

ASYMPTOTIC BEHAVIOUR OF RANDOM TRIDIAGONAL MARKOV CHAINS IN BIOLOGICAL APPLICATIONS

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Dedicated to the memory of Alexei Pokrovsky

ABSTRACT. Discrete-time discrete-state random Markov chains with a tridiagonal generator are shown to have a random attractor consisting of singleton subsets, essentially a random path, in the simplex of probability vectors. The proof uses the Hilbert projection metric and the fact that the linear cocycle generated by the Markov chain is a uniformly contractive mapping of the positive cone into itself. The proof does not involve probabilistic properties of the sample path ω and is thus equally valid in the nonautonomous deterministic context of Markov chains with, say, periodically varying transitions probabilities, in which case the attractor is a periodic path.

1. Introduction. Markov chains with a tridiagonal generator are common in biological applications, see, e.g., [1, 13, 21]. Their asymptotic behaviour is well understood when the transitions probabilities are constant, i.e, the Markov chain is homogeneous or, equivalently, autonomous in the language of dynamical systems. In this paper we consider the case where the transitions probabilities can vary in time, e.g., periodically, or randomly due to a periodically or randomly changing environment. The Markov chains are then nonautonomous or random dynamical systems [2, 8, 15] and the concepts of autonomous dynamical systems such as equilibria are inadequate. Thus a new concept of nonautonomous or random attractors is needed.

The results in this paper are presented in the context of random Markov chains and random dynamical systems, although the proofs are do not depend at all on probabilistic properties of the sample path parameter ω and are thus equally valid in

2000 *Mathematics Subject Classification.* Primary: 15B48, 15B52, 37H10; Secondary: 15B51, 60J10, 92C99.

Key words and phrases. Random Markov chain, positive cones, Hilbert metric, uniformly contracting cocycles, linear cocycles, random attractors.

The first author was DFG grants KL 1203/7-1, the Spanish Ministerio de Ciencia e Innovación project MTM2011-22411, the Consejería de Innovación, Ciencia y Empresa (Junta de Andalucía) under the Ayuda 2009/FQM314 and the Proyecto de Excelencia P07-FQM-02468. The second author was partially supported by the Russian Foundation for Basic Research, project no. 10-01-93112.

the nonautonomous deterministic context of Markov chains with, say, periodically varying transitions probabilities.

Tridiagonal Markov chains, both deterministic and random, are presented in Section 2. The long term dynamical behaviour of the autonomous deterministic case is then given for completeness in Section 3, although it follows as a special case, because it provides useful background information for the random case. The proof uses the fact that random Markov chains generate contractive linear cocycles which map a positive cone into itself. First, in Section 4, a general theorem on existence of a random attractor with singleton subsets in a metric space is formulated and proved. The assumptions seem rather restrictive at the first sight, but are just what is needed later. Then, in Section 5 some preliminaries from the theory of positive linear operators are recalled, the essence of which is that in a quite a general situation, for linear maps positive with respect to the same invariant cone, there exists a *common* metric, the Hilbert projective metric, in which all these linear mappings are *uniformly contractive*. Finally, in Section 6, it is shown that the linear cocycles generated by the random Markov chains in Section 2 satisfy the conditions of the abstract theorem from Section 4 under uniform upper and lower positivity bounds on the tridiagonal transitions probabilities and thus have a random attractor consisting of singleton subsets. The random attractor is essentially a randomly varying path in the simplex of probability vectors which pathwise attracts all other iterates of the Markov chain. In the nonautonomous deterministic setting with periodical transitions probabilities it is a periodic path.

There is an extensive literature on products of random Markov chains, see e.g., [10, 11, 12, 17, 18, 22]. Although the problem investigated here does not seem to have been addressed directly as such yet in the literature, the results could probably be obtained by extending the proofs in [11, 22] after similar computations to those that are needed below. The proof given here is preferable since it is direct and is written in the language of random dynamical systems. In particular, the paper demonstrates that the effect of a random attractor to be singleton is valid not only for some classes of monotone random systems, as it is reported earlier in [9], but also for the class of random Markov chains studied below.

2. Tridiagonal Markov chains in biological models. Markov chains with tridiagonal transition matrices are common in biological models, for example, birth-and-death processes [1], cell-cell communication [13] and cancer dynamics [21], to name just a few.

To fix ideas, consider the distance $d(t_n)$ between two cells at time $t_n = n\Delta$, which is supposed to take discrete values in $\{1, \dots, N\}$, essentially the distance that they can move in one unit of time, where $d(t_n)$ can stay unchanged or change to $d(t_n) \pm 1$ with certain probabilities. This can be formulated as an N state discrete-time Markov chain with states $\{1, \dots, N\}$ corresponding to the value of $d(t_n)$.

Let $\mathbf{p}(t_n) = (p_1(t_n), \dots, p_N(t_n))^T$ be the probability vector for the state of the system at time t_n . The dynamics are described by the system of difference equations

$$\begin{aligned} p_1(t_{n+1}) &= [1 - q_1\Delta] p_1(t_n) + q_2 p_2(t_n)\Delta \\ p_j(t_{n+1}) &= q_{2j-3} p_{j-1}(t_n)\Delta + [1 - (q_{2j-2} + q_{2j-1})\Delta] p_j(t_n) + q_{2j} p_{j+1}(t_n)\Delta, \\ &\hspace{15em} j = 2, \dots, N-1, \\ p_N(t_{n+1}) &= q_{2N-3} p_{N-1}(t_n)\Delta + [1 - q_{2N-2}\Delta] p_N(t_n), \end{aligned}$$

where

$$q_j > 0, \quad j = 1, \dots, 2N - 2,$$

and the p_j satisfy the probability constraints

$$\sum_{j=1}^N p_j = 1, \quad p_j \geq 0, \quad j = 1, \dots, N$$

This is a vector-valued difference equation

$$\mathbf{p}(t_{n+1}) = [I_N + \Delta Q] \mathbf{p}(t_n)$$

on the simplex Σ_N in \mathbb{R}^N defined by

$$\Sigma_N = \left\{ \mathbf{p} = (p_1, \dots, p_N)^T : \sum_{j=1}^N p_j = 1, p_1, \dots, p_N \in [0, 1] \right\},$$

where I_N is the $N \times N$ identity matrix and Q is the tridiagonal $N \times N$ -matrix

$$Q = \begin{bmatrix} -q_1 & q_2 & & & & & & & \circ \\ q_1 & -(q_2 + q_3) & q_4 & & & & & & \\ & \ddots & \ddots & \ddots & \ddots & & & & \\ & & & & q_{2N-5} & -(q_{2N-4} + q_{2N-3}) & q_{2N-2} & & \\ \circ & & & & & q_{2N-3} & -q_{2N-2} & & \end{bmatrix} \quad (1)$$

This is a discrete-time finite-state Markov chain

$$\mathbf{p}^{(n+1)} = [I_N + \Delta Q] \mathbf{p}^{(n)} \quad (2)$$

with the transition matrix $[I_N + \Delta Q]$. It is a first order linear difference equation on Σ_N corresponding to the Euler numerical scheme for the ordinary algebraic-differential equation

$$\frac{d\mathbf{p}}{dt} = Q\mathbf{p}, \quad \mathbf{p} \in \Sigma_N,$$

with the constant time step $\Delta > 0$.

2.1. Random Markov chains. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and suppose now that the coefficients in the Q matrix are random, i.e., the $q_j : \Omega \rightarrow \mathbb{R}$ are \mathcal{F} -measurable mappings or, equivalently, $Q : \Omega \rightarrow \mathbb{R}^{N \times N}$ is an \mathcal{F} -measurable $N \times N$ -matrix valued mapping.

This corresponds to a random environment, which is supposed to vary or be driven by a stochastic process modelled by a metrical (i.e., measurable) dynamical system $\Theta = \{\theta_n, n \in \mathbb{Z}\}$ on Ω generated by a bi-measurable invertible mapping $\theta : \Omega \rightarrow \Omega$. In particular, Θ satisfies $\theta_0\omega = \omega$ and

$$\theta_{m+n}\omega \equiv \theta_m(\theta_n\omega), \quad \forall m, n \in \mathbb{Z}, \omega \in \Omega.$$

See Arnold [2] for more information.

Define $L_\omega := I_N + \Delta Q(\omega)$ and assume that $\Delta > 0$ is sufficiently small, so that the eigenvalues of each matrix for given ω lie in the unit disc of the complex plane (see next section).

This gives the *random Markov chain*

$$\mathbf{p}^{(n+1)} = L_{\theta^n\omega} \mathbf{p}^{(n)}, \quad (3)$$

which is a random linear difference equation on Σ_N , see [2, 8, 14] for random difference equations. The iterates of (3) are random probability vectors in Σ_N , i.e., \mathcal{F} -measurable mappings $\mathbf{p} : \Omega \rightarrow \Sigma_N$.

3. Dynamical behaviour: deterministic case. It is well-known that, under certain nondegeneracy conditions, the “deterministic” Markov chain (2) has a unique equilibrium state which is globally asymptotically stable in Σ_N . This result follows as a special case of the main result of this paper below. A direct proof using elementary methods will now be given, since it also provides useful background information for the general “random” case.

Let $\mathbf{1}_N$ be the column vector in \mathbb{R}^N with all components equal to 1. Then

$$\mathbf{1}_N^T Q = \mathbf{0}, \quad (4)$$

i.e. each column of Q adds to zero. Moreover, $\mathbf{1}_N^T I_N = \mathbf{1}_N$, so $\mathbf{1}_N$ is a left eigenvector corresponding to eigenvalue $\lambda = 1$ of the matrix $I_N + \Delta Q$. Note that the matrix $I_N + \Delta Q$ is a stochastic matrix.

The Perron-Frobenius theorem applies to the matrix $L_\Delta := I_N + \Delta Q$ when $\Delta > 0$ is chosen sufficiently small. In particular, it has eigenvalue $\lambda = 1$ and there is a positive eigenvector $\bar{\mathbf{x}}$ which can be normalized (in the $\|\cdot\|_1$ norm) to give a probability vector $\bar{\mathbf{p}}$, i.e. $[I_N + \Delta Q]\bar{\mathbf{p}} = \bar{\mathbf{p}}$, so $Q\bar{\mathbf{p}} = \mathbf{0}$. In fact, we can show these properties directly for the given matrix.

One can solve $Q\bar{\mathbf{x}} = \mathbf{0}$ uniquely in \mathbb{R}_+^N (up to a scalar multiplier) since by assumption the $q_j > 0$. Specifically,

$$\bar{x}_{j+1} = \frac{q_{2j-1}}{q_{2j}} \bar{x}_j, \quad j = 1, \dots, N-1.$$

Taking $\bar{x}_1 = 1$ yields

$$\bar{x}_{j+1} = \prod_{i=1}^j \frac{q_{2i-1}}{q_{2i}}, \quad j = 1, \dots, N-1,$$

and, hence, the probability vector

$$\bar{\mathbf{p}}_1 = \frac{1}{\|\bar{\mathbf{x}}\|_1}, \quad \bar{\mathbf{p}}_{j+1} = \frac{1}{\|\bar{\mathbf{x}}\|_1} \prod_{i=1}^j \frac{q_{2i-1}}{q_{2i}}, \quad j = 1, \dots, N-1,$$

where

$$\|\bar{\mathbf{x}}\|_1 = \sum_{j=1}^N \bar{x}_j = 1 + \sum_{j=1}^{N-1} \prod_{i=1}^j \frac{q_{2i-1}}{q_{2i}}$$

The corresponding Markov chain is, in fact, ergodic since by assumption all the q_j are positive. In particular, when $q_{2j-1} = q_{2j}$ for each j , then $\bar{\mathbf{p}}$ is the uniformly distributed probability vector with identical components $\bar{p}_i = \frac{1}{N}$, $i = 1, \dots, N$.

Theorem 3.1. *Let $q_i > 0$ for $j = 1, 2, \dots, 2N-2$. Then the probability eigenvector $\bar{\mathbf{p}}$ is an asymptotically stable steady state of the difference equation (2) on the simplex Σ_N .*

Proof. First note that by Geshgorin’s theorem applied to columns the eigenvalues of the matrix Q lie in the union of the closed discs centered on $-q_1 + 0i, \dots, -q_{2j-2} - q_{2j-1} + 0i, \dots, -q_{2N-2} + 0i$ in the complex plane with respective radii $q_1, \dots, q_{2j-2} + q_{2j-1}, \dots, q_{2N-2}$. These all contain the origin $0+0i$ on their boundary, but otherwise lie in the negative real part of the complex plane. It is already known that 0 is an eigenvalue, so all other eigenvalues have strictly negative real parts. Moreover, 0 is a simple eigenvalue with the positive eigenvector $\bar{\mathbf{p}}$.

It is easy to show that no generalized eigenvectors exist, since one would satisfy the equation $Q\bar{\mathbf{x}} = 0\bar{\mathbf{x}} + \bar{\mathbf{p}}$, i.e., $Q\bar{\mathbf{x}} = \bar{\mathbf{p}}$, which is impossible since the sum of

components on the left hand side is equal to 0, while the sum on the right side is equal to 1.

It follows that $\bar{\mathbf{p}}$ is an eigenvector of the matrix $L = I_N + \Delta Q$ corresponding to the simple eigenvalue $\lambda = 1$. Moreover, if Δ is small enough, then all other eigenvalues of the matrix L lie inside the unit disc in the complex plane, i.e., satisfy $|\lambda| < 1$. It then follows that all solutions $\mathbf{p}^{(n)}$ of the difference equation (2) on Σ_N converge to $\bar{\mathbf{p}}$.

This can be shown by adapting the proof of Theorem 10.9 in [19]. Consider the Jordan canonical decomposition $LQ = QJ$, where Q is the matrix of eigenvectors and generalized eigenvectors of the matrix L with $\bar{\mathbf{p}}$ as its first column corresponding to a 1×1 Jordan block [1] and the other Jordan blocks corresponding to the other eigenvalues with $|\lambda| < 1$. Then J^k converges to an $N \times N$ matrix Z with $z_{1,1} = 1$ and all other components $z_{i,j} = 0$. This implies that

$$L^k \mathbf{p}^{(0)} = QJ^k Q^{-1} \mathbf{p}^{(0)} \rightarrow QZQ^{-1} \mathbf{p}^{(0)}$$

for any $\mathbf{p}^{(0)} \in \Sigma_N$. Now $QZ = [\bar{\mathbf{p}}|\mathbf{0}|\dots|\mathbf{0}]$, so $QZQ^{-1} \mathbf{p}^{(0)}$ is a scalar multiple of $\bar{\mathbf{p}}$. But L maps Σ_N into itself, so this scalar multiple of $\bar{\mathbf{p}}$ is, in fact, $\bar{\mathbf{p}}$ itself, i.e., $L^k \mathbf{p}^{(0)} \rightarrow \bar{\mathbf{p}}$ as $k \rightarrow \infty$ for all $\mathbf{p}^{(0)} \in \Sigma_N$. \square

4. Random attractors of uniformly contracting cocycles. Let M be a complete metric space equipped with the metric ρ .

Definition 4.1. A map $F : \mathbb{Z}^+ \times \Omega \times M \rightarrow M$ is called a (discrete-time) cocycle on M with respect to the driving system Θ if it satisfies the *initial condition*

$$F(0, \omega, x) = x, \quad x \in M, \omega \in \Omega,$$

and the *cocycle property*

$$F(n + m, \omega, x) = F(n, \theta_m \omega, F(m, \omega, x)), \quad x \in M, \omega \in \Omega, m, n \in \mathbb{Z}^+. \quad (5)$$

The pair (Θ, F) is called a (discrete-time) random dynamical system in [2].

Define $f(\omega, x) := F(1, \omega, x)$. Then, clearly, due to cocycle property (5) for any $n \in \mathbb{Z}^+$ the map $F(n, \omega, x)$ can be expressed as a superposition of maps $f(\omega, x)$ for different ω and x :

$$F(n, \omega, x) = f(\theta_{n-1} \omega, F(n-1, \omega, x)) = f(\theta_{n-1} \omega, \dots f(\theta_1 \omega, f(\theta_0 \omega, x)) \dots). \quad (6)$$

The map $f(\omega, x)$ is called the *generator* of the cocycle $F(n, \omega, x)$.

In what follow it will be supposed that the cocycle $F(n, \omega, x)$ is continuous in x for every $n \in \mathbb{Z}^+$ and $\omega \in \Omega$. By (6) the cocycle $F(n, \omega, x)$ is continuous in x if its generator $f(\omega, x)$ is continuous in x for every $\omega \in \Omega$.

An \mathcal{F} -measurable family $\mathcal{A} = \{A_\omega, \omega \in \Omega\}$ of nonempty compact subsets of M is the family of image sets of an \mathcal{F} -measurable set valued mapping $\omega \mapsto A_\omega$, i.e., for which the real valued mapping $\omega \mapsto \text{dist}_\rho(x, A_\omega)$ is \mathcal{F} -measurable for each $x \in M$, see [4, Theorem 8.1.4]. It is called *f-invariant* if $f(\omega, A_\omega) = A_{\theta\omega}$, and hence

$$F(n, \omega, A_\omega) = A_{\theta_n \omega}, \quad \omega \in \Omega, n \in \mathbb{Z}^+.$$

Recall that the *Hausdorff separation*, or semi-metric, $H_\rho^*(X, Y)$, of the nonempty compact subsets X and Y of M is defined by

$$H_\rho^*(X, Y) := \max_{x \in X} \text{dist}_\rho(x, Y),$$

where $\text{dist}_\rho(x, Y) := \min_{y \in Y} \rho(x, y)$, and the *Hausdorff metric* $H_\rho(X, Y)$ for the nonempty compact subsets X and Y of M is

$$H_\rho(X, Y) := \max \{H_\rho^*(X, Y), H_\rho^*(Y, X)\}.$$

Finally, the *diameter* of a subset X of M is defined by $\text{diam}(X) := \sup_{x, y \in X} \rho(x, y)$.

Definition 4.2. An \mathcal{F} -measurable family $\mathcal{A} = \{A_\omega, \omega \in \Omega\}$ of nonempty compact subsets of M is called a *random attractor* if it (i) is F -invariant, (ii) pullback attracts nonempty bounded subsets of M , i.e.,

$$H_\rho^*(F(n, \theta_{-n}\omega, D), A_\omega) \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

for all $\omega \in \Omega$ and nonempty bounded subsets D of M , and (iii) is the minimal family (under inclusion) satisfying (i) and (ii).

Definition 4.3. A cocycle $F : \mathbb{Z}^+ \times \Omega \times M \rightarrow M$ is called *uniformly dissipative* if there exist a number $N_d \in \mathbb{Z}^+$ and a closed bounded set $M_0 \subset M$ such that

$$F(N_d, \omega, M) \subseteq M_0, \quad \forall \omega \in \Omega. \quad (7)$$

Definition 4.4. A cocycle $F : \mathbb{Z}^+ \times \Omega \times M \rightarrow M$ is called *uniformly contractive* if there exist a number $N_c \in \mathbb{Z}^+$ and a number $\lambda \in [0, 1)$ such that

$$\rho(F(N_c, \omega, x), F(N_c, \omega, y)) \leq \lambda \rho(x, y), \quad \forall \omega \in \Omega, x, y \in M. \quad (8)$$

We can now formulate an abstract theorem on existence of a random attractor.

Theorem 4.5. *Let $F : \mathbb{Z}^+ \times \Omega \times M \rightarrow M$ be a uniformly dissipative and uniformly contractive cocycle. Then it has a random attractor $\mathcal{A} = \{A_\omega, \omega \in \Omega\}$. Moreover, the set A_ω consists of a single point for each $\omega \in \Omega$.*

Proof. Fix numbers $N_d, N_c \geq 1$ and $\lambda < 1$ and a closed bounded set M_0 for which (7) and (8) hold.

First, it will be shown that for any bounded set $D \subseteq M$ and any $\omega \in \Omega$ the sets $F(n, \theta_{-n}\omega, D)$ converge to some single-point set A_ω as $n \rightarrow \infty$. To do this, in fact, an even stronger statement will be proved: for any $\omega \in \Omega$ the sets

$$M_n(\omega) := F(n, \theta_{-n}\omega, M)$$

converge to some single-point set A_ω as $n \rightarrow \infty$.

Note that for each ω the sets $M_n(\omega)$ are closed as images of the closed set M_0 under continuous maps $F(n, \theta_{-n}\omega, \cdot)$. Moreover, for any $\omega \in \Omega$ the sequence of sets $M_n(\omega)$ is nested under inclusion:

$$M_{n+1}(\omega) \subseteq M_n(\omega), \quad n \geq 1. \quad (9)$$

Indeed, by (5)

$$M_{n+1}(\omega) := F(n+1, \theta_{-(n+1)}\omega, M) = F(n, \theta_1(\theta_{-(n+1)}\omega), F(1, \theta_{-(n+1)}\omega, M)).$$

Clearly $\theta_1(\theta_{-(n+1)}\omega) = \theta_{-n}\omega$ and $F(1, \theta_{-(n+1)}\omega, M) \subseteq M$, so

$$M_{n+1}(\omega) := F(n+1, \theta_{-(n+1)}\omega, M) \subseteq F(n, \theta_{-n}\omega, M) =: M_n(\omega),$$

and (9) holds.

Set

$$d_n = \sup_{\omega \in \Omega} \text{diam}(M_n(\omega)).$$

It will be shown that

$$d_{n+1} \leq d_n \leq \text{diam}(M_0), \quad n \geq N_d. \quad (10)$$

The inequality $d_{n+1} \leq d_n$ follows from (9) provided that the both numbers d_{n+1} and d_n are finite. Thus (10) follows if it can be shown that

$$d_{N_d} \leq \text{diam}(M_0).$$

This last inequality readily follows from the inclusion

$$M_{N_d}(\omega) = F(N_d, \theta_{-N_d}\omega, M) \subseteq M_0,$$

which is a direct corollary of (7).

The inequalities (10) are thus proved, but now they will be strengthened to

$$d_{n+N_c} \leq \lambda d_n, \quad n \geq N_d. \quad (11)$$

To prove this inequality note that by (5)

$$\begin{aligned} M_{N_c+n}(\omega) &= F(N_c + n, \theta_{-(N_c+n)}\omega, M) \\ &= F(N_c, \theta_n(\theta_{-(N_c+n)}\omega), F(n, \theta_{-(N_c+n)}\omega, M)). \end{aligned}$$

Here $F(n, \theta_{-(N_c+n)}\omega, M) = M_n(\theta_{-N_c}\omega)$, so

$$M_{N_c+n}(\omega) = F(N_c, \theta_{-N_c}\omega, M_n(\theta_{-N_c}\omega))$$

and, by the uniform contractivity of the cocycle $F(n, \omega, x)$,

$$\text{diam}(M_{N_c+n}(\omega)) \leq \lambda \text{diam}(M_n(\theta_{-N_c}\omega)).$$

Taking the supremum over all $\omega \in \Omega$ in the above inequality then gives (11).

To finalize the proof of the theorem it remains to note that, for any given $\omega \in \Omega$, the sequence of closed sets $\{M_n(\omega)\}$ is nested under inclusion and, by (10) and (11), the diameters of the sets $M_n(\omega)$ tend to zero as $n \rightarrow \infty$. Then, by the Cantor Intersection Theorem (or Property), see, e.g., [20, Th. 13.65], the intersection

$$A_\omega = \bigcap_{n \geq 1} M_n(\omega)$$

is nonempty and consists of exactly one point. \square

Remark 1. The proof of Theorem 4.5 implies that the component sets A_ω of the random attractor $\mathcal{A} = \{A_\omega, \omega \in \Omega\}$ satisfy the inclusion $A_\omega \subseteq M_0$ for all $\omega \in \Omega$.

Remark 2. To prove Theorem 4.5 it would suffice to require that (8) holds only for $x, y \in M_0$, provided that the cocycle F is uniformly dissipative.

Remark 3. It would be preferable to formulate properties of dissipativity and contractivity for a cocycle not in terms of the map $F(n, x, y)$, but in terms of its generator $f(\omega, x)$. As will be seen below, in general, this is not possible in some interesting and natural applications, where the arising cocycle is uniformly dissipative and contractive, whereas neither the dissipativity for $N_d = 1$ nor the contractivity for $N_c = 1$ holds.

Remark 4. Theorem 4.5 has been formulated under rather severe assumptions. These can be essentially weakened, but serve perfectly well for the purposes of this paper.

5. The Hilbert metric and Birkhoff's theorem. To apply the results from Section 4 to linear cocycles generated by tridiagonal Markov chains (3), first recall some auxiliary facts (see, e.g., [5, 6, 16]) following the work [3].

Denote by \mathbb{K}_+^N the cone¹ of elements $x = (x_1, x_2, \dots, x_N)^T \in \mathbb{R}^N$ with nonnegative components and by $\overset{\circ}{\mathbb{K}}_+^N$ the interior of \mathbb{K}_+^N , which is clearly non-empty. Then the quantity

$$\vartheta(x, y) = \inf \{t : tx - y \in \mathbb{K}_+^N\}$$

is finite valued for any $x, y \in \overset{\circ}{\mathbb{K}}_+^N$.

Definition 5.1. The function

$$\rho_H(x, y) = \left| \ln \left(\frac{\vartheta(x, y)}{\vartheta(y, x)} \right) \right|$$

is called the *Hilbert projective metric* (or, sometimes, the Birkhoff metric [16]).

Remark 5. Definition 5.1 is applicable to cones in a general Banach space. For the cone \mathbb{K}_+^N in the finite-dimensional space \mathbb{R}^N , it can be shown to be equal to

$$\rho_H(x, y) = \left| \ln \left(\frac{\max_i y_i/x_i}{\max_i x_i/y_i} \right) \right| \quad (12)$$

or

$$\rho_H(x, y) = \left| \max_i \{\ln y_i - \ln x_i\} - \max_i \{\ln x_i - \ln y_i\} \right|$$

for vectors $x = (x_1, x_2, \dots, x_N)^T$ and $y = (y_1, y_2, \dots, y_N)^T$ in \mathbb{K}_+^N .

Observe that $\rho_H(x, y)$ satisfies the triangle inequality

$$\rho_H(x, z) \leq \rho_H(x, y) + \rho_H(y, x), \quad x, y, z \in \overset{\circ}{\mathbb{K}}_+^N,$$

whereas the relation $\rho_H(x, y) = 0$ with $x, y \in \overset{\circ}{\mathbb{K}}_+^N$ does not implies the equality $x = y$, but only the equality $x = ty$ for some $t > 0$. Moreover,

$$\rho_H(sx, ty) = \rho_H(x, y), \quad \forall s, t > 0, \quad x, y \in \overset{\circ}{\mathbb{K}}_+^N.$$

Thus, strictly speaking, $\rho_H(x, y)$ is not a metric on $\overset{\circ}{\mathbb{K}}_+^N$, but only a semi-metric. It becomes a metric, however, on a projective space. An important way to make it a proper metric is covered by the following theorem.

Theorem 5.2. *Let $X \subseteq \mathbb{R}^N$ be a closed, bounded set such that $0 \notin X$ and any ray $\{tx : t > 0\}$ with $x \in \overset{\circ}{\mathbb{K}}_+^N$ intersects X in at most one point. Then $(X \cap \overset{\circ}{\mathbb{K}}_+^N, \rho_H)$ is a metric space.*

Moreover, if any ray $\{tx : t > 0\}$ with $x \in \overset{\circ}{\mathbb{K}}_+^N$ intersects X in exactly one point, then the metric space $(X \cap \overset{\circ}{\mathbb{K}}_+^N, \rho_H)$ is complete.

Corollary 1. *The interior $\overset{\circ}{\Sigma}_N$ of the probability simplex Σ_N is the complete metric space with the Hilbert projective metric.*

¹Recall, see, e.g. [16], that a set K in a Banach space is called a *cone* if it is convex, closed and $tK \subseteq K$ for any real $t \geq 0$, and $K \cap -K = \{0\}$.

Remark 6. In general, the set $X \cap \overset{\circ}{\mathbb{K}}_+^N$ in Theorem 5.2 is unbounded in metric ρ_H . Convergence in the metric space $(X \cap \overset{\circ}{\mathbb{K}}_+^N, \rho_H)$ implies convergence with respect to any norm in \mathbb{R}^N .

Definition 5.3. A linear operator (matrix) $L : \mathbb{R}^N \rightarrow \mathbb{R}^N$ is called *nonnegative* if $L : \mathbb{K}_+^N \rightarrow \mathbb{K}_+^N$, and *positive* if $L : \mathbb{K}_+^N \setminus \{0\} \rightarrow \overset{\circ}{\mathbb{K}}_+^N$.²

Definition 5.4. If a linear operator (matrix) $L : \mathbb{R}^N \rightarrow \mathbb{R}^N$ is positive, then its *projective diameter* $\delta(L)$ is defined by

$$\delta(L) = \sup \left\{ \rho_H(Lx, Ly) : x, y \in \overset{\circ}{\mathbb{K}}_+^N \right\}$$

and the *contraction ratio* $\varkappa(L)$ of L is defined by

$$\varkappa(L) = \inf \left\{ \lambda : \rho_H(Lx, Ly) \leq \lambda \rho_H(x, y), x, y \in \overset{\circ}{\mathbb{K}}_+^N \right\}$$

Theorem 5.5 (Birkhoff). *If $L : \mathbb{R}^N \rightarrow \mathbb{R}^N$ is a positive linear operator (matrix), then*

$$\varkappa(L) \leq \tanh \left(\frac{1}{4} \delta(L) \right) < 1.$$

Given a positive linear operator (matrix) $L : \mathbb{R}^N \rightarrow \mathbb{R}^N$ denote by $\tilde{L}(\cdot)$ the (nonlinear) operator defined by

$$\tilde{L}(x) = P(Lx), \quad x \in \Sigma_N,$$

where P is the projection operator to the simplex Σ_N defined by

$$P(x) = \frac{1}{x_1 + x_2 + \cdots + x_N} x, \quad x \in \mathbb{K}_+^N.$$

Theorem 5.6. $\tilde{L}(\cdot)$ is a contracting mapping with the contraction constant $\varkappa(L)$ on the metric space $(\overset{\circ}{\Sigma}_N, \rho_H)$.

Remark 7. If the simplex Σ_N is invariant for the positive linear operator L then the assertion of Theorem 5.6 can be simplified to: *if $L : \mathbb{R}^N \rightarrow \mathbb{R}^N$ is a positive linear operator (matrix) for which $L\Sigma_N \subseteq \Sigma_N$, then it is a contracting mapping with the contraction constant $\varkappa(L)$ on the metric space $(\overset{\circ}{\Sigma}_N, \rho_H)$.*

6. Attractors of linear cocycles with a tridiagonal generator. The result of Sections 4 and 5 will be applied here to the linear system generated by the tridiagonal Markov chains that were introduced in Section 2.

Let \mathcal{L} be a set of linear operators $L_\omega : \mathbb{R}^N \rightarrow \mathbb{R}^N$ parametrized by the parameter ω taking values in some set Ω and let $\{\theta_n, n \in \mathbb{Z}\}$ be a (discrete-time) group of maps of Ω onto itself. The maps $L_\omega x$ serve as the generator of a linear cocycle

$$F_{\mathcal{L}}(n, \omega)x = L_{\theta_{n-1}\omega} \cdots L_{\theta_1\omega} L_{\theta_0\omega} x.$$

In particular, consider the case where

$$L_\omega := I_N + \Delta Q(\omega) \tag{13}$$

²It is straightforward to show that a matrix L is nonnegative iff its components are nonnegative, and positive iff its components are strictly positive.

with the tridiagonal matrices $Q(\omega)$ of the form (1) for which the band entries q_i depend on the parameter $\omega \in \Omega$, i.e., $q_i = q_i(\omega)$ for $i = 1, 2, \dots, 2N - 2$.

In the sequel the following basic assumption will be used.

Assumption 1. *There exist numbers $0 < \alpha \leq \beta < \infty$ such that the uniform estimates*

$$\alpha \leq q_i(\omega) \leq \beta, \quad \omega \in \Omega, \quad i = 1, 2, \dots, 2N - 2, \quad (14)$$

hold.

Lemma 6.1. *Let $\Delta < \frac{1}{2\beta}$. Then the entries of all of the matrices L_ω , $\omega \in \Omega$, are nonnegative, and thus*

$$L_\omega \mathbb{K}_+^N \subseteq \mathbb{K}_+^N, \quad \omega \in \Omega.$$

Proof. This follows directly from the representation $L_\omega := I_N + \Delta Q(\omega)$ and the form (1) of the matrices $Q(\omega)$. \square

If $\Delta < \frac{1}{2\beta}$, then $\gamma := \min\{\Delta\alpha, 1 - 2\Delta\beta\} > 0$. Define the subset $\Sigma_N(\gamma)$ of the simplex Σ_N as

$$\Sigma_N(\gamma) = \left\{ x = (x_1, x_2, \dots, x_N) : \sum_{i=1}^N x_i = 1, \quad x_1, x_2, \dots, x_N \geq \gamma^{N-1} \right\},$$

and the subcone $\mathbb{K}_+^N(\gamma)$ of the cone \mathbb{K}_+^N as

$$\mathbb{K}_+^N(\gamma) = \{tx : t > 0, \quad x \in \Sigma_N(\gamma)\}.$$

Clearly

$$\mathbb{K}_+^N(\gamma) \setminus \{0\} \subseteq \mathring{\mathbb{K}}_+^N.$$

Lemma 6.2. *Let $\Delta < \frac{1}{2\beta}$ and $\gamma = \min\{\Delta\alpha, 1 - 2\Delta\beta\}$. Then,*

$$F_{\mathcal{L}}(N-1, \omega) \Sigma_N \subseteq \Sigma_N(\gamma), \quad \omega \in \Omega, \quad (15)$$

and, hence,

$$F_{\mathcal{L}}(N-1, \omega) (\mathbb{K}_+^N \setminus \{0\}) \subseteq \mathbb{K}_+^N(\gamma) \setminus \{0\} \subseteq \mathring{\mathbb{K}}_+^N, \quad \omega \in \Omega.$$

Proof. Fix an $\omega \in \Omega$. By induction, it can be shown for each $n = 1, 2, \dots, N-1$ that the matrix

$$F_{\mathcal{L}}(n, \omega) = L_{\theta_{n-1}\omega} \cdots L_{\theta_1\omega} L_{\theta_0\omega}$$

is $(2n+1)$ -diagonal and that its components belonging to the main diagonal and to the first n sub and super diagonals are greater than or equal to γ^n , while others vanish. Thus

all components of the matrix $F_{\mathcal{L}}(N-1, \omega)$ are positive and exceed γ^{N-1} . (16)

By (4) and (13)

$$\mathbf{1}_N^T L_\omega = \mathbf{1}_N^T, \quad \omega \in \Omega,$$

and (16) implies that

$$F_{\mathcal{L}}(N-1, \omega) \mathring{\Sigma}_N \subseteq \Sigma_N(\gamma) \subset \mathring{\Sigma}_N, \quad \omega \in \Omega.$$

The inclusion (15) is thus established. \square

Theorem 6.3. *Let $F_{\mathcal{L}}(n, \omega)x$ be the linear cocycle*

$$F_{\mathcal{L}}(n, \omega)x = L_{\theta_{n-1}\omega} \cdots L_{\theta_1\omega} L_{\theta_0\omega} x.$$

with matrices $L_{\omega} := I_N + \Delta Q(\omega)$, where the tridiagonal matrices $Q(\omega)$ are of the form (1) with the entries $q_i = q_i(\omega)$ satisfying the uniform estimates (14) in Assumption 1. In addition, suppose that $\Delta < \frac{1}{2\beta}$.

Then, the set Σ_N is invariant under $F_{\mathcal{L}}(N-1, \omega)$, i.e.,

$$F_{\mathcal{L}}(N-1, \omega)\Sigma_N \subseteq \Sigma_N, \quad \omega \in \Omega.$$

Moreover, the restriction of $F_{\mathcal{L}}(n, \omega)x$ to the set Σ_N is a uniformly dissipative and uniformly contractive cocycle (with respect to the Hilbert metric), which has a random attractor $\mathcal{A} = \{A_{\omega}, \omega \in \Omega\}$ such that each set A_{ω} , $\omega \in \Omega$, consists of a single point.

Proof. Under the assumptions, $\gamma := \min\{\Delta\alpha, 1 - 2\Delta\beta\} > 0$. Define

$$\delta = \sup \{\rho_H(x, y) : x, y \in \Sigma_N(\gamma)\}.$$

It follows by formula (12) for the Hilbert metric in \mathbb{K}_+^N that $\delta < \infty$, so

$$\varkappa \leq \tanh\left(\frac{1}{4}\delta\right) < 1. \quad (17)$$

Hence, by Lemmata 6.1 and 6.2, for each $\omega \in \Omega$ the matrix $F_{\mathcal{L}}(N-1, \omega)$ satisfies the conditions of Theorem 5.5 and is thus uniformly contractive in the metric space $(\overset{\circ}{\Sigma}_N, \rho_H)$ with the contraction constant \varkappa . Moreover, by Lemma 6.2, each of the matrices $F_{\mathcal{L}}(N-1, \omega)$, $\omega \in \Omega$, maps the set $\overset{\circ}{\Sigma}_N$ to the set $\Sigma_N(\gamma) \subset \overset{\circ}{\Sigma}_N$, which is bounded in the Hilbert metric ρ_H .

Setting $M = \overset{\circ}{\Sigma}_N$ and $\rho = \rho_H$, $M_0 = \Sigma_N(\gamma)$ and $N_d = N_c = N-1$, it follows that all of the conditions of Theorem 4.5 are satisfied. Hence the restriction of the cocycle $F_{\mathcal{L}}(n, \omega)x$ to the set $\overset{\circ}{\Sigma}_N$ has a random attractor $\mathcal{A} = \{A_{\omega}, \omega \in \Omega\}$ such that each set A_{ω} , $\omega \in \Omega$, consists of a single point.

To complete the proof it remains to note only that due to Lemma 6.2 all the matrices $F_{\mathcal{L}}(N-1, \omega)$, $\omega \in \Omega$, not only map the set $\overset{\circ}{\Sigma}_N$ to the set $\Sigma_N(\gamma) \subset \overset{\circ}{\Sigma}_N$, but also map the larger set Σ_N to the set $\Sigma_N(\gamma) \subset \overset{\circ}{\Sigma}_N$. Hence, the family of sets $\mathcal{A} = \{A_{\omega}, \omega \in \Omega\}$ is a random attractor for the cocycle $F_{\mathcal{L}}(n, \omega)x$ considered not only on the set $\overset{\circ}{\Sigma}_N$, but also on its closure Σ_N . \square

Remark 8. The proof of Theorem 6.3 implies that the component sets A_{ω} of the random attractor $\mathcal{A} = \{A_{\omega}, \omega \in \Omega\}$ satisfy the inclusion $A_{\omega} \subseteq \Sigma_N(\gamma)$ for every $\omega \in \Omega$.

Henceforth write $A_{\omega} = \{a_{\omega}\}$ for the singleton component subsets of the random attractor \mathcal{A} . Then the random attractor is an entire random sequence $\{a_{\theta_n\omega}, n \in \mathbb{Z}\}$ in $\Sigma_N(\gamma) \subset \overset{\circ}{\Sigma}_N$, which attracts other iterates of the random Markov chain in the pullback sense. Pullback convergence involves starting at earlier initial times with a fixed end time, see [7, 15]. It is, generally, not the same as forward convergence in the sense usually understood in dynamical systems, but in this case it is the same due to the uniform boundedness of the contractive rate with respect to ω . By (17)

$$\varkappa(L_{\omega}) \leq \nu := \tanh\left(\frac{1}{4}\delta\right) < 1, \quad \forall \omega \in \Omega.$$

Now let $\mathbf{p}^{(n)}(\omega)$ be the n th iterate of the random Markov chain (2). Then

$$\rho_H \left(\mathbf{p}^{(n+1)}(\omega), a_{\theta_{n+1}\omega} \right) \leq \nu \rho_H \left(\mathbf{p}^{(n)}(\omega), a_{\theta_n\omega} \right)$$

for all $n \geq 0$ and every $\omega \in \Omega$, since

$$\mathbf{p}^{n+1}(\omega) = L_{\theta_n\omega} \mathbf{p}^{(n)}(\omega), \quad a_{\theta_{n+1}\omega} = L_{\theta_n\omega} a_{\theta_n\omega}.$$

Hence,

$$\rho_H \left(\mathbf{p}^{(n)}(\omega), a_{\theta_n\omega} \right) \leq \nu^n \rho_H \left(\mathbf{p}^{(0)}, a_\omega \right)$$

for all $n \geq 0$ and every $\omega \in \Omega$, from which follows the pathwise forward convergence with respect to the Hilbert projective metric.

$$\rho_H \left(\mathbf{p}^{(n)}(\omega), a_{\theta_n\omega} \right) \rightarrow 0 \quad \text{as } n \rightarrow \infty, \quad \forall \omega \in \Omega.$$

By Remark 6, convergence in the Hilbert projective metric implies convergence in any norm on \mathbb{R}^N . This gives

Corollary 2. *For any norm $\|\cdot\|$ on \mathbb{R}^N , $\mathbf{p}^{(0)} \in \Sigma_N$ and $\omega \in \Omega$*

$$\left\| \mathbf{p}^{(n)}(\omega) - a_{\theta_n\omega} \right\| \rightarrow 0 \quad \text{as } n \rightarrow \infty, \quad \forall \omega \in \Omega.$$

The random attractor is, in fact, asymptotic Lyapunov stable in the conventional forward sense.

6.1. Deterministic nonautonomous Markov chains. The above proofs make no use of probabilistic properties of the sample path parameter ω (apart from \mathcal{F} -measurability considerations, which are not an essential part of the proof). It applies immediately to deterministic nonautonomous Markov chains in which the transition probabilities vary, say, periodically in time.

As described in [15], this time variation can be modelled by letting ω be an bi-infinite sequence $\omega = (\omega_n)_{n \in \mathbb{Z}} \in \Lambda^{\mathbb{Z}}$, i.e., with $\omega_n \in \Lambda$, $n \in \mathbb{Z}$, for some compact metric space (Λ, ρ_Λ) . Then $\Omega = \Lambda^{\mathbb{Z}}$ is a compact metric space with the metric

$$\rho_\Omega(\omega, \bar{\omega}) = \sum_{n \in \mathbb{Z}} 2^{-|n|} \rho_\Lambda(\omega_n, \bar{\omega}_n)$$

and the shift operator $\theta(\omega_n)_{n \in \mathbb{Z}} = (\omega_{n+1})_{n \in \mathbb{Z}}$ is continuous in the metric ρ_Ω . It turns out then that $\omega \mapsto a_\omega$ is continuous here (in general, the set-valued mapping $\omega \mapsto A_\omega$ is only upper semi-continuous). These topological properties of the driving system replace the measurability properties in the random dynamical systems.

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