On games of continuous and discrete randomized forecasting

V. V. V’yugin

Institute for Information Transmission Problems
Russian Academy of Sciences

Game-theoretic approach to forecasting and probability

Shafer G., Vovk V. Probability and Finance. It's Only a Game! New York: Wiley, 2001.

BINARY FORECASTING GAME II

FOR $n = 1, 2, \dots$
Skeptic announces $S_n : [0,1] \to \mathscr{R}$ (set of all real numbers).
Forecaster announces a probability distribution $P_n \in \mathscr{P}[0,1]$.
Reality announces $\omega_n \in \{0,1\}$.
Forecaster announces $f_n : [0,1] \to \mathscr{R}$ such that
$\int f_n(p) P_n(dp) \leq 0$.
Random number generator announces $p_n \in [0,1]$.
Sceptic updates his total gain $\mathscr{K}_n = \mathscr{K}_{n-1} + S_n(p_n)(\omega_n - p_n)$.
Forecaster updates his total gain $\mathscr{F}_n = \mathscr{F}_{n-1} + f_n(p_n)$.
ENDFOR

Restriction on Skeptic: Skeptic must choose the S_n so that his total gain $\mathscr{K}_n$ is nonnegative for all n no matter how the other players move; $\mathscr{K}_0 = 1$.
Restriction on Forecaster: Forecaster must choose the P_n and f_n so that his total gain $\mathscr{F}_n$ is nonnegative for all n no matter how the other players move; $\mathscr{F}_0 = 1$.

Winners:

Forecaster wins if either (i) his total gain $\mathscr{F}_n$ is unbounded or (ii) Skeptic's total gain $\mathscr{K}_n$ stays bounded; otherwise the other players win.

Theorem

Forecaster has a winning strategy in Binary Forecasting Game II.

Vovk V., Shafer G., Good randomized sequential probability forecasting is always possible // J. Royal Stat. Soc. B. 67 (2005) 747-763.

The von Neumann minimax theorem is used on each round n.

Sketch of the proof

Zero-sum auxiliary game: **Forecaster** announces $p_n \in [0,1]$, **Nature** announces $\omega_n \in \{0,1\}$.

$F(\omega_n, p_n) = S(p_n)(\omega_n - p_n)$ – **Forecaster's** loss (**Nature** gain)

For each **Nature's** mixed strategy $Q_n \in \mathscr{P}\{0,1\}$ a **Forecaster's** pure strategy $p_n = Q\{1\}$ exists such that

$$F(Q_n, P_n) = E_{Q,P}(F(\omega_n, p_n)) = \int S_n(p_n))(\omega - p_n)dQ = 0.$$

Hence, $\max_Q \min_P F(Q,P) \leq 0$.

After discretization by P

$$\max_Q \min_P F(Q,P) \leq \Delta.$$

By minimax theorem

$$\min_P \max_Q F(Q,P) = \max_Q \min_P F(Q,P) \leq \Delta.$$

Equivalently, P_Δ exists such that

$$\forall Q: \; F(Q, P_\Delta) \leq \Delta, \text{ or}$$

Forecaster has a mixed strategy P_Δ on a discrete set such that

$$\int S_n(p)(\omega_n - p) P_\Delta(dp) \leq \Delta$$

for $\omega_n = 0$ and $\omega_n = 1$.

For $\Delta \to 0$ we obtain

$$\int S_n(p)(\omega_n - p)P_n(dp) \leq 0$$

for $\omega_n = 0$ and $\omega_n = 1$,

P_n is a limit point of $\{P_\Delta\}$ in the week topology

Forecaster's winning strategy:

Forecaster's Move 1: P_n

Forecaster's Move 2: $f_n(p) = S_n(p)(\omega_n - p)$

Then $\mathscr{F}_n = \mathscr{K}_n$.

Forecaster wins since $\sup_n \mathscr{K}_n < \infty$ or $\sup_n \mathscr{F}_n = \infty$

Universal forecasting requires unrestrictedly increasing degree of accuracy.

We present some results showing that discrete universal forecasting is impossible.

Level of discreteness

Measure P_n is concentrated on a finite subset $D_n \subset [0,1]$
$D_n = \{p_{n,1}, \ldots, p_{n,m_n}\}$.

$\Delta_n = \inf\{|p_{n,i} - p_{n,j}| : i \neq j\}$;

$\Delta = \liminf_{n\to\infty} \Delta_n$ is called the strategy's level of discreteness.

A typical example is the uniform rounding of $[0,1]$.

PROBABILISTIC BINARY FORECASTING GAME II

FOR $n = 1, 2, \dots$
Skeptic announces $S_n : [0,1] \to \mathscr{R}$.
Forecaster announces a probability distribution $P_n \in \mathscr{P}[0,1]$.
Reality announces $\omega_n \in \{0,1\}$.
Random Number Generator announces $p_n \in [0,1]$.
Skeptic updates his total gain
$\mathscr{K}_n = \mathscr{K}_{n-1} + S_n(p_n)(\omega_n - p_n)$.
ENDFOR

Restriction on Skeptic: Skeptic must choose the S_n so that his total gain $\mathscr{K}_n$ is nonnegative for all n no matter how the other players move; $\mathscr{K}_0 = 1$.

Realty and Skeptic win if Skeptic's total gain $\mathscr{K}_n$ is unbounded; otherwise Forecaster wins.

Pr – overall probability distribution on infinite paths $p_1, p_2, \ldots$ of Forecaster's moves (there exists by Ionescu-Tulcea theorem)

Theorem

If Forecaster uses a randomized strategy with a positive level of discreteness then Realty and Skeptic win in Probabilistic Binary Forecasting Game II with Pr-probability 1. Otherwise, Forecaster wins with Pr-probability 1.

Game-theoretic counterparts

SYMMETRIC BINARY FORECASTING GAME II

FOR $n = 1, 2, \dots$
Skeptic announces $S_n : [0,1] \to \mathscr{R}$ (set of all real numbers).
Forecaster announces a probability distribution $P_n \in \mathscr{P}[0,1]$.
Reality announces $\omega_n \in \{0,1\}$.
Forecaster announces $f_n : [0,1] \to \mathscr{R}$ such that
$\int f_n(p) P_n(dp) \le 0$.
Sceptic announces $h_n : [0,1] \to \mathscr{R}$ such that $\int h_n(p) P_n(dp) \le 0$.
Random Number Generator announces $p_n \in [0,1]$.
Skeptic updates both his total gains:
$\mathscr{K}_n = \mathscr{K}_{n-1} + S_n(p_n)(\omega_n - p_n)$.
$\mathscr{G}_n = \mathscr{G}_{n-1} + h_n(p_n)$ (statistical gain).
Forecaster updates his total statistical gain:
$\mathscr{F}_n = \mathscr{F}_{n-1} + f_n(p_n)$.
ENDFOR

Restriction 1 on Skeptic: Skeptic must choose the S_n so that his total gain $\mathscr{K}_n$ is nonnegative for all n no matter how the other players move; $\mathscr{K}_0 = 1$.

Restriction 2 on Skeptic: Skeptic must choose the h_n and S_n so that his total gain $\mathscr{G}_n$ is nonnegative for all n no matter how the other players move; $\mathscr{G}_0 = 1$.

Restriction on Forecaster: Forecaster must choose the P_n and f_n so that his total gain $\mathscr{F}_n$ is nonnegative for all n no matter how the other players move; $\mathscr{F}_0 = 1$.

Three parties:

1) **Sceptic** and **Realty** against 2) **Forecaster**

3) **Random Number Generator** – neutral player

Random Number Generator is **fair** in the game if both statistical total gains are bounded $\sup_n G_n < \infty$ and $\sup_n F_n < \infty$.

Assume that Random Number Generator is **fair**.
Winners in this case:

Sceptic and **Realty** win if the Skeptic's total gain is unbounded: $\sup_n \mathscr{K}_n = \infty$; otherwise **Forecaster** wins.

The following theorem shows that in case where Random Number Generator is fair Forecaster wins if and only if it can use a randomized strategy with zero level of discreteness.

Theorem

Assume Random Number Generator is fair. If Forecaster's uses a randomized strategy with a positive level of discreteness.[a] *then Realty and Skeptic win in the Symmetric Binary Forecasting Game II. Otherwise, Forecaster wins.*

[a] A value of this level of discreteness is unknown for Realty and Skeptic.

Two parties

1) **Sceptic**, **Realty**, and **Random Number Generator**

against

2) **Forecaster**

ASYMMETRIC BINARY FORECASTING GAME II – Simplification

FOR $n = 1, 2, \ldots$
Skeptic announces $S_n : [0,1] \to \mathscr{R}$ (set of all real numbers).
Forecaster announces a probability distribution $P_n \in \mathscr{P}[0,1]$.
Reality announces $\omega_n \in \{0,1\}$.
Forecaster announces $f_n : [0,1] \to \mathscr{R}$ such that $\int f_n(p) P_n(dp) \le 0$.
Sceptic announces $h_n : [0,1] \to \mathscr{R}$ such that $\int h_n(p) P_n(dp) \le 0$.
Random Number Generator announces $p_n \in [0,1]$.
Skeptic updates both his gains
$\mathscr{K}_n = \mathscr{K}_{n-1} + S_n(p_n)(\omega_n - p_n) + h_n(p_n)$.
Forecaster updates his total statistical gain:
$\mathscr{F}_n = \mathscr{F}_{n-1} + f_n(p_n)$.
ENDFOR

ASYMMETRIC BINARY FORECASTING GAME II

$\mathscr{K}_0 = 1$.
FOR $n = 1, 2, \ldots$
Skeptic announces $S_n : [0,1] \to \mathscr{R}$.
Forecaster announces a probability distribution $P_n \in \mathscr{P}[0,1]$.
Reality announces $\omega_n \in \{0,1\}$.
Skeptic announces $h_n : [0,1] \to \mathscr{R}$ such that $\int h_n(p) P_n(dp) \leq 0$.
Random Number Generator announces $p_n \in [0,1]$.
Skeptic updates his total gain
$\mathscr{K}_n = \mathscr{K}_{n-1} + S_n(p_n)(\omega_n - p_n) + h_n(p_n)$.
ENDFOR

Realty and Skeptic win if Skeptic's total gain $\mathscr{K}_n$ is unbounded; otherwise Forecaster and Random Number Generator win.

Theorem

Assume Forecaster's uses a randomized strategy with a positive level of discreteness. Then Realty and Skeptic win in the Asymmetric Binary Forecasting Game II.

Sketch of the proof

Strategy for Realty: at any step n Realty announces an outcome

$$\omega_n = \begin{cases} 0 \text{ if } P_n((0.5,1]) > 0.5 \\ 1 \text{ otherwise.} \end{cases}$$

Strategy for Sceptic: Move 1 and Move 2 (below).

Skeptic's capitals:

Sceptic's capital for Move 1:
$\mathscr{K}_n = \mathscr{K}_{n-1} + S_n(p_n)(\omega_n - p_n)$

Sceptic's (statistical) capital for Move 2:
$\mathscr{G}_n = \mathscr{G}_{n-1} + g_n(p_n)$ for all $n > 0$.

Forecaster's (statistical) capital for Move 2:
$\mathscr{F}_n = \mathscr{F}_{n-1} + f_n(p_n)$ for all $n > 0$.

Initially, $\mathscr{K}_0 = 1$, $\mathscr{G}_0 = 1$, and $\mathscr{F}_0 = 1$.

$$\vartheta_{n,1} = \sum_{j=1}^{n} \xi(p_j > 0.5)(\omega_j - p_j)$$

$$\vartheta_{n,2} = \sum_{j=1}^{n} \xi(p_j \leq 0.5)(\omega_j - p_j)$$

where $\xi(true) = 1$ and $\xi(false) = 0$.

We have $\vartheta_{n,2} - \vartheta_{n,1} = \sum_{j=1}^{n} g_j(p_j)$, where

$g_j(p) = \xi(p \leq 0.5)(\omega_j - p) - \xi(p > 0.5)(\omega_j - p)$.

For any discrete Forecaster's strategy $\{P_j\}$, in the mean :

$$E(\vartheta_{n,2} - \vartheta_{n,1}) = \sum_{j=1}^{n} E_{P_j}(g_j) \geq 0.5\Delta n.$$

Since Random Number Generator is fair, $\sup \mathscr{G}_n < \infty$.

Move 2 of Sceptic's strategy forces:

$$\sup_n \mathscr{G}_n < \infty \Rightarrow \liminf_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} (g_j(p_j) - E_{P_j}(g_j) \geq -\varepsilon.$$

Then

$$\liminf_{n\to\infty} \frac{1}{n} (\vartheta_{n,2} - \vartheta_{n,1}) \geq 0.5\Delta - \varepsilon,$$

where $\varepsilon > 0$ is arbitrary small.

Move 1 of Sceptic's strategy forces:

$$2^{\frac{\ln \mathscr{Q}_n}{n}} \geq \varepsilon(\vartheta_{n,2} - \vartheta_{n,1}) - 2\varepsilon^2 \geq \varepsilon(0.5\Delta - \varepsilon) - 2\varepsilon^2 > \varepsilon^2$$

for infinitely many n, where $\varepsilon > 0$ is arbitrary small fixed real number (tuned in the game to be much smaller than Δ : $\varepsilon < \Delta/8$).

Therefore,

$$\limsup_{n\to\infty} \frac{\ln \mathscr{Q}_n}{n} > \varepsilon/2.$$

Hence, Sceptic's capital is unbounded

$$\sup_n \mathscr{K}_n = \infty.$$

Calibration: Kakade and Foster' result - 2004

Theorem

For any sequence of outcomes $\omega_1 \omega_2 \ldots$, an observer can only randomly round the deterministic forecast up to Δ in order to calibrate with the internal probability 1 *:*

$$\left| \frac{1}{n} \sum_{i=1}^{n} I(p_i)(\omega_i - p_i) \right| \leq \Delta$$

for all n, where Δ is the calibration error, $I(p)$ is the indicator function of an arbitrary subinterval of $[0,1]$.

A lower bound of calibration error:

Corollary

Assume Forecaster uses a randomized strategy with a positive level of discreteness Δ. *Then Realty (without using information on a value of* Δ*) can announce an infinite binary sequence* $\omega_1 \omega_2 \dots$ *such that one of two possibilities holds:*

$$\limsup_{n\to\infty} \left| \frac{1}{n} \sum_{j=1}^{n} I(p_j > 0.5)(\omega_j - p_j) \right| \geq 0.25\Delta$$

$$\limsup_{n\to\infty} \left| \frac{1}{n} \sum_{j=1}^{n} I(p_j \leq 0.5)(\omega_j - p_j) \right| \geq 0.25\Delta$$

More details:

Auxiliary Skeptic's strategies for Move 1:

$$S_n^{1,k}(p) = -\varepsilon_k \mathscr{Q}_{n-1}^{1,k} \xi(p > 0.5), \quad (1)$$

$$S_n^{2,k}(p) = \varepsilon_k \mathscr{Q}_{n-1}^{2,k} \xi(p \leq 0.5), \quad (2)$$

where $\xi(\mathit{true}) = 1$, $\xi(\mathit{false}) = 0$, and $n \geq 1$

Auxiliary Skeptic's capital for Move 1:

$$\mathscr{Q}_n^{1,k} = \mathscr{Q}_{n-1}^{1,k} + S_n^{1,k}(p_n)(\omega_n - p_n)),$$

$$\mathscr{Q}_n^{2,k} = \mathscr{Q}_{n-1}^{2,k} + S_n^{2,k}(p_n)(\omega_n - p_n)).$$

Skeptic's strategy for Move 1:

$$S_n(p) = \frac{1}{2}(S_n^1(p) + S_n^2(p)),$$

where

$$S_n^1(p) = \sum_{k=1}^{\infty} \varepsilon_k S_n^{1,k}(p)$$

$$S_n^2(p) = \sum_{k=1}^{\infty} \varepsilon_k S_n^{2,k}(p).$$

Skeptic's capital for Move 1:

$$\mathcal{Q}_n = \frac{1}{2}\sum_{k=1}^{\infty} \varepsilon_k(\mathcal{Q}_n^{1,k} + \mathcal{Q}_n^{2,k}).$$

Define $g_n(p) = \xi(p \le 0.5)(\omega_n - p) - \xi(p > 0.5)(\omega_n - p)$

Auxiliary Skeptic's strategy and capital for Move 2:

Define recursively by n, $\mathscr{F}_0^k = 1$, $g_0^k(p) = 0$;

$g_n^k(p) = -\varepsilon_k \mathscr{F}_{n-1}^k (g_n(p) - E_{P_n}(g_n))$,

$\mathscr{F}_n^k = \mathscr{F}_{n-1}^k + g_n^k(p_n)$

for $n \ge 1$, where $\varepsilon_k = 2^{-k}$ and P_n – Forecaster's move.

Skeptic's strategy for Move 2:

$$h_n(p) = \sum_{k=1}^{\infty} \varepsilon_k g_n^k(p).$$

By definition $\int h_n(p) P_n(dp) \leq 0$.

Skeptic's (statistical) capital for Move 2:

$$\mathcal{G}_n = \sum_{k=1}^{\infty} \varepsilon_k \mathcal{G}_n^k.$$

Also, $\mathcal{G}_n \geq 0$ for all n.

We have for each k,

$$\ln \mathscr{G}_n^k \geq -\varepsilon_k \sum_{j=1}^{n} (g_j(p_j) - E_{P_j}(g_j)) - n\varepsilon_k^2.$$

Since $\sup_n \mathscr{G}_n < C$, we have for any k

$$\frac{1}{n}\sum_{j=1}^{n} (g_j(p_j) - E_{P_j}(g_j)) \geq \frac{-\ln C + \ln(\varepsilon_k)}{n\varepsilon_k} - \varepsilon_k \geq -2\varepsilon_k$$

Hence,

$$\frac{1}{n}\sum_{j=1}^{n} (g_j(p_j) - E_{P_j}(g_j)) \geq -2\varepsilon_k$$

for all sufficiently large n.

Result of Sceptic's Move 2

$$\frac{1}{n}(\vartheta_{n,2} - \vartheta_{n,1}) = \frac{1}{n}\sum_{j=1}^{n} g_j(p_j) \geq$$

$$\geq \frac{1}{n}\sum_{j=1}^{n} E_{P_j}(g_j) - 2\varepsilon_k \geq 0.5\Delta - 2\varepsilon_k.$$

Sceptic's Move 1

$$\ln \mathcal{Q}_n^{1,k} \geq -\varepsilon_k \vartheta_{n,1} - \varepsilon_k^2 n,$$
$$\ln \mathcal{Q}_n^{2,k} \geq \varepsilon_k \vartheta_{n,2} - \varepsilon_k^2 n.$$

Hence,

$$\frac{\ln \mathcal{Q}_n^{1,k} + \ln \mathcal{Q}_n^{2,k}}{n} \geq \varepsilon_k \frac{1}{n}(\vartheta_{n,2} - \vartheta_{n,1}) - 2\varepsilon_k^2 \geq$$
$$\geq \varepsilon_k(0.5\Delta - 2\varepsilon_k) - 2\varepsilon_k^2 = 0.5\varepsilon_k\Delta - 2\varepsilon_k^2 \geq 2\varepsilon_k^2$$

for all sufficiently large n, where $\varepsilon_k \leq \frac{1}{8}\Delta$.

From this, we obtain

$$\limsup_{n\to\infty} \frac{\ln \mathcal{Q}_n^{i,k}}{n} \geq \varepsilon_k^2$$

for $i = 1$ or for $i = 2$.
Hence,

$$\sup_n \mathcal{Q}_n = \infty$$

no matter how Forecaster moves if Realty uses her strategy defined above.