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On the 60th birthday of A. A. Bolibrukh

On deformations of linear differential systems

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Abstract. This article concerns deformations of meromorphic linear differential systems. Problems relating to their existence and classification are reviewed, and the global and local behaviour of solutions to deformation equations in a neighbourhood of their singular set is analysed. Certain classical results established for isomonodromic deformations of Fuchsian systems are generalized to the case of integrable deformations of meromorphic systems.

Bibliography: 40 titles.

Keywords: holomorphic bundle, meromorphic connection, integrability, monodromy, Painlevé property, isomonodromic deformation.

Contents

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Introduction

The text has the following structure. The first section is introductory. We review there some standard general concepts widely used in investigations involving meromorphic and especially Fuchsian linear systems. This involves an introduction to the concept of monodromy, some basics of Levelt's theory, and a short introduction to the integrability of Pfaffian systems, which will appear throughout the text. Special attention is paid to resonant singularities of Fuchsian systems and their relevance to uniqueness problems.

The second section is devoted to integrable deformations of meromorphic connections. We give a proof of the existence of such a deformation for any initial holomorphic vector bundle with a meromorphic connection over the Riemann sphere $\overline{\mathbb{C}}$.

In the third section we treat isomonodromic deformations of Fuchsian systems, or connections that have only poles of the first order as singular points. Such deformations are a special case of integrable ones. The fact that any isomonodromic deformation is naturally related to some matrix differential 1-form possessing a number of special properties allows us to construct a complete classification of isomonodromic deformations of Fuchsian systems starting from the simplest case of the Schlesinger deformation and moving to the most general resonant deformation with additional parameters. The classification has a rather clear geometric interpretation in terms of resonances and their dynamics, which we discuss in Appendix A.

In the fourth section we study properties of integrable deformations. Introducing the machinery of Fredholm theory, we investigate the (non-)triviality of the deformed bundle, the properties of the theta-divisor, meromorphic continuation, and the Painlevé property in the more general setting of integrable deformations, thus deriving analogues of classical results concerning isomonodromic deformations of Fuchsian systems. To conclude we apply results on the behaviour of Schlesinger isomonodromic families in a neighbourhood of the theta-divisor to estimate the orders of moveable poles of Garnier integrable systems. Appendix B reviews the symplectic nature of isomonodromic deformations.

1. Preliminaries

1.1. Linear systems and monodromy. Consider a *meromorphic linear system* on the Riemann sphere $\overline{\mathbb{C}}$, that is, a system of p linear ordinary differential equations with singularities $a_1^0,\dots,a_n^0\in\mathbb{C}$ and possibly at ∞. Such a system can be written in the form

$$\frac{dy}{dz}=B(z)\,y,\qquad B(z)=\sum_{i=1}^{n}\sum_{l\geqslant 1}\frac{B_{il}^0}{(z-a_i^0)^l}+\sum_{l\geqslant 0}B_{\infty l}^0 z^l, \tag{1}$$

where $y(z)\in\mathbb{C}^p$ and B_{il}^0, $B_{\infty l}^0$ are $(p\times p)$-matrices equal to zero for sufficiently large l. Using a conformal map $z\mapsto\dfrac{az+b}{cz+d}$, one can always arrange that all the singularities are in the complex plane only. This means that one can reduce the system (1) to one where all $B^0_{\infty l}$ are equal to zero and

$$\sum_{i=1}^{n} B^0_{i1} = 0, \tag{2}$$

to ensure that ∞ is not a singular point.

The system (1) can be interpreted in more geometrical terms as follows. Consider the open set $U = \overline{\mathbb{C}} \setminus \{a^0_1, \dots, a^0_n\}$ and the trivial vector bundle $E^0 = U \times \mathbb{C}^p$ of rank p over U with the basis of trivializing sections $s_1, \dots, s_p$ defined by $s_i(z) = e_i$ for all $z \in U$. Here e_i denotes the standard ith basis vector of $\mathbb{C}^p$. One has the holomorphic connection

$$\nabla^0 \colon \Gamma^0(E^0) \to \Gamma^1(E^0)$$

in E^0, a linear map from the space $\Gamma^0(E^0)$ of holomorphic sections of E^0 to the space $\Gamma^1(E^0)$ of holomorphic differential 1-forms on U with values in E^0 satisfying

$$\nabla^0(fs) = s\,df + f\nabla^0(s),$$

where $s \in \Gamma^0(E^0)$ and f is a holomorphic function on U. This connection is determined by the connection form $\omega^0 = -B(z)\,dz$ with respect to the basis $\{s_i\}$, which means that $(s_1, \dots, s_p)$ maps to $(s_1, \dots, s_p)\omega^0$ under ∇^0. For a small disk $D \subset U$ the local holomorphic vector function $y = \sum_{i=1}^{p} y_i s_i$ in equation (1), considered in D, corresponds in this picture to a section of $E^0|_D$. Such sections are said to be *horizontal* because they satisfy $\nabla^0(y) = 0$.

An important characteristic of a meromorphic linear system (or holomorphic connection in a trivial vector bundle over U) is the monodromy or monodromy representation defined below. Assuming again that one has merely singularities at the points $a^0_1, \dots, a^0_n$ in the complex plane, we consider in a neighbourhood D of a non-singular point z_0 a germ of a fundamental matrix $Y(z)$ of the system (1). Operating again with geometric terms, one says that the columns of $Y(z)$ form a basis in the space of horizontal (with respect to ∇^0) sections of the bundle $E^0|_D$. An analytic continuation of $Y(z)$ along an arbitrary loop γ outgoing from z_0 and lying in $\overline{\mathbb{C}} \setminus \{a^0_1, \dots, a^0_n\}$ transforms the matrix $Y(z)$ into (in general) a different matrix $\widetilde{Y}(z)$. The two fundamental matrices are connected via a non-singular transition matrix G_γ corresponding to the loop γ:

$$\widetilde{Y}(z) = Y(z)G_\gamma.$$

The map $[\gamma] \mapsto G^{-1}_\gamma$ depends only on the homotopy class $[\gamma]$ of the loop γ and thus defines a representation

$$\chi \colon \pi_1(\overline{\mathbb{C}} \setminus \{a^0_1, \dots, a^0_n\}, z_0) \to \mathrm{GL}(p, \mathbb{C})$$

of the fundamental group of $\overline{\mathbb{C}} \setminus \{a^0_1, \dots, a^0_n\}$ into the space of non-singular complex matrices of size $p \times p$. This representation is called the *monodromy* of the linear system (or connection ∇^0).

By the *monodromy matrix* of the system (1) at a singular point a_i^0 with respect to the fundamental matrix $Y(z)$ we mean the matrix G_i corresponding to a simple loop γ_i encircling a_i^0, so that $G_i^{-1} = \chi([\gamma_i])$. The matrices $G_1, \dots, G_n$ are the generators of the *monodromy group* of the system (1). They satisfy the condition $G_1 \cdots G_n = \mathrm{Id}$ implied by the relation $\gamma_n \cdots \gamma_1 = e$ for appropriately ordered generators of the fundamental group.

If one takes another fundamental matrix $Y'(z) = Y(z)C$, $C \in \mathrm{GL}(p, \mathbb{C})$, of the system (1), then the corresponding monodromy matrices are $G_i' = C^{-1}G_iC$. In a similar way the matrices G_i depend on the choice of an initial point z_0. So one sees that the monodromy of a meromorphic linear system is defined up to an overall conjugation by a constant non-singular matrix. Thus, it is more correct to say that the monodromy is an element in the space of conjugacy classes of p-dimensional representations of the group $\pi_1(\overline{\mathbb{C}} \setminus \{a_1^0, \dots, a_n^0\})$, which we denote by

$$\mathscr{M}_{a^0} = \mathrm{Hom}\bigl(\pi_1(\overline{\mathbb{C}} \setminus \{a_1^0, \dots, a_n^0\}),\, \mathrm{GL}(p, \mathbb{C})\bigr) / \,\mathrm{GL}(p, \mathbb{C}).$$

We conclude this subsection by the following observation. Having a holomorphic connection in a trivial holomorphic vector bundle of rank p over $U = \overline{\mathbb{C}} \setminus \{a_1^0, \dots, a_n^0\}$, one defines its monodromy as a p-dimensional representation of the group $\pi_1(U)$. The converse also holds: given a p-dimensional representation χ of the group $\pi_1(U)$, one can construct a unique holomorphic vector bundle of rank p over U (which is trivial as a holomorphic vector bundle over a non-compact Riemann surface; see [1], Proposition 30.4) with a holomorphic connection whose monodromy coincides with χ (see, for example, [2], Lecture 8).

1.2. Fuchsian systems: local and global theory. Our special interest is in the so-called *Fuchsian systems* on the Riemann sphere $\overline{\mathbb{C}}$, that is, meromorphic linear systems (1) whose coefficient matrix has only poles of first order as singularities. Such a system with singular points $a_1^0, \dots, a_n^0$ in the complex plane is of the following form:

$$\frac{dy}{dz} = \Biggl(\sum_{i=1}^{n} \frac{B_i^0}{z - a_i^0}\Biggr) y, \qquad y(z) \in \mathbb{C}^p, \quad B_i^0 \in \mathrm{Mat}(p, \mathbb{C}), \tag{3}$$

where the relation (2) holds for the matrices B_i^0.

Let us recall some facts about the local structure of fundamental matrices of the Fuchsian system (3) in a neighbourhood of a singular point a_i^0. Let G_i be the monodromy matrix of the system at the point a_i^0. According to Levelt [3], the system has a fundamental matrix of the form

$$Y_i(z) = U_i(z)(z - a_i^0)^{\Lambda_i}(z - a_i^0)^{E_i}, \tag{4}$$

where

a) the matrix $U_i(z)$ is holomorphically invertible at the point a_i^0;

b) the matrix $E_i = \dfrac{1}{2\pi\sqrt{-1}} \log G_i = \mathrm{diag}(E_i^1, \dots, E_i^m)$ is block-diagonal, each block E_i^j is an upper triangular matrix with the unique eigenvalue ρ_i^j, and the branch of the logarithm is chosen such that the eigenvalues of the matrix E_i satisfy the condition

$$0 \leqslant \operatorname{Re} \rho_i^j < 1; \tag{5}$$

c) the matrix $\Lambda_i = \operatorname{diag}(\Lambda_i^1, \dots, \Lambda_i^m)$ is a diagonal integer-valued matrix of the same block structure as E_i and such that the diagonal elements of each block Λ_i^j form a non-increasing sequence.

The matrix $Y_i(z)$ is called the (*weakly*) *Levelt fundamental matrix*. Its columns form the (*weakly*) *Levelt basis*.

The eigenvalues β_i^j of the residue matrix B_i^0 are said to be the *exponents* of the Fuchsian system (3) at the singular point a_i^0. Using the relation

$$B_i^0 = \operatorname{res}_{a_i^0}\left(\frac{dY_i}{dz}\, Y_i^{-1}\right),$$

it is not difficult to check that the exponents coincide with the eigenvalues of the matrix $\Lambda_i + E_i$. The matrix Λ_i is called the *valuation matrix* of the Fuchsian system (3) at the singularity a_i^0. According to (5) its diagonal elements coincide with the integer parts of the numbers $\operatorname{Re}\beta_i^j$.

A singular point $z = a_i^0$ of the Fuchsian system (3) is said to be *resonant* if for some pair β_i^j, β_i^k of the exponents at this point their difference $\beta_i^j - \beta_i^k$ is a non-zero integer. Otherwise a singularity is said to be *non-resonant*. As follows from the above, the singular point a_i^0 is non-resonant if and only if all blocks Λ_i^j of the matrix Λ_i in the Levelt decomposition (4) are scalar. Resonances play an important role in describing and parametrizing Fuchsian systems via their local data (monodromy and asymptotics or exponents).

Two Fuchsian systems with a non-resonant singular point a_i^0 that have the same monodromy matrix and the same exponents at this point are locally holomorphically equivalent (that is, they are locally connected by a gauge transformation $y' = \Gamma(z)y$, where $\Gamma(z)$ is a matrix holomorphically invertible at the point a_i^0). Indeed, the Levelt fundamental matrices $Y_i(z)$, $Y_i'(z)$ of such systems at the point a_i^0 are of the form

$$Y_i(z) = U_i(z)(z - a_i^0)^{\Lambda_i}(z - a_i^0)^{E_i}, \qquad Y_i'(z) = U_i'(z)(z - a_i^0)^{\Lambda_i}(z - a_i^0)^{E_i'},$$

where $E_i' = SE_iS^{-1}$ is conjugate to E_i by a matrix $S = \operatorname{diag}(S^1, \dots, S^m)$ of the same block-diagonal form as the matrices E_i, E_i'. Since the matrices Λ_i and S commute, one has

$$Y_i'(z)S = U_i'(z)S(z - a_i^0)^{\Lambda_i}(z - a_i^0)^{E_i} = U_i'(z)SU_i^{-1}(z)Y_i(z),$$

which implies the local holomorphic equivalence of the two systems.

Globally, two Fuchsian systems with the same singularities, which are *all non-resonant*, and having the same monodromy and exponents (at all singular points) are connected by a constant gauge transformation $y' = Cy$, $C \in \mathrm{GL}(p, \mathbb{C})$. (Using analogous reasoning, one gets that such systems are globally holomorphically equivalent, but a matrix holomorphic on the whole Riemann sphere is constant.) We do not distinguish such systems, thus considering a Fuchsian system with n fixed singularities $a_1^0, \dots, a_n^0$ as an element of the space

$$\mathscr{M}_{a^0}^* = \{(B_1^0, \dots, B_n^0) \mid B_i^0 \in \operatorname{Mat}(p, \mathbb{C}),\ B_1^0 + \dots + B_n^0 = 0\} / \mathrm{GL}(p, \mathbb{C}).$$

In this sense *there can exist only one Fuchsian system with the given non-resonant singularities*, *monodromy*, *and exponents*.

In the resonant case essentially different Fuchsian systems with the same singularities, monodromy, and exponents can occur, and thus a moduli space arises. We illustrate this with an example.

Example 1. Consider a Fuchsian system

$$\frac{dy}{dz} = \left(\begin{pmatrix} 1 & t_1 \\ 0 & 0 \end{pmatrix} \frac{1}{z-a_1} + \begin{pmatrix} 0 & -t_1 \\ 0 & -1 \end{pmatrix} \frac{1}{z-a_2} + \begin{pmatrix} -1 & t_2 \\ 0 & 0 \end{pmatrix} \frac{1}{z-a_3} + \begin{pmatrix} 0 & -t_2 \\ 0 & 1 \end{pmatrix} \frac{1}{z-a_4} \right) y$$

of two equations with four resonant singular points a_1, a_2, a_3, a_4, where t_1 and t_2 are parameters. In fact, the pair (t_1, t_2) is projective since (t_1, t_2) and $(\lambda t_1, \lambda t_2)$, $\lambda \neq 0$, define the same Fuchsian system in the above sense. At the same time it can easily be checked that non-proportional pairs of parameters define essentially different Fuchsian systems.

Since the coefficient matrix of this system is upper triangular, one can easily integrate and obtain a fundamental matrix of the following form:

$$Y(z) = \begin{pmatrix} \dfrac{z-a_1}{z-a_3} & 0 \\ 0 & \dfrac{z-a_4}{z-a_2} \end{pmatrix} \begin{pmatrix} 1 & \alpha \log \dfrac{z-a_1}{z-a_2} + \dfrac{Ct_1}{z-a_1} + \dfrac{Dt_1}{z-a_2} \\ 0 & 1 \end{pmatrix},$$

$$\alpha = At_1 + Bt_2,$$

where A, B, C, D are some rational functions of a_1, a_2, a_3, a_4. The matrix $Y(z)$ has no ramification at the points a_3, a_4, while it goes to matrices $\widetilde{Y}_1(z)$ and $\widetilde{Y}_2(z)$ after analytic continuations around the points a_1 and a_2, respectively:

$$\widetilde{Y}_1(z) = \begin{pmatrix} \dfrac{z-a_1}{z-a_3} & 0 \\ 0 & \dfrac{z-a_4}{z-a_2} \end{pmatrix} \begin{pmatrix} 1 & \alpha \log \dfrac{z-a_1}{z-a_2} + \dfrac{Ct_1}{z-a_1} + \dfrac{Dt_1}{z-a_2} + 2\pi i \alpha \\ 0 & 1 \end{pmatrix},$$

$$\widetilde{Y}_2(z) = \begin{pmatrix} \dfrac{z-a_1}{z-a_3} & 0 \\ 0 & \dfrac{z-a_4}{z-a_2} \end{pmatrix} \begin{pmatrix} 1 & \alpha \log \dfrac{z-a_1}{z-a_2} + \dfrac{Ct_1}{z-a_1} + \dfrac{Dt_1}{z-a_2} - 2\pi i \alpha \\ 0 & 1 \end{pmatrix}.$$

Thus, the monodromy matrices of $Y(z)$ are the following:

$$G_1 = \begin{pmatrix} 1 & 2\pi i \alpha \\ 0 & 1 \end{pmatrix}, \qquad G_2 = \begin{pmatrix} 1 & -2\pi i \alpha \\ 0 & 1 \end{pmatrix}, \qquad G_3 = G_4 = \mathrm{Id}.$$

For all $(t_1, t_2) \in \mathbb{C}^2$ such that $\alpha = At_1 + Bt_2 \neq 0$ the set $\{G_1, \ldots, G_4\}$ can be transformed into the set

$$G'_1 = \begin{pmatrix} 1 & 2\pi i \\ 0 & 1 \end{pmatrix}, \qquad G'_2 = \begin{pmatrix} 1 & -2\pi i \\ 0 & 1 \end{pmatrix}, \qquad G'_3 = G'_4 = \mathrm{Id}$$

by simultaneous conjugation by the diagonal matrix $\mathrm{diag}(\alpha, 1)$. This implies that there is the whole space

$$\{(t_1 : t_2) \in \mathbb{P}^1(\mathbb{C}) \mid At_1 + Bt_2 \neq 0\}$$

of Fuchsian systems having the same four resonant singularities, monodromy, and exponents. It is a complex projective line punctured at the point $(-B : A)$.

There is another case of local holomorphic equivalence of two Fuchsian systems having the same monodromy matrix G_i and the same exponents at a singular point a_i^0 (either non-resonant or resonant). This is the case when the condition

$$\operatorname{rank}(G_i - \lambda \operatorname{Id}) \geqslant p - 1 \quad \forall \lambda \in \mathbb{C}, \tag{6}$$

holds, which means that for each eigenvalue of the matrix G_i there is a unique Jordan block with this eigenvalue. In this case a Jordan basis of the monodromy operator in the space of local solutions to a system forms a (weakly) Levelt basis (see [2], Example 5.2). Using this fact, one proves that *if all monodromy matrices of a Fuchsian system satisfy the condition* (6), *then there is no other Fuchsian system with the same singularities*, *monodromy*, *and exponents*. (Note that the monodromy in Example 1 does not satisfy the condition (6) since the matrices $G_3 = G_4$ are identities.)

1.3. Integrability of Pfaffian systems. The systems mainly considered in this paper are particular cases of so-called *Pfaffian systems*, that is, systems of first-order partial (or ordinary) differential equations of the form

$$\begin{array}{l} dy_1 - (f_1^1\, dz_1 + \cdots + f_r^1\, dz_r) = 0, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ dy_p - (f_1^p\, dz_1 + \cdots + f_r^p\, dz_r) = 0, \end{array} \tag{7}$$

where $z_1, \dots, z_r$ are independent variables, $y_1, \dots, y_p$ are unknowns, and $f_j^i = f_j^i(z, y)$ are holomorphic functions in a domain $D \subset \mathbb{C}^r \times \mathbb{C}^p$. The system (7) is said to be (*completely*) *integrable* if for any $(z^0, y^0) \in D$ there exists a unique solution $y(z)$ such that $y(z^0) = y^0$.

The integrability of the system (7) can also be expressed in terms of the r-dimensional holomorphic distribution S on D determined by this system. For an arbitrary $(z, y) \in D$ consider the basis $\dfrac{\partial}{\partial z_1}, \dots, \dfrac{\partial}{\partial z_r}, \dfrac{\partial}{\partial y_1}, \dots, \dfrac{\partial}{\partial y_p}$ of the tangent space $T_{(z,y)}D$ and let $S_{(z,y)} \subset T_{(z,y)}D$ be the r-dimensional subspace spanned by the vectors $\left\{ \sum_{i=1}^{p} f_j^i \dfrac{\partial}{\partial y_i} + \dfrac{\partial}{\partial z_j} \right\}_{j=1}^{r}$ (thus, S is generated by r holomorphic vector fields on D). Then the integral manifolds of the system (7) are those of the distribution S. The integrability of the system is then equivalent to the condition that for any two local holomorphic vector fields X and Y satisfying $X_{(\tilde z, \tilde y)}$, $Y_{(\tilde z, \tilde y)} \in S_{(\tilde z, \tilde y)}$ for all nearby $(\tilde z, \tilde y)$ the commutator also satisfies $[X, Y]_{(\tilde z, \tilde y)} \in S_{(\tilde z, \tilde y)}$.

This can also be formulated on the level of the cotangent bundle. Denote by $\theta_1, \dots, \theta_p$ the differential 1-forms on the left-hand side of (7) and by $\mathscr{I}$ the ideal in the differential algebra of holomorphic k-forms $(1 \leqslant k \leqslant r + p)$ on D generated by the 1-forms $\theta_1, \dots, \theta_p$. Then one has the *Frobenius integrability condition* for the Pfaffian system (7) (or the distribution S).

Theorem 1. *The system* (7) *is integrable if and only if*

$$d\theta_1, \dots, d\theta_p \in \mathscr{I}.$$

The proof can be found in [4], Theorem 5.1, p. 13 (in the particular case $r = 2$, $p = 1$), or in [5].[1] The Frobenius integrability condition may be thought of as the generalization of the Cauchy theorem for ordinary differential equations to the case of partial differential equations. It is not difficult to check that the integrability condition is always fulfilled in the case $r = 1$.

Example 2. Consider the algebra of rational functions in the variables $t = (t_1, \dots, t_r) \in \mathbb{C}^r$, $(p, q) = (p_1, \dots, p_n, q_1, \dots, q_n) \in \mathbb{C}^{2n}$ and the *Poisson bracket* $\{\cdot, \cdot\}$ on it, which maps a pair of functions f, g into the function

$$\{f, g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right).$$

Then a Pfaffian system

$$\begin{aligned} dp_1 &= \sum_{j=1}^{r} \{H_j, p_1\}\, dt_j, \quad \dots, \quad dp_n = \sum_{j=1}^{r} \{H_j, p_n\}\, dt_j, \\ dq_1 &= \sum_{j=1}^{r} \{H_j, q_1\}\, dt_j, \quad \dots, \quad dq_n = \sum_{j=1}^{r} \{H_j, q_n\}\, dt_j \end{aligned} \tag{8}$$

is called a *Hamiltonian system* with *Hamiltonians* $H_j = H_j(t, p, q)$. This system can also be written in the form

$$\frac{\partial p_i}{\partial t_j} = \frac{\partial H_j}{\partial q_i}, \quad \frac{\partial q_i}{\partial t_j} = -\frac{\partial H_j}{\partial p_i}, \qquad i = 1, \dots, n, \quad j = 1, \dots, r.$$

The integrability condition for the Hamiltonian system (8) is formulated as follows (see [4], Proposition 6.4, p. 23). Put

$$\Gamma_{ij} = \frac{\partial H_j}{\partial t_i} - \frac{\partial H_i}{\partial t_j} + \{H_i, H_j\}.$$

Then the system (8) is integrable if and only if

$$\{\Gamma_{jk}, p_i\} = \{\Gamma_{jk}, q_i\} = 0, \qquad i = 1, \dots, n, \quad j, k = 1, \dots, r,$$

that is, all the Γ_{jk} are independent of (p, q).

An important particular case of Pfaffian systems is the linear system (written in the matrix form)

$$dy = \Omega y, \qquad y(z) \in \mathbb{C}^p, \tag{9}$$

where Ω is a holomorphic matrix differential 1-form on $U \subset \mathbb{C}^r$. For example, the Fuchsian system (3) is a system of the form (9), where $r = 1$ and Ω is meromorphic on $\overline{\mathbb{C}}$, with poles $a_1^0, \dots, a_n^0$ of the first order.

Obviously, the set of local solutions to the system (9) near any point $z^0 \in U$ forms a vector space, and the integrability of the system is equivalent to the fact that these spaces are p-dimensional. On the other hand, the Frobenius integrability condition for the system (9) transforms as follows.

[1] [5] concerns smooth rather than holomorphic distributions, but the proof is also suitable for the latter.

Corollary 1. *The system* (9) *is integrable if and only if*

$$d\Omega = \Omega \wedge \Omega. \tag{10}$$

Proof. If the system (9) is integrable, then there exists a local holomorphic $(p \times p)$-matrix $Y(z)$ such that $\Omega = dY \cdot Y^{-1}$. Therefore [2]

$$d\Omega = -dY \wedge d(Y^{-1}) = dY \wedge Y^{-1} \cdot dY \cdot Y^{-1} = dY \cdot Y^{-1} \wedge dY \cdot Y^{-1} = \Omega \wedge \Omega.$$

Conversely, if the relation (10) holds, then the differential 1-forms $\theta_i = dy_i - \sum_{j=1}^{p} \Omega_{ij} y_j$ satisfy the condition of Theorem 1 and the system (9) is integrable. Indeed,

$$d\theta_i = -\sum_{j=1}^{p} (d\Omega_{ij}) y_j + \sum_{j=1}^{p} \Omega_{ij} \wedge dy_j = -\sum_{j=1}^{p} (d\Omega_{ij}) y_j + \sum_{j=1}^{p} \Omega_{ij} \wedge \left(\theta_j + \sum_{k=1}^{p} \Omega_{jk} y_k \right)$$
$$= \sum_{k=1}^{p} \left(-d\Omega_{ik} + \sum_{j=1}^{p} \Omega_{ij} \wedge \Omega_{jk} \right) y_k + \sum_{j=1}^{p} \Omega_{ij} \wedge \theta_j = \sum_{j=1}^{p} \Omega_{ij} \wedge \theta_j .$$

The corollary is proved.

The linear Pfaffian system (9) again has a description in terms of holomorphic vector bundles and connections. One considers the trivial vector bundle $E = U \times \mathbb{C}^p$ of rank p over U and an analogous basis $\{s_i\}$ of trivializing sections corresponding to the standard basis $\{e_i\}$ of $\mathbb{C}^p$. A matrix

$$\Omega = \sum_{i=1}^{r} C_i \, dz_i$$

of holomorphic differential 1-forms on U defines with respect to the basis $\{s_i\}$ a holomorphic connection ∇ in E:

$$\nabla \colon (s_1, \ldots, s_p) \mapsto -(s_1, \ldots, s_p)\, \Omega.$$

The connection $\nabla \colon \Gamma^0(E) \to \Gamma^1(E)$ has a natural extension $\widetilde{\nabla} \colon \Gamma^1(E) \to \Gamma^2(E)$ from the space $\Gamma^0(E)$ of holomorphic sections of E to the space $\Gamma^1(E)$ of holomorphic differential 1-forms with values in E, given by the formula

$$\widetilde{\nabla}(s\,\sigma) = s\, d\sigma + \nabla(s) \wedge \sigma,$$

where $s \in \Gamma^0(E)$ and σ is a holomorphic differential 1-form on U. The space of horizontal sections (that is, those $y = \sum_{i=1}^{p} y_i s_i$ with $\nabla(y) = 0$) is p-dimensional if and only if the curvature $\widetilde{\nabla} \circ \nabla \colon \Gamma^0(E) \to \Gamma^2(E)$ of ∇ is zero, that is, the curvature 2-form (with respect to the basis $\{s_i\}$) is

$$-d\Omega + (-\Omega) \wedge (-\Omega) = 0,$$

[2] One should remember that for a function f and a differential 1-form σ one has $d(f\sigma) = f d\sigma + df \wedge \sigma$, while $d(\sigma f)$ (which is the same) has the form $d(\sigma f) = (d\sigma) f - \sigma \wedge df$.

which is equivalent to the Pfaffian system (9) being integrable. The connection ∇ is then also said to be *integrable* (or *flat*). In terms of the matrices C_i the integrability of ∇ amounts to the so-called *zero curvature* equations

$$\frac{\partial C_j}{\partial z_i} - \frac{\partial C_i}{\partial z_j} - [C_i, C_j] = 0, \qquad i, j = 1, \dots, r.$$

By considering the above description appropriately, one can introduce for a (non-trivial) holomorphic vector bundle E of rank p over a complex manifold Z notions such as (holomorphic) connection, integrable connection, horizontal sections, and monodromy of an integrable connection. Defining the last object requires the space of local horizontal sections of the connection to be p-dimensional. This is provided by the integrability of the connection.

2. Integrable deformations of meromorphic connections

For the space $U = \overline{\mathbb{C}} \setminus \{a_1^0, \dots, a_n^0\}$ above (see the end of § 1.1) a link was established between an integrable holomorphic connection ∇^0 in a (trivial) holomorphic vector bundle of rank p over U and a p-dimensional representation of the fundamental group $\pi_1(U)$. This relation holds in general.

Let E be a holomorphic vector bundle of rank p over a complex manifold X and ∇ an integrable connection in E. There exists between such pairs (E, ∇) and $\mathrm{GL}(p, \mathbb{C})$-representations of the fundamental group of X a crucial relation which we shall use widely: namely, the following statement holds (see [6], Proposition 2.5).

Theorem 2. *If X is connected, then there is an equivalence between the category of pairs (E, ∇) consisting of a holomorphic vector bundle E of rank p over X and an integrable connection ∇, and that of p-dimensional representations of $\pi_1(X)$. This correspondence is given by the natural action of $\pi_1(X)$ on the fibre of E over some base point of X.*

The connections from (1) are not defined on the whole Riemann sphere $\overline{\mathbb{C}}$, but possess singularities that are mild in the sense that the matrix $B(z)$ is meromorphic on $\overline{\mathbb{C}}$. Thus, one can say that the system (1) defines a *meromorphic connection* ∇^0 in the trivial vector bundle $E^0 = \overline{\mathbb{C}} \times \mathbb{C}^p$ of rank p over $\overline{\mathbb{C}}$, a linear map from the space $M^0(E^0)$ of meromorphic sections of E^0 to the space $M^1(E^0)$ of meromorphic differential 1-forms on $\overline{\mathbb{C}}$ with values in E^0. We now introduce this notion of meromorphic connection in a very general setting. Let X be a complex manifold of dimension m, $Y \subset X$ an analytic subset of codimension one, and E a holomorphic vector bundle of rank p over X. If X is connected, then the space $X \setminus Y$ is also connected (see [7], Theorem 16.2). Assume that the restriction $E|_{X\setminus Y}$ admits an integrable connection ∇.

Definition 1. The connection ∇ is said to be *meromorphic over* Y if for any $y \in Y$ there exists a neighbourhood U of y such that $E|_U$ is trivial and the connection form of ∇ with respect to a basis of holomorphic trivializing sections of $E|_U$ is meromorphic on U, with a polar locus $U \cap Y$.

Note that this notion is independent of the choice of local holomorphic trivializing sections. Recall that a function f is *meromorphic* on U if it is holomorphic on $U \setminus P$, cannot be extended to P holomorphically, and is represented as a quotient

$f(t) = \varphi(t)/\psi(t)$ of holomorphic functions in a neighbourhood of every point $t^* \in P$ (hence $\psi(t^*) = 0$), where $P \subset U$ is an analytic subset of codimension one (it is defined locally by the equation $\psi(t) = 0$). The set P is called the *polar locus* of the meromorphic function f. The points of this set are divided into *poles* (at which the function φ does not vanish) and *ambiguous points* (at which $\varphi = 0$).

The connection defined by the Fuchsian system (3) serves as an example for the following class of connections ∇ meromorphic over Y.

Definition 2. A connection ∇ that is meromorphic over Y is said to have a *logarithmic pole over* Y if for any regular point $y \in Y$ there exists a neighbourhood U of y with local coordinates $(t_1, \dots, t_m)$ such that $Y \cap U$ is given by $\{t_1 = 0\}$ and the connection form Ω_U of ∇ with respect to a basis of holomorphic trivializing sections of $E|_U$ has the form

$$\Omega_U = \frac{A_1}{t_1}\, dt_1 + \sum_{i=2}^{m} A_i\, dt_i,$$

where all the A_i are holomorphic $(p \times p)$-matrices in U.

We will briefly say that ∇ is a meromorphic (respectively, logarithmic) connection in the holomorphic vector bundle E. The set Y is called the *polar locus* of the connection ∇.

The starting point for integrable deformations is a meromorphic linear system on $\overline{\mathbb{C}}$ that only has singularities $a_1^0, \dots, a_n^0 \in \mathbb{C}$. In more geometrical terms this amounts to:

- a trivial holomorphic vector bundle E^0 of rank p over $\overline{\mathbb{C}}$,
- an integrable meromorphic connection ∇^0 in E^0 with singularities $a_1^0, \dots, a_n^0$.

Since there are no non-zero holomorphic differential 1-forms on $\overline{\mathbb{C}}$, the connection form ω^0 of ∇^0 can be written as

$$\omega^0 = \sum_{i=1}^{n} \sum_{l \geqslant 1} \frac{B_{il}^0}{(z - a_i^0)^l}\, dz, \qquad \sum_{i=1}^{n} B_{i1}^0 = 0.$$

The deformations we are interested in consist in moving the poles $\{a_i^0\}$ of ω^0 inside $\mathbb{C}$. More precisely, the way the poles move is determined by a connected complex manifold T, called the *deformation space* for short, and a set of holomorphic functions

$$a_i \colon T \to \mathbb{C}, \qquad i = 1, \dots, n,$$

the so-called *deformation functions.* Furthermore, there has to be a base point $t^0 \in T$ such that

$$a_i(t^0) = a_i^0, \qquad i = 1, \dots, n.$$

We will restrict ourselves to deformations for which the poles never collide, that is, $a_i(t) \neq a_j(t)$ for all $t \in T$ and $i \neq j$. Below are some examples of deformation spaces which will be used later.

Example 3. 1) $T = D(a^0)$, a disk of small radius centred at the point $a^0 = (a_1^0, \dots, a_n^0)$ of the space

$$Z = \mathbb{C}^n \setminus \bigcup_{i \neq j} \{a_i = a_j\}.$$

In this case the parameter $a = (a_1, \dots, a_n) \in D(a^0)$ of the deformation is a set of poles and the deformation functions are the projections on all coordinates, so that we simply write a_i instead of $a_i(a)$.

2) $T = D(a^0) \times \mathscr{W}$, where $\mathscr{W}$ is a connected complex manifold. In this case we distinguish the variables $a \in D(a^0)$ and $w \in \mathscr{W}$ for convenience. The ith deformation function is $(a, w) \mapsto a_i$ and we again write a_i instead of $a_i(a, w)$.

3) $T = \widetilde{Z}$, the universal covering space of Z. The ith deformation function is then the composition of the cover $\widetilde{Z} \to Z$ with the projection on the ith coordinate.

Since $\widetilde{Z}$ is a universal covering space, its fundamental group is trivial. Using the exact sequence of homotopy groups for the fibre bundle

$$\mathbb{C}^n \setminus \bigcup_{i \neq j} \{a_i = a_j\} \to \mathbb{C}^{n-1} \setminus \bigcup_{i \neq j} \{a_i = a_j\}$$

determined by the projection along the last coordinate, and induction with respect to n, one can check that all the homotopy groups $\pi_k(\widetilde{Z}) = \pi_k(Z)$, $k \geqslant 2$, are also trivial (see, for example, [2], Appendix 4). It follows then from the Whitehead theorem (see [8], § 11.5) that the space $\widetilde{Z}$ is contractible.

The space $\widetilde{Z}$ possesses another useful property, namely it is a Stein manifold, being the universal covering space of the Stein manifold Z (according to [9] every unramified covering space of a Stein manifold is again a Stein manifold). Since $\widetilde{Z}$ is contractible, Grauert's theorem applies to it, and hence every holomorphic vector bundle over $\widetilde{Z}$ is holomorphically trivial (see [10]).

Let now X be the manifold $\overline{\mathbb{C}} \times T$. Inside X we consider the submanifold $Y = Y_1 \cup \dots \cup Y_n$ given by

$$Y_i = \{(z, t) \in X \mid z - a_i(t) = 0\}, \qquad i = 1, \dots, n.$$

This brings us to the introduction of the following notion.

Definition 3. An *integrable deformation* (E, ∇) of the pair (E^0, ∇^0) with the deformation space T, deformation functions $\{a_i(t)\}$, and base point $t^0 \in T$ consists of

- a holomorphic vector bundle E of rank p over $X = \overline{\mathbb{C}} \times T$,
- an integrable meromorphic connection ∇ in E with Y as a polar locus, such that the restriction of (E, ∇) to $\overline{\mathbb{C}} \times \{t^0\}$ is isomorphic to (E^0, ∇^0).

Given the deformation space T, deformation functions $\{a_i(t)\}$, and base point $t^0 \in T$, a relevant question is whether there exists an integrable deformation with these deformation data. Here one might encounter topological obstructions.

Let $i \colon \overline{\mathbb{C}} \hookrightarrow \overline{\mathbb{C}} \times T$ be the embedding, $i(z) = (z, t^0)$. It induces a natural map

$$i_* \colon \pi_1(\overline{\mathbb{C}} \setminus \{a_1^0, \dots, a_n^0\}) \to \pi_1(X \setminus Y).$$

In view of Theorem 2, the existence of an integrable deformation (E, ∇) of (E^0, ∇^0) implies that the representation χ of $\pi_1(\overline{\mathbb{C}} \setminus \{a_1^0, \dots, a_n^0\})$ corresponding to (E^0, ∇^0) is isomorphic to $\widetilde{\chi} \circ i_*$, where $\widetilde{\chi}$ is the representation of $\pi_1(X \setminus Y)$ corresponding to (E, ∇). Consider the fibre bundle

$$p_2 \colon X \setminus Y \to T, \qquad p_2(z, t) = t,$$

over T with the fibre $p_2^{-1}(t^0) = \overline{\mathbb{C}} \setminus \{a_1^0, \ldots, a_n^0\}$. The corresponding long exact sequence

$$\cdots \longrightarrow \pi_2(T) \xrightarrow{\delta} \pi_1(\overline{\mathbb{C}} \setminus \{a_1^0, \ldots, a_n^0\}) \xrightarrow{i_*} \pi_1(X \setminus Y) \longrightarrow \pi_1(T) \tag{11}$$

of homotopy groups implies that χ, besides being extendable to $\widetilde{\chi}$, also has to be trivial on the image of δ, since $\chi \circ \delta = \widetilde{\chi} \circ (i_* \circ \delta) \equiv 1$. This question does not arise if $\pi_1(T) = \pi_2(T) = \{1\}$. Thus, for the spaces $T = D(a^0)$ or $T = \widetilde{Z}$ in Example 3 and the corresponding deformation functions, there are no topological obstructions to the existence of an integrable deformation with these deformation data. This turns out to be sufficient for the following statement to hold.

Theorem 3. *For any pair (E^0, ∇^0) there exists an integrable deformation (E, ∇) of (E^0, ∇^0) with $\widetilde{Z}$ as a deformation space and the corresponding $\{a_i(t)\}$ as deformation functions.*

Proof. First we construct (E, ∇) over $X \setminus Y$. From the exact sequence (11) of homotopy groups we know that i_* is an isomorphism. Hence the p-dimensional representation of $\pi_1(\overline{\mathbb{C}} \setminus \{a_1^0, \ldots, a_n^0\})$ corresponding to the pair $(E^0|_{\overline{\mathbb{C}} \setminus \{a_1^0, \ldots, a_n^0\}}, \nabla^0)$ determines a p-dimensional representation of $\pi_1(X \setminus Y)$, and thus we obtain a pair (E, ∇) over $X \setminus Y$. What remains is to extend the vector bundle E to X and show that the connection ∇ is meromorphic over Y. This will be done on disjoint neighbourhoods of the components of Y. It requires some notation.

For each $t \in \widetilde{Z}$ one denotes by $\delta(t)$ the minimum of the distances $\{|a_i(t) - a_j(t)|, |a_i^0 - a_j^0|\}_{i \neq j}$ and by $D_j(t)$ the disk

$$D_j(t) = \left\{ z \in \mathbb{C} \;\middle|\; |z - a_j(t)| < \frac{\delta(t)}{4} \right\}$$

of radius $\delta(t)/4$ about the point $a_j(t) \in \mathbb{C}$. Next, one defines the neighbourhood V_j of Y_j by

$$V_j = \{(z, t) \in X \mid z \in D_j(t)\}, \qquad j = 1, \ldots, n.$$

By construction these open subsets are disjoint. The projection $(z, t) \mapsto t$ maps the sets $V_j \setminus Y_j$ onto $\widetilde{Z}$ so that the fibre over $t^0 \in \widetilde{Z}$ is equal to $D_j(t^0) \setminus \{a_j^0\}$. Hence these neighbourhoods are fibre bundles with the base space $\widetilde{Z}$ and fibre $D_j(t^0) \setminus \{a_j^0\}$. The corresponding long exact sequence

$$\cdots \longrightarrow \pi_2(\widetilde{Z}) \longrightarrow \pi_1(D_j(t^0) \setminus \{a_j^0\}) \xrightarrow{i_*} \pi_1(V_j \setminus Y_j) \longrightarrow \pi_1(\widetilde{Z})$$

of homotopy groups implies then that the map i_* induces an isomorphism between the groups $\pi_1(D_j(t^0) \setminus \{a_j^0\})$ and $\pi_1(V_j \setminus Y_j)$.

Consider the surjective maps

$$p_j \colon V_j \to D_j(t^0), \qquad p_j(z, t) = z - a_j(t) + a_j^0, \qquad j = 1, \ldots, n,$$

and the vector bundles E_j over V_j obtained by pulling back through p_j the bundle $E^0|_{D_j(t^0)}$. Let ω^0 be the connection form of ∇^0 with respect to the basis $\{s_1, \ldots, s_p\}$ of holomorphic trivializing sections of the latter bundle. One defines the connection ∇_j in $E_j|_{V_j \setminus Y_j}$ as the connection whose connection form Ω_j with respect to

$\{p_j^*(s_1), \ldots, p_j^*(s_p)\}$ is $p_j^*(\omega^0)$. In particular, one sees that Ω_j can be written with respect to any basis of holomorphic trivializing sections of E_j as

$$\Omega_j = \sum_{l \geqslant 1} \frac{B_{jl}(t)}{(z - a_j(t))^l} \, d(z - a_j(t)) + w_j, \tag{12}$$

where w_j is a holomorphic matrix differential 1-form on V_j and $\{B_{jl}\}$ are holomorphic $(p \times p)$-matrices on $\widetilde{Z}$ satisfying $B_{jl} = 0$ for all sufficiently large l. Since

$$dp_j^*(\omega^0) = p_j^*(d\omega^0) = p_j^*(\omega^0 \wedge \omega^0) = p_j^*(\omega^0) \wedge p_j^*(\omega^0),$$

the form $\Omega_j = p_j^*(\omega^0)$ satisfies (10) and the connection ∇_j is integrable. Further, ∇_j is meromorphic over Y_j since Ω_j is, and the pull-back through i of the restriction of (E_j, ∇_j) to $D_j(t^0) \times \{t^0\}$ is isomorphic to $(E^0|_{D_j(t^0)}, \nabla^0)$. One sees that the monodromy representations of the pairs $(E|_{V_j \setminus Y_j}, \nabla)$ and $(E_j|_{V_j \setminus Y_j}, \nabla_j)$ are the same, being both equal to that of $(E^0|_{D_j(t^0)}, \nabla^0)$. Hence these pairs are isomorphic and in this way one has extended (E, ∇) to X. This completes the proof of the theorem.[3]

Remark 1. As follows from the proof of the theorem, if the initial connection ∇^0 is *logarithmic*, then the connection ∇ is logarithmic as well.

Remark 2. Clearly, if one replaces $\widetilde{Z}$ by an arbitrary deformation space T with $\pi_1(T) = \pi_2(T) = \{1\}$, such as a small disk $D(a^0)$ around a^0, then the same construction works. Integrable deformations also exist for a deformation space $T = D(a^0) \times \mathscr{W}$, where $\mathscr{W}$ is a connected complex manifold. In that case one takes the integrable deformation on $\overline{\mathbb{C}} \times D(a^0)$ and pulls it back through the standard projection $\overline{\mathbb{C}} \times T \to \overline{\mathbb{C}} \times D(a^0)$.

If a deformation space T is *simply connected*, then the deformation functions $a_i(t)$ provide a holomorphic map

$$a \colon T \to Z, \qquad a(t) = \big(a_1(t), \ldots, a_n(t)\big),$$

which is lifted to $\tilde{a} \colon T \to \widetilde{Z}$. Therefore one has the holomorphic map

$$f \colon \overline{\mathbb{C}} \times T \to \overline{\mathbb{C}} \times \widetilde{Z}, \qquad (z, t) \mapsto (z, \tilde{a}(t)).$$

The pull-back $\big(f^*(E), f^*(\nabla)\big)$ of the integrable deformation (E, ∇) obtained in Theorem 3 is an integrable deformation as well (the set $\displaystyle\bigcup_{i=1}^{n} \{z - a_i(t) = 0\}$ goes to the polar locus of ∇ under the map f).

Remark 3. Let (E, ∇) be an integrable deformation as defined above. Then it determines by restriction an integrable connection $\nabla(\infty)$ in $E|_{\{\infty\} \times T}$.

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[3] Note that each p_j maps the corresponding hypersurface Y_j into the singular point a_j^0 of the connection ∇^0, which is why we obtain the set $Y = \bigcup_{i=1}^n Y_i$ as the polar locus of the constructed connection ∇. If instead of the local maps one used a simple global construction with the projection $\overline{\mathbb{C}} \times \widetilde{Z} \to \overline{\mathbb{C}}$ and pulled back the pair (E^0, ∇^0) through it, then one would not obtain the required polar locus.

We note that restrictions of the integrable deformation (E, ∇) to $\overline{\mathbb{C}} \times \{t\}$, $t \in T$, determine the family $(E_t, \nabla_t)_{t \in T}$ of holomorphic vector bundles of rank p over $\overline{\mathbb{C}}$ with meromorphic connections. If the deformation space T is simply connected, then the monodromy of the connection ∇_t can be extended holomorphically to T as a function of the parameter t. As shown in the proof of Theorem 3, in the case of trivial $\pi_2(T)$ this function is constant, that is, the deformation preserves the monodromy of ∇_t (the latter coincides with the monodromy of the connection ∇ for all $t \in T$).

Further, we consider the two main goals of the paper:

1) Let (E, ∇) be the integrable deformation in Theorem 3 (and Remark 2). The first goal is to describe the set of those $t \in T$ for which the bundle E_t is holomorphically trivial.

2) Let (E, ∇) be an integrable deformation of the *logarithmic* connection ∇^0 such that the bundle E is holomorphically trivial for some deformation space (this is the case for isomonodromic deformations of Fuchsian systems, which will be considered in the next section). The second goal is to describe the connection form Ω of ∇.

For simplicity we do not consider isomonodromic deformations of differential systems (1) with *irregular* (non-Fuchsian) singularities in this paper. Such deformations are traditionally required to preserve not only the monodromy but also the Stokes matrices of an initial system. These deformations are presented and partially studied in [11]–[14].

3. Isomonodromic deformations of Fuchsian systems

3.1. General theory. Isomonodromic deformations of Fuchsian systems can be defined as follows. Consider a family

$$\frac{dy}{dz} = \left(\sum_{i=1}^{n} \frac{B_i(t)}{z - a_i(t)}\right) y, \qquad y(z) \in \mathbb{C}^p, \quad B_i(t) \in \mathrm{Mat}(p, \mathbb{C}), \tag{13}$$

of Fuchsian systems holomorphically depending on the parameter $t \in T$, where T is a *simply connected* complex manifold with a base point t^0. The family (13) is defined on the space

$$(\overline{\mathbb{C}} \times T) \setminus \bigcup_{i=1}^{n} \{z - a_i(t) = 0\}.$$

We assume that the singularities of (13) do not collide, that is, $a_i(t) \neq a_j(t)$ for all $t \in T$ and $i \neq j$.

Definition 4. One says that the family (13) is *isomonodromic* or more elaborately, it is an *isomonodromic deformation* of the Fuchsian system (3), if $a_i(t^0) = a_i^0$, $B_i(t^0) = B_i^0$, and for any $t \in T$ the monodromy representations

$$\chi\colon \pi_1(\overline{\mathbb{C}} \setminus \{a_1(t), \dots, a_n(t)\}) \to \mathrm{GL}(p, \mathbb{C})$$

of the corresponding systems are the same. This means that for every $t \in T$ there exists a fundamental matrix $Y(z, t)$ of the corresponding system (13) such that its monodromy does not depend on t. The matrix $Y(z, t)$ is called an *isomonodromic fundamental matrix*.

We give some comments on this definition. Let $Y^0(z,t)$ be the fundamental matrix of (13) which satisfies the initial condition $Y^0(z_0,t) = \mathrm{Id}$ and is thus analytically dependent on t by the corresponding theorem. Since T is simply connected, the monodromy matrices $G_1(t), \dots, G_n(t)$ corresponding to $Y^0(z,t)$ are holomorphic functions on the whole space T. Then the family (13) is isomonodromic if for every $t \in T$ there exists a matrix $C(t) \in \mathrm{GL}(p,\mathbb{C})$ such that

$$C^{-1}(t)G_i(t)C(t) = \mathrm{const}.$$

In other words, the monodromy matrices corresponding to the fundamental matrix $Y(z,t) = Y^0(z,t)C(t)$ do not depend on t. If T is a *contractible Stein manifold*, then by Bolibrukh's result (see [15] or [2], Theorem 13.1 for details) the matrix function $C(t)$ can be chosen to be holomorphic.[4] Hence the isomonodromic fundamental matrix $Y(z,t)$ is holomorphically dependent on t as well.

The Fuchsian system (3) determines a logarithmic connection ∇^0 in the holomorphically trivial vector bundle E^0 of rank p over $\overline{\mathbb{C}}$. According to Theorem 3 and Remarks 1 and 2 there exists an integrable deformation (E,∇) of (E^0,∇^0) with T as deformation space and the $\{a_i(t)\}$ as deformation functions, and furthermore, the connection ∇ is logarithmic. As mentioned above (see the end of § 2), the restrictions of the pair (E,∇) to $\overline{\mathbb{C}} \times \{t\}$, $t \in T$, determine the family $(E_t,\nabla_t)_{t\in T}$ of holomorphic vector bundles over $\overline{\mathbb{C}}$ with logarithmic connections, having the same monodromy. If all the bundles E_t are holomorphically trivial, then we have an isomonodromic deformation of the initial Fuchsian system. Conversely, under certain additional conditions on the deformation space T any isomonodromic deformation of the Fuchsian system (3) determines an integrable deformation (E,∇) of the pair (E^0,∇^0) such that the vector bundle E is holomorphically trivial, although the connection ∇ may be non-logarithmic, as we will see later. This is provided by the following theorem.

Theorem 4. *Let T be a contractible Stein manifold. Then the family* (13) *of Fuchsian systems is isomonodromic if and only if there exists a meromorphic matrix differential* 1*-form Ω on $X = \overline{\mathbb{C}} \times T$ (with the polar locus $\bigcup_{i=1}^{n}\{z - a_i(t) = 0\}$) such that*

i) $\Omega = \displaystyle\sum_{i=1}^{n} \frac{B_i(t)}{z - a_i(t)}\, dz$ *for any fixed $t \in T$;*

ii) $d\Omega = \Omega \wedge \Omega$.

Proof. If the family (13) is isomonodromic, then it has an isomonodromic fundamental matrix $Y(z,t)$ that is analytically dependent on both variables z and t, as mentioned above. It follows that $\Omega = dY \cdot Y^{-1}$ is a matrix differential 1-form meromorphic on X and satisfying conditions i) and ii). (In fact, the last condition was checked in the proof of Corollary 1.)

Conversely, if there exists a meromorphic matrix differential 1-form Ω on X that satisfies the conditions i) and ii), then it defines an integrable connection ∇ in a holomorphically trivial vector bundle E of rank p over X, such that the pair (E,∇)

[4] Bolibrukh [15] proved this result by using isomonodromic families with the deformation space $T = D(a^0)$ (see Example 3), but his proof can be applied to any contractible Stein manifold T.

is an integrable deformation of the pair (E^0, ∇^0). The restrictions of the pair (E, ∇) to $\overline{\mathbb{C}} \times \{t\}$, $t \in T$, determine the family $(E_t, \nabla_t)_{t \in T}$ of holomorphically trivial vector bundles over $\overline{\mathbb{C}}$ with logarithmic connections, having the same monodromy. By virtue of the condition i) the connection form of ∇_t coincides with $\sum_{i=1}^{n} \frac{B_i(t)}{z - a_i(t)} dz$, hence the family (13) is isomonodromic. The theorem is proved.

An isomonodromic deformation preserves not only the monodromy but also the exponents of the initial system (so that the eigenvalues of the matrices $B_i(t)$ do not depend on the parameter). Indeed, the set of their possible values is discrete by the isomonodromy. At the same time, they depend continuously on t as the roots of polynomials with coefficients holomorphically dependent on t. Therefore, the exponents are constant.

We should note that the 1-form in Theorem 4 is not quite unique. For instance, one can consider another 1-form $\Omega' = \Omega + (df)\mathrm{Id}$, where f is a holomorphic function on T. Then Ω' and Ω both determine the same family, since they coincide for every fixed $t \in T$, and $d\Omega' = \Omega' \wedge \Omega'$.

However, if the monodromy of the isomonodromic family (13) is *irreducible*, then the only ambiguity in the choice of a 1-form determining this family is that described above. More precisely, in this case the matrix differential 1-form Ω in Theorem 4 is uniquely defined by the family up to a summand $(df)f^{-1}\mathrm{Id}$, where f is a non-vanishing holomorphic function on T (see [2], Proposition 15.1).

If the monodromy of the isomonodromic family (13) is *reducible*, then the latter can have non-trivial symmetries of the form

$$y' = \Gamma(z,t)y, \qquad \Gamma(z,t) = Y(z,t)C(t)Y^{-1}(z,t),$$

where $Y(z,t)$ is an analytic isomonodromic fundamental matrix of the family and the matrix $C(t)$, which is holomorphic on T, belongs to the centralizer of the monodromy group of the family. Indeed, in this case the matrix $\Gamma(z,t)$ is meromorphic on the space X and the transformed matrix

$$Y'(z,t) = \Gamma(z,t)Y(z,t) = Y(z,t)C(t)$$

is an analytic isomonodromic fundamental matrix of the same family (13). Such symmetries (and only these) give different 1-forms

$$\Gamma\Omega\Gamma^{-1} + (d\Gamma)\Gamma^{-1},$$

determining the fixed isomonodromic family (13)

Given the criterion in Theorem 4 it is natural to investigate, describe, and classify matrix differential 1-forms Ω satisfying it. That is the main aim of this section. As a result we obtain for isomonodromic deformations of Fuchsian systems a complete description and classification that turn out to correspond to certain natural geometric properties of deformations.

3.2. Schlesinger isomonodromic deformations. Let us begin with isomonodromic deformations whose deformation space $T = D(a^0)$ is a disk of small radius centred at the point $a^0 = (a_1^0, \dots, a_n^0)$ of the space $Z = \mathbb{C}^n \setminus \bigcup_{i \neq j} \{a_i = a_j\}$ (see Example 3). Consider a family

$$\frac{dy}{dz} = \left(\sum_{i=1}^{n} \frac{B_i(a)}{z - a_i} \right) y, \qquad B_i(a^0) = B_i^0, \tag{14}$$

of Fuchsian systems which is holomorphically dependent on the parameter $a = (a_1, \dots, a_n) \in D(a^0)$. This family is defined on the space

$$\big(\overline{\mathbb{C}} \times D(a^0)\big) \setminus \bigcup_{i=1}^{n} \{z - a_i = 0\}.$$

When looking for a meromorphic matrix differential 1-form on $\overline{\mathbb{C}} \times D(a^0)$ restricting to $\sum_{i=1}^{n} B_i(a) dz/(z - a_i)$ for $\{a_i = \text{const}\}$, it seems quite natural to start from a form like

$$\Omega_s = \sum_{i=1}^{n} \frac{B_i(a)}{z - a_i}\, d(z - a_i).$$

Such an ansatz has a long history and is known as the *Schlesinger form*. This approach proves to be rather successful. The statement below claims that for any Fuchsian system one can always construct an isomonodromic deformation of it.

Theorem 5 (Schlesinger [16], [17]). *For any initial data $\{a_1^0, \dots, a_n^0, B_1^0, \dots, B_n^0\}$ there exists a unique Schlesinger form*

$$\Omega_s = \sum_{i=1}^{n} \frac{B_i(a)}{z - a_i}\, d(z - a_i)$$

on $\overline{\mathbb{C}} \times D(a^0)$ such that
(i) $B_i(a^0) = B_i^0$, $i = 1, \dots, n$,
(ii) $d\Omega_s = \Omega_s \wedge \Omega_s$.

Written down in coordinate form, condition (ii) looks like

$$dB_i = - \sum_{j=1,\, j \neq i}^{n} \frac{[B_i, B_j]}{a_i - a_j}\, d(a_i - a_j), \qquad i = 1, \dots, n, \tag{15}$$

and is well known under the name of the *Schlesinger equation*, being another example of a Pfaffian system (for the equivalence of the condition $d\Omega_s = \Omega_s \wedge \Omega_s$ to the Schlesinger equation see, for example, [2], Theorem 14.1). To prove the theorem it is sufficient to show the complete integrability of this equation. That can be achieved by a straightforward check that the distribution defined by the matrix differential 1-forms

$$\Theta_i = dB_i + \sum_{j=1,\, j \neq i}^{n} \frac{[B_i, B_j]}{a_i - a_j}\, d(a_i - a_j), \qquad i = 1, \dots, n,$$

satisfies the Frobenius integrability condition (see details in [4], Proposition 5.3, p. 200, or [2], Theorem 14.2). Thus any Fuchsian system (3) can be included in a *Schlesinger isomonodromic family*, that is, a family of the form (14) whose residue matrices $B_i(a)$ satisfy the Schlesinger equation.

Below we list some special and nice properties of the Schlesinger equation (15):

- it is completely integrable;
- it admits Hamiltonian ([18], Appendix 5, [19], § 5) and Lax ([11], Example 3.1) representations;
- it has the Painlevé property (see § 4.3);
- for $n \geqslant 4$ and $p = 2$ it is equivalent to a Garnier system (which coincides with the sixth Painlevé equation in the particular case $n = 4$; see § 4.4).

Theorem 5 is a statement on the local existence of isomonodromic deformations of a Fuchsian system. Further, one can ask about the uniqueness of such a deformation and the universality of the Schlesinger ansatz: are there any other 1-forms satisfying the criterion in Theorem 4 and determining non-Schlesinger deformations, or are there no other possibilities? In the original papers of Schlesinger [16], [17] there is a statement that the family (14) is isomonodromic if and only if the matrices $B_i(a)$ satisfy the Schlesinger equation (15). In general this is not true, as we discuss later. Below are two results concerning Schlesinger isomonodromic deformations.

Theorem 6 ([4], Theorem 5.1, p. 196). *If the initial Fuchsian system* (3) *is non-resonant, then its isomonodromic deformation* (3) *can be Schlesinger only* (*that is, a matrix differential* 1*-form* Ω *giving the deformation can be chosen to be of the form* $\Omega = \Omega_s$).

Theorem 7 (Bolibruch [15], [20]). *If for each monodromy matrix* G_i *of the initial Fuchsian system* (3) *one has*

$$\operatorname{rank}(G_i - \lambda\,\mathrm{Id}) \geqslant p - 1 \quad \forall\,\lambda \in \mathbb{C},$$

then its isomonodromic deformation (14) *can be Schlesinger only.*

Remark 4. Here and in the next subsection, § 3.3, we ignore trivial gauge transformations of a family that do not depend on z and only depend on parameters of the deformation. These gauge transformations contribute the summand $\sum_{i=1}^{n} C_i(a)\,da_i$ to a 1-form Ω and transform 1-forms Ω_s to the so-called *non-normalized* Schlesinger forms (and vice versa) giving non-normalized Schlesinger isomonodromic deformations.

The idea of the proof of Theorems 6 and 7 follows immediately from the construction of the moduli space of Fuchsian systems and the integrability of the Schlesinger equation. The latter implies the existence of a Schlesinger deformation of the initial system. But a possible alternative to the Schlesinger isomonodromic family must coincide with it, since in the non-resonant case as well as under the condition of Theorem 7 there exists for any set $\{a_1, \dots, a_n\}$ of poles at most one Fuchsian system with the given poles, monodromy, and exponents (see § 1.2).

To conclude the subsection on Schlesinger isomonodromic deformations we note that the deformation parameter $a = (a_1, \dots, a_n)$ turns out to be quite natural for them. Indeed, if we consider a family (13) depending on the parameter $t \in T$, where $\dim T > n$ and the residue matrices $B_i(t)$ satisfy the Schlesinger equation

$$dB_i(t) = -\sum_{j=1,\, j\neq i}^{n} \frac{[B_i(t), B_j(t)]}{a_i(t) - a_j(t)}\, d\big(a_i(t) - a_j(t)\big),$$

then for the analytic subset

$$A = \big\{\big(a_1(t), \dots, a_n(t)\big) = \mathrm{const}\big\} \subset T$$

of positive dimension we have $B_i|_A = \mathrm{const}$ in view of the Schlesinger equation. Thus, the residue matrices of such a family can be regarded as functions of the singular points: $B_i = B_i(a)$.

3.3. Resonant isomonodromic deformations. However, the question still remains whether there exist isomonodromic deformations (and matrix differential 1-forms Ω in Theorem 4) other than the Schlesinger ones. It turns out that they really do exist. The first explicit example of a non-Schlesinger isomonodromic deformation was constructed by Bolibrukh (others were later found; see [21], [22])

Example 4 (Bolibrukh [15]). Consider a family

$$\frac{dy}{dz} = \Bigg(\begin{pmatrix} 1 & 0 \\ -\dfrac{2a}{a^2-1} & 0 \end{pmatrix} \frac{1}{z+a} + \begin{pmatrix} 0 & -6a \\ 0 & -1 \end{pmatrix} \frac{1}{z} + \begin{pmatrix} 2 & 3+3a \\ \dfrac{1}{a+1} & -1 \end{pmatrix} \frac{1}{z-1} + \begin{pmatrix} -3 & -3+3a \\ \dfrac{1}{a-1} & 2 \end{pmatrix} \frac{1}{z+1}\Bigg) y \tag{16}$$

of Fuchsian systems with singular points $a_1 = -a,\ a_2 = 0,\ a_3 = 1,\ a_4 = -1$, where $a \in \mathbb{C} \setminus \{0, 1, -1\}$ is a parameter.

This family is isomonodromic since it is determined by the matrix differential 1-form

$$\Omega = \begin{pmatrix} 1 & 0 \\ -\dfrac{2a}{a^2-1} & 0 \end{pmatrix} \frac{d(z+a)}{z+a} + \begin{pmatrix} 0 & -6a \\ 0 & -1 \end{pmatrix} \frac{dz}{z} + \begin{pmatrix} 2 & 3+3a \\ \dfrac{1}{a+1} & -1 \end{pmatrix} \frac{d(z-1)}{z-1} + \begin{pmatrix} -3 & -3+3a \\ \dfrac{1}{a-1} & 2 \end{pmatrix} \frac{d(z+1)}{z+1} + \begin{pmatrix} 0 & 0 \\ \dfrac{2a}{a^2-1} & 0 \end{pmatrix} \frac{da}{z+a},$$

which satisfies the conditions of Theorem 4. The family (16), however, is not a Schlesinger one (neither normalized, nor non-normalized), since its monodromy is *irreducible*, and therefore any matrix differential 1-form determining this family differs from Ω by a holomorphic summand of the form $df(a) f^{-1}(a)\mathrm{Id}$, where f is a non-vanishing holomorphic function. Thus, one cannot eliminate the last summand of Ω.

The irreducibility of the monodromy of the family (16) can be explained as follows. The exponents β_i^j of the family are

$$\beta_1^{1,2} = 0, 1, \qquad \beta_2^{1,2} = 0, -1, \qquad \beta_3^{1,2} = \frac{1}{2} \pm \frac{\sqrt{21}}{2}, \qquad \beta_4^{1,2} = -\frac{1}{2} \pm \frac{\sqrt{37}}{2},$$

therefore the eigenvalues $\mu_i^j = e^{2\pi\sqrt{-1}\,\beta_i^j}$ of the corresponding monodromy matrices G_i are

$$\mu_1^{1,2} = 1, \qquad \mu_2^{1,2} = 1, \qquad \mu_3^{1,2} = -e^{\pm\pi\sqrt{-21}}, \qquad \mu_4^{1,2} = -e^{\pm\pi\sqrt{-37}}.$$

Thus, $\mu_1^{j_1}\mu_2^{j_2}\mu_3^{j_3}\mu_4^{j_4} \neq 1$ for any j_1, j_2, j_3, j_4, which implies the irreducibility of the monodromy (recall that the relation $G_1G_2G_3G_4 = \mathrm{Id}$ holds for the monodromy matrices). Note also that one of the monodromy matrices G_1, G_2 is the identity by virtue of Theorem 7.

A general form of a matrix differential 1-form determining the isomonodromic family (14) of Fuchsian systems with resonant singular points was found by Bolibrukh [15], [20] (see also [2], Theorem 15.2). His analytical approach can be applied to any contractible Stein manifold T as well. So we pass to isomonodromic deformations of the general form (13) and describe a matrix differential 1-form Ω in Theorem 4 determining such deformations.

Let the dimension of the deformation space T, which is a contractible Stein manifold, be equal to s.

Theorem 8. *Any meromorphic matrix differential 1-form Ω determining the isomonodromic deformation (13) has the form*

$$\Omega = \sum_{i=1}^{n} \frac{B_i(t)}{z - a_i(t)}\, d(z - a_i(t)) + \sum_{k=1}^{s} \left(\sum_{i=1}^{n} \sum_{j=1}^{r_i} \frac{\gamma_{ijk}(t)}{(z - a_i(t))^j} \right) dt_k \tag{17}$$

in local coordinates $(z, t_1, \dots, t_s)$ of the space $X = \overline{\mathbb{C}} \times T$, where the γ_{ijk} are holomorphic matrix functions on T and r_i is the maximal i-resonance of the family (13).

The *maximal i-resonance* in this theorem is the maximal difference among all *integer* pairwise differences of the exponents at the point a_i (recall that the exponents are equal to the eigenvalues of B_i). Such integer differences always exist at resonant singular points. If a_i is non-resonant, then $r_i = 0$, and one can easily see that Theorem 6 emerges as a corollary of Theorem 8. Moreover, any matrix differential 1-form Ω determining a non-resonant isomonodromic family is of the form $\Omega = \Omega_s$ (modulo Remark 4).

Proof. We use Bolibrukh's generalization of Levelt's decomposition (4) to the case of isomonodromic families ([20]; see also [2], Theorem 15.1). He established a decomposition of an analytic isomonodromic fundamental matrix $Y(z,t)$ in a neighbourhood of the hypersurface $\{z - a_i(t) = 0\}$. If the monodromy matrix $G_i = \mathrm{diag}(G_i^1, \dots, G_i^m)$ of $Y(z, t^0)$ at the point a_i^0 is block-diagonal (with respect to the decomposition into the direct sum of the root subspaces), then for any $t^* \in T$ there exists a neighbourhood $D(t^*) \subset T$ and a block-diagonal matrix $S_i(t) = \mathrm{diag}(S_i^1(t), \dots, S_i^m(t))$ which is holomorphically invertible in $D(t^*)$ such that the decomposition analogous to (4) holds:

$$Y(z,t)S_i^{-1}(t) = U_i(z,t)\big(z - a_i(t)\big)^{\Lambda_i}\big(z - a_i(t)\big)^{E_i(t)}, \tag{18}$$

where the matrix $U_i(z,t)$ is holomorphically invertible in a neighbourhood of the hypersurface $\{z - a_i(t) = 0\}$ and $E_i(t) = S_i(t)E_iS_i^{-1}(t)$ is an upper triangular matrix.

Let $\widetilde{Y}(z,t)$ be the analytic isomonodromic fundamental matrix of the family (13) such that $\Omega = (d\widetilde{Y})\widetilde{Y}^{-1}$. One can take an invertible constant matrix C_i such that the monodromy matrix G_i of $\widetilde{Y}(z,t)C_i$ is of block-diagonal form. Then according to (18) one has

$$\widetilde{Y}(z,t)C_i = U_i(z,t)\big(z-a_i(t)\big)^{\Lambda_i} S_i(t)\big(z-a_i(t)\big)^{E_i}$$

in a neighbourhood of the hypersurface $\{z - a_i(t) = 0\}$. It follows that for the 1-form Ω one has there

$$\Omega = (d\widetilde{Y})\widetilde{Y}^{-1} = (d\widetilde{Y}C_i)(\widetilde{Y}C_i)^{-1} = \Omega_1 + \Omega_2,$$

where

$$\begin{aligned}\Omega_1 &= \frac{dU_i}{dz}\,U_i^{-1}\,d(z-a_i(t)) + U_i\biggl(\frac{\Lambda_i}{z-a_i(t)} + (z-a_i(t))^{\Lambda_i}\,\frac{E_i(t)}{z-a_i(t)}(z-a_i(t))^{-\Lambda_i}\biggr)\\ &\qquad \times U_i^{-1}\,d(z-a_i(t)) + \text{a holomorphic 1-form}\\ &= \frac{B_i(t)}{z-a_i(t)}\,d(z-a_i(t)) + \text{a holomorphic 1-form},\\ \Omega_2 &= U_i(z-a_i(t))^{\Lambda_i}(dS_i)S_i^{-1}\big(z-a_i(t)\big)^{-\Lambda_i}U_i^{-1}\\ &= \sum_{k=1}^{s}\biggl(\sum_{j=1}^{r_i}\frac{\gamma_{ijk}(t)}{(z-a_i(t))^j}\biggr)\,dt_k + \text{a holomorphic 1-form},\end{aligned}$$

with all the γ_{ijk} holomorphic matrix functions in a coordinate neighbourhood $D(t^*)$ of the space T. The form of Ω_2 follows from the block-diagonal structure of the matrices Λ_i, S_i and the definition of the maximal i-resonance r_i of the family (13).

Therefore, the matrix differential 1-form

$$\widetilde{\Omega} = \Omega - \sum_{i=1}^{n}\frac{B_i(t)}{z-a_i(t)}\,d(z-a_i(t)) - \sum_{k=1}^{s}\biggl(\sum_{i=1}^{n}\sum_{j=1}^{r_i}\frac{\gamma_{ijk}(t)}{(z-a_i(t))^j}\biggr)\,dt_k$$

is holomorphic on $\overline{\mathbb{C}} \times D(t^*)$, and so $\widetilde{\Omega} = \sum_{k=1}^{s}\gamma_k(t)\,dt_k$ (where the γ_k are matrix functions holomorphic on $D(t^*)$) and one obtains the statement of the theorem modulo Remark 4.

Remark 5. In the proof of Theorem 8 one encounters the matrices $C_iS_i^{-1}(t)$ connecting the analytic isomonodromic fundamental matrix $\widetilde{Y}(z,t)$ of the family (13) with the Levelt fundamental matrices at the points $a_i(t)$. The isomonodromic family (13) is a Schlesinger one if and only if these connection matrices are independent of t (see [2], Exercise 15.3). This statement is formulated in a more rigorous setting in [22], where such deformations (that is, when the connection matrices are independent of t) are said to be *isoprincipal*.

A particular case of Theorem 8 concerns resonant isomonodromic families (14) with the deformation space $T = D(a^0)$. It follows from this theorem that a matrix differential 1-form Ω determining such a deformation on the space $\overline{\mathbb{C}} \times D(a^0)$ has the form

$$\Omega = \sum_{i=1}^{n} \frac{B_i(a)}{z-a_i}\, d(z-a_i) + \sum_{k=1}^{n}\biggl(\sum_{i=1}^{n}\sum_{j=1}^{r_i} \frac{\gamma_{ijk}(a)}{(z-a_i)^j}\biggr) da_k, \tag{19}$$

where the γ_{ijk} are holomorphic matrix functions on $D(a^0)$ and r_i is the maximal i-resonance of the family (14).

Theorem 8 also helps one to understand the result of Poberezhnyi [23], who studied the local geometry of a deformation space T in a general resonant case when there are additional deformation parameters besides singular points. Assume that locally T looks like a direct product $D(a^0) \times \mathscr{W}$, where $\mathscr{W} \subset \mathscr{M}^*_{a^0,\chi,\exp}$ is a continuous component of the moduli space $\mathscr{M}^*_{a^0,\chi,\exp}$ of Fuchsian systems having the singularities $a_1^0,\dots,a_n^0$, monodromy χ, and fixed exponents. Generically one has $d = \dim \mathscr{W} > 0$. Then the following statement holds.

Any meromorphic matrix differential 1*-form* Ω *on the space* $\overline{\mathbb{C}} \times D(a^0) \times \mathscr{W}$ *giving the local isomonodromic deformation* (13) *has the form*

$$\begin{aligned}\Omega = \sum_{i=1}^{n} \frac{B_i(a,w)}{z-a_i}\, d(z-a_i) &+ \sum_{k=1}^{n}\biggl(\sum_{i=1}^{n}\sum_{j=1}^{r_i} \frac{\gamma_{ijk}(a,w)}{(z-a_i)^j}\biggr) da_k \\ &+ \sum_{l=1}^{d}\biggl(\sum_{i=1}^{n}\sum_{j=1}^{r_i} \frac{\beta_{ijl}(a,w)}{(z-a_i)^j}\biggr) dw_l, \end{aligned} \tag{20}$$

where the γ_{ijk} *and* β_{ijl} *are holomorphic matrix functions on* $D(a^0)\times\mathscr{W}$, $w_1,\dots,w_d$ *are local coordinates on* $\mathscr{W}$, *and* r_i *is the maximal* i*-resonance of the family* (13).

4. Integrable deformations and the Painlevé property

In this section we investigate singularities of deformation equations. We describe the set of moveable singularities of the deformation (the theta-divisor) and the local behavior of the coefficients of the family in a neighbourhood of this set. We generalize the classical results (established for Schlesinger isomonodromic deformations of Fuchsian systems) on meromorphic continuation of solutions to the Schlesinger equation and the global structure of the theta-divisor, to the case of the integrable deformations considered in Theorem 3. Following [24], [25], our approach makes wide use of the theory of Fredholm operators briefly presented in the next subsection.

4.1. Fredholm operators. The Fredholm operators reviewed here play a role in the analysis of families of vector bundles in the next subsection. Let H be a separable complex Hilbert space. Denote by $B(H)$ the space of bounded linear operators from H to H.

Definition 5. An operator $A \in B(H)$ is called a *Fredholm operator* if it satisfies the following properties:

- the kernel $\ker A$ is a finite-dimensional subspace of H;
- the image $\operatorname{Im} A$ is a closed subspace in H of finite codimension

$$\dim \operatorname{coker} A = \dim(H/\operatorname{Im} A).$$

The set of all Fredholm operators is denoted by $\Phi(H)$. For $A \in \Phi(H)$ one defines the index $\operatorname{ind} A$ by

$$\operatorname{ind} A = \dim \operatorname{coker} A - \dim \ker A.$$

The set of all invertible operators is an open part of $B(H)$ and clearly lies in the set $\Phi(H)$, but also $\Phi(H)$ is an open part of $B(H)$ (see, for instance, [26]). The set $\Phi(H)$ possesses a number of properties which we list in a theorem. Their proofs can be found, for instance, in the book [27] (see Theorem 1.4 in Chap. 4 and Theorems 2.3 and 3.1 in Chap. 5).

Theorem 9. *Let $\mathscr{K}(H)$ be the ideal of compact operators in $B(H)$. Then*

1) *if $A \in \Phi(H)$ and $K \in \mathscr{K}(H)$, then their sum $A+K$ also belongs to $\Phi(H)$ and the index does not change,*

$$\operatorname{ind} A = \operatorname{ind}(A+K);$$

2) *for any $A \in \Phi(H)$ such that $\operatorname{ind} A = 0$ there exists a finite-dimensional operator $K \in B(H)$ such that $A+K$ is an invertible operator*;

3) *for any two operators $A, B \in \Phi(H)$ the product AB belongs to $\Phi(H)$ and its index is given by the formula*

$$\operatorname{ind} AB = \operatorname{ind} A + \operatorname{ind} B.$$

Here we treat a natural context related to the constructions in this paper, where one meets Fredholm operators. Let S^1 be the unit circle in the complex plane and $H = L^2(S^1, \mathbb{C}^p)$ the Hilbert space of square-integrable $\mathbb{C}^p$-valued functions. Elements of H are Fourier series

$$h = \sum_{n\in\mathbb{Z}} b_n e^{in\varphi}, \qquad b_n \in \mathbb{C}^p.$$

The space H decomposes as the direct sum $H_+ \oplus H_-$, where

$$H_+ = \left\{ \sum_{n\geqslant 0} b_n z^n \right\}$$

is the closure with respect to the L^2-norm of the subspace of continuous functions on S^1 that extend holomorphically to the unit disk, and

$$H_- = \left\{ \sum_{n<0} b_n z^n \right\}$$

is the closure with respect to the same norm of the subspace of continuous functions on S^1 that extend holomorphically to the exterior of the unit circle and are zero at infinity.

Denote by $\Gamma(U)$ the group of all holomorphic maps γ from a connected neighbourhood U of S^1 to $\mathrm{GL}(p,\mathbb{C})$ and let Γ be its inductive limit with respect to U. This group is the collection of transition functions for holomorphic vector bundles of rank p over $\overline{\mathbb{C}}$. The elements $\gamma \in \Gamma$ act on H by left multiplication as linear operators M_γ.

For $\gamma \in \Gamma$ let us decompose M_γ with respect to $H = H_+ \oplus H_-$ as

$$M_\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

We will show that A is a Fredholm operator.

According to the Birkhoff–Grothendieck theorem [28], [29], every holomorphic vector bundle E of rank p over $\overline{\mathbb{C}}$ is a direct sum of line bundles,

$$E \cong \mathscr{O}(m_1) \oplus \cdots \oplus \mathscr{O}(m_p), \qquad m_i \in \mathbb{Z},$$

which means that a matrix function $\gamma(z)$ decomposes as

$$\gamma(z) = \gamma_-(z) \begin{pmatrix} z^{m_1} & & 0 \\ & \ddots & \\ 0 & & z^{m_p} \end{pmatrix} \gamma_+(z) = \gamma_-(z)\gamma_0(z)\gamma_+(z),$$

where γ_+ is an element of Γ that extends holomorphically to zero, and γ_- is an element of Γ that extends holomorphically to infinity and satisfies $\gamma_-(\infty) = \mathrm{Id}$. Therefore $M_\gamma = M_{\gamma_-} M_{\gamma_0} M_{\gamma_+}$, where the restrictions $M_{\gamma_-}|_{H_-}$ and $M_{\gamma_+}|_{H_+}$ are isomorphisms of the subspaces H_- and H_+, respectively. Hence the operators M_{γ_-} and M_{γ_+} can be decomposed with respect to $H = H_+ \oplus H_-$ as

$$M_{\gamma_-} = \begin{pmatrix} A_- & 0 \\ C_- & D_- \end{pmatrix}, \qquad M_{\gamma_+} = \begin{pmatrix} A_+ & B_+ \\ 0 & D_+ \end{pmatrix},$$

where $A_-, A_+ \colon H_+ \to H_+$ are isomorphisms,[5] and thus Fredholm operators of zero index. Decomposing M_{γ_0} with respect to $H = H_+ \oplus H_-$ as

$$M_{\gamma_0} = \begin{pmatrix} A_0 & B_0 \\ C_0 & D_0 \end{pmatrix},$$

we obtain the relation $A = A_- A_0 A_+$. One can check that A_0 is a Fredholm operator as follows.

Let $e_1, \dots, e_p$ be the standard basis of $\mathbb{C}^p$. The explicit form of the matrix function $\gamma_0(z)$ implies that $M_{\gamma_0}(e_i z^n) = e_i z^{n+m_i}$, therefore

$$\ker A_0 = \langle e_i z^n \mid m_i < 0,\ 0 \leqslant n < -m_i \rangle, \qquad \operatorname{Im} A_0 = \langle e_i z^n \mid n \geqslant m_i \geqslant 0 \rangle.$$

Hence

$$\dim \ker A_0 = -\sum_{\{i : m_i < 0\}} m_i \quad \text{and} \quad \dim \operatorname{coker} A_0 = \sum_{\{i : m_i > 0\}} m_i.$$

This means that A_0 is a Fredholm operator of index $\sum_{i=1}^{p} m_i$. According to Theorem 9 the same is true for the operator $A = A_- A_0 A_+$. Note that the index $\operatorname{ind} A = \sum_{i=1}^{p} m_i$ of the operator A equals the degree of the holomorphic vector bundle determined by the transition function $\gamma(z)$.

[5] One can easily check that for any $h_+ \in H_+$ the equation $A_-(x_+) = h_+$ has a unique solution $x_+ \in H_+$. It is the projection of the element $\gamma_-^{-1} h_+ \in H$ on the subspace H_+.

4.2. Non-triviality of the deformed bundle. Here we return to the integrable deformations (E,∇) of the pair (E^0,∇^0) that were constructed in § 2. Recall that the original vector bundle E^0 of rank p over $\overline{\mathbb{C}}$ was holomorphically trivial. This implies that the restriction $E|_{\overline{\mathbb{C}}\times\{t^0\}}$ is also trivial. One might wonder whether $E_t = E|_{\overline{\mathbb{C}}\times\{t\}}$ is trivial throughout the deformation space T. Consider therefore the subset Θ of T where this property has been lost, that is,

$$\Theta = \{t \in T \mid E_t \text{ is non-trivial}\}.$$

Proposition 1 (see also [24] and [30]). *The set Θ is either empty or $\Theta \subset T$ is an analytic subset of codimension one.*

Proof. Choose real numbers ρ_1 and ρ_2 such that $0 < \rho_1 < 1 < \rho_2$ and consider the sets

$$D_1 = \bigl\{z \in \overline{\mathbb{C}} \bigm| |z| > \rho_1\bigr\}, \qquad D_2 = \bigl\{z \in \overline{\mathbb{C}} \bigm| |z| < \rho_2\bigr\}.$$

Choose also an open neighbourhood U of $t \in T$ that is a contractible Stein manifold. Then the vector bundles $E|_{D_1\times U}$ and $E|_{D_2\times U}$ are holomorphically trivial. (If $T = \widetilde{Z}$ as in Theorem 3, then one could take $U = \widetilde{Z}$.) Let $\{f_1,\dots,f_p\}$ and $\{g_1,\dots,g_p\}$ be bases of holomorphic trivializing sections of $E|_{D_1\times U}$ and $E|_{D_2\times U}$, respectively. Then there is a holomorphic transition function

$$S\colon (D_1 \cap D_2)\times U \to \mathrm{GL}(p,\mathbb{C}),$$

which expresses $\{f_i\}$ in terms of $\{g_i\}$, that is, $(g_1,\dots,g_p) = (f_1,\dots,f_p)S$. Take any $t \in U$. Then by the Birkhoff–Grothendieck theorem one deduces that there are holomorphic maps

$$S_+(\,\cdot\,,t)\colon D_2 \to \mathrm{GL}(p,\mathbb{C}), \qquad S_-(\,\cdot\,,t)\colon D_1 \to \mathrm{GL}(p,\mathbb{C})$$

such that the above U-dependent family of elements in the group Γ decomposes as

$$S(z,t) = S_-^{-1}(z,t)\,\mathrm{diag}(z^{m_1},\dots,z^{m_p})S_+(z,t), \tag{21}$$

where $m_1 \geqslant \cdots \geqslant m_p$ are integers and one may assume that $S_-(\infty,t) = \mathrm{Id}$. The bundle E_t is holomorphically trivial if and only if all the m_i equal zero, that is, for all $z \in D_1 \cap D_2$ one has

$$S(z,t) = S_-^{-1}(z,t)S_+(z,t). \tag{22}$$

One associates with each of the members in (21) a bounded linear operator on the space $H = L^2(S^1,\mathbb{C}^p)$. As we saw in § 4.1, multiplication by $S(\cdot\,,t)$ determines for a fixed $t \in U$ a bounded linear operator $\mathbb{S}(t)$ on H whose decomposition with respect to $H = H_+ \oplus H_-$ is given by

$$\mathbb{S}(t) = \begin{pmatrix} A(t) & B(t) \\ C(t) & D(t) \end{pmatrix},$$

where $A(t)\colon H_+ \to H_+$ is a Fredholm operator of index $\sum_{i=1}^{p} m_i$. One sees that the map $t \mapsto A(t)$ is holomorphic since the matrix function $S(z,t)$ depends holomorphically on t.

Now all the factors on the right-hand side of the relation (21) determine bounded linear operators on H that are decomposed with respect to $H = H_+ \oplus H_-$ as

$$\begin{pmatrix} A_- & 0 \\ C_- & D_- \end{pmatrix}, \quad \begin{pmatrix} A_0 & B_0 \\ C_0 & D_0 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} A_+ & B_+ \\ 0 & D_+ \end{pmatrix}.$$

respectively, and thus

$$A(t) = A_- A_0 A_+.$$

Since A_- and A_+ are invertible, the invertibility of $A(t)$ is equivalent to that of A_0, and from § 4.1 one sees directly that this holds if and only if all the m_i equal zero, that is, the bundle E_t is holomorphically trivial and the decomposition (22) holds.

Since the set U is connected and the operator $A(t^0)$ is invertible (the bundle E_{t^0} is holomorphically trivial), $\operatorname{ind} A(t) = \deg E_t = 0$ for all $t \in U$ (in other words, the degree of the bundles E_t, $t \in U$, does not change throughout an integrable deformation[6]). Because $A(t)$ is a Fredholm operator, the last fact implies (see Theorem 9) that for any $t \in U$ there is a finite-dimensional operator $K(t) \in B(H_+)$ such that $A(t) + K(t)$ is invertible. Locally one can even choose $K(t) = K$ independent of t. Indeed, if $A(t^1) + K(t^1)$ is invertible, then $t \mapsto A(t) + K(t^1)$ is a continuous map from U to $\Phi(H_+)$. Since the set of invertible operators is an open part of $\Phi(H_+)$, the operators $A(t) + K(t^1)$ are invertible for all t close to t^1. Thus we have

$$A(t)(A(t) + K)^{-1} = \mathrm{Id} - K(A(t) + K)^{-1},$$

that is, for any $t_\alpha \in U$ there exist an open neighbourhood U_α and a holomorphic map

$$Q_\alpha \colon U_\alpha \to \mathrm{Aut}(H_+)$$

such that $A(t)Q_\alpha^{-1}(t) - \mathrm{Id}$ is a finite-dimensional operator. According to [29] any bounded linear operator of the form Id + {finite-dimensional} possesses a well defined determinant, and such an operator is invertible if and only if its determinant is not equal to zero. Hence locally in a neighbourhood U_α we have

$$U_\alpha \cap \Theta = \bigl\{t \in U_\alpha \mid \tau_\alpha(t) := \det\bigl(A(t)Q_\alpha^{-1}(t)\bigr) = 0\bigr\},$$

which proves the proposition.

On every non-empty intersection $U_\alpha \cap U_\beta$ one has

$$\bigl(A(t)Q_\alpha^{-1}(t)\bigr)\bigl(Q_\alpha(t)Q_\beta^{-1}(t)\bigr) = \bigl(A(t)Q_\beta^{-1}(t)\bigr).$$

---

[6] Indeed, from the relation (21) it follows that

$$\frac{d_z \det S(z,t)}{\det S(z,t)} = \frac{\sum_{i=1}^p m_i}{z}\,dz + \frac{d_z \det S_+(z,t)}{\det S_+(z,t)} - \frac{d_z \det S_-(z,t)}{\det S_-(z,t)}\,,$$

therefore

$$\deg E_t = \sum_{i=1}^p m_i = \int_{|z|=1} \frac{d_z \det S(z,t)}{\det S(z,t)}$$

(the differential 1-forms $\dfrac{d_z \det S_+(z,t)}{\det S_+(z,t)}$ and $\dfrac{d_z \det S_-(z,t)}{\det S_-(z,t)}$ are holomorphic on D_2 and D_1, respectively, so their integrals over the unit circle equal zero). Thus the integer-valued function $\deg E_t$ depends holomorphically on the parameter t, hence it is constant.

Since the operators $A(t)Q_\alpha^{-1}(t)$ and $A(t)Q_\beta^{-1}(t)$ are of the form $\mathrm{Id} + \{\text{finite-dimensional}\}$, so is the invertible operator $Q_\alpha(t)Q_\beta^{-1}(t)$. Hence it possesses a non-vanishing determinant, and the functions τ_α and τ_β differ by a factor equal to the non-vanishing holomorphic function

$$\det\bigl(Q_\alpha(t)Q_\beta^{-1}(t)\bigr).$$

The set of such functions defines a line bundle over U which is holomorphically trivial, since U is a contractible Stein manifold. Therefore, the local functions τ_α describing Θ can be glued together to form a global holomorphic function $\tau\colon U \to \mathbb{C}$ for which $U \cap \Theta$ is the zero set. Thus, for the deformation spaces $T = \widetilde{Z}$ or $T = D(a^0)$ there exists a global holomorphic function $\tau\colon T \to \mathbb{C}$ such that the set Θ is its zero set. Then the analogous functions for the other integrable deformations constructed in § 2 (with the deformation space $T = D(a^0) \times \mathscr{W}$, where $\mathscr{W}$ is a connected complex manifold, or with an arbitrary *simply connected* deformation space T) have the form $\tau \circ f$, where f is the corresponding map $D(a^0) \times \mathscr{W} \to D(a^0)$ or $T \longrightarrow \widetilde{Z}$. Thus one obtains the following statement.

Corollary 2. *For every integrable deformation* (E, ∇) *constructed in* § 2 *and for the corresponding deformation space* T *there is a global holomorphic function* τ : $T \to \mathbb{C}$ *such that*

$$\Theta = \{t \in T \mid \tau(t) = 0\}.$$

The function τ is usually referred to as the (*global*) τ*-function* of the integrable deformation (E, ∇).

Let us investigate the character of the t-dependence of the matrices $S_+(z,t)$ and $S_-(z,t)$ expressing the decomposition (22) in terms of the operator $A(t)$. If $t \notin \Theta$, then one defines the matrix $\Sigma_+(z,t)$ by

$$\Sigma_+(z,t) = A^{-1}(t)\,\mathrm{Id},$$

where the columns of the $(p \times p)$ identity matrix are seen as elements of H_+ and $A^{-1}(t)$ acts in the natural way on them. Thus, the matrix function Σ_+ is holomorphic (in z) in a neighbourhood of the disk $\bigl\{z \in \mathbb{C} \bigm| |z| \leqslant 1\bigr\}$. From the definition of $\Sigma_+(z,t)$ we see that

$$A(t)\Sigma_+(z,t) = \mathrm{Id},$$

where $A(t)$ acts on the columns of the matrix $\Sigma_+(z,t)$ seen as elements of H_+. Therefore

$$S(z,t)\Sigma_+(z,t) = \mathrm{Id} + \Sigma_-(z,t),$$

where the matrix function Σ_- is holomorphic (in z) in a neighbourhood of $\bigl\{z \in \overline{\mathbb{C}} \bigm| |z| \geqslant 1\bigr\}$, and $\Sigma_-(\infty,t) = 0$ because its columns are elements of H_-. Since constants are the only holomorphic functions on $\overline{\mathbb{C}}$, we get from $S(z,t) = S_-^{-1}(z,t)S_+(z,t)$ that

$$S_+(z,t)\Sigma_+(z,t) = S_-(z,t)\{\mathrm{Id} + \Sigma_-(z,t)\} = S_-(\infty,t)\{\mathrm{Id} + \Sigma_-(\infty,t)\} = \mathrm{Id}.$$

This implies the invertibility of the matrices $\Sigma_+(z,t)$ and $\mathrm{Id} + \Sigma_-(z,t)$. Since they depend holomorphically on $t \in U \setminus \Theta$, so do $S_+(z,t)$ and $S_-(z,t)$. Let us show that the latter matrices are meromorphic along $D_2 \times \Theta$ and $D_1 \times \Theta$, respectively.

From [29] we know that the inverse of an operator of the form $\mathrm{Id} + F$, where F is a finite-dimensional operator in $B(H_+)$, is given by the formula

$$(\mathrm{Id} + F)^{-1} = \frac{1}{\det(\mathrm{Id} + F)}\, R(F).$$

Here the operator $R(F) \in B(H_+)$ is defined by the series

$$R(F) = \sum_{n=0}^{\infty} R_n(F), \tag{23}$$

where $R_0(F) = \mathrm{Id}$, and for each $n \geqslant 1$ the operator $R_n(F) \in B(H_+)$ is defined as

$$R_n(F) = R_n(F, \dots, F),$$

with R_n a multilinear map from $B_{\mathrm{f}}(H_+)^n$, the n-fold Cartesian product of the space $B_{\mathrm{f}}(H_+) \subset B(H_+)$ of finite-dimensional operators, to $B(H_+)$. The multilinear map R_n is defined by the pairing

$$\mathrm{Tr}\big(F_0 \circ R_n(F_1, \dots, F_n)\big) = (n+1)\,\mathrm{Tr}(F_1 \wedge \cdots \wedge F_n \wedge F_0) \quad \forall\, F_0 \in B_{\mathrm{f}}(H_+),$$

where Tr denotes the trace of the operators involved. Note that the sum in (23) is finite for every $F \in B_f(H_+)$ since the n-fold exterior power of such an F is zero as soon as $n > \dim \mathrm{Im}\, F$. Moreover, if F depends holomorphically on the parameter t, then the same holds for $R(F)$. We apply this to the present context as follows. As above, let

$$Q_\alpha \colon U_\alpha \to \mathrm{Aut}(H_+)$$

be a holomorphic map such that $F_\alpha(t) := A(t)Q_\alpha^{-1}(t) - \mathrm{Id}$ is a finite-dimensional operator. Then the following holds:

$$\begin{aligned}\Sigma_+(z,t) &= A^{-1}(t)\,\mathrm{Id} = Q_\alpha^{-1}(t)\big(A(t)Q_\alpha^{-1}(t)\big)^{-1}\,\mathrm{Id} = Q_\alpha^{-1}(t)\big(\mathrm{Id} + F_\alpha(t)\big)^{-1}\,\mathrm{Id}\\ &= \frac{1}{\det(\mathrm{Id} + F_\alpha(t))}\, Q_\alpha^{-1}(t) R\big(F_\alpha(t)\big)\,\mathrm{Id}.\end{aligned}$$

This implies that the map $t \mapsto \tau_\alpha(t)\Sigma_+(z,t)$ is holomorphic on U_α and thus the matrix $\Sigma_+(z,t)$ is meromorphic (in t) along Θ. The same holds for its inverse $S_+(z,t)$. The matrix $\mathrm{Id}+\Sigma_-(z,t)$ is by definition equal to $S(z,t)\Sigma_+(z,t)$. Therefore the inverse $S_-(z,t)$ of $\mathrm{Id} + \Sigma_-(z,t)$ is also meromorphic (in t) along Θ. Thus one comes to the following statement.

Proposition 2. (see also [24] and [25]). *Let T be a connected complex manifold, let $t^0 \in T$, and let*

$$S \colon (D_1 \cap D_2) \times T \to \mathrm{GL}(p, \mathbb{C})$$

be a holomorphic map such that

$$S(z, t^0) = F_-^{-1}(z) F_+(z),$$

where F_- and F_+ are holomorphically invertible matrix functions on D_1 and D_2, respectively. Then there is a subset $\Theta \subset T$ which is either empty or an analytic subset of T of codimension one, and there exist holomorphic maps

$$S_+ \colon D_2 \times (T \setminus \Theta) \to \mathrm{GL}(p, \mathbb{C}), \qquad S_- \colon D_1 \times (T \setminus \Theta) \to \mathrm{GL}(p, \mathbb{C})$$

such that:

1) $S(z,t)=S_-^{-1}(z,t)S_+(z,t)$ *and* $S_-(\infty,t)=\mathrm{Id}$;
2) *the maps* S_+ *and* S_- *are meromorphic along* $D_2\times\Theta$ *and* $D_1\times\Theta$, *respectively.*

Let us return to the proof of Proposition 1. Consider an open contractible Stein manifold $U\subset T$ and the bases $\{f_i\}$, $\{g_i\}$. Then the holomorphic sections on $\overline{\mathbb{C}}\times(U\setminus\Theta)$

$$(h_1,\dots,h_p)=(f_1,\dots,f_p)S_-^{-1}=(g_1,\dots,g_p)S_+^{-1}$$

form a basis of global trivializing sections (linearly independent at each point) of a bundle that is holomorphically equivalent to $E|_{\overline{\mathbb{C}}\times(U\setminus\Theta)}$. Thus the bundle $E|_{\overline{\mathbb{C}}\times(T\setminus\Theta)}$ is holomorphically trivial in the case of the integrable deformation (E,∇) with deformation space $T=\widetilde{Z}$ or $T=D(a^0)$. Since the other integrable deformations in § 2 were obtained as pull-backs of the latter, the analogous statement holds for them.

Corollary 3. *For every integrable deformation* (E,∇) *constructed in* § 2 *the vector bundle* E *is holomorphically trivial over* $\overline{\mathbb{C}}\times(T\setminus\Theta)$.

4.3. Meromorphic continuation. Consider an open contractible Stein manifold $U\subset T$ and the integrable connection $\nabla(\infty)$ in the holomorphically trivial vector bundle $E|_{\{\infty\}\times U}$ in Remark 3. Let $\{s_1,\dots,s_p\}$ be a basis of horizontal sections of this connection. Now we choose the basis $\{f_1,\dots,f_p\}$ of holomorphic sections of the restriction $E|_{D_1\times U}$ such that $f_j|_{\{\infty\}\times U}=s_j$. Let $\{g_1,\dots,g_p\}$ be a basis of holomorphic sections of the restriction $E|_{D_2\times U}$ and let $S(z,t)$, as in the previous subsection, be the transition matrix relating $\{g_j\}$ and $\{f_j\}$.

According to Proposition 2 the transition matrix $S(z,t)$ decomposes over $(D_1\cap D_2)\times(U\setminus\Theta)$ as $S=S_-^{-1}S_+$. Then the trivializing basis $\{h_j\}$ in Corollary 3 consists of trivial extensions of $\{s_j\}$. Let Ω be the connection form of ∇ with respect to the basis $\{h_j\}$. From the formula (12) we see that in a neighbourhood of the hypersurface

$$Y_i=\{(z,t)\in\overline{\mathbb{C}}\times U\mid z-a_i(t)=0\}$$

the matrix differential 1-form Ω looks like

$$\Omega=\sum_{l\geqslant1}\frac{B_{il}(t)}{(z-a_i(t))^l}\,d(z-a_i(t))+w_i,$$

where w_i is a holomorphic matrix differential 1-form near Y_i and the matrices B_{il} are holomorphic on $U\setminus\Theta$. We now introduce the matrix differential 1-form

$$\Omega_f=\sum_{i=1}^n\sum_{l\geqslant1}\frac{B_{il}(t)}{(z-a_i(t))^l}\,d(z-a_i(t)),$$

which is meromorphic on $\overline{\mathbb{C}}\times(U\setminus\Theta)$. Then $\Omega_\infty=\Omega-\Omega_f$ is a matrix differential 1-form which is holomorphic on $\overline{\mathbb{C}}\times(U\setminus\Theta)$. Hence it depends just on t. From the definition of $\nabla(\infty)$ one sees directly that Ω_∞ is the connection form of $\nabla(\infty)$ with respect to the basis $\{s_j\}$. Since the sections s_j are horizontal, we obtain $\Omega_\infty=0$, that is, $\Omega=\Omega_f$.

Since the pair (E_{t^0}, ∇_{t^0}) is isomorphic to the pair (E^0, ∇^0), we may conclude that there is a matrix $C \in \mathrm{GL}(p, \mathbb{C})$ such that

$$B_{il}(t^0) = C B^0_{il} C^{-1}$$

for all i and l. Hence (by applying a linear transformation to the basis $\{h_j\}$) we may assume that the matrices $B_{il}(t)$ satisfy the initial conditions $B_{il}(t^0) = B^0_{il}$ for all i and l. To see that these matrices are meromorphic on U let us consider the connection forms Ω_1 and Ω_2 of ∇ in the bases $\{f_j\}$ and $\{g_j\}$, respectively. Then Ω_1 and Ω_2 are holomorphic on $(D_1 \times U) \setminus Y$ and $(D_2 \times U) \setminus Y$ and meromorphic along $Y = Y_1 \cup \dots \cup Y_n$. They relate to Ω respectively according to

$$\Omega = S_- \Omega_1 S_-^{-1} + (dS_-) S_-^{-1} \quad \text{on } D_1 \times (U \setminus \Theta),$$
$$\Omega = S_+ \Omega_2 S_+^{-1} + (dS_+) S_+^{-1} \quad \text{on } D_2 \times (U \setminus \Theta).$$

Taking into account Proposition 2, we may summarize the results of this subsection as follows.

Consider a meromorphic linear system (1) with singularities $a^0_1, \dots, a^0_n$ in the complex plane, so that all the $B^0_{\infty l}$ are equal to zero and the relation (2) holds. Regarding this system as an equation for horizontal sections of the meromorphic connection ∇^0 in the holomorphically trivial vector bundle E^0, one can construct the integrable deformation (E, ∇) of the pair (E^0, ∇^0) according to Theorem 3 (and Remark 2).

Theorem 10. *Every integrable deformation (E, ∇) constructed in § 2 possesses the following properties*:

1) *there is a neighbourhood V_0 of any $t^0 \in T$ such that the restriction $E|_{\overline{\mathbb{C}} \times V_0}$ is holomorphically trivial and one can find a basis of its sections in which the connection form Ω of ∇ looks like*

$$\Omega = \sum_{i=1}^{n} \sum_{l \geqslant 1} \frac{B_{il}(t)}{(z - a_i(t))^l}\, d(z - a_i(t)),$$

and $B_{il}(t^0) = B^0_{il}$ for all i, l;

2) *the solution $\{B_{il}(t)\}$ to the integrability equation extends holomorphically to $T \setminus \Theta$ and is meromorphic along Θ.*

For the integrable deformation constructed in Theorem 3 the above theorem implies that the solution $\{B_{il}\}$ extends meromorphically to $\widetilde{Z}$. In the case that the connection ∇^0 is logarithmic one obtains a famous theorem of Malgrange.

Theorem 11 (Malgrange [24], [19]). *Let $\{B_1(a), \dots, B_n(a)\}$ be a local solution to the Schlesinger equation (15) with arbitrary initial conditions $B_i(a^0) = B^0_i$. Then the matrices $B_i(a)$ can be extended to the universal cover $\widetilde{Z}$ of the space $\mathbb{C}^n \setminus \bigcup_{i \neq j} \{a_i = a_j\}$ as meromorphic functions.*

The statement of this theorem is commonly referred to as the *Painlevé property* of the Schlesinger equation. Usually one says that a differential equation (or a system of differential equations) possesses the Painlevé property if its *moveable* singularities[7] can only be poles, or equivalently, all solutions extend meromorphically to the universal cover of the space of the independent variable punctured at the fixed singularities.

The polar locus $\Theta \subset \widetilde{Z}$ of the extended matrix functions $B_i(a)$ is called the *Malgrange Θ-divisor*[8] (Θ depends on the initial conditions $B_i(a^0) = B_i^0$). According to Corollary 2 there exists a function τ holomorphic on the whole space $\widetilde{Z}$ and whose zero set coincides with Θ. This is the global τ-function of the Schlesinger equation. By Miwa's theorem [31], [11] (see also [2], Lecture 17) the following holds:

$$d\log\tau(t) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1,\,j\neq i}^{n}\frac{\operatorname{Tr}(B_i(t)B_j(t))}{a_i(t)-a_j(t)}\,d\big(a_i(t)-a_j(t)\big).$$

If we consider the Fuchsian system (3) as an equation for horizontal sections of the logarithmic connection ∇^0 (with singularities $a_1^0,\ldots,a_n^0$) in the holomorphically trivial vector bundle E^0 of rank p over $\overline{\mathbb{C}}$, then the integrable deformation (E,∇) of (E^0,∇^0) constructed in Theorem 3 (with the deformation space $T=\widetilde{Z}$) can be naturally called the *Schlesinger integrable deformation*, since the restriction $(E,\nabla)|_{\overline{\mathbb{C}}\times D(a^0)}$ determines the Schlesinger isomonodromic deformation of this system. But the construction in Theorem 3 can also be used to produce an integrable deformation (E,∇) of an initial pair (E^0,∇^0) with holomorphically non-trivial bundle E^0. If $\deg E^0 = 0$, the question is whether the bundles $E_t = E|_{\overline{\mathbb{C}}\times\{t\}}$ are holomorphically trivial for almost all $t\in T$. This question should be asked in the case when the monodromy representation of the connection ∇^0 is irreducible, since in the opposite case there are examples of representations that cannot be realized by a Fuchsian system for any positions of poles (see [2], Example 11.1, [32]).

Let ∇^0 be a logarithmic connection in a (non-trivial) holomorphic vector bundle E^0 of rank p and *zero* degree over $\overline{\mathbb{C}}$ such that the monodromy representation of ∇^0 is *irreducible*. Consider the Schlesinger integrable deformation (E,∇) of the pair (E^0,∇^0) provided by Theorem 3.

Question. Is the set

$$\Theta = \{t\in T \mid E_t = E|_{\overline{\mathbb{C}}\times\{t\}} \textit{ is non-trivial}\}$$

an analytic subset of codimension one (if it is non-empty)?

The answer is known to be positive in the particular two-dimensional case ($p=2$); see [33] or [34].

Now we recall the definition of the divisor of a meromorphic function $f: T\to\mathbb{C}$. Denote by $A = N\cup P$ the union of its zero set N and the polar locus P. Any regular point t^0 of the set A can belong to only one irreducible component of N or P. Thus,

[7] A differential equation has moveable singularities if the singular points of its solutions fill some domain in the space of the independent variable. Points of this domain are called moveable singular points of the equation. In other words, moveable singularities are those whose position depends on the initial conditions of an equation. For instance, the singular points $z=c$ of the solutions $y=1/(c-z)$ to the equation $dy/dz=y^2$ fill the whole complex plane $\mathbb{C}$.

[8] Here the term 'divisor' is not precise enough; a traditional definition of a divisor is given later on.

one can define the *order* of this component as the degree (taken with + if $t^0 \in N$, and with − if $t^0 \in P$) of the corresponding factor in the decomposition of the function φ or ψ into irreducible factors (here φ and ψ locally define the sets N and P, respectively). Then the *divisor* of the meromorphic function f is the pair (A, κ), where $\kappa = \kappa(t)$ is an integer-valued function on the set A^0 of regular points of A (which takes a constant value on each irreducible component of A^0, this value being equal to the order of a component). The pair (P, κ) is called the *polar divisor* of the meromorphic function f. By $(f)_\infty$ we will mean the restriction of κ to the regular points of P.

Notation. Let P be the polar locus of the function f and $t^0 \in P$ an arbitrary point of this set. We denote by $\Sigma_{t^0}(f)$ the sum of the orders of all the irreducible components of $P \cap D(t^0)$ (that is, the irreducible components of P that go through t^0).

In the particular case presented below, which is important for applications, one can describe the behaviour of a general solution to the Schlesinger equation near the Θ-divisor.

Theorem 12 (Bolibrukh [35], [2], Theorem 16.1). *Consider the two-dimensional* ($p = 2$) *Schlesinger isomonodromic family* (14) *whose monodromy is irreducible. Let a^* be an arbitrary point of its Θ-divisor and*

$$E_{a^*} = E|_{\overline{\mathbb{C}}\times\{a^*\}} \cong \mathscr{O}(k) \oplus \mathscr{O}(-k).$$

Then $\Sigma_{a^*}(B_i) \geqslant -2k,\ i = 1, \dots, n.$

Remark 6. In the paper [35] and the textbook [2] the above theorem is formulated and proved in the particular case $k = 1$, but using the technique of the paper [36] one can consider also the general case (see [34]).

It is known (see [2], Theorem 11.1) that $2k \leqslant n - 2$. Thus, the estimate of Theorem 12 can be written in the form $\Sigma_{a^*}(B_i) \geqslant 2-n$; furthermore $\Sigma_{a^*}(B_i) \geqslant 3-n$ in the case of odd n.

There also exists a kind of description (though not as direct as in Theorem 12) of the local behaviour of solutions to the Schlesinger equation near the Malgrange Θ-divisor in the case of arbitrary dimension p and arbitrary monodromy of the family; see [37].

We should note that the Painlevé property can be violated in the general case of a non-Schlesinger isomonodromic deformation of a Fuchsian system. Here we return to Example 1 to illustrate this phenomenon (see also [22] and references therein, where the illustration is based on the theory of holomorphic families of Fuchsian systems whose fundamental solutions are generic rational matrices with respect to z).

Example 5. On the basis of Example 1 we consider an isomonodromic family of Fuchsian systems

$$\frac{dy}{dz} = \left(\left(\begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix}\frac{1}{z-a_1} + \begin{pmatrix}0 & -1\\ 0 & -1\end{pmatrix}\right)\frac{1}{z-a_2} + \begin{pmatrix}-1 & w\\ 0 & 0\end{pmatrix}\frac{1}{z-a_3} + \begin{pmatrix}0 & -w\\ 0 & 1\end{pmatrix}\frac{1}{z-a_4}\right)y, \tag{24}$$

depending on the parameters

$$a = (a_1, a_2, a_3, a_4) \in \mathbb{C}^4 \setminus \bigcup_{i \neq j} \{a_i = a_j\}, \qquad w \in \mathbb{P}^1(\mathbb{C}) \setminus \{(-B : A)\}$$

(see Example 1). Recall that for any fixed a, different values of the parameter w define essentially different Fuchsian systems.

1) Fix a value $w = w^0$ and consider the family (24) depending on the parameter a only. This isomonodromic family is neither Schlesinger (since its residue matrices B_i^0 are constant but do not commute) nor non-normalized Schlesinger, since it cannot be transformed to a Schlesinger family via a gauge transformation $y' = C^{-1}(a)y$. Indeed, if the latter were to hold, then the matrices $C^{-1}(a)B_i^0 C(a)$ would satisfy the Schlesinger equation, or equivalently, the matrix differential 1-form $\Psi = (dC)C^{-1}$ would satisfy the system of linear algebraic equations

$$[B_i^0, \Psi] = -\sum_{j=1,\, j \neq i}^{4} \frac{[B_i^0, B_j^0]}{a_i - a_j}\, d(a_i - a_j), \qquad i = 1, \dots, 4.$$

But this system is overdetermined and, as can be checked, has no solution.

However, the constructed non-Schlesinger isomonodromic family obviously possesses the Painlevé property.

2) On the space $\mathbb{C}^4 \setminus \bigcup_{i \neq j} \{a_i = a_j\}$ one can take any holomorphic function f and consider the isomonodromic family (24), where $w = w(a) = e^{1/(f(a)-c)}$ is a function of the parameter a and $c = c(a^0, w^0)$ is a constant depending on the initial condition $w(a^0) = w^0$. Then the hypersurfaces $\{f(a) = c\}$ form the set of moveable singularities for isomonodromic deformations of initial systems in (24). Thus, such non-Schlesinger deformations do not possess the Painlevé property.

3) Regarding a and w as independent parameters, we have in (24) an example of an isomonodromic family of the general form (13). One can obtain the matrix differential 1-form Ω of the form (20) corresponding to this family with all resonances r_j equal to 1; this is left to the reader.

One should also note that the singularities of a Schlesinger isomonodromic family are moveable singularities of the Schlesinger equation and only these, while non-Schlesinger isomonodromic families may be undefined even at points where the residue matrices are holomorphic. For the present example these are points $\{(a, w) \mid Aw + B = 0\}$ at which the monodromy changes.

4.4. Application to Garnier systems. The Garnier system $\mathscr{G}_n(\theta)$ depending on $n+3$ complex parameters $\theta_1, \dots, \theta_{n+2}, \theta_\infty$ was obtained by Garnier [38] as a completely integrable system of non-linear partial differential equations of the second order. Later it was described by Okamoto [39] in an equivalent Hamiltonian form (see Example 2)

$$\frac{\partial u_i}{\partial a_j} = \frac{\partial H_j}{\partial v_i}, \quad \frac{\partial v_i}{\partial a_j} = -\frac{\partial H_j}{\partial u_i}, \qquad i, j = 1, \dots, n, \tag{25}$$

with certain Hamiltonians $H_j = H_j(a, u, v, \theta)$ which are rationally dependent on the independent variable $a = (a_1, \dots, a_n)$, the unknowns $u = (u_1, \dots, u_n)$ and $v = (v_1, \dots, v_n)$, and the parameter $\theta = (\theta_1, \dots, \theta_{n+2}, \theta_\infty)$.

In the case $n=1$, the Garnier system $\mathscr{G}_1(\theta_1,\theta_2,\theta_3,\theta_\infty)$ is an equivalent (Hamiltonian) form of the sixth Painlevé equation $\mathrm{P_{VI}}(\alpha,\beta,\gamma,\delta)$, where

$$\alpha=\frac{1}{2}\,\theta_\infty^2,\quad \beta=-\frac{1}{2}\,\theta_2^2,\quad \gamma=\frac{1}{2}\,\theta_3^2,\quad \delta=\frac{1}{2}\,(1-\theta_1^2).$$

As is well known, the Painlevé equations possess the Painlevé property.

For $n>1$ the Garnier system does not, in general, have the Painlevé property. However, by virtue of Garnier's theorem, the elementary symmetric polynomials $\sigma_i\bigl(u_1(a),\dots,u_n(a)\bigr)$ depending on local solutions to the Garnier system extend to meromorphic functions F_i on the universal cover $\widetilde{Z}'$ of the space $Z'=(\mathbb{C}\setminus\{0,1\})^n\setminus\bigcup\limits_{i\neq j}\{a_i=a_j\}$. One can study the behaviour of the functions F_i near their polar loci using the relationship between Schlesinger isomonodromic deformations and Garnier systems, which we recall below.

Consider a two-dimensional Schlesinger isomonodromic family

$$\frac{dy}{dz}=\biggl(\sum_{i=1}^{n+2}\frac{B_i(a)}{z-a_i}\biggr)y,\qquad B_i(a^0)=B_i^0\in\mathrm{sl}(2,\mathbb{C}),\tag{26}$$

of Fuchsian systems with singular points $a_1,\dots,a_n$, $a_{n+1}=0$, $a_{n+2}=1$, $a_{n+3}=\infty$ which depends holomorphically on the parameter $a=(a_1,\dots,a_n)\in D(a^0)$, where $D(a^0)$ is a disk of small radius centred at the point $a^0=(a_1^0,\dots,a_n^0)$ of the space Z'. Denote by $\pm\beta_i$ the eigenvalues of the matrices $B_i(a)$, respectively. Recall that the isomonodromic deformation preserves the eigenvalues of the residue matrices $B_i(a)$. As follows from the Schlesinger equation, the matrix residue at infinity is constant. We assume that it is a diagonalizable matrix, that is, $\sum\limits_{i=1}^{n+2}B_i(a)=-B_\infty=\mathrm{diag}(-\beta_\infty,\beta_\infty)$.

By Malgrange's theorem the matrix functions

$$B_i(a)=\begin{pmatrix}b_i^{11}(a)&b_i^{12}(a)\\ b_i^{21}(a)&b_i^{22}(a)\end{pmatrix}$$

can be extended to the universal cover $\widetilde{Z}'$ of the space Z' as meromorphic functions (holomorphic off the analytic subset Θ of codimension one).

Since the upper-right element $-\sum\limits_{i=1}^{n+2}b_i^{12}(a)$ of the matrix B_∞ equals zero, the function

$$P_n(z,a)=(z-a_1)\cdots(z-a_{n+2})\sum_{i=1}^{n+2}\frac{b_i^{12}(a)}{z-a_i}\tag{27}$$

for every fixed a is a polynomial of degree n in z. We denote by $u_1(a),\dots,u_n(a)$ the roots of this polynomial and define the functions $v_1(a),\dots,v_n(a)$:

$$v_j(a)=\sum_{i=1}^{n+2}\frac{b_i^{11}(a)+\beta_i}{u_j(a)-a_i}\,,\qquad j=1,\dots,n.$$

Then the following statement holds: *the pair* $(u, v) = (u_1, \dots, u_n, v_1, \dots, v_n)$ *satisfies the Garnier system* (25) *with the parameters* $2\beta_1, \dots, 2\beta_{n+2}, 2\beta_\infty - 1$ (see the proof of Proposition 3.1 in [39], or [4], Corollary 6.2.2, p. 207).

Since the coefficients of the polynomial $P_n(z, a)$ are expressed rationally via the upper-right elements $b_i^{12}(a)$ of the matrices $B_i(a)$, so are the elementary symmetric polynomials of the roots of $P_n(z, a)$. In addition to formulae for the transition from a two-dimensional Schlesinger isomonodromic family with $\mathrm{sl}(2, \mathbb{C})$-residues to a Garnier system, there also exist formulae for the inverse transition (see [39], Proposition 3.2). It follows that the Garnier theorem (which claims that the elementary symmetric polynomials $F_i = \sigma_i(u_1, \dots, u_n)$ of solutions to a Garnier system are meromorphic on $\widetilde{Z}'$) can be regarded as a corollary to the Malgrange theorem (Theorem 11). Moreover, applying Theorem 12 carefully, one can obtain the supplement presented below.

By the *linear monodromy* of a solution to a Garnier system we will mean the monodromy of the corresponding two-dimensional Schlesinger isomonodromic family.

Theorem 13. *Let* $\big(u(a), v(a)\big)$ *be a solution to the Garnier system* (25) *that has an irreducible linear monodromy, and let* $\Delta_i \subset \widetilde{Z}'$ *denote the polar locus of the function* $F_i,\ i = 1, \dots, n$. *Then*

a) *in the case* $\theta_\infty = 0$ *and* $u_i \not\equiv u_j$ *for* $i \neq j$ *one has* $\Sigma_{a^*}(F_i) \geqslant -n - 1$ *for any point* $a^* \in \Delta_i$;

b) *in the case* $\theta_\infty \neq 0$ *one has* $\Sigma_{a^*}(F_i) \geqslant -n$ *for any point* $a^* \in \Delta_i$, *with the possible exception of some subset* $\Delta^0 \subset \Delta_i$ *of positive codimension*; *in any case* $(F_i)_\infty \geqslant -n$.

Proof. Consider the family (26) with the irreducible monodromy corresponding to the given solution, and the polynomial $P_n(z, a)$ constructed from the residue matrices $B_i(a)$ (see (27)). Its coefficients can be expressed as follows.

Let us consider the elementary symmetric polynomials

$$\sigma_1(a) = \sum_{i=1}^{n+2} a_i, \quad \sigma_2(a) = \sum_{1 \leqslant i < j \leqslant n+2} a_i a_j, \quad \dots, \quad \sigma_{n+1}(a) = a_1 \cdots a_n$$

of $a_1, \dots, a_n,\ a_{n+1} = 0,\ a_{n+2} = 1$ and the polynomial $Q(z) = \prod_{i=1}^{n+2} (z - a_i)$. Then

$$P_n(z, a) = \sum_{i=1}^{n+2} b_i^{12}(a) \frac{Q(z)}{z - a_i} =: b(a) z^n + f_1(a) z^{n-1} + \dots + f_n(a)$$

(recall that $\sum_{i=1}^{n+2} b_i^{12}(a)=0$). By the Viète theorem one has

$$b(a)=\sum_{i=1}^{n+2} b_i^{12}(a)(-\sigma_1(a)+a_i)=\sum_{i=1}^{n+2} b_i^{12}(a)a_i=\sum_{i=1}^{n} b_i^{12}(a)a_i+b_{n+2}^{12}(a),$$

$$f_1(a)=\sum_{i=1}^{n+2} b_i^{12}(a)\left(\sigma_2(a)-\sum_{j=1,\,j\neq i}^{n+2} a_ia_j\right)=-\sum_{1\leqslant i<j\leqslant n+2}(b_i^{12}(a)+b_j^{12}(a))a_ia_j.$$

In a similar way

$$f_k(a)=(-1)^k\sum_{1\leqslant i_1<\cdots<i_{k+1}\leqslant n+2}(b_{i_1}^{12}(a)+\cdots+b_{i_{k+1}}^{12}(a))a_{i_1}\cdots a_{i_{k+1}}$$

for each $k=1,\dots,n$.

Now, by the Viète theorem, $F_i(a)=(-1)^i f_i(a)/b(a)$. According to Theorem 12 and Remark 6, for each function f_i and any point a^* of the Θ-divisor of the family (26) one has $\Sigma_{a^*}(f_i)\geqslant -n-1$.

It is not difficult to check (using the Schlesinger equation) that $db(a)=-\theta_\infty\times\sum_{i=1}^{n} b_i^{12}(a)\,da_i$, where $\theta_\infty=2\beta_\infty-1$.

a) In the case $\theta_\infty=0$ one has $db(a)\equiv 0$, hence $b(a)\equiv\text{const}\neq 0$. Indeed, if $b(a)\equiv 0$, then $P_n(z,a)$ is a polynomial of degree $n-1$ in z, and $u_i(a)\equiv u_j(a)$ for some $i\neq j$, which contradicts the conditions of the theorem. Thus, $\Sigma_{a^*}(F_i)=\Sigma_{a^*}(f_i)\geqslant -n-1$ in this case.

b) In the case $\theta_\infty\neq 0$ one has

$$b_i^{12}(a)=-\frac{1}{\theta_\infty}\,\frac{\partial b(a)}{\partial a_i}\,,\qquad i=1,\dots,n;$$

$$b_{n+2}^{12}(a)=b(a)-\sum_{i=1}^{n} b_i^{12}(a)a_i,\qquad b_{n+1}^{12}(a)=-b_{n+2}^{12}(a)-\sum_{i=1}^{n} b_i^{12}(a).\tag{28}$$

Thus, if the function b is holomorphic at a point $a'\in\widetilde{Z}'$, then so are the functions b_i^{12}, $i=1,\dots,n+2$, and hence the functions f_i. Therefore, points $a^*\in\Delta_i$ can be of two types: those such that $b(a^*)=0$ (then $\Sigma_{a^*}(F_i)\geqslant -1$, since the function b is irreducible[9]) or those that belong to the polar locus $\Delta\subset\Theta$ of the function b.

Denote by $\Delta^0\subset\Delta$ the set of ambiguous points of the function b. Then in a neighbourhood of any point $a^*\in\Delta\setminus\Delta^0$ the function can be represented in the form

$$b(a)=\frac{h(a)}{\tau_1^{j_1}(a)\cdots\tau_r^{j_r}(a)}\,,\qquad j_1\geqslant 1,\ \dots,\ j_r\geqslant 1,\tag{29}$$

[9]Indeed, if for some $a'\in\{b(a)=0\}$ one has $db(a')\equiv 0$, then $\sum_{i=1}^{n} b_i^{12}(a')da_i\equiv 0$ and $b_1^{12}(a')=\cdots=b_n^{12}(a')=0$. Taking into consideration the relations (28), one also obtains $b_{n+2}^{12}(a')=0$ and $b_{n+1}^{12}(a')=0$. This contradicts the irreducibility of the monodromy of the family (28).

where the functions τ_l, h are holomorphic near a^*, $h(a^*) \neq 0$, and furthermore, the functions τ_l are irreducible at a^*, and at the same time

$$f_i(a) = \frac{g_i(a)}{\tau_1^{k_1}(a)\cdots\tau_r^{k_r}(a)}, \quad k_1 + \cdots + k_r \leqslant n+1, \tag{30}$$

where the function g_i is holomorphic near a^* $(i = 1, \dots, n)$. Thus,

$$\frac{f_i(a)}{b(a)} = \frac{g_i(a)}{\tau_1^{k_1}(a)\cdots\tau_r^{k_r}(a)} \Big/ \frac{h(a)}{\tau_1^{j_1}(a)\cdots\tau_r^{j_r}(a)} = \frac{g_i(a)/h(a)}{\tau_1^{k_1-j_1}(a)\cdots\tau_r^{k_r-j_r}(a)},$$

therefore

$$\Sigma_{a^*}(F_i) = -\sum_\alpha (k_\alpha - j_\alpha) \geqslant -n$$

(the sum is taken over indices α such that $k_\alpha - j_\alpha > 0$), which proves the first part of the statement b).

In a neighbourhood of a point $a^* \in \Delta^0$ the decompositions (29), (30) are valid for the functions b, f_i, respectively, but $h(a^*) = 0$. However, due to the irreducibility of b, all the irreducible factors of h in its decomposition $h(a) = h_1(a)\cdots h_s(a)$ near a^* are distinct (we can assume also that none of the h_i coincides with any function τ_l). Since $k_l - j_l \leqslant n$ for all $l = 1, \dots, r$, the second part of statement b) follows from the decomposition

$$\frac{f_i(a)}{b(a)} = \frac{g_i(a)}{h_1(a)\cdots h_s(a)\,\tau_1^{k_1-j_1}(a)\cdots\tau_r^{k_r-j_r}(a)}.$$

The theorem is proved.

Remark 7. As follows from Remark 6, in all the estimates in Theorem 13 one can replace n by $n-1$ in the case of even n.

In particular, the polar loci of the functions $F_1(a) = u_1(a) + u_2(a)$ and $F_2(a) = u_1(a)u_2(a)$, where (u_1, u_2, v_1, v_2) represents a solution to the Garnier system $\mathscr{G}_2(\theta_1, \dots, \theta_4, \theta_\infty)$ corresponding to a two-dimensional Schlesinger isomonodromic family with five singular points and irreducible monodromy, are analytic submanifolds with $(F_i)_\infty \geqslant -2$. (Note that the bundle E_{a^*} corresponding to a point a^* of the Θ-divisor of this family has the form $E_{a^*} \cong \mathscr{O}(1) \oplus \mathscr{O}(-1)$, which implies the regularity of the Θ-divisor; see [2], Theorem 16.2 or [34].)

Appendix A. Results of § 3.3 revisited

Here we consider the general case of families (13) of Fuchsian systems and give an interpretation of the results in § 3.3, based on a moduli space approach. We will describe a general scheme for an alternative proof of the representation (20). More details can be found in [23].

The construction uses the technique of Schlesinger transformations as gauge transformations of a matrix differential 1-form ω^0 regarded as a connection form in a holomorphically trivial vector bundle over $\overline{\mathbb{C}}$. This approach is widely used in work on the Riemann–Hilbert problem and related topics. The gauge transformations under consideration preserve the monodromy of the system but change its exponents. Since the sum of the residues of a 1-form is an invariant, any local changes of exponents should be compensated at the same or another singular point. It follows that the simplest possible Schlesinger transformation (we call it elementary) that increases one of the exponents at some singular point should decrease one at another singular point. Such a gauge transformation can be constructed explicitly (see [23] for more details).

Now assume that we have an initial resonant Fuchsian system. First let us note that using the sequence of elementary gauge transformations described above, one can resolve all the resonances of the system by introducing some auxiliary fixed singular point a^*. Indeed, any elementary gauge transformation deals with at most two singular points, hence one can remove all the resonances by compensating every change of valuations by an appropriate change of exponents at the point a^*. Now there exists a Schlesinger family defined by this auxiliary system with the only resonant singular point a^*.

Note that the gauge transformations of Fuchsian systems we used can be generalized to isomonodromic families as well. Since they have a rather explicit and simple form, one can check directly the action of the reverse gauge transformations on the Schlesinger 1-form. The next important observation is that during this reverse gauge transforming from the Schlesinger family to the initial one there are two different steps: the creation of resonances and their amplification. When creating a resonance one chooses the direction of the solution from the monodromy eigenspace according to the data of our initial system. When disengaging the parameters $\{w_1, \dots, w_d\}$ encoding these directions, we obtain the whole connected component $\mathscr{W}$ of the moduli space of Fuchsian systems, the set of all possible Fuchsian systems having the prescribed monodromy and exponents. Finally, it remains to add these parameters to the gauge transformations as new variables, and to modify the action of the gauge transformations to include differentiation with respect to these new variables. Since the new variables determine the Fuchsian system uniquely among all systems having the same monodromy and exponents, any isomonodromic family can be represented as a gauge transformed Schlesinger family. This also shows that the proof is independent of the position of a^*. This position can be expressed in terms of the new variables and is inessential. Explicit calculations give the representation (20).

It is important to note that representing any possible isomonodromic family as a result of certain elementary gauge transformations of the Schlesinger family enables one to estimate the growth of its coefficients under the deformation and demonstrate the possible satisfaction or violation of the Painlevé property for the deformation equations. Indeed, assuming the new variables w_i to be functions of $a_1, \dots, a_n$ reduces the 1-form Ω to the representation (19), and if these functions have greater than polynomial growth, then the Painlevé property must obviously be violated for the resulting family (as we saw in Example 5). Moreover, this is the only possible construction of a violation of the Painlevé property for isomonodromic deformations (14) of Fuchsian systems. On the other hand, investigation of the explicit form of the gauge transformations used in the above arguments can ensure the satisfaction of the Painlevé property for the resulting family if there are no dependences of the type $w = w(a)$ or if the functions $w_i(a)$ exhibit an appropriate growth.

Appendix B. Symplectic structures

In this appendix we try to elucidate the geometry underlying isomonodromic deformations and to give a short review of the links between these deformations and certain symplectic structures on moduli spaces of connections and representations.

We begin with Fuchsian systems. The starting point is an investigation of the properties of the monodromy map. This highly transcendental map has a number of notable properties. Starting from a monodromy which maps a meromorphic differential equation to its monodromy data, the map extends to the (appropriately defined) moduli space of meromorphic connections, taking values in the moduli space of fundamental group representations. The moduli space of connections is naturally fibred over the space of deformation parameters; in addition there is a bundle over the same base which is naturally associated with the moduli space of representations. More precisely, that is the bundle obtained by extending the moduli space by including certain additional (rather formal) data, namely, the positions of the punctures that the generators of the fundamental group encircle. Clearly one can easily define a monodromy map between these two bundles. Both bundles have natural symplectic structures, and the monodromy map gives a symplectomorphism. The natural flat Ehresmann connection on the target bundle, corresponding to the representations, is quite easy to describe: horizontal sections are those passing through the same class of monodromy representations over each base point. The pull-backs of these sections are just isomonodromic families of Fuchsian systems. Thus, the equations of isomonodromic deformations describe the natural flat symplectic connection on the moduli space of Fuchsian systems fibred over singular point positions. Expressed in terms of coordinates, the connection form varies from (15) in the simplest case to (20) in the most general settings.

This approach was generalized by Boalch [40]. He described the corresponding moduli spaces, symplectic (Poisson in the general case) structures, and the corresponding maps for the systems with singular points of positive Poincaré rank (non-Fuchsian singular points). He has also shown that general symplectic geometry also lies at the basis of such generalized deformations: the monodromy map remains a symplectomorphism between the appropriately defined generalized moduli spaces.

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