta,$$

where β is the sum of the multiplicities of the negative eigenvalues of the symmetric matrix $A + A^*$.

An important criterion for (12) follows from the Schnirelman-Borsuk-Hopf theorem, which states that the index $ind(W, \infty)$ of every even function $W(x)$ is odd and hence differs from zero.

1.4. We formulate in this section the main results of the paper.

Theorem 1. *Assume that system* (1) *has a* t_0*-guiding function satisfying* (12). *Then the system has at least one* T*-periodic solution. Moreover, any* T*-periodic solution* $x_*(t)$ *satisfies*

$$|x_*(t_0)| \le \varphi(\rho_0), \quad \max_{-\infty < t < +\infty} |x_*(t)| \le \varphi(\varphi(\rho_0)). \tag{13}$$

Theorem 2. *Assume that system* (1) *has a generalized* t_0*-guiding function satisfying* (12). *Then the system has at least one* T*-periodic solution* $x_*(t)$ *satisfying* (13).

The function $\varphi(\cdot)$ is defined by (5), the number ρ_0 comes from the definition of the t_0-guiding function.

2 Proof of Theorem 1

2.1. Consider the sphere S_r of fixed radius r satisfying

$$r > \varphi(\rho_0). \tag{14}$$

The operators (4) define the vector fields

$$\Phi(t)x = U(t, t_0)x - x, \quad t_0 < t \leq t_0 + T, \ x \in S_r.$$

Let $x_0 \in S_r$. We consider the value

$$\tau(x_0) = \max\{\tau \in [t_0, t_0 + T] : \ t_0 \leq t \leq \tau \ \Rightarrow \ |p(t; t_0, x_0)| \geq \rho_0\}, \tag{15}$$

Inequality (14) implies that $\tau(x_0) > t_0$.

Let $W(x)$ be a t_0-guiding function satisfying (12). This $W(x)$ generates an auxilliary function

$$\mu(t; x_0) = W(p(t; t_0, x_0)), \quad t_0 \leq t \leq \tau(x_0). \tag{16}$$

Since

$$\frac{d\mu(t; x_0)}{dt} = [\nabla W(p(t; t_0, x_0)), f(t, p(t; t_0, x_0))]$$

the auxiliary function (16) satisfies the differential inequality

$$\frac{d\mu(t; x_0)}{dt} > a(t, \mu(t; x_0)), \quad t_0 < t \leq \tau(x_0),$$

and by (16), the equality

$$\mu(t_0; x_0) = W(x_0).$$

Assumption (C) and the classical theorems on differential inequalities imply the estimate

$$\mu(t; x_0) > \xi^*(t, t_0, W(x_0)), \quad t_0 < t \leq \tau(x_0). \tag{17}$$

Here ξ^* is the upper solution (8). It follows from (17) together with (10) that

$$\mu(t;x_0) > W(x_0), \quad t_0 < t \le \tau(x_0).$$

Therefore

$$W(p(t;t_0,x_0)) > W(x_0), \quad t_0 < t \le \tau(x_0).$$

and

$$p(t;t_0,x_0) \neq x_0, \quad t_0 < t \le \tau(x_0). \tag{18}$$

Let $\tau(x_0) < t_0+T$; according to definition (15) we have $|p(\tau(x_0);t_0,x_0)| = \rho_0$. Let $t \in (\tau(x_0), t_0+T]$; the semigroup identity

$$p(t;t_0,x_0) = p(t;\tau(x_0), p(\tau(x_0);t_0,x_0))$$

and (5) imply

$$|p(t;t_0,x_0) - x_0| \ge |x_0| - |U(t,\tau(x_0))p(\tau(x_0);t_0,x_0))| \ge r - \varphi(\rho_0) > 0.$$

This together with (18) implies that

$$\Phi(t)x_0 \neq 0, \quad t_0 < t \le t_0+T, \; x_0 \in S_r. \tag{19}$$

2.2. Since $r > \varphi(\rho_0)$ is arbitrary, (19) guarantees the first relation of the a priori estimate (13), and the second relation follows from the first one.

The following arguments are standard in the guiding function method. The vector fields $\Phi(t)$ for $t \in (t_0, t_0+T]$ are homotopic on S_r for $r > \varphi(\rho_0)$. Therefore the value $\gamma(\Phi(t), r)$ of the rotation of the field $\Phi(t)$ on S_r does not depend on t and on r, and we denote it by γ_0.

From (10) follows the inequality $a(t_0,\xi) \ge 0$, which in turn implies

$$[\nabla W(x), f(t_0,x)] > 0, \quad |x| \ge \rho_0,$$

i.e. the vector $f(t_0,x)$ differs from zero and the angle between it and $\nabla W(x)$ is less than $\pi/2$. It means that for $t > t_0$ and t close to t_0 this angle is also less than $\pi/2$. Therefore there exist a small positive δ such that the vector fields $\Phi(t_0+\delta)$ and ∇W are homotopic. Consequently $\gamma_0 = ind(W,\infty)$, which implies that the rotation of the field $\Phi(t_0+T)$ on the sphere S_r is different from zero and hence at least one point $x^* \subset B_r$ exists such that $\Phi(t_0+T)x^* \neq 0$. This x^* is the initial value of the required periodic solution $p(t;t_0,x^*)$ of equation (1).

3 Proof of Theorem 2

Theorem 2 follows from Theorem 1 and a limit argument. Indeed, consider the equations

$$\frac{dx}{dt} = g_n(t,x), \quad x \in R^N, \ n = 1, 2, \ldots \tag{20}$$

where

$$g_n(t,x) = f(t,x) + \frac{1}{n}\nabla W(x).$$

The function $W(x)$ is a t_0-guiding function for every equation (20):

$$\begin{gathered} [\nabla W(x), g_n(t,x)] \geq a(t, W(x)) + \tfrac{1}{n}|\nabla W(x)|^2, \\ -\infty < t < +\infty, \ x \in R^N, \ |x| \geq \rho_0 \end{gathered} \tag{21}$$

and $\frac{1}{n}|\nabla W(x)|^2 > 0$.

Unfortunately, even for n sufficiently large, we cannot state the uniqueness and continuability of solutions of system (20). But by (21) it is possible to construct new systems

$$\frac{dx}{dt} = h_n(t,x), \quad x \in R^N, \ n = 1, 2, \ldots \tag{22}$$

satisfying the following four properties:

(i) the right-hand sides of (22) are T-periodic in t: $h_n(t,x) \equiv h_n(t+T,x)$;

(ii) the function W is a t_0-guiding function for the system (22) with the same $a(t,\xi)$:

$$[\nabla W(x), h_n(t,x)] > a(t, W(x)), \quad -\infty < t < +\infty, \ x \in R^N, \ |x| \geq \rho_0;$$

(iii) the right-hand sides of (22) are close to $f(t,x)$:

$$|h_n(t,x) - f(t,x)| \leq \varepsilon_n, \quad -\infty < t < +\infty, \ x \in R^N, \ |x| \geq \rho_0 \tag{23}$$

with $\varepsilon_n \to 0$;

(iv) any Cauchy problem (22), (2) has a unique solution which is defined for $t \in [t_0, +\infty[$.

Such functions $h_n(t,x)$ can be constructed by some smoothing of the right-hand sides in (20). This smoothing is a rather standard but cumbersome procedure, and we will not concentrate on the technical difficulties of the construction of the $h_n(t;x)$.

3.2. Denote by $q_n(t;t_1,x_0)$ the solution of (22) satisfying the initial condition $x(t_1)=x_0$. Consider the functions

$$\varphi_n(\rho) = \max\{|q_n(t;t_1,x)|:\ 0\le t_1\le T,\ t_1\le t\le t_1+T,\ x\in R^N,\ |x|\le\rho\} \tag{24}$$

similar to (5). Any function (24) is well-defined for $\rho>0$, continuous and not decreasing. The estimates (23) imply the equality

$$\lim_{n\to\infty}\varphi_n(\rho)=\varphi(\rho),\qquad \rho>0. \tag{25}$$

Every system (22) satisfies all the assumptions of Theorem 1. Therefore every system (22) has at least one T-periodic solution

$$x_n(t)=q_n(t;t_0,x_n^0).$$

From Theorem 1, these periodic solutions satisfy the inequality (13):

$$|x_n^0|\le\varphi(\rho_0),\qquad \max_{-\infty<t<+\infty}|x_n(t)|\le\varphi_n(\varphi_n(\rho_0)). \tag{26}$$

The sequence $x_n^0\in R^N$ is bounded by (25), and we can choose a subsequence $x_{n_k}^0$ such that

$$\lim_{k\to\infty}x_{n_k}^0=x^*. \tag{27}$$

By (25) and the first inequality in (26) we have $|x^*|\le\varphi(\rho_0)$.

The uniform convergence of the T-periodic functions $x_{n_k}(t)$ to a function $x^*(t)$ follows from the relations (23) and (27). Passing to the limit in the equalities

$$x_{n_k}(t)=x_{n_k}^0+\int_{t_0}^t h_{n_k}(s,x_{n_k}(s))ds$$

one obtains the equality

$$x^*(t) = x^* + \int_{t_0}^{t} f(s, x^*(s))ds, \quad t_0 \le t \le t_0 + T.$$

Consequently, $x^*(t)$ is a T-periodic solution of (1). The first estimate in (13) was already mentioned, the second one follows from (25) and the definitions.

4 Remarks

4.1. The conclusions of Theorems 1 and 2 remain valid if we replace, in the definitions of t_0-guiding function and generalized t_0-guiding function, the conditions (7) and (11) respectively by

$$[\nabla W(x), f(t,x)] < b(t, W(x)), \quad -\infty < t < +\infty, \ x \in R^N, \ |x| \ge \rho_0$$

and

$$[\nabla W(x), f(t,x)] \le b(t, W(x)), \quad -\infty < t < +\infty, \ x \in R^N, \ |x| \ge \rho_0.$$

In this case, instead of condition (C), it is necessary to consider the following analogous condition: for every initial value ξ_0 the lower solution

$$\xi_*(t) = \xi_*(t, t_0, \xi_0), \quad t_0 \le t \le t_0 + T$$

of the Cauchy problem

$$\frac{d\xi}{dt} = b(t, \xi), \quad \xi(t_0) = \xi_0$$

satisfies the inequality

$$\xi_*(t) \le \xi_0, \quad t_0 \le t \le t_0 + T.$$

4.2. The analogs of Theorems 2 and 3 can be easily formulated where non-local continuability to the right is replaced by non-local continuability to the left. The proofs can be simplified if every solution of each Cauchy problem for (1) is defined for $t \in (-\infty, +\infty)$.

4.3. With the use of relationship principles (see [2]), Theorems 1 and 2 can be extended to various classes of differential equations with delays and hysteresis nonlinearities. Those relationship theorems also justify the applicability of the harmonic balance method: under the assumptions of Theorem 1, the method is applicable and converges.

5 Examples

5.1. To apply Theorems 1 and 2 it is necessary to obtain first the estimates (7) and (11). Then the problem arises to find the initial time t_0 such that condition (C) holds. Conditions of existence of such initial times allow the formulation of useful corollaries of Theorems 1 and 2.

Theorem 3. *Assume that each Cauchy problem for equation* (1) *has a unique solution which is continuable to the right. Let* $W(\cdot)$ *be a continuously differentiable function satisfying*

$$\nabla W(x) \neq 0, \quad x \in R^N, \ |x| \geq \rho_0; \quad ind(W, \infty) \neq 0. \tag{28}$$

Assume moreover that

$$[\nabla W(x), f(t,x)] \geq \alpha(t) b(W(x)), \quad -\infty < t < +\infty, \ x \in R^N, \ |x| \geq \rho_0 \tag{29}$$

for some T*-periodic integrable* $\alpha(t)$ *and some positive continuous* $b(\xi)$*. Let*

$$\bar{\alpha} = \frac{1}{T} \int_0^T \alpha(s) ds > 0.$$

Then the system (1) *has at least one* T*-periodic solution* $x^*(t)$*. Moreover, any* T*-periodic solution* $x^*(t)$ *of* (1) *satisfies the a priori estimate*

$$|x^*(t)| \leq c < \infty, \quad -\infty < t < +\infty \tag{30}$$

with some positive c*. The constant* c *is determined by the number* ρ_0 *and by the function* $\varphi(\rho)$*.*

The simplest example corresponds to the case where $b(\xi) \equiv 1$.

Theorem 4. *Assume that each Cauchy problem for equation* (1) *has a unique solution which is continuable to the right. Let* $W(\cdot)$ *be a continuously differentiable function satisfying* (28)*. Let* (29) *be valid for some* T*-periodic integrable* $\alpha(t)$ *and some nonnegative continuous* $b(\xi)$*, and assume that*

$$\bar{\alpha} = \frac{1}{T} \int_0^T \alpha(s) ds \geq 0. \tag{31}$$

Then the system (1) *has at least one* T*-periodic solution* $x^*(t)$*, satisfying*

(30).

To prove Theorems 3 and 4 we shall prove that if condition (31) holds, then there exists some t_0 such that all the solutions $\xi(t)$ of Cauchy problem

$$\frac{d\xi}{dt} = \alpha(t)b(\xi), \quad \xi(t_0) = \xi_0 \tag{32}$$

satisfy the inequality

$$\xi(t) \geq \xi_0, \quad t \geq t_0. \tag{33}$$

Then we shall prove both Theorems. Theorem 4 is an immediate corollary of Theorem 2, and to prove Theorem 3 it is only necessary to observe that

$$[\nabla W(x), f(t,x)] \geq \alpha(t)b(W(x)) > (\alpha(t) - \frac{1}{2}\bar{\alpha})b(W(x))$$

and that the average of the function $\alpha(t) - \frac{1}{2}\bar{\alpha}$ is positive together with $\bar{\alpha}$.

Let the function $b(\xi)$ be strictly positive. Set

$$A'(t) = \alpha(t) - \bar{\alpha}, \quad B'(\xi) = \frac{1}{b(\xi)}.$$

The function $A(t)$ is T-periodic; let t_0 be a point of global minimum of $A(t)$, i.e. $A(t_0) \leq A(t)$ for any t. The function $B(\xi)$ is strictly increasing. Every solution $\xi(t)$ of (32) satisfies

$$B(\xi(t)) - B(\xi_0) = \bar{\alpha}(t - t_0) + A(t) - A(t_0) \geq 0, \quad t \geq t_0.$$

The last relation prove (33) for positive $b(\xi)$. For nonnegative $b(\xi)$, the proof can be obtained by limiting constructions.

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