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A Pre-Encoding–Post-Decoding Technique for Stabilizing Asynchronously Operating Systems ⋆

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Abstract: In the paper, a new method allowing to 'asynchronously stabilize' a broad class of control systems is developed. Viability of the proposed approach is demonstrated by examples among which an application to stabilizing unmanned aerial vehicle systems. Pros and cons of the proposed approach are discussed.

Keywords: Asynchronous systems, stability, convergence analysis

1. INTRODUCTION

The theory of asynchronous systems Asarin et al. (1992); Bertsekas and Tsitsiklis (1989); Kaszkurewicz and Bhaya (2000) got its rather distinctive shape about 20 years ago and nowadays has many connections with switching and discrete-event systems Shorten et al. (2007) and other fields of control theory. It was grounded on quite practical problems concerning functioning of distributed computational networks and from the very beginning demonstrated plenty of mathematically difficult, though easily formulated, problems. Examples of systems, for which the problem of synchronization is acute, are complex digital electronic devices, multiprocessor systems, distributed digital networks, discrete-time models of market economy, various problems of control theory, etc. So, from different points of view it should be very attractive field of investigation for mathematicians. Nevertheless, until now only few practical methods to deal with asynchronous systems are known.

The paper is devoted to discussion of a new method allowing to 'asynchronously stabilize' a broad class of control systems. Advantages of the proposed approach are demonstrated by some examples among which an application to stabilizing unmanned aerial vehicle systems.

2. STATEMENT OF A PROBLEM

Given a system W consisting of subsystems (components, elements, parts) $W_1, W_2, \ldots, W_N$ which, in the course of functioning, can exchange information between each other at time moments $\{t_n\}$ and are influenced by the outdoor environment.

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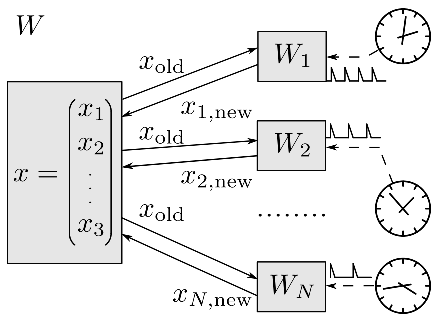

Fig. 1. Example of an asynchronous system

Then, in general, the dynamics of a synchronous system, i.e., the system each components of which are switched simultaneously, is governed by the following equation:

$$x(n+1) = Ax(n) + f(n), \tag{1}$$

where $x(n), x(n+1)$ are the state vectors of a system at the time moments t_n and t_{n+1}, respectively, while A is the transition matrix of a system and $f(n)$ is the vector of external actuations.

In the case when not all the components of a system are switched simultaneously at the time moment t_n, the behavior of the system is described by the following equation:

$$x(n+1) = A_{\omega(n)}x(n) + f_{\omega(n)}, \tag{2}$$

where $\omega(n)$ is the set of indices of components switching at the moment t_n, and $A_{\omega(n)}$ is the matrix row of which with the indices $i \in \omega(n)$ coincide with the corresponding rows of the matrix A whereas the rows with indices $i \notin \omega(n)$ coincide with the corresponding rows of the identity matrix.

One of the principal problems in the theory of asynchronous systems is as follows: what are conditions under which the asynchronous system (2) related to its synchronous counterpart (1) converges to the solution of the linear equation

$$x = Ax + f, \tag{3}$$

for arbitrary choice of the index sequences $\{\omega(n)\}$?

In general, convergence of the synchronous system (1) does not imply convergence of its asynchronous counterpart (2). Moreover, there are examples demonstrating that all possible combinations of stability/instability are possible for the pair 'synchronous system + its asynchronous counterpart' Asarin et al. (1992); Kozyakin (2004). At the same time, classes of matrices A are known for which convergence of the synchronous system (1) implies convergence of its asynchronous counterpart (2). For example, symmetric matrices as well as matrices with non-negative entries the spectral radius of which is less than one form such classes.

It is worth mentioning that the transition from an asynchronous system to asynchronous one is coordinate-dependent Asarin et al. (1992). This observation rises a possibility, in some situations, to bring equations (1) and (2), by means of a change of variables $x = Qy$, to the form

$$y(n+1) = Q^{-1}AQy(n) + Q^{-1}f \tag{4}$$

and

$$y(n+1) = (Q^{-1}AQy(n) + Q^{-1}f)_{\omega(n)}, \tag{5}$$

respectively, where equation (5) becomes convergent to the solution of the linear equation

$$y = Q^{-1}AQy + Q^{-1}f$$

under arbitrary choice of the index sequences $\{\omega(n)\}$, which allows to recover the solution $x = Qy$ of the initial equation $x = Ax + f$.

Unfortunately, not for each pair of systems (1) and (2) one can find a change of variables under which the pair of systems (4) and (5) becomes convergent. Moreover, even in the case when such a desirable change of variables exists it is technically difficult to find an appropriate matrix Q.

Nevertheless, it is possible, at least theoretically, to overcome this seemingly unresolvable situation using the following observation by Daimond and Opoitsev (2001): *if the spectral radius $\rho(A)$ of a $(d \times d)$-matrix A is strictly less than 1 then, for some natural number D, there exist a $(D \times d)$-matrix L, $(d \times D)$-matrix P, and $(D \times D)$-matrix B with non-negative entries such that*

$$LA = BL, \quad AP = PB, \tag{6}$$

with

$$\rho(B) < 1.$$

Relations (6) mean that the matrices L and P are pseudoinverse to each other.

The Diamond-Opoitsev observation provides a ground for developing the following scheme of 'transition of the matrix A to space of higher dimension' which can be treated as the scheme of pre-encoding (transition from the vector x to the vector y of higher dimension to make computations) and post-decoding (transition from the vector y to the original vector $x = Qy$ of lower dimension to make interpretation of computed results).

3. ALGORITHM OF PRE-ENCODING AND POST-DECODING

To be short, let us present a toy example demonstrating how the algorithm of construction of the matrix B works. Consider a system transition between states of which are governed by a 'rotation matrix with contraction':

$$x(n+1) = Ax(n) + f(n),$$

where

$$A = \lambda \cdot \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}, \quad \lambda \in (0,1),\ \alpha \in \mathbb{R}.$$

For a chosen matrix A, we have $\rho(A) = \lambda < 1$. Then there exists a norm $\|\cdot\|_*$ such that $\|A\|_* < 1$. In our case such a norm may be chosen explicitly as the matrix norm generated by the usual Euclidean vector norm $\|x\|_2 = \sqrt{\sum_i x_i^2}$. It's well-known (see, e.g., Horn and Johnson (1994)) $\|A\|_2 = \sqrt{\rho(A^*A)} = \lambda < 1$.

On the plain, the unit ball in the Euclidean norm $\Omega = \{x : \|x\|_2 \le 1\}$ is the circle of radius 1 centered at the origin $(0,0)$. The matrix A maps this ball into itself or, more specifically, into the ball $\Omega_\lambda = \{x : \|x\| \le \lambda\}$, which is the circle of radius λ centered at the origin $(0,0)$.

Let us inscribe an arbitrary convex polygon J in the ring between the balls Ω and Ω_λ, and denote by D the number of vertices of this polygon. Of course the number D will depend on the value of the gap between the balls Ω and Ω_λ, and will grow as the value of the gap decrease. Then the matrix A will map the polygon J onto a polygon $\tilde{J}$ lying inside the ball Ω_λ, see Fig. 2.

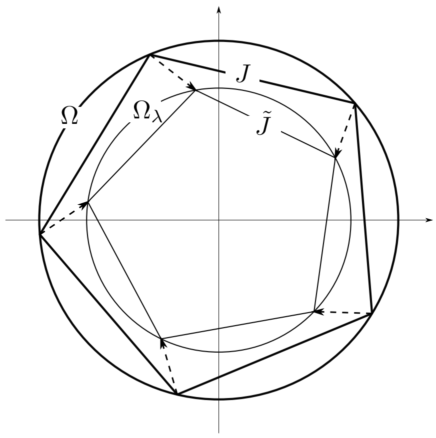

Fig. 2. Polygons J and $\tilde{J}$; case $D = 5$

Now, let us embed the polygon J into D-dimensional space $\mathbb{L} = \mathbb{R}^D$ and define a basis in $\mathbb{L}$ in such a way that the vertices of J were the basis vectors $e_1, e_2, \dots, e_D$, see Fig. 3. Then the convex hull of vectors $e_1, e_2, \dots, e_D$, passing the vertices of J, will form the cone K_+ of vectors with non-negative coordinates in $\mathbb{L}$. Finally, let us define the $(D \times D)$-matrix B as such a matrix which maps the basis vectors $e_1, e_2, \dots, e_D$ in $\mathbb{L}$ (equiv. vertices of J) to appropriate vertices of the polygon $\tilde{J}$. Then it is straightforward to show that B will be desired matrix with nonnegative entries since it maps the cone K_+ into itself.

Summarizing the above reasoning, we need to find a convex polygon J lying in the gap between the balls Ω and Ω_λ, and

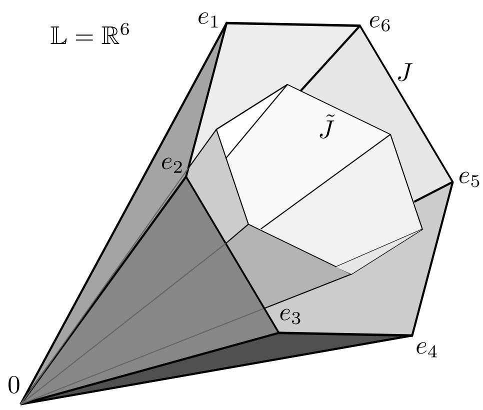

Fig. 3. Polygons J and $\tilde{J}$ embedded in $\mathbb{L}$; case $D = 6$

then to build a matrix B which transforms the polygon J to polygon $\tilde{J}$. Then this operation may be treated as a 'pre-encoding' of the matrix A, and the matrix B automatically will be non-negative. The procedure of computation of the entries of the matrix B is straightforward.

So, describing the work with the iteration process (1) in a bit more formal terms, we should behave as follows:

Step 1: Preparation of a procedure. Given a matrix A with the spectral radius satisfying $\rho(A) < 1$, it is needed to find the number D and matrices L, P and B. Remark that, for any eigenvalue λ of the matrix B, the equality $\lambda x = Bx$ implies

$$|\lambda| \cdot \|x\| = \|Bx\| \leq \|B\| \cdot \|x\| \quad \Rightarrow \quad |\lambda| \leq \|B\|,$$

that is any norm of the matrix B is not less than λ. In particular,

$$\rho(B) \leq \|B\|_1.$$

Step 2: Pre-encoding. To find solutions of the equation

$$x = Ax + f$$

we perform the change of variables $y = Lx, \tilde{f} = Lf$, where the vectors y and $\tilde{f}$ belong to space $\mathbb{R}^D$ of rather high dimension.

Step 3: Asynchronous computations with encoded data. To construct successive approximations we consider the following asynchronous procedure

$$y(n+1) = B_{\omega(n)} y(n) + \tilde{f}_{\omega(n)}.$$

By construction of the matrix B, this last iteration procedure is convergent for any choice of the index sequence $\{\omega(n)\}$.

Step 4: Post-decoding. To compute successive approximations to the solution of the initial equation

$$x = Ax + f$$

it suffices to perform the 'backward' change of variables $x(n) = Py(n)$ (post-decoding).

4. TESTING

Let us consider the iteration procedure

$$x(n+1) = Ax(n) + f(n),$$

where

$$A = \lambda \cdot \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

and

$$\lambda = 0.99, \quad \alpha = \frac{2\pi}{3}, \quad x_0 = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}, \quad f = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}$$

Synchronous computations for this procedure are convergent whereas asynchronous computations are divergent which is demonstrated by Fig. 4.

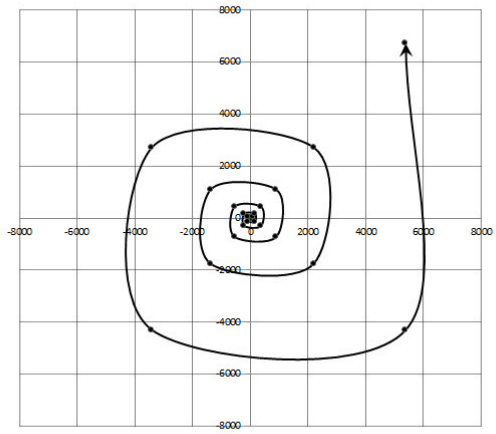

Fig. 4. Divergence of asynchronous procedure

To implement the asynchronous procedure with pre-encoding we first computed dimension D of the matrix B which is in our case equals to 23. We omit the entries of the matrix B due to its high dimension, and only remark that

$$\rho(B) \leq \|B\|_1 = \max_j \sum_i |b_{ij}| = 0.9999 < 1$$

The rate and details of convergence for the both synchronous and asynchronous procedures are plotted in Figs. 5 and 6. More specifically, Fig. 6 plots the second coordinate of the vectors $x(n)$ in the procedure

$$x(n+1) = Ax(n) + a(n) \begin{pmatrix} \cos\alpha \\ 0 \end{pmatrix}$$

with a periodic quantity $a(n)$.

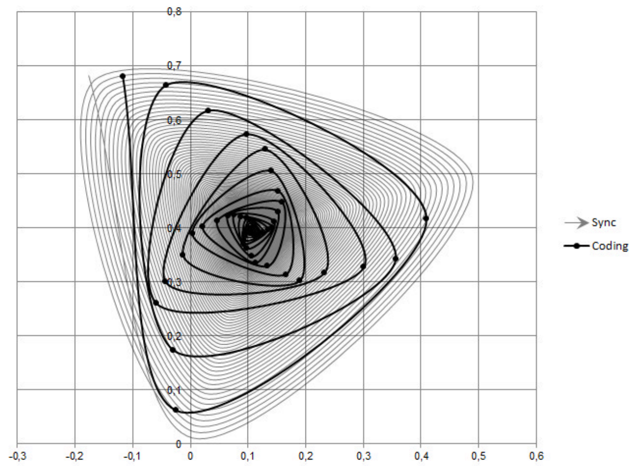

Fig. 5. Successive approximations for synchronous and pre-encoded asynchronous processes

5. APPLICATION TO STABILIZING UNMANNED AERIAL 2-VEHICLE SYSTEM

In investigation of distributed computations modeling dynamics of the autonomous flight of an unmanned aerial

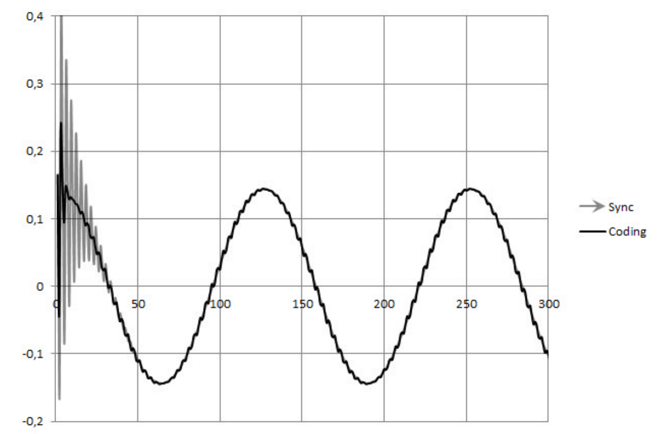

Fig. 6. Convergence of synchronous and pre-encoded asynchronous processes with periodic external excitation

vehicle system (UAVs) consisting of several inter-operating unmanned aerial vehicles (UAVs), the problem of tracking of one vehicle by another is of great importance. One of the main difficulties here is that re-formation of a flown UAV relative to a lead UAV is affected by considerable time lags.

In a simplified form, the dynamics of the UAV's flight altitude is governed by a third-order differential equation. The related difference scheme for such an equation is as follows:

$$\begin{aligned} x_1(t+1) &= \alpha x_1(t) \\ &\quad - k_1 x_2(t) + (x_{in}(t+1) - x_3(t)) \\ x_2(t+1) &= x_2(t) + x_1(t)\delta t \\ x_3(t+1) &= x_3(t) + x_2(t)\delta t \end{aligned} \quad (7)$$

where x_3 is the observed flight altitude of an UAV, $x_2 = \frac{dx_3}{dt}$ is the velocity of changing of the flight altitude, and $\delta t = 0.1$ sec. is the sampling period for coordinates.

The tracking system of the flown UAV tracks the coordinate value x_3 of the lead UAV. Typically, this is accompanied by a considerable time lag in tracking of x'_3 (flown UAV flight altitude) comparing with x_3 (lead UAV flight altitude). The related results of computer testing are presented in Fig. 7.

We modeled the dynamics of the flight of two UAVs by distributed computations, where the dynamics of UAVs is described by six-order difference equations with independent quantization periods close to $\delta t = 0.1$ sec. and independent tracking of the flown UAV coordinates with period close to $\delta t = 0.5$ sec. Our aim was to try to change the dynamics of UAVs consisting of two vehicles in such a way to improve the accuracy of tracking the coordinates of the lead UAV by the flown UAV. In order to do it, we extended the phase space of the system. For the sake of convenience, we enlarged only dimension of the phase space of the flown UAV. Let us note that in our case the related asynchronous system is stable. Therefore it was reasonable to try to correct dynamics of the system by a moderate expanding of the phase space. It turned out that in our case the double enlargement of space dimension was enough. The difference equations for the enlarging filter of the second order are as follows:

$$\begin{aligned} \hat{x}_2(t+1) &= \hat{x}_2(t) + \gamma_1(x_3(t+1) - \hat{x}_3(t)), \\ \hat{x}_3(t+1) &= \hat{x}_3(t) + \hat{x}_2(t)\delta t + \gamma_2(x_3(t+1) - \hat{x}_3(t)), \end{aligned}$$

where the coordinate $\hat{x}_2$ tracks the velocity x_2 of the changing of the altitude x_3, while the coordinate $\hat{x}_3$ tracks the altitude x_3.

This, together with the difference scheme (7), yields the difference equation of the fifth order. Our goal was to minimize the lead UAV altitude tracking error

$$|x_3(t+1) - x'_3(t)|$$

by adjusting the coefficients γ_1 and γ_2 in the above equations. Results of modeling of the tracking dynamics in the extended space are presented in Fig. 8.

Relying on the presented example, we can conclude that the technique of extending the phase space might be an efficient tool of improving the quality of transient processes in asynchronous systems. Contrary to the case of synchronous systems, in the case of asynchronous systems there arise a need to synchronize computations in the extended system with those in the original one. Unfortunately, in some cases this may lead to essential increasing of dimension of the phase space of asynchronous system comparing with that of synchronous one. Also it is worth mentioning that dynamics of the extended system depends on the sampling periods for coordinates, and it is required to take care of physical realizability of the enlarging filter.

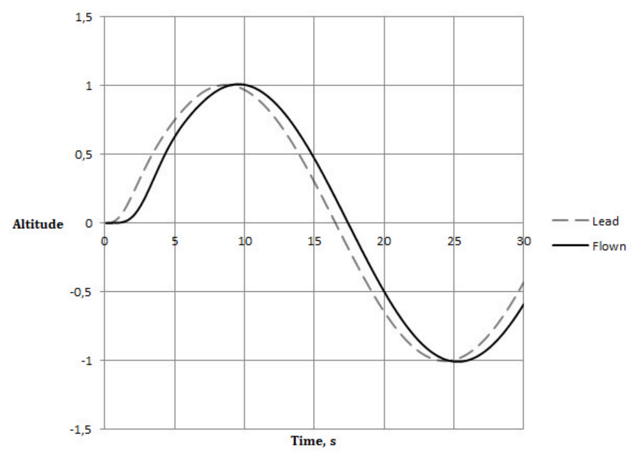

Fig. 7. Tracking the flight altitude of a host UAV

6. DISCUSSION

Let us mention briefly principal pros and cons for the approach proposed in the paper.

6.1 Pros

The main advantage of the proposed approach is that it justifies principal possibility to use asynchronous procedures to compute solutions of equation (1), for arbitrary matrix with the spectral radius not exceeding 1.

Another possible advantage of the proposed approach is that the operands of a computational procedure, i.e., processors computing the vectors y_3 on Step 3, will work with encoded data whereas the initial encoding of the data on Steps 1 and 2 and further decoding on Step 4 will be

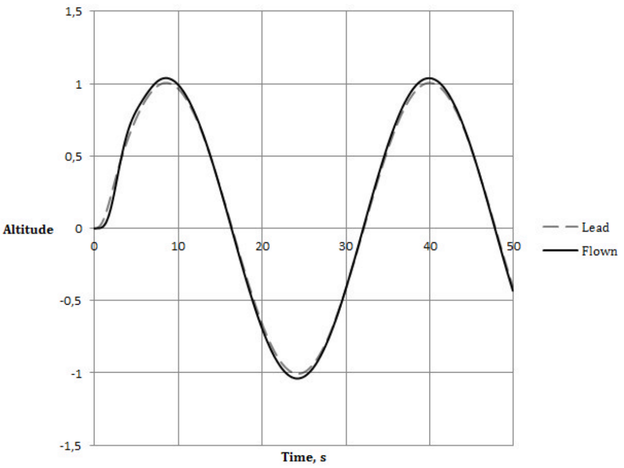

Fig. 8. Tracking the flight altitude of a host UAV in an extended phase space

done by a 'problem originator'. In some cases the fact that real computational data are hidden from view of operands of a computational procedure might be essential.

6.2 Cons

The main disadvantage of the proposed approach is that dimension D of the matrix B is much higher than the dimension d of the initial matrix A. A rough estimate shows that $D \simeq (1 - \rho(A))^{-(d-1)/2}$.

At the same time, it is worth mentioning that the matrix B is turned out to be a sparse matrix with each its row and column containing no more than $d + 1$ non-vanishing entries. As is known, this essentially lowers the computing load at work with such matrices. Besides, more elaborate means of construction of the matrix B might also to lower the value of D essentially.

7. CONCLUSION

The proposed approach of transition to asynchronous procedures with guaranteed convergence is far from being perfect and from practical implementation. Nevertheless, in this work, we tried to demonstrate that, at least theoretically, construction of asynchronous procedures with guaranteed convergence *is possible* in rather general situations. This means that intensive investigations in the indicated direction might be fruitful.

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