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The dynamics of epidemiological systems with nonautonomous and random coefficients

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Abstract. Steady state solutions are important in characterizing the asymptotic behaviour of epidemiological systems such as the ubiquitous SIR system, but they need not exist when the coefficients vary with time. Recent developments in the theory of nonautonomous dynamical systems provide the appropriate counterpart, time varying nonautonomous equilibria, i.e., entire solutions determined by pullback attraction. These will be illustrated here through explicit solutions for the simpler SI system. For the more complicated SIR system it will be shown how pullback attractors provide additional insight into dynamics, especially the chaotic dynamics induced by periodical forcing.

1 Introduction

Epidemiological models, of which the SIR is prototypical, have been investigated extensively in the literature, particular in the autonomous case [2], where steady state solutions play an important role in the analysis. In recent years the nonautonomous case has attracted attention [4, 6, 11, 14, 15, 16]

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due to the seasonal variations in many diseases. For example, for the spread of infectious childhood diseases Thieme [16] cites arguments that the school system induces a time-heterogeneity in the per capita/capita infection rate, because the chain of infections is interrupted or at least weakened by the vacations and new individuals are recruited into a scene with higher infection risk at the beginning of each school year. This requires the inclusion of time variable coefficients or forcing terms in the models, which could even be random. As a consequence steady state solutions may no longer exists.

Recent developments in the theory of nonautonomous dynamical systems [10] offer new concepts, which allow the asymptotic dynamics to be characterized. Instead of the steady state equilibria of autonomous ordinary differential equations there are now time varying nonautonomous equilibria that are entire solutions determined by pullback attraction. These are special cases of more general nonautonomous pullback attractors. They will be introduced and illustrated here in terms of simpler SI models since the nonautonomous equilibria solutions can be determined explicitly in analytical form.

There are two formalisms for abstract nonautonomous dynamical systems, with the process or two-parameter semigroup formalism based on the solution map of a nonautonomous initial value problem being somewhat simpler. The other, the skew product formalism, includes a driving system that models time variation in the system. The complications mentioned in [16] can be avoided by defining the driving system as the shift operator on the hull formed by the time varying coefficients rather than of the full vector field. The advantage then is that, unlike the process formalism, the driving system already contains all possible asymptotic behaviour. Random dynamical systems [1, 3] have a similar formalism, but with a measure theoretic rather than topological driving system. In the final part of the paper it will be shown how this theory and its attractor concepts provide additional information about the chaotic dynamics detected in the periodically driven SIR systems investigated in [4, 12, 14, 15] as well as in the switching model in [7].

2 The SIR model

The SIR equations have variables S (susceptibles), I (infected) and R (recovered). They will be investigated here time variable parameters or forcing (deterministic or random) in the model, so the total population $N(t) = S(t) + I(t) + R(t)$ need not be constant. This will be achieved first through a temporal forcing term given by a continuous function $q : \mathbb{R} \to \mathbb{R}$ taking positive bounded values, i.e., $q(t) \in [q^-, q^+]$ for all $t \in \mathbb{R}$, where $0 < q^- \leq q^+$, and later with a time variable interaction coefficient γ. To fix ideas, think of a periodic or almost periodic function, although it could be more complicated and even random.

The SIR model is given by the system of ODEs

$$\frac{dS}{dt} = aq(t) - aS + bI - \gamma\frac{SI}{N}, \tag{2.1}$$

$$\frac{dI}{dt} = -(a+b+c)I + \gamma\frac{SI}{N}, \tag{2.2}$$

$$\frac{dR}{dt} = cI - aR, \tag{2.3}$$

where the parameters a, b, c and γ are positive constants. (Later they will also be allowed to vary in time within suitable positive bounds). This means that solutions with nonnegative initial values remains nonnegative.

This system has no steady state solutions when $q(t)$ is not identically equal to a constant.

Adding both sides of the ODEs gives the scalar nonautonomous ODE

$$\frac{dN}{dt} = a(q(t) - N), \tag{2.4}$$

which has the solution

$$N(t) = N_0 e^{-a(t-t_0)} + ae^{-at} \int_{t_0}^{t} q(s)e^{as}\, ds.$$

Now

$$ae^{-at} \int_{t_0}^{t} e^{as}\, ds = 1 - e^{-a(t-t_0)},$$

so given the positivity bounds on $q(t)$ one sees that the integral takes values between

$$q^{-}\left(1 - e^{-a(t-t_0)}\right) \leq ae^{-at} \int_{t_0}^{t} q(s)e^{as}\, ds \leq q^{+}\left(1 - e^{-a(t-t_0)}\right).$$

Hence the total population is bounded above and below, specifically

$$q_{-} + (N_0 - q_{-})\, e^{-a(t-t_0)} \leq N(t) \leq q^{+} + \left(N_0 - q^{+}\right) e^{-a(t-t_0)}.$$

From this one concludes that the simplex slab

$$\Sigma_3^{\pm} = \left\{(S,I,R) :\ S,I,R \geq 0,\ N = S+I+R \in [q^{-},q^{+}]\right\} \subset \mathbb{R}^3$$

attracts all populations starting outside it and that populations starting within it remain there. When the forcing term $q(t)$ is not identically equal to a constant, the simplex slab $\Sigma_3^{\pm}$ will in fact absorb outside populations in a finite time and will be positive invariant (rather than strictly invariant). One can thus restrict attention to the dynamics in $\Sigma_3^{\pm}$.

The ODE (2.4) has no steady state solutions, but it has what Chueshov [3] called a *nonautonomous equilibrium solution*, which is found be taking the *pullback limit* (i.e., as $t_0 \to -\infty$ with t held fixed, see [10]), namely

$$\widehat{N}(t) = ae^{-at} \int_{-\infty}^{t} q(s)e^{as}\, ds. \tag{2.5}$$

This also forward attracts all other solution of the ODE (2.4), i.e.

$$\left|N(t) - \widehat{N}(t)\right| = e^{-a(t-t_0)} \left|N(t_0) - \widehat{N}(t_0)\right| \to 0 \quad \text{as} \quad t \to \infty.$$

3 The SI equations with variable population

Assume for simplicity that the R term is not present and $c = 0$, as solutions can then be determined explicitly. The equations reduce to

$$\frac{dS}{dt} = aq(t) - aS + bI - \gamma\frac{SI}{N}, \tag{3.1}$$

$$\frac{dI}{dt} = -(a+b)I + \gamma\frac{SI}{N}. \tag{3.2}$$

If $I_0 = 0$, then $I(t) \equiv 0$. This means that the S face of

$$\Sigma_2^{\pm} = \left\{(S,I) : S,I \geq 0, N = S+I \in [q^-,q^+]\right\} \subset \mathbb{R}^2$$

is invariant and the equations reduce to

$$\frac{dS}{dt} = aq(t) - aS, \tag{3.3}$$

which has the solution

$$S(t) = S_0 e^{-a(t-t_0)} + ae^{-at}\int_{t_0}^{t} q(s)e^{as}\,ds.$$

It has no steady state solution (when $q(t)$ is not a constant), but it has a nonautonomous equilibrium solution (in the sense of Chueshov) found be taking the pullback limit (i.e., as $t_0 \to -\infty$ with t held fixed), namely

$$\widehat{S}_1(t) = ae^{-at}\int_{-\infty}^{t} q(s)e^{as}\,ds.$$

This also forward attracts all other solution of the S equation in the $I = 0$ face, i.e.

$$\left|S(t) - \widehat{S}_1(t)\right| \to 0 \quad \text{as} \quad t \to \infty.$$

In particular, the SI dynamics has no steady state but a nonautonomous equilibrium solution $(\widehat{S}_1(t),0)$.

Lemma 1. The nonautonomous equilibrium solution $(\widehat{S}_1(t),0)$ is globally asymptotically stable in $\Sigma_2^{\pm}$ when $\gamma \leq a+b$. It is unstable when $\gamma > a+b$.

Proof. Suppose that $\gamma < a+b$. Multiply the I equation by $2I$ to obtain

$$\frac{d}{dt}I^2 = -2(a+b)I^2 + 2\gamma\frac{SI^2}{N} \leq -2(a+b)I^2 + 2\gamma\frac{SI^2}{N} \leq 2(\gamma - a - b)I^2$$

because $I^2 \geq 0$ and

$$0 \leq \frac{S}{N} \leq 1.$$

Hence

$$I^2(t) \leq I^2(0)e^{-2|\gamma-a-b|t} \to 0 \quad \text{as} \quad t \to \infty.$$

If $\gamma = a+b$, the I equation gives

$$\frac{d}{dt}I = -(a+b)I + \gamma\frac{SI}{N} = -(a+b)I + \gamma\frac{(N-I)I}{N} = (\gamma - a - b)I - \gamma\frac{I^2}{N} = -\gamma\frac{I^2}{N} < -\gamma\frac{I^2}{\gamma^-} < 0$$

for all $I > 0$, so

$$0 \leq I(t) \leq \frac{I(0)\gamma^-}{\gamma^- + \gamma I(0)t} \to 0 \quad \text{as} \quad t \to \infty.$$

In both cases

$$S(t) - \widehat{S}_1(t) = \left(N(t) - \widehat{N}(t)\right) - I(t) \to 0 \quad \text{as} \quad t \to \infty,$$

since $\widehat{S}_1(t) \equiv \widehat{N}(t)$ and both terms on the right converge to zero with increasing time. The nonautonomous equilibrium solution is thus globally asymptotically stable when $\gamma \leq a+b$.

Now let $\gamma > a+b$ and suppose that $I \leq \varepsilon N$ in the I equation, so $S \geq (1-\varepsilon)N$. Then

$$\frac{d}{dt}I = -(a+b)I + \gamma\frac{SI}{N} \geq -(a+b)I + \gamma(1-\varepsilon)I \geq (\gamma(1-\varepsilon)-a-b)I,$$

so $I(t)$ is strictly increasing if $0 < I \leq \varepsilon N$, provided $\varepsilon < 1 - \frac{a+b}{\gamma}$. In particular, the nonautonomous equilibrium solution $(\widehat{S}_1(t), 0)$ is unstable when $\gamma > a+b$. □

3.1 Nontrivial limiting solutions

Now consider the possibility of a nontrivial equilibrium, i.e. with $I \neq 0$. The limiting population will be used here, $\widehat{N} = S+I$, in the SI equations (3.1)–(3.2). Then the equation for I can be rewritten

$$\frac{dI}{dt} = -(a+b)I + \gamma\frac{I}{\widehat{N}}(\widehat{N}-I),$$

that is as

$$\frac{dI}{dt} = (\gamma-a-b)I - \gamma\frac{1}{\widehat{N}}I^2.$$

This is a Bernoulli equation which can be solved with the substitution $V = I^{-1}$, to give the linear ODE

$$\frac{dV}{dt} + (\gamma-a-b)V = \gamma\frac{1}{\widehat{N}}.$$

It has the explicit solution

$$V(t) = e^{-(\gamma-a-b)(t-t_0)}V_0 + \gamma e^{-(\gamma-a-b)t}\int_{t_0}^{t}\frac{e^{(\gamma-a-b)s}}{\widehat{N}(s)}\,ds$$

The pullback limit as $t_0 \to -\infty$ exists only if $\gamma - a - b > 0$ and is then equal to

$$\widehat{V}(t) = \gamma e^{-(\gamma-a-b)t}\int_{-\infty}^{t}\frac{e^{(\gamma-a-b)s}}{\widehat{N}(s)}\,ds.$$

This gives a nontrivial nonautonomous equilibrium

$$\widehat{S}(t) = \widehat{N}(t) - \widehat{I}(t), \quad \widehat{I}(t) = \frac{e^{(\gamma-a-b)t}}{\gamma\int_{-\infty}^{t}\frac{e^{(\gamma-a-b)s}}{\widehat{N}(s)}\,ds}, \tag{3.4}$$

where

$$\widehat{N}(t) = ae^{-at}\int_{-\infty}^{t} q(s)e^{as}\,ds.$$

Remark 1. When $\gamma - a - b < 0$, then "$V = \infty$", which corresponds to $\widehat{I}(t) \equiv 0$.

Remark 2. The $\widehat{S}(t)$, $\widehat{I}(t)$ and $\widehat{N}(t) = \widehat{S}(t) + \widehat{I}(t)$ defined here satisfy the SI equations (3.1)–(3.2) at the start of the section.

Lemma 2. The nonautonomous equilibrium solution $(\widehat{S}(t), \widehat{I}(t))$ is globally asymptotically stable in $\Sigma_2^{\pm}$ when $\gamma > a+b$.

Proof. A general solution of the SI equations (3.1)–(3.2) will be compared with the nonautonomous equilibrium solution. From above

$$\left|N(t)-\widehat{N}(t)\right|^2 = e^{-2a(t-t_0)}\left|N(t_0)-\widehat{N}(t_0)\right| \to 0 \quad \text{as} \quad t\to\infty.$$

Now consider two solutions $I(t)$ and $\widehat{I}$ of the I equations with $I(0)>0$, i.e.,

$$\frac{dI}{dt} = (\gamma-a-b)I - \gamma\frac{1}{N}I^2.$$

and

$$\frac{d\widehat{I}}{dt} = (\gamma-a-b)\widehat{I} - \gamma\frac{1}{N}\widehat{I}^2,$$

respectively. These are Bernoulli equations which can be solved with the substitution $V = I^{-1}$, to give the linear ODE

$$\frac{dV}{dt} + (\gamma-a-b)V = \gamma\frac{1}{N}.$$

Writing $\widehat{V} = \widehat{I}^{-1}$, it follows that $\Delta V := V - \widehat{V}$ satisfies the linear ODE

$$\frac{d\Delta V}{dt} + \rho\Delta V = \gamma\left(\frac{1}{\widehat{N}} - \frac{1}{N}\right), \tag{3.5}$$

where

$$\rho := \gamma - a - b,$$

which is positive here.

This gives

$$\begin{aligned}\frac{d}{dt}(\Delta V)^2 + 2\rho(\Delta V)^2 &= 2\gamma\left(\frac{1}{\widehat{N}} - \frac{1}{N}\right)\Delta V \le 2\gamma\frac{|N-\widehat{N}|}{N\widehat{N}}|\Delta V| \\ &\le \frac{2\gamma}{(\gamma^-)^2}|N-\widehat{N}||\Delta V| \le \rho(\Delta V)^2 + \frac{4\gamma^2}{\rho(\gamma^-)^4}|N-\widehat{N}|^2,\end{aligned}$$

so

$$\frac{d}{dt}(\Delta V)^2 + \rho(\Delta V)^2 \le \frac{4\gamma^2}{\rho(\gamma^-)^4}|N(0)-\widehat{N}(0)|^2 e^{-2at}.$$

Integrating gives

$$\begin{aligned}(\Delta V)(t)^2 &= (\Delta V)(0)^2 e^{-\rho t} + \frac{4\gamma^2}{\rho(\gamma^-)^4}|N(0)-\widehat{N}(0)|^2 e^{-\rho t}\int_0^t e^{-2as}e^{\rho s}\,ds \\ &\le (\Delta V)(0)^2 e^{-\rho t} + \frac{4\gamma^2}{\rho(\gamma^-)^4}|N(0)-\widehat{N}(0)|^2 e^{-\rho t}\int_0^t e^{(\rho-2a)s}\,ds \\ &\le (\Delta V)(0)^2 e^{-\rho t} + \frac{4\gamma^2}{\rho(\rho-2a)(\gamma^-)^4}|N(0)-\widehat{N}(0)|^2\left(e^{-\rho t} - e^{-2at}\right),\end{aligned}$$

so

$$\Delta V(t) \to 0 \quad \text{as} \quad t \to 0.$$

(Note if $\rho = 2a$ the argument holds with a slight change in the use of Young's inequality). This says that

$$\frac{1}{I(t)} - \frac{1}{\widehat{I}(t)} \to 0 \quad \text{as} \quad t \to 0$$

and hence

$$I(t) - \widehat{I}(t) \to 0 \quad \text{as} \quad t \to 0$$

since $I(t) > 0$ and $\widehat{I}(t) > 0$ for all t. Now $|N(t) - \widehat{N}(t)| \to 0$ as $t \to 0$, so

$$\begin{aligned}\left|S(t) - \widehat{S}(t)\right| &= \left|(N(t) - I(t)) - \left(\widehat{N}(t) - \widehat{I}(t)\right)\right| \\ &\le \left|N(t) - \widehat{N}(t)\right| + \left|I(t) - \widehat{I}(t)\right| \to 0 \quad \text{as} \quad t \to 0.\end{aligned}$$

This completes the proof of lemma 2. □

4 The SI equations with variable interaction

Consider now the SI equations for a constant limiting population $N = S + I \equiv 1$, but with a time-variable interaction term $\gamma = \gamma(t)$. It will be assumed that $\gamma \in C(\mathbb{R}, [\gamma^-, \gamma^+])$, where $0 < \gamma^- \le \gamma^+ < \infty$. The equations now have the form

$$\frac{dS}{dt} = a - aS + bI - \gamma(t)SI, \tag{4.1}$$

$$\frac{dI}{dt} = -(a+b)I + \gamma(t)SI. \tag{4.2}$$

On adding the equations reduce to

$$\frac{dN}{dt} = a - aN,$$

which has the globally asymptotic solution $N(t) \equiv 1$, so the analysis can be restricted to the compact set

$$\Sigma_2 = \{(S,I) : S, I \ge 0,\ S + I \equiv 1\} \subset \mathbb{R}^2.$$

The equations (4.1)–(4.2) have an equilibrium solution $(\bar{S}, \bar{I}) = (1,0)$ for all parameter values.

Lemma 3. The nonautonomous equilibrium solution $(\bar{S}, \bar{I}) = (1,0)$ is globally asymptotically stable in Σ_2 when $\gamma^+ \le a+b$. It is unstable when $\gamma^- > a+b$.

Proof. Suppose that $\gamma^+ < a+b$. Multiply the I equation by $2I$ to obtain

$$\frac{d}{dt}I^2 = -2(a+b)I^2 + 2\gamma(t)SI^2 \le -2(a+b)I^2 + 2\gamma^+ I^2 \le 2(\gamma^+ - a - b)I^2$$

because $0 \le S \le 1$ and $\gamma^+ > 0$. Hence

$$I^2(t) \le I^2(0)e^{-2|\gamma^+ - a - b|t} \to 0 \quad \text{as} \quad t \to \infty.$$

The case $\gamma^+ = a+b$ is the same as in Lemma 1 with γ replaced by γ^+.

Let $\gamma^- > a+b$ and suppose that $I \le \varepsilon$ in the I equation, so $S \ge (1-\varepsilon)$. Then

$$\frac{d}{dt}I = -(a+b)I + \gamma(t)SI \ge -(a+b)I + \gamma^-(1-\varepsilon)I \ge (\gamma^-(1-\varepsilon) - a - b)I,$$

so $I(t)$ is strictly increasing if $0 < I \le \varepsilon N$, provided $\varepsilon < 1 - \frac{a+b}{\gamma^-}$. In particular, the equilibrium solution $(1,0)$ is unstable when $\gamma^- > a+b$. □

4.1 Nontrivial limiting solutions

Now consider the possibility of a nontrivial equilibrium, i.e., with $I \neq 0$, when $\gamma^- > a+b$. It cannot exist as a steady state, so a nonautonomous equilibrium will be sought.

The equation (4.2) for I can be rewritten

$$\frac{dI}{dt} = -(a+b)I + \gamma(t)I(1-I),$$

that is as

$$\frac{dI}{dt} = (\gamma(t) - a - b)I - \gamma(t)I^2.$$

This is a Bernoulli equation which can be solved with the substitution $V = I^{-1}$, to give the linear ODE

$$\frac{dV}{dt} + (\gamma(t) - a - b)V = \gamma(t).$$

It has the explicit solution

$$\begin{aligned} V(t) = e^{-\int_{t_0}^t \rho(s)\,ds} V_0 + e^{-\int_{t_0}^t \rho(s)\,ds} \int_{t_0}^t \gamma(\tau) e^{\int_{t_0}^\tau \rho(s)\,ds}\, d\tau, \\ = e^{-\int_{t_0}^t \rho(s)\,ds} V_0 + \int_{t_0}^t \gamma(\tau) e^{\int_{t_0}^\tau \rho(s)\,ds \int_{t_0}^t \rho(s)\,ds}\, d\tau, \\ = e^{-\int_{t_0}^t \rho(s)\,ds} V_0 + \int_{t_0}^t \gamma(\tau) e^{-\int_\tau^t \rho(s)\,ds}\, d\tau, \end{aligned}$$

where $\rho(t) := \gamma(t) - a - b > 0$. Now

$$\int_{t_0}^t \rho(s)\,ds = \int_{t_0}^t (\gamma(s) - a - b)\,ds \geq \left(\gamma^- - a - b\right)(t - t_0) \geq 0.$$

The pullback limit as $t_0 \to -\infty$ exists only if $\gamma - a - b > 0$ and is then equal to

$$\widehat{V}(t) = \int_{-\infty}^t \gamma(\tau) e^{-\int_\tau^t \rho(s)\,ds}\, d\tau.$$

These integrals exists due to the assumptions on the coefficients. The nontrivial nonautonomous equilibrium

$$\widehat{S}(t) = 1 - \widehat{I}(t), \qquad \widehat{I}(t) = \left(\int_{-\infty}^t \gamma(\tau) e^{-\int_\tau^t \rho(s)\,ds}\, d\tau\right)^{-1}.$$

For γ a positive constant, they integrate out to give the nontrivial steady state solution

$$\bar{S} = \frac{a+b}{\gamma}, \qquad \bar{I} = \frac{\gamma - a - b}{\gamma}.$$

The proof of the following lemma is very similar to that of Lemma 2, but is given for completeness.

Lemma 4. The nonautonomous equilibrium solution $(\widehat{S}(t), \widehat{I}(t))$ is globally asymptotically stable in $\Sigma_2^\pm$ when $\gamma^- > a+b$.

Proof. Consider an arbitrary solution $I(t)$ of the I equation (4.2) with $I(0) > 0$ and the solution, i.e.,

$$\frac{dI}{dt} = (\gamma(t) - a - b)I - \gamma(t)I^2,$$

and

$$\frac{d\widehat{I}}{dt} = (\gamma(t) - a - b)\widehat{I} - \gamma(t)\widehat{I}^2,$$

respectively. These are Bernoulli equations which can be solved with the substitution $V = I^{-1}$ to give the linear ODE

$$\frac{dV}{dt} + (\gamma(t) - a - b)V = \gamma(t).$$

Writing $\widehat{V} = \widehat{I}^{-1}$, it follows that $\Delta V := V - \widehat{V}$ satisfies the linear ODE

$$\frac{d\Delta V}{dt} + \rho(t)\Delta V = 0, \tag{4.3}$$

where $\rho(t) := \gamma(t) - a - b \geq \gamma^- - a - b > 0$. This gives

$$(\Delta V)(t)^2 = (\Delta V)(0)^2 e^{-2\int_0^t \rho(s)\,ds} \leq (\Delta V)(0)^2 e^{-2(\gamma^- - a - b)t},$$

so

$$\Delta V(t) \to 0 \quad \text{as} \quad t \to 0.$$

This says that

$$\frac{1}{I(t)} - \frac{1}{\widehat{I}(t)} \to 0 \quad \text{as} \quad t \to 0$$

and hence

$$I(t) - \widehat{I}(t) \to 0 \quad \text{as} \quad t \to 0$$

since $I(t) > 0$ and $\widehat{I}(t) > 0$ for all t. Similarly

$$\left|S(t) - \widehat{S}(t)\right| = \left|I(t) - \widehat{I}(t)\right| \to 0 \quad \text{as} \quad t \to 0.$$

This completes the proof of Lemma 2. □

4.2 Critical case

The situation for $a + b \in (\gamma^-, \gamma^+]$ depends on how the function $\gamma(t)$ varies in time. Specifically, it will depend on the asymptotic behaviour of the integral

$$\frac{1}{t - t_0} \int_{t_0}^{t} \gamma(s)\,ds$$

as either $t \to \infty$ or $t_0 \to -\infty$. The limits usually will not exists, so the upper and lower limits should be used. These exist and lie between γ^- and γ^+.

In fact, here the disease free steady state $(1, 0)$ undergoes a nonautonomous bifurcation to the nonautonomous equilibrium solution $(\widehat{S}(t), \widehat{I}(t))$, see [10].

5 The SIR model with variable population

The SIR equations with variable population (2.1)–(2.3) will be analysed here for the asymptotically stable limiting population $\widehat{N}(t)$ given by (2.5)

$$\frac{dS}{dt} = aq(t) - aS + bI - \gamma \frac{SI}{\widehat{N}(t)}, \tag{5.1}$$

$$\frac{dI}{dt} = -(a+b+c)I + \gamma \frac{SI}{\widehat{N}(t)}, \tag{5.2}$$

$$\frac{dR}{dt} = cI - aR. \tag{5.3}$$

These have a nonautonomous disease free equilibrium as for the SI equations for all parameter values. The dynamics after it becomes unstable and can be very complicated.

It is convenient here to eliminate the S variable and just consider the equations for the I and R variables in (5.1)–(5.3). Essentially, replace S by $S = \widehat{N} - I - R$ in the I equation (5.2) to obtain the IR system

$$\frac{dI}{dt} = (\gamma - a - b - c)I - \gamma \frac{I(I+R)}{\widehat{N}(t)} \tag{5.4}$$

$$\frac{dR}{dt} = cI - aR, \tag{5.5}$$

where $0 \leq I, R \leq \widehat{N}(t)$, i.e., with solutions in time dependent triangular sets

$$T_2(t) := \left\{(I,R) \,:\, I,R \geq 0,\ 0 \leq I + R \leq \widehat{N}(t)\right\},$$

hence in the common compact subset of $\mathbb{R}^2$

$$T_2(t)\} \subset \left\{(I,R) \,:\, I,R \geq 0,\ 0 \leq I + R \leq g^+\right\}.$$

5.1 Disease free nonautonomous equilibrium solution

Note that $I(t) = R(t) \equiv 0$ is a solution. The S equation (5.1) then reduces to

$$\frac{dS}{dt} = aq(t) - aS$$

which is exactly the same as the equation for the total population, so it has pullback limiting

$$\widehat{S}_1(t) = \widehat{N}(t) = ae^{-at} \int_{-\infty}^{t} q(s)e^{as}\, ds.$$

The proof of the following lemma is very similar to that of Lemma 1, using reduced IR equations (5.4)–(5.5) for the stability proof and the original I equation (5.2) for the instability proof.

Lemma 5. The disease free nonautonomous equilibrium solution $(\widehat{S}_1(t), 0, 0)$ is globally asymptotically stable in $\Sigma_3^{\pm}$ when $\gamma \leq a + b + c$. It is unstable when $\gamma > a + b - c$.

5.2 Nontrivial dynamics

Of interest here is the case $\gamma > a+b-c$ where the disease free nonautonomous equilibrium solution is unstable. For this the reduced IR equations (5.4)–(5.5) will be used, which will be rewritten as

$$\frac{dI}{dt} = \rho I - \gamma \frac{I(I+R)}{\widehat{N}(t)}, \qquad \frac{dR}{dt} = cI - aR, \tag{5.6}$$

where $\rho := \gamma - a - b - c > 0$. Since $R \geq 0$, the solution of the I equation in (5.6) satisfies the differential inequality

$$\frac{dI}{dt} \leq \rho I - \gamma \frac{I^2}{\widehat{N}(t)}.$$

This inequality written as an equality is a Bernoulli equation, which analogously with the derivation of the pullback solution (3.4) has the pullback limiting solution

$$\widehat{I}^+(t) = \frac{e^{\rho t}}{\gamma \int_{-\infty}^{t} \frac{e^{\rho s}}{\widehat{N}(s)}\, ds}, \qquad \widehat{N}(t) = ae^{-at} \int_{-\infty}^{t} q(s) e^{as}\, ds.$$

By a monotonicity argument, any pullback limiting solution of the reduced IR equations (5.6) thus has the upper bound

$$\widehat{I}(t) \leq \widehat{I}^+(t), \qquad \widehat{R}(t) \leq be^{-at} \int_{-\infty}^{t} e^{as} \widehat{I}^+(s)\, ds.$$

Such a pullback limiting solution arises as a nonautonomous bifurcation as ρ becomes positive. However, it need not be unique, especially for larger ρ. Indeed, O'Regan *et al.* [12] (see also [5, Ch. 9]) use complicated topological arguments show that chaotic dynamics is possible when the population varies periodically. A similar situation was encountered by Stone *et al.* [14, 15] for the SIR equations with constant population and time periodic interaction parameter $\gamma(t)$. To proceed with the analysis results from the theories of nonautonomous and random dynamical systems are needed.

6 Nonautonomous and random attractors

The nonautonomous equilibrium solution above that were obtained by pullback convergence are simple examples of nonautonomous attractors. There have been some major developments in the past decade or so on the theory of nonautonomous and random dynamical systems, see, e.g., the monographs by Arnold [1], Chueshov [3] and Kloeden & Rasmussen [10]. These provide new concepts and results that provide a better understanding of the dynamics of SIR and related epidemiological systems with nonautonomous or random coefficients, an example of which are the explicit nonautonomous equilibrium solutions considered above. The more complicated behaviour requires a deeper look into these theories.

The nonautonomous systems of ODEs above involve initial value problems of the form

$$\frac{dx}{dt} = f(x,t), \quad x(t_0) = x_0,$$

in $\mathbb{R}^d$ for $d = 1$, 2 or 3 with a unique solution given by $\phi(t;t_0,x_0)$. Such solution mappings ϕ are the basis of the *process* or 2-parameter semigroup formalism of abstract nonautonomous dynamical systems [10]. This quite intuitive formalism. however, does not always allow the whole asymptotic behaviour to be revealed without additional assumptions, see [9], in contrast to the somewhat more complicated *skew product flow* formalism that already contains more built-in information in terms of what is called a driving system.

Let (X,d_X) and (P,d_P) be metric spaces. A *skew product flow* (θ,φ) is defined in terms of a cocycle mapping φ on a state space X which is driven by an autonomous dynamical system θ acting on a base or parameter space P and the time set $\mathbb{R}$. Specifically, the driving system θ on P is a group of homeomorphisms $(\theta_t)_{t\in\mathbb{R}}$ under composition on P with the properties that

(i) $\theta_0(p) = p$ for all $p \in P$,
(ii) $\theta_{s+t} = \theta_s(\theta_t(p))$ for all $s,\, t \in \mathbb{R}$,
(iii) the mapping $(t,p) \mapsto \theta_t(p)$ is continuous,

and the *cocycle mapping* $\varphi : \mathbb{R}_0^+ \times P \times X \to X$, where $\mathbb{R}_0^+$ is the set of nonnegative real numbers, satisfies

(i) $\varphi(0,p,x) = x$ for all $(p,x) \in P \times X$,
(ii) $\varphi(t+s,p,x) = \varphi(t,\theta_s(p),\varphi(s,p,x))$ for all $s,t \in \mathbb{R}_0^+$, $(p,x) \in P \times X$,
(iii) the mapping $(t,p,x) \mapsto \varphi(t,p,x)$ is continuous.

In terms of the SIR and SI systems above, $\varphi(t,p,x)$ is the unique solution for $t \in \mathbb{R}_0^+$ of an initial value problem

$$\frac{dx}{dt} = f(x,\theta_t(p)) \tag{6.1}$$

in $\mathbb{R}^d$ for $d = 1$, 2 or 3 with the initial value $x(0) = x_0$ for the driving system starting at p. Here P can be taken as the *hull* of the forcing term $q : \mathbb{R} \to \mathbb{R}$ or interaction term $\gamma : \mathbb{R} \to \mathbb{R}$ in the space $C(\mathbb{R},\mathbb{R})$, i.e., in the first case

$$P = \overline{\{q(t+\cdot)\,:\, t \in \mathbb{R})\}}^{C(\mathbb{R},\mathbb{R})},$$

and θ_t is the shift operator defined by $\theta_t(q(\cdot)) = q(t+\cdot)$. The advantage is that when q or γ are periodic or almost periodic then P is a compact metric space.

Note that a process can be represented a skew product flow with the parameter set $P = \mathbb{R}$, $p = t_0$ the initial time and the shift operator $\theta_t(t_0) = t + t_0$.

Random dynamical systems are defined analogously, but with the metric space P replaced by the sample space Ω of a probability space $(\Omega,\mathcal{F},\mathbb{P})$ and the continuity of $p \to \theta_t(p)$ by the measurability of $\omega \to \theta_t(\omega)$. In this case the ordinary differential equation (6.1) becomes a random ordinary differential equation. Random dynamical systems are also generated by the solutions of Itô stochastic differential equations.

6.1 Nonautonomous and random attractors

It is too restrictive to insist that nonautonomous attractors consist of a single compact set. Instead one considers a families of nonempty compact subsets $\mathcal{A} = \{A_p\,:\; p \in P\}$ of X, which are φ-invariant $\varphi(t,p,A_p) = A_{\theta_t(p)}$ for all $t \in \mathbb{R}_0^+$ and $p \in P$. Such families are made up of *entire* solutions, i.e., continuous functions $\xi : \mathbb{R} \to \mathbb{R}$ satisfying $\xi(t+s) = \varphi(t,p,\xi(s))$ for all $t \geq 0$ and $s \in \mathbb{R}$.

A φ-invariant family of nonempty compact subsets $\mathcal{A}$ of X is called a *pullback attractor* of a skew product flow (θ,φ) if the *pullback convergence*

$$\lim_{t\to\infty} \operatorname{dist}_X\left(\varphi(t,\theta_{-t}(p),D),A_p\right) = 0 \quad (p \text{ fixed})$$

holds for every nonempty bounded subset D of X and $p \in P$, and a *forward attractor* if the *forward convergence*

$$\lim_{t\to\infty} \operatorname{dist}_X \left(\varphi(t,p,D), A_{\theta_t(p)}\right) = 0$$

holds for every nonempty bounded subset D of X and $p \in P$.

In general, pullback and forward convergence are not equivalent, but it is when the convergence is uniform in p.

The existence of a pullback attractor is ensured by that of a *pullback absorbing set*, i.e., a nonempty compact subset B of X so that for each $p \in P$ and every bounded subset D of X, there exists a $T(p,D) > 0$ such that

$$\varphi(t,\theta_{-t}(p),D) \subset B \quad \text{for all} \quad t \geq T(p,D).$$

Then the components subsets A_p of the pullback attractor are given by

$$A_p = \bigcap_{s>0} \overline{\bigcup_{t>s} \varphi(t,\theta_{-t}(p),B)}$$

for each $p \in P$.

In terms of a process ϕ, ϕ-invariance is written $\phi(t,t_0,A_{t_0}) = A_t$ for all $t \geq t_0$, while pullback attraction is given by

$$\lim_{t_0\to-\infty} \operatorname{dist}_X \left(\phi\left(t,t_0,D\right),A_t\right) = 0 \quad (t \text{ fixed}),$$

where $\mathcal{A} = \{A_t \,:\, t \in \mathbb{R}\}$ is the pullback attractor. It starts progressively earlier in time and thereby involves the past history of the dynamics. This is quite realistic in biological applications when the dynamics is driven by a temporally recurrent system, e.g., periodic, almost periodic or stochastic stationary.

6.2 Application to SIR systems

Pullback convergence was used to construct the explicit nonautonomous entire solutions $\widehat{N}(t)$, $\widehat{S}_1(t)$, $\widehat{S}(t)$, $\widehat{I}(t)$, and $\widehat{R}(t)$ for the SIR and SI systems, which were considered as processes above. They form the coordinate components of the corresponding pullback attractors, which thus consist of singleton subsets. In some cases they were also forward attractors, but need not be as the fact that the disease free entire solution could lose (forward) stability and bifurcate shows. It was clear from the analysis there that the simplicial slabs $\Sigma_3^{\pm}$ and $\Sigma_2^{\pm}$ and simplex Σ_2 are positive invariant under the corresponding solution process. Moreover, after the disease free entire solution loses stability, the corresponding sets minus a very small neighbourhood of the disease free entire solution absorbs the dynamics in finite time (in both pullback and forwards senses). Thus it contains a pullback attractor distinct from the disease free entire solution.

It is clear from the chaotic dynamics under a periodic population reported by O'Regan *et al.* [12] and the periodic interaction parameter in Stone *et al.* [14, 15] that the component sets of the pullback attractor need not be singleton valued. They do, however, contain the chaotic dynamics.

The skew product representation of the SIR and SI dynamics provides more insight into how the pullback attractor component subsets may vary in time, in particular for the just mentioned chaotic dynamics. For example, with periodic time-dependent parameters $g(t)$ or $\gamma(t)$ of period T, the driving system θ_t is periodic with period T, so by φ-invariance, the pullback attractor component sets are also T periodic, since

$$\varphi(T,p,A_p) = A_{\theta_T(p)} = A_{\theta_0(p)} = A_p.$$

Although chaotic, the trajectories in [12] and [14, 15] move around in these periodic subsets of the state space. Even without chaotic dynamics, this also explains the possible existence of subharmonic periodic trajectories under periodic forcing in [13]. (In [12] a sinusoidal interaction parameter is replaced by an alternating piecewise expression, so the system is essentially switching periodically between two autonomous systems; see [8] for the appropriate base space P and topology).

A similar analysis is possible for the random attractors of SIR systems with randomly varying parameters.

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