# Generalized Guiding Functions in a Problem on High Frequency Forced Oscillations

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**Abstract**

New conditions are presented for the existence of forced periodic oscillations for systems of ordinary differential equations. The new results are based on two approaches. The first one is the use of generalized quiding functions instead of usual ones; it makes possible to study some degenerate cases. The second approach allows to use an unique guiding function in cases when we have no information about the non-local continuity of solutions. The possible values of the period are restricted in the case of a single guiding function. The new method introduced here makes it possible to obtain generalization of the analogous results from [1] [1].

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[1] In [1] in formulae (14) sup must be replaced by inf

# 1 Main result

Consider the system of ordinary differential equations

$$\frac{dx}{dt} = f(t,x), \quad x \in I\!R^N \tag{1.1}$$

with the right-hand side $T$-periodic in t,

$$f(t+T,x) \equiv f(t,x), \quad t \in I\!R, \ x \in I\!R^N. \tag{1.2}$$

We discuss the problem of the existence of $T$-periodic solutions $x(t)$ for system (1.1).

For simplicity suppose that the vector-function $f(t,x)$ is continuous with respect to all variables and satisfies local Lipschitz condition in $x$. In this case every initial value

$$x(t_0) = x_0 \tag{1.3}$$

defines an unique solution

$$x(t) = p(t; t_0, x_0) \tag{1.4}$$

of system (1.1). Every solution (1.4) is determined on some maximal open interval denoted by $(\tau_-(t_0,x_0), \tau_+(t_0,x_0))$. Evidently, $\tau_-(t_0,x_0) < t_0 < \tau_+(t_0,x_0)$; the symbol $\tau_-(t_0,x_0)$ (resp. $\tau_+(t_0,x_0)$) means either a finite number or $-\infty$ (resp. $+\infty$). If $\tau_-(t_0,x_0) = -\infty$ then solution (1.4) is called **infinitely continuable to the left**, if $\tau_+(t_0,x_0) = +\infty$ then it is called **infinitely continuable to the right**.

The Euclidean norm and inner product in $I\!R^N$ are denoted by $|\cdot|$ and $(\cdot,\cdot)$ respectively.

We shall consider scalar-valued functions $V(x)$ which are continuously differentiable on $I\!R^N$ and satisfy the assumption:

$$\mathbf{grad}V(x) \equiv V'(x) \neq 0, \quad |x| \geq \rho_0, \ x \in I\!R^0. \tag{1.5}$$

The value $\rho_0 > 0$ is supposed to be fixed in the paper.

Condition (1.5) implies that the continuous vector-field $V'(x)$ is non-degenerate on any sphere

$$S(\rho) = \{x \in I\!R^N : \ |x| = \rho\}, \quad \rho \geq \rho_0.$$

Therefore on $S(\rho)$ $(\rho \geq \rho_0)$ the rotation of $V'(x)$ is defined (cf [2]); this rotation does not depend on $\rho$. The common rotation is called **the index (at infinity)** of the function $V(x)$ and is denoted by $\mathbf{ind}(V;\infty)$. Below we shall consider cases when

$$\mathbf{ind}(V;\infty) \neq 0. \tag{1.6}$$

Relation (1.6) is always valid if $V(-x) \equiv V(x)$ (for example $V(x)$ is a non-degenerate quadratic form, i.e. $V(x) = (Ax,x)$, where $A$ is an invertible symmetric matrix) since the gradient of every even function is an odd vector-field and odd non-degenerate vector-fields on $S(\rho)$ (according to the Shnirelman-Borsuk-Hopf theorem) have an odd rotation. The index of every non-degenerate quadratic form $(Ax,x)$ is equal to $+1$ or $-1$ depending on the sum of multiplicities of the negative eigenvalues of the symmetric matrix $A$).

The function $V(x)$ is called a generalized guiding function for system (1.1) if

$$(f(t,x), V'(x)) \geq 0, \quad |x| \geq \rho_0. \tag{1.7}$$

In conditions (1.5) and (1.7) $\rho_0$ has the same value.

Assumption (1.7) was used by M.A.Krasnosel'skii, A.I.Perov, V.V.Strygin and other authors (see references in [2, 3]) in the more restrictive form

$$(f(t,x), V'(x)) > 0, \quad |x| \geq \rho_0.$$

Assumption (1.7) was used by J.Mawhin (see [4] and the list of references therein) for functions $V(x)$ satisfying

$$\lim_{|x|\to\infty} V(x) = \infty. \tag{1.8}$$

In this case $\mathbf{ind}(V;\infty) = 1$. Assumptions (1.7) for functions $f(t,x)$ of an arbitrary index were used in [1].

Note that in our paper condition (1.8) is not used. Functions $V(x)$ can take values of both signs and, moreover, can satisfy

$$\liminf_{|x|\to\infty} V(x) = -\infty, \quad \limsup_{|x|\to\infty} V(x) = +\infty.$$

This type of functions $V(x)$ can effectively be used for study non-dissipative systems (1.1).

**Theorem 1** *Let the $T$-periodic system* (1.1) *have a generalized guiding function $V(x)$ ((1.7) holds) and $\mathbf{ind}(V;\infty) \neq 0$. Let every solution* (1.4) *be infinitely continuable at least to one of the sides (or both to the left and to the right) for any initial values* (1.3) *from the ball $\{|x_0| \leq \rho_0\}$. Then the system* (1.1) *has at least one $T$-periodic solution.*

In the case when some solutions (1.4) are defined only on finite intervals the intervals $(\tau_-(t_0,x_0), \tau_+(t_0,x_0))$ play the leading role. Consider the value

$$\alpha(\rho_0) = \inf_{0\leq t_0\leq T;\ |x_0|\leq\rho_0} [\tau_-(t_0,x_0) - \tau_+(t_0,x_0)]. \tag{1.9}$$

This value can be equal to either $+\infty$ or some positive number. Below without special notice we shall use the equalities

$$\alpha(\rho_0) = \inf_{\tau\leq t_0\leq T+\tau;\ |x_0|\leq\rho_0} [\tau_-(t_0,x_0) - \tau_+(t_0,x_0)], \quad \tau \in I\!R.$$

and

$$\alpha(\rho_0) = \inf_{t_0\in I\!R;\ |x_0|\leq\rho_0} [\tau_-(t_0,x_0) - \tau_+(t_0,x_0)],$$

which follow from (1.2).

Let us note that the finiteness of the numbers $\tau_\pm(t_0,x_0)$ follows usually from a fast increasing of $|f(t,x)|$ for $|x| \to \infty$.

**Theorem 2** *Let the $T$-periodic system* (1.1) *have a generalized guiding function $V(x)$ and $\mathbf{ind}(V;\infty) \neq 0$. Let*

$$T < \alpha(\rho_0), \tag{1.10}$$

*where $\alpha(\rho_0)$ is defined by* (1.9). *Then the system* (1.1) *has at least one $T$-periodic solution.*

Theorem 1 is a corollary of Theorem 2 (under assumptions of Theorem 1 $\alpha(\rho_0) = \infty$ and (1.10) holds).

Theorem on the existence of $T$-periodic solutions can also be established without any information about the value of the maximal intervals of definition of Cauchy problem solutions for (1.1).

Let the continuously differentiable functions

$$V_1(x),\ V_2(x), \ldots, V_k(x), \quad x \in I\!R^N \tag{1.11}$$

satisfy the conditions

$$|V_j^{'}(x)| \neq 0, \quad j = 1, \ldots, k;\ |x| \geq \rho_0$$

and

$$\Big((ft, x), V_j^{'}(x)\Big) \geq 0, \quad j = 1, \ldots, k;\ |x| \geq \rho_0, \tag{1.12}$$

i.e. each of the function (1.11) is a generalized guiding function for (1.1). The set of generalized guiding function is called **complete** if

$$\lim_{|x|\to\infty} \sum_{j=1}^{k} |V_j(x)| = \infty. \tag{1.13}$$

This set is called **sharp** if for every fixed $x \in I\!R^N,\ |x| \geq \rho_0$ the set

$$K(x) = \left\{ y \in I\!R^N :\ y = \sum_{j=1}^{k} \alpha_j V_j^{'}(x),\ \alpha_1, \ldots, \alpha_k \geq 0 \right\}$$

is a cone in sense by M.G.Krein (i.e. $y, -y \in K(x)$ implies $y = 0$).

If instead of (1.12) the strong inequalities hold:

$$\Big(f(t, x), V_j^{'}(x)\Big) > 0, \quad j = 1, \ldots, k;\ |x| \geq \rho_0$$

then the set (1.11) is sharp. If $k = 1$ then the set of a single function $V(x)$ is sharp; the condition of completeness in this case has the form (1.8).

**Theorem 3** *Let the $T$-periodic system* (1.1) *have a complete and sharp set of generalized guiding functions* (1.11). *Let*

$$\mathbf{ind}(V_1, \infty) \neq 0. \tag{1.14}$$

*Then the system* (1.1) *has at least one $T$-periodic solution.*

In the formulation of Theorem 3 we do not use any restrictions on the period $T$.

The next result concerns the existence of bounded solutions of (1.1). In its formulation we do not use the periodicity of the function $f(t, x)$ in $t$.

**Theorem 4** *Let the system* (1.1) *have a complete and sharp set of generalized guiding functions* (1.11). *Let* (1.14) *be valid. Then the system* (1.1) *has at least one solution $x(t)$ defined on* $(-\infty, +\infty)$ *and satisfying*

$$\sup_{-\infty<t<+\infty} |x(t)| < \infty.$$

Theorems 2–4 are proved in the following sections of the paper. Section 6 contains examples of applications of Theorem 2. Proofs of Theorems 3 and 4 are given in Section 7.

Note, that Theorems 1–4 are valid for systems (1.1) with non-Lipschitzian functions $f(t, x)$. The proofs become more complicated.

## 2 A lemma on auxiliary vector-function

Let the continuous vector-functions

$$\varphi_1(x),\ \varphi_2(x), \dots, \varphi_k(x), \quad x \in I\!R^N \tag{2.1}$$

be defined on $I\!R^N$, with $k$ a natural number. The set (2.1) is called sharp if for every $x \in I\!R^N$, $|x| \geq \rho_0$ one can choose a vector $\psi(x) \in I\!R^N$ such that

$$(\varphi_j(x), \psi(x)) > 0, \quad j = 1, \dots, k;\ |x| \geq \rho_0. \tag{2.2}$$

The estimates

$$|\varphi_j(x)| > 0, \quad j = 1, \dots, k;\ |x| \geq \rho_0$$

follow from (2.2). The function $\psi(x)$ is not supposed to be continuous.

**Lemma 1** *Let the set* (2.1) *be sharp. Then there exists a locally Lipschitzian function* $g(x) :\ I\!R^N \to I\!R^N$ *such that*

$$(\varphi_j(x), g(x))\,, \quad j = 1, \dots, k;\ |x| \geq \rho_0. \tag{2.3}$$

**Proof.** Denote by $B(x, r)$ the open ball $\{y \in I\!R^N :\ |x - y| < r\}$. The continuity of (2.1) and estimates (2.2) imply that for every $x$, $|x| \geq \rho_0$ there is a $r(x) \in (0, 1]$ such that

$$(\varphi_j(y), \psi(x)) > 0, \quad j = 1, \dots, k;\ y \in B(x, r(x)). \tag{2.4}$$

Open balls $B(x, r(x))$ form a cover of the closed set $F = \{x :\ |x| \geq \rho_0\} \subset I\!R^N$. Let us choose from this cover a countable subcover

$$B(x_i, r(x_i)), \quad i = 1, 2, \dots \tag{2.5}$$

satisfying the following condition: every point $x$ belongs to a finite number only of balls (2.5).

Define for every $i = 1, 2, \dots$ some continuously differentiable function $\beta_i(x)$ $(x \in I\!R^N)$ satisfying $\beta_i(x) = 0$ for $x \notin B(x_i, r(x_i))$ and $\beta_i(x) > 0$ for $x \in B(x_i, r(x_i))$. Put

$$g(x) = \sum_{i=1}^{\infty} \beta_i(x), \quad x \in I\!R^N. \tag{2.6}$$

In this sum, for every $x$ a finite number of values $\beta_i(x)$ only differ from 0. The function (2.6) is obviously continuously differentiable on $I\!R^N$, (2.3) follows from (2.4). The lemma is proved.■

## 3 A lemma on the non-existence of $(T, \rho_0, \rho_1)$-solutions

Let $g(x)$ $(x \in I\!R^N)$ be a locally Lipschitzian $I\!R^N$-valued function. Consider for $\varepsilon \geq 0$ the perturbed systems

$$\frac{dy}{dt} = f(t, y) + \varepsilon g(y) \tag{3.1}$$

generated by system (1.1) and the function $g(y)$. Denote by

$$y(t) = q(t; t_0, y_0, \varepsilon) \tag{3.2}$$

an unique solution of (3.1) satisfying the initial condition

$$y(t_0) = y_0.$$

A solution $y(t) = y(t;\varepsilon)$ of (3.1) is called a $(T, \rho_0, \rho_1)$-solution $(\rho_1 > \rho_0)$ if it is defined on some closed bounded interval $[t_1, t_2]$ and

(i)

$$|y(t_1)| = |y(t_2)| = \max_{t_1 \le t \le t_2} |y(t)| = \rho_1,$$

(ii) for some $t_0 \in (t_1, t_2)$

$$|y(t_0)| \le \rho_0,$$

(iii)

$$t_2 - t_1 \le T.$$

**Remark 1** According to the periodicity of the function $f(t, x)$ we can assume without loss of generality that the values $t_0,\ t_1\ t_2$ satisfy the inequalities

$$0 \le t_1 \le T, \quad 0 \le t_1 < t_0 < t_2 \le 2T.$$

**Remark 2** If (3.1) for some $\varepsilon > 0$ has some $(T, \rho_0, \rho_1)$-solution $y(t)$ $(t_1 \le t \le t_2)$ then (3.1) for the same $\varepsilon$ has a $(T, \rho_0, \rho)$-solution for any $\rho \in (\rho_0, \rho_1)$. Consider the values

$$t_{1,\rho} = \max\left\{t \in [t_1, t_0],\ |y(t)| = \rho\right\}, \quad t_{2,\rho} = \min\left\{t \in [t_0, t_2],\ |y(t)| = \rho\right\}.$$

The function $y(t)$ considered on the interval $[t_{1,\rho}, t_{2,\rho}]$ is the $(T, \rho_0, \rho)$-solution of (3.1).

**Lemma 2** *Under the assumptions of Theorem* 2 *there exist a* $\rho_* > \rho_0$ *and* $\varepsilon_* > 0$ *such that for every* $\varepsilon \in [0, \varepsilon_*]$ *system* (3.1) *has no* $(T, \rho_0, \rho_1)$*-solution for* $\rho_1 \ge \rho_*$.

**First step of the proof.** Due to Remark 2 it is sufficient to prove the existence of $\rho_* > \rho_0$ and $\varepsilon_* > 0$ such that for $\varepsilon \in [0, \varepsilon_*]$ the system (3.1) has no $(T, \rho_0, \rho_*)$-solution.

**Second step of the proof.** Consider an arbitrary $r > \rho_0$ and suppose that there exists a sequence of positive $\varepsilon_n,\ \varepsilon_n \to 0$ such that every system

$$\frac{dy}{dt} = f(t, y) + \varepsilon_n g(y), \quad n = 1, 2, \ldots$$

has at least one $(T, \rho_0, r)$-solution $y_n(t)$ $(t_1^n \le t \le t_2^n)$. Due to Remark 1 we can assume

$$0 \le t_1^n \le T, \quad 0 \le t_1^n < t_0^n < t_2^n \le 2T.$$

By definition of $(T, \rho_0, \rho_1)$-solution there exist $t_0^n \in (t_1^n, t_2^n)$ such that

$$|y_n(t_0^n)| \le \rho_0, \quad n = 1, 2, \ldots \tag{3.3}$$

Since the sets $[0, 2T]$ and $\{|x| \le \rho_0\}$ are compact we can without loss of generality assume that all the sequences $t_1^n, t_0^n, t_2^n$ and $y_n(t_0^n)$ converge to some limits:

$$\lim_{n\to\infty} t_1^n = t_1^*, \quad \lim_{n\to\infty} t_0^n = t_0^*, \quad \lim_{n\to\infty} t_2^n = t_2^*, \quad \lim_{n\to\infty} y_n(t_0^n) = y^*.$$

Inequalities (3.3) imply $|y^*| \le \rho_0$, and $t_2^n - t_1^n \le T \quad (n = 1, 2, \dots)$ imply $t_2 - t_1 \le T$.

Consider the solution $x(t) = p(t; t_0^*, y^*)$ of (1.1) with initial value $x(t_0^*) = y^*$. Let us show that this solution is defined on $[t_1^*, t_2^*]$ and

$$|x(t_1^*)| = r = |x(t_2^*)|. \tag{3.4}$$

Suppose $x(t)$ is not defined on $[t_1^*, t_2^*]$. This means that at least one of the two following inequalities is valid:

$$\tau_-(t_0^*, y^*) > t_1^*, \quad \tau_+(t_0^*, y^*) < t_2^*.$$

Let $\tau_+(t_0^*, y^*) < t_2^*$. In this case $|x(t)| > r$ for some $t \in (t_0^*, t_2^*)$. The uniqueness of solution of (1.1)–(1.3) according to the general theorems on continuity of Cauchy problem solution with respect to parameters and initial values implies that for sufficiently large $n$ estimate $|y_n(t)| > r$ holds. This means that $y_n(t)$ is not a $(T, \rho_0, r)$-solution of (3.1) for $\varepsilon = \varepsilon_n$. The case $\tau_-(t_0^*, y^*) > t_1^*$ can be studied by analogous way.

Thus $x(t)$ is defined on $[t_1^*, t_2^*]$.

Let us prove now the left equality in (3.4) (the another one can be proved analogously). Since $t_1^n \to t_1^*$ at least one of the following two statements is valid: (a) for an infinite number of $n$'s: $n_1, n_2, \dots$ we have $t_1^{n_k} \le t_1^*$ or (b) for an infinite number of $n$'s: $n_1, n_2, \dots$ we have $t_1^{n_k} \ge t_1^*$.

If statement (a) is true then the functions $y_{n_k}(t)$ are defined for $t = t_1^*$. In this case

$$|x(t_1^*)| = \lim_{n\to\infty} |y_{n_k}(t_1^*)|,$$

and according to uniform continuity

$$\lim_{n\to\infty} |y_{n_k}(t_1^{n_k})| = \lim_{n\to\infty} |y_{n_k}(t_1^*)|;$$

therefore $|y_{n_k}(t_1^{n_k})| = r$ implies $|x(t_1^*)| = r$.

In the case (b)

$$|x(t_1^*)| = \lim_{n\to\infty} |x(t_1^{n_k})| = \lim_{n\to\infty} |y_{n_k}(t_1^{n_k})| = r.$$

We completely proved that under the assumptions of this step $x(t)$ (or its restriction on a shorter interval) is a $(T, \rho_0, r)$-solution of (1.1). Hence to prove Lemma 2 it is sufficient to prove that for some $\rho_* > \rho_0$ the system (1.1) has no $(T, \rho_0, \rho_*)$-solutions.

**Third step of the proof.** Let $\rho_n > \rho_0$ and $\rho_n \to \infty$. Let us assume that for every $n$ the system (1.1) has a $(T, \rho_0, \rho_n)$-solution $x_n(t)$ $(t_1^n \le t \le t_2^n)$. It means that

$$|x_n(t_1^n)| = |x_n(t_2^n)| = \max_{t_1^n \le t \le t_2^n} |x_n(t)| = \rho_n$$

and for some $t_0^n \in (t_1^n, t_2^n)$

$$|x_n(t_0^n)| \le \rho_0.$$

Due to Remark 1 we can assume that

$$0 \le t_1^n \le T, \quad 0 \le t_1^n < t_0^n < t_2^n \le 2T, \quad t_2^n - t_1^n \le T.$$

Without loss of generality (consider if necessary some subsequence $x_{n_k}(t)$) we suppose that all the sequences $t_1^n, t_0^n, t_2^n$ and $x_n(t_0^n)$ converge to some limits $t_1^*, t_0^*, t_2^*, x_0^*$. Obviously, $0 \le t_1^* \le T,\ t_1^* \le t_0^* \le t_2^* \le 2T,\ t_2^* - t_1^* \le T,\ |x_0^*| \le \rho_0$.

Consider the solution $x^*(t) = p(t; t_0^*, x_0^*)$ of (1.1). Since by (1.10) the inequality

$$T < \tau_+(t_0^*, x_0^*) - \tau_-(t_0^*, x_0^*)$$

holds then the solution $x^*(t)$ is defined at least on one of the closed intervals $[t_1^*, t_0^*],\ [t_0^*, t_2^*]$, say the interval $[t_1^*, t_0^*]$. The theorem on the continuity of solutions with respect to initial values implies that for sufficiently large $n$ the estimate

$$|x_n(t_1^n) - x^*(t_1^n)| < 1$$

is valid; therefore

$$\rho_n = |x_n(t_1^n)| \le 1 + |x^*(t_1^*)| < \infty.$$

This contradiction proves the lemma. ■

# 4 A lemma on *a priory* estimate of periodic solutions

According to Lemma 1 one can choose a locally Lipschitzian function $g(x): \ I\!R^N \to I\!R^n$ such that

$$(g(x), V'(x)) > 0, \quad x \in I\!R^N,\ |x| \ge \rho_0. \tag{4.1}$$

**Lemma 3** *Let the assumptions of Theorem* 2 *be valid and* $\rho_*,\ \varepsilon_*$ *be the values introduced in Lemma* 2. *Then all* $T$*-periodic solutions* $y(t)$ *of system* (3.1) *for* $\varepsilon \in (0, \varepsilon_*]$ *satisfy the* a priory *estimate*

$$|y(t)| < \rho_*, \quad t \in I\!R.$$

**Proof.** Let the statement of the lemma be false. It means that for some $\varepsilon_0 \in (0, \varepsilon_*]$ the system

$$\frac{dy}{dt} = f(t, y) + \varepsilon_0 g(y) \tag{4.2}$$

has a $T$-periodic solution $y(t)$ and

$$\max_{0 \le t \le T} |y(t)| = r \ge \rho_*. \tag{4.3}$$

We have one of the two following cases: either

$$\min_{0 \le t \le T} |y(t)| \ge \rho_0, \tag{4.4}$$

or for some $t_0 \in [0, T]$

$$|y(t_0)| < \rho_0. \tag{4.5}$$

Let (4.4) be valid. Put

$$v(t) = V[y(t)], \quad t \in I\!R. \tag{4.6}$$

Due to the $T$-periodicity of $y(t)$, the scalar-valued function (4.6) is also $T$-periodic. On the another hand

$$v'(t) = (f[t, y(t)], V'[y(t)]) + \varepsilon_0 \, (V'(y), g(y))\,;$$

therefore (by (1.7) and (4.1))

$$v'(t) > 0, \quad t \in I\!R.$$

The last inequality contradicts to periodicity of $v(t)$.

Let (4.5) be valid for some $t_0 \in [0, T]$. Put

$$t_1 = \max\{t < t_0 : \ |y(t)| = \rho_*\}\,, \quad t_2 = \min\{t > t_0 : \ |y(t)| = \rho_*\}$$

(these numbers can be defined according to the assumption (4.3)). Since $y(t)$ is $T$-periodic, $t_2 - t_1 \le T$. Therefore $y(t)$ is a $(T, \rho_0, \rho_*)$-solution of (4.2). But Lemma 2 implies that (4.2) has no $(T, \rho_0, \rho_*)$-solutions. This contradiction proves the lemma. ■

Note that Lemma 3 says nothing about the existence of $T$-periodic solutions. We only obtained that for small $\varepsilon_0$ the system (4.2) has no $T$-periodic solutions $y(t)$ with $\max\{|y(t)|\} \ge \rho_*$.

**Lemma 4** *Under the assumptions of Lemma* 3 *the equation*

$$\frac{dy}{dt} = f(t, y) + \varepsilon g(y) \tag{4.7}$$

*has for* $0 < \varepsilon \le \varepsilon_*$ *at least one* $T$*-periodic solution.*

**Proof.** Let

$$\beta(r) = \max_{0 \le t \le T,\ |y| \le r} |f(t, y) + \varepsilon g(y)|, \quad 0 \le r < \infty.$$

This function is non-decreasing and since, by (1.7)

$$(f(t, y) + \varepsilon g(y), V'(y)) \ge \varepsilon \, (V'(y), g(y))\,, \quad |y| \ge \rho_0$$

its values are positive for $r \ge \rho_0$. Put

$$\varphi(y) = \begin{cases} 1 & \text{if } 0 \le |y| \le \rho_*, \\ \text{linear} & \text{if } \rho_* \le |y| \le 1 + \rho_*, \\ (1 + \beta(|y|))^{-1} & \text{if } 1 + \rho_* \le |y|. \end{cases}$$

Consider the auxiliary equation

$$\frac{dy}{dt} = \varphi(y)[f(t, y) + \varepsilon g(y)]. \tag{4.8}$$

Since the right-hand side of (4.8) is uniformly bounded we see that all the solutions of (4.8) are infinitely continuable both to the left and to the right. Since, by (1.7) and (4.1)

$$\begin{gathered} (\varphi(y)[f(t, y) + \varepsilon V'(y)], g(y)) = \varphi(y)\,[(f(t, y), V'(y)) + \varepsilon\,(g(y), V'(y))] \\ \ge \varepsilon \varphi(y)\,(V'(y), g(y)) > 0, \qquad |y| \ge \rho_0 \end{gathered}$$

the function $V(y)$ is a guiding function for system (4.8) and the index at infinity of the guiding function $V(y)$ is different from 0. Therefore the general theorems on guiding functions for equations with infinitely continuable solutions (cf [1, 2]) imply the existence of at least one $T$-periodic solution of (4.8).

Let us show that every $T$-periodic solution $y(t)$ of (4.8) is a $T$-periodic solution of (4.7), i.e. let us prove the estimate

$$|y(t)| \leq \rho_*, \quad t \in I\!R. \tag{4.9}$$

We shall use constructions which has been already used in the proof of Lemma 3. Assume (4.9) is not valid.

If

$$\min_{0 \leq t \leq T} |y(t)| \geq \rho_0$$

then the $T$-periodic scalar-valued function $w(t) = V[y(t)]$ $(t \in I\!R)$ satisfies the estimate

$$\frac{dw(t)}{dt} \geq \varepsilon\varphi[y(t)]\,(V'[y(t)], g[y(t)]) > 0;$$

this is impossible.

Now, if for some $t_0 \in [0, T]$

$$|y(t_0)| < \rho_0,$$

then $y(t)$ $(t_1 \leq t \leq t_2)$ is a $(T, \rho_0, \rho_*)$-solution of (4.8). Therefore $|y(t)| \leq \rho_*$ for $t \in [t_1, t_2]$ and $y(t)$ is a $(T, \rho_0, \rho_*)$-solution of system (4.7). This contradicts Lemma 2.

Hence (4.9) is proved, which implies that every $T$-periodic solution of (4.8) is a $T$-periodic solution of (4.7). Lemma 4 is proved. ∎

# 5 End of the proof

Consider a sequence of equations

$$\frac{dz}{dt} = f(t, z) + \varepsilon_n g(z) \tag{5.1}$$

with $\varepsilon_n \in (0, \varepsilon_*)$ and $\varepsilon_n \to 0$. According to Lemma 4 equation (5.1) has at least one $T$-periodic solution $z_n(t)$ for every $n$. This solution satisfies

$$|z_n(t)| \leq \rho_*, \quad t \in I\!R;\ n = 1, 2, \ldots \tag{5.2}$$

Since all the derivatives $z_n^{'}(t)$ are uniformly bounded:

$$|z_n^{'}(t)| \leq \max_{0 \leq t \leq T,\ |z| \leq \rho_*} |f(t, z)| + \varepsilon_* \max_{|z| \leq \rho_*} |g(z)|, \quad t \in I\!R;\ n = 1, 2, \ldots$$

the sequence $z_n(t)$ is compact in $C = C[0, T]$. Therefore one can choose a subsequence $z_{n_k}(t)$ $(k = 1, 2, \ldots)$ converging in $C$ to some function $z(t)$. Every function $z_{n_k}(t)$ satisfies the integral equation

$$z_{n_k}(t) = z_{n_k}(T) + \int_0^t f[s, z_{n_k}(s)]ds + \varepsilon_{n_k} \int_0^t g[z_{n_k}(s)]ds.$$

Consequently the $z(t)$ is a $T$-periodic solution of (1.1).

Theorem 2 is completely proved. ■

**Remark 3** Estimates (5.2) imply the estimate $|z(t)| \leq \rho_*$ ($t \in I\!R$). This estimate does not mean any *a priori* estimate of all the solutions of equation (1.1). In some cases, under assumptions of Theorem 2, $T$-periodic solutions of (1.1) can have an arbitrary large norm in $C$. In those cases some $T$-periodic solutions of (1.1) cannot be constructed by the procedure used above in the proof of Theorem 2.

# 6 Additional statements and constructions

a. Consider the system

$$\frac{dx}{dt} = Ax + bf(t, x), \quad x \in I\!R^N. \tag{6.1}$$

Here $A$ is a $N \times N$ matrix, $b \in I\!R^N$, $f(t, x)$ is a scalar-valued continuous function which is Lipschitzian in $x$. Such type of systems are usual in control theory.

Denote by $\mathcal{F}(\gamma; \rho_0)$ the set of functions $f(t, x)$ satisfying

$$|f(t, x)| \leq \gamma |x|, \quad |x| \geq \rho_0.$$

A function $V(x)$ is called a generalized guiding function for (6.1) uniformly with respect to $f(t, x) \in \mathcal{F}(\gamma; \rho_0)$ if $V'(x) \neq 0$ for $|x| \geq \rho_0$ and

$$(Ax + bf(t, x), V'(x)) \geq 0, \quad f \in \mathcal{F}(\gamma; \rho_0),\ |x| \geq \rho_0$$

Some precise results of Kalman and Jakubovitch (see [5]) on Ljapunov functions for absolute stability problems imply the following statement.

**Lemma 5** *Let the matrix $A$ have no eigenvalues of the type $\xi i$, $\xi \in I\!R$, and*

$$\gamma \max_{-\infty < \omega < \infty} |(i\omega I - A)^{-1} b| \leq 1. \tag{6.2}$$

*Then there exists a non-degenerate quadratic form $V(x)$ which is a generalized quiding function for* (6.1) *uniformly with respect to $f(t, x) \in \mathcal{F}(\gamma; \rho_0)$ and has non-zero index at infinity.*

As always for non-degenerate quadratic forms the index at infinity of the quiding function $V(x)$ is equal to $+1$ or $-1$.

If $f(t, x) \in \mathcal{F}(\gamma; \rho_0)$ then all the solutions of (6.1) are infinitely continuable both to the left and to the right. Therefore the following theorem follows from Theorem 1 and Lemma 5.

**Statement 1** *Let* (6.2) *be valid. Then every system* (6.1) *with a $T$-periodic in $t$ function $f(t, x) \in \mathcal{F}(\gamma; \rho_0)$ has at least one $T$-periodic solution.*

Here there is no necessity to use any restrictions for the period $T$.

b. In applications of Theorem 2 the natural question arises about the most precise estimates of the value $\alpha(\rho_0)$. Some approaches to this problem are connected with theorems on differential inequalities (see e.g. [3, 6]). Consider the following examples.

Let $M(u)$ $(u \geq 0)$ be a positive and continuous function. Natural examples of $M(u)$ are the functions

$$a + bu^p \ (p \geq 1), \quad a + bu \ln(1+u), \quad a + be^u, \quad a + be^{u^2} \tag{6.3}$$

etc. Denote by $u(t)$ $(t \geq 0)$ a solution of Cauchy problem

$$\frac{du}{dt} = M(u), \quad u(0) = \rho_0^2.$$

The solution $u(t)$ increases (since $M(u) > 0$). Let $[0, m(\rho_0))$ be the maximal interval where $u(t)$ is defined.

Assume that

$$(f(t,x), x) \leq \frac{1}{2} M(|x|^2), \quad |x| \geq \rho_0. \tag{6.4}$$

Consider some solution $x(t)$ of (1.1) satisfying

$$x(t_0) = x_0, \quad |x_0| = \rho_0, \quad |x(t)| \geq \rho_0, \quad (t \geq t_0).$$

Assumption (6.4) (for $t > t_*$) implies

$$\frac{d}{dt}|x_0(t)|^2 = 2\left(\frac{dx_0}{dt}, x_0(t)\right) = 2\left(f[t, x_0(t)], x_0(t)\right) \geq M\left[|x(t)|\right].$$

Therefore the estimate

$$|x_0(t)|^2 \leq u(t - t_0)$$

holds for $t \in [t_0, t_0 + m(\rho_0))$. We have proved the following result.

**Statement 2** *The estimate* (6.4) *guarantees an estimate*

$$\tau_+(t_0, x_0) \geq t_0 + m(\rho_0)$$

*for every* $t_0 \in I\!R$ *and* $|x_0| \geq \rho_0$.

The next statement is dual to Statement 2 and can be proved analogously.

Consider one function $M_1$ of type (6.3). Let the function $M_1(u)$ $(u \geq 0)$ be positive and continuous. Denote by $u_1(t)$ $(t \geq 0)$ a solution of Cauchy problem

$$\frac{du_1}{dt} = M_1(u), \quad u_1(0) = \rho_0^2.$$

Let $[0, m_1(\rho_0))$ be the maximal interval where $u_1(t)$ is defined.

**Statement 3** *The estimate*

$$(f(t,x), x) \geq -\frac{1}{2} M_1(|x|^2), \quad |x| \geq \rho_0 \tag{6.5}$$

*guarantees an estimate*

$$\tau_-(t_0, x_0) \leq t_0 - m_1(\rho_0)$$

*for every* $t_0 \in I\!R$ *and* $|x_0| \geq \rho_0$.

Condition (6.4) implies $\alpha(\rho_0) \geq m(\rho_0)$, condition (6.5) implies $\alpha(\rho_0) \geq m_1(\rho_0)$. If both conditions (6.4) and (6.5) hold then

$$\alpha(\rho_0) \geq m(\rho_0) + m_1(\rho_0),$$

In this case assumption (1.10) of Theorem 2 can be rewritten as

$$T < m(\rho_0) + m_1(\rho_0).$$

A successful choice of functions $M(u)$ and $M_1(u)$ makes it possible to obtain sharp estimates for the period $T$.

c. Suppose we cannot estimate the value of $\alpha(\rho_0)$. In this case Theorem 2 has the following meaning: the existence of guarantees the existence of a generalized guiding function of non-zero index at infinity $T$-periodic solutions for sufficiently small $T > 0$.

d. Consider the system

$$\frac{dx}{dt} = Ax + g(t,x), \quad x \in I\!R^N. \tag{6.6}$$

Here $A$ is a $N \times N$ matrix having no eigenvalues of the type $\xi i$ ($\xi \in I\!R$); $g(t,x) : I\!R \times I\!R^N \to I\!R^N$ is continuous and $T$-periodic in $t$.

Let $N_1$ be the sum of the multiplicities of all eigenvalues of $A$ having negative real part, let $N_2$ be the sum of the multiplicities of all eigenvalues of $A$ having positive real part.

The system (6.6) can be rewritten in the form

$$\begin{array}{l} \frac{d\xi}{dt} = A_1\xi + g_1(t, \xi + \eta), \\ \frac{d\eta}{dt} = A_2\eta + g_2(t, \xi + \eta). \end{array}$$

Here $\xi \in I\!R^{N_1}$, $\eta \in I\!R^{N_2}$, $x = \xi + \eta$. The matrix $A_1$ is of order $N_1 \times N_1$; all its eigenvalues have negative real parts. The matrix $A_2$ is of order $N_2 \times N_2$; all its eigenvalues have positive real parts. Without loss of generality we can assume that $I\!R^{N_1}$ and $I\!R^{N_2}$ are orthogonal subspaces of $I\!R^B$.

There exist quadratic forms $(B_1\xi, \xi)$ ($\xi \in I\!R^{N_1}$) and $(B_2\eta, \eta)$ ($\eta \in I\!R^{N_2}$) such that $(B_1\xi, \xi) \geq 0$, $(B_2\eta, \eta) \geq 0$ and

$$(B_1\xi, A_1\xi) \leq -(\xi, \xi), \quad (B_2\eta, A_2\eta) \geq (\eta, \eta), \quad (\xi \in I\!R^{N_1},\ \eta \in I\!R^{N_2}). \tag{6.7}$$

**Statement 4** *Let*

$$(g(t,x), B_1\xi - B_2\eta) \leq |x|^2 \quad (x = \xi + \eta,\ \xi \in I\!R^{N_1},\ \eta \in I\!R^{N_2},\ |x_0| \geq \rho_0). \tag{6.8}$$

*Then the non-degenerate quadratic form*

$$V(x) = -(B_1\xi, \xi) + (B_2\eta, \eta)$$

*is a generalized guiding function for system* (6.6).

This conclusion follows immediately from (6.7) and (6.8). Under the assumption of Statement 4 $\mathbf{ind}(V; \infty) = (-1)^N$.

Statement 4 with Theorems 1 and 2 imply various concrete criteria for the existence of $T$-periodic solutions for (6.1).

# 7 Proof of Theorems 3 and 4

The main part of the proof is common for both theorems ($T$-periodicity is not used).

According to Lemma 1 it is possible to construct a locally Lipschitzian function $g(x) : \mathbb{R}^N \to \mathbb{R}^N$ such that

$$(g(x), V_j'(x)) > 0, \quad j = 1, \ldots, k;\ |x| \geq \rho_0.$$

Consider for $\varepsilon \geq 0$ the equations

$$\frac{dx}{dt} = f(t, x) + \varepsilon g(x). \tag{7.1}$$

Put

$$m_j = \min_{|z| \leq \rho_0} V_j(z), \quad M_j = \max_{|z| \leq \rho_0} V_j(z)$$

and consider the family of open sets

$$G_j^- = \left\{ x \in \mathbb{R}^N :\ V_j(x) < m_j \right\}, \quad G_j^+ = \left\{ x \in \mathbb{R}^N :\ V_j(x) > M_j \right\}. \tag{7.2}$$

The main property of these sets is the following one. If any solution of system (7.1) starts at $t = t_0$ in a set $G_j^+$ for some $j$, then it will stay in the same $G_j^+$ for all values of $t > t_0$. If any solution of system (7.1) was in a set $G_j^-$ at $t = t_0$ for some $j$, then it stayed in the same $G_j^-$ for all values of $t < t_0$.

This property follows from the inequalities

$$(f(t, x) + \varepsilon g(x), V_j'(x)) \geq \varepsilon (g(x), V_j'(x)) > 0, \quad j = 1, \ldots, k;\ |x| \geq \rho_0. \tag{7.3}$$

These inequalities guarantee the increasing character of the function $V_j(x)$ in the set $G_j^+$ along the trajectory of an arbitrary solution of (7.1), and the decreasing character of the function $V_j(x)$ in the set $G_j^-$ along this trajectory for the time "moving back".

The completeness of the system $V_1(x), \ldots, V_k(x)$ (condition (1.13)) implies that there exists s $\rho_* > \rho_0$ such that the closed set $\{x \in \mathbb{R}^N :\ |x| \geq \rho_*\}$ is completely covered by the union of all the sets (7.2).

**Lemma 6** *Let system* (1.1) *have a complete and sharp set of generalized guiding functions* (1.11) *and let* (1.14) *be valid. Then for every $\varepsilon > 0$ and $a > 0$ there exist a solution $x(t) = x(t; \varepsilon, a)$ of* (7.1) *defined for $t \in [-a, a]$ and such that $x(-a) = x(a)$ and*

$$|x(t)| \leq \rho_*, \quad -a \leq t \leq a. \tag{7.4}$$

**Proof.** Consider a positive continuously differentiable function $r \mapsto \psi(r; \varepsilon)$ $(r \geq 0)$ such that $\psi(r; \varepsilon) \equiv 1$ $(0 \leq r \leq \rho_*)$ and

$$\psi(r; \varepsilon) \geq \max_{|t| \leq a,\ |z| \leq r} |f(t, z) + \varepsilon g(z)|.$$

Then the right-hand sides of the systems

$$\frac{dx}{dt} = \frac{f(t, x) + \varepsilon g(x)}{\psi(|x|; \varepsilon)} \tag{7.5}$$

are locally Lipschitzian and uniformly bounded for $|t| \leq a$. Consequently an unique solution of (7.5) with initial value $x(-a) = x_0 \in I\!R^N$ is defined on $[-a, a]$. It means that on $I\!R^N$ the translation operators $U(-a, t)$ are defined along the trajectories of system (7.5) during the time from $-a$ till $t \in [-a, a]$ (for the definition, properties etc of such operators see, cf [3]).

Consider on the sphere $S = \{x \in I\!R^N,\ |x| = \rho_* + 1\}$ the vector-fields

$$\Phi(x; t) = U(-a, t)x - x$$

Since (7.3) hold, these fields are non-degenerate for $t \in (-a, a]$. Therefore the rotation of $\Phi(x; t)$ on $S$ is defined ([2]), and does not depend on $t$. Denote it by $\gamma$.

For values of $t$ sufficiently closed to $-a$ the angle between $f(-a, x) + \varepsilon g(x)$ and $\Phi(x; t)$ is less than $\frac{\pi}{2}$; the angle between $f(-a, x) + \varepsilon g(x)$ and $V_1'(x)$ is also less than $\frac{\pi}{2}$. Hence $\gamma = \mathbf{ind}(V_1; \infty)$, and by (1.14) $\gamma \neq 0$. Consequently the operator $U(-a, a)$ has at least one fixed point $x_*$. An unique solution $x_*(t)$ of Cauchy problem for (7.5) with initial condition $x(-a) = x_*$ satisfies $x_*(-a) = x_*(a)$.

Let $x_* \in G_j^+$. Then on one hand the function $V_j[x_*(t)]$ takes the same values in the points $-a$ and $a$, and on the other hand this function has derivative

$$\frac{dV_j[x_*(t)]}{dt} = \left( \frac{f[t, x_*(t)] + \varepsilon g[x_*(t)]}{\psi(|x_*(t)|; \varepsilon)}, V_j'[x_*(t)] \right)$$

which is greater than 0.

Let $x_* \in G_j^-$. Then on one hand the function $V_j[x_*(-t)]$ takes the same values in the points $-a$ and $a$, and on the other hand this function has derivative

$$\frac{dV_j[x_*(-t)]}{dt} = -\left( \frac{f[-t, x_*(-t)] + \varepsilon g[x_*(-t)]}{\psi(|x_*(-t)|; \varepsilon)}, V_j'[x_*(-t)] \right)$$

which is less than 0.

These contradictions mean that

$$x_* \notin \bigcup_{j=1}^{k} [G_j^+ \cup G_j^-].$$

According the main property of the sets (7.2) we see that

$$x_*(t) \notin \bigcup_{j=1}^{k} [G_j^+ \cup G_j^-] \quad (-a \leq t \leq a).$$

Therefore (7.4) holds and $\psi(|x_*(t)|; \varepsilon) \equiv 1$. The function $x_*(t)$ is a solution not only of (7.5) but also of (7.1).

Lemma 6 is proved. ■

Proof of Theorem 3.

Put $a = T/2,\ \varepsilon_n = 1/n \ \ (n = 1, 2, \ldots)$. According to Lemma 6, for every $n \in I\!N$ and $\varepsilon = \varepsilon_n$, equation (7.1) has at least one $T$-periodic solution $x_n(t)$ such that $|x_n(t)| \leq \rho_*$ ($t \in I\!R$). All the derivatives $x_n'(t)$ are uniformly bounded;

$$|x_n'| \leq \sup_{|x| \leq \rho_*,\ -T/2 \leq t \leq T/2,\ 0 \leq \varepsilon \leq 1} |f(t, x) + \varepsilon g(x)| < \infty.$$

Therefore we can choose a subsequence $x_{n_k}(t)$ which converges uniformly to some $T$-periodic function $x_*(t)$. This function is a $T$-periodic solution of (1.1). ■

Proof of Theorem 4.

Put $a = n,\ \varepsilon_n = 1/n \quad (n = 1, 2, \ldots)$. According to Lemma 6, for every $n \in I\!N$ and $\varepsilon = \varepsilon_n$, equation (7.1) has at least one solution $x_n(t)$ such that $|x_n(t)| \le \rho_*$ $(-n \le t \le n)$. All the derivatives $x_n^{'}(t)$ are uniformly bounded;

$$|x_n^{'}| \le \sup_{|x| \le \rho_*,\ |t| \le c,\ 0 \le \varepsilon \le 1} |f(t, x) + \varepsilon g(x)| < \infty$$

on every finite interval of the time. Therefore we can choose a subsequence $x_{n_k}(t)$ which converges uniformly to some $x_*(t)$ on every finite interval of the time. This function $x_*(t)$ is defined for $t \in I\!R$, $|x_*(t)| \le \rho_*$. ■

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