ARNOLD TONGUES FOR BIFURCATION FROM INFINITY

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ABSTRACT. We consider discrete time systems $x_{k+1} = U(x_k; \lambda)$, $x \in \mathbb{R}^N$, with a complex parameter λ. The map $U(\cdot; \lambda)$ at infinity contains a principal linear term, a bounded positively homogeneous nonlinearity, and a smaller part. We describe the sets of parameter values for which the large-amplitude n-periodic trajectories exist for a fixed n. In the related problems on small periodic orbits near zero, similarly defined parameter sets, known as Arnold tongues, are more narrow.

1. **Introduction.** We study bifurcations of large periodic trajectories of the system

$$x_{k+1} = U(x_k; \lambda), \qquad x \in \mathbb{R}^N, \; N \geq 2, \tag{1}$$

with the complex parameter $\lambda \in D \subset \mathbb{C}$. We suppose that for sufficiently large $|x|$ the map U is continuous with respect to the set of its arguments and has the form

$$U(x; \lambda) = A(\lambda)x + \Phi(x; \lambda) + \xi(x; \lambda) \tag{2}$$

where $A(\lambda)x$ is the principal linear part, $\Phi(\cdot; \lambda)$ is a bounded positively homogeneous nonlinearity of order 0, and $\xi(\cdot; \lambda)$ tends to zero at infinity.

Given a $\lambda \in D$ consider the set $\mathfrak{P}_\lambda^n \subset \mathbb{R}^N$ of all n-periodic points of system (1). Representation (2) implies that if the matrices $A(\lambda)$ for $\lambda \in D$ have no eigenvalues in a fixed vicinity of the finite set $\{e^{2\pi mi/n}, \; m \in \mathbb{Z}\}$, then the union $\mathfrak{P}^n = \bigcup_{\lambda \in D} \mathfrak{P}_\lambda^n$ is bounded. If $A(\lambda)$ has a pair of simple complex conjugate eigenvalues $\mu(\lambda), \bar{\mu}(\lambda)$ and the range of $\mu(\lambda)$ has a limit point $e^{2\pi mi/n}$, then the set $\mathfrak{P}^n$ may be unbounded.

The union $\mathfrak{P} = \bigcup_n \mathfrak{P}^n$ of all periodic points is unbounded in the two cases:

(i) At least one set $\mathfrak{P}^n$ of periodic points of a fixed period n is unbounded;

(ii) The set $\mathfrak{P}$ is unbounded, while each set $\mathfrak{P}^n$ is bounded. Hence, the periods increase to infinity together with the norms of periodic points.

Cases (i) and (ii) are referred to as subharmonic bifurcation and subfurcation [1], respectively. We shall present sufficient conditions for the subharmonic bifurcation.

Let us recall some relevant well-known facts about bifurcations of periodic trajectories from the zero equilibrium of a smooth system (1). Let for small $|x|$ (we denote by $|\cdot|$ a norm in $\mathbb{R}^N$ as well as the module of complex numbers)

$$U(x;\lambda) = A(\lambda)x + A_2(x;\lambda) + A_3(x;\lambda) + \ldots, \tag{3}$$

where A_s are polynomials of degree s with respect to x-variables. Then, generically, for any rational $q = m/n$ with coprime positive integer m and n there exists an open connected set $\mathfrak{A}_q$ called the *Arnold tongue*[1] [2] with $\lambda_q = e^{2\pi qi} \in \overline{\mathfrak{A}}_q$ such that system (1) has a periodic point of the period n whenever $A(\lambda)$ has an eigenvalue $\mu(\lambda)$ in $\mathfrak{A}_q$ and the norms of the n-periodic points tend to zero as $\mu(\lambda)$ tends to λ_q along the set $\mathfrak{A}_q$ (see, e.g. [3]). The values $q = m/n$ with $n = 1, 2, 3, 4$ are called strong resonances. For any q with $n > 4$, the Arnold tongue $\mathfrak{A}_q$ has the shape shown on the left part of Fig. 1. Locally, $\mathfrak{A}_q$ is a cusp between two curves that meet at the point λ_q and have the same tangent straight line at this point. Moreover, the width of the cusp at a distance d from the vertex λ_q is of order $d^{n/2-1}$ as $d \to 0$ [4].

FIGURE 1. Typical shapes of the Arnold tongues at zero and at infinity.

Here, we prove that at infinity the Arnold tongues are thick sectors shown on the right picture of Fig. 1. The angle between the boundary lines of any tongue $\mathfrak{A}_q$ at its vertex λ_q is positive except for non-generic degenerate situations.

Assume that the parameter λ varies so that the eigenvalue $\mu(\lambda)$ of $A(\lambda)$ goes along a smooth curve Γ through a point λ_q of the unit circle with a rational q, and ℓ is a tangent line of Γ at this point. According to the approach of [3], one observes the subharmonic bifurcation if some arc of the curve Γ with one end at λ_q is contained in the Arnold tongue $\mathfrak{A}_q$, and the subfurcation if the curve Γ crosses infinitely many Arnold tongues in any neighborhood of the point λ_q. Fig. 1 shows that the subharmonic bifurcation can occur for a unique direction of ℓ in the local problem at an equilibrium point of smooth system (1) (and thus the subharmonic bifurcation is a specific non-generic case here except for points of strong resonances), while there is a bunch of such directions in the problem at infinity.

The shapes of the Arnold tongues at the points of strong resonances in the bifurcation problem at the origin are similar to that of the generic Arnold tongues $\mathfrak{A}_q$ at infinity. Moreover, there are indications that this analogy can be extended to other aspects of dynamics at infinity (at least for finite ranges of the denominator n of $q = m/n$, where a particular range is defined by the nonlinearity Φ in representation (2) and can be arbitrarily large). Our numerical experiments with systems (1) near the points λ_q reveal a rich variety of dynamical scenarios at infinity,

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[1]The wording "Arnold tongue" is used in various bifurcation problems to describe two dimensional parameter sets of special form.

which are typical for strong resonances of smooth systems at the origin. This paper concentrates on periodic trajectories and triangle shapes of the Arnold tongues.

A simple change of variables of the type $x \to Bx/|x|^2$ transforms the bifurcation problem at infinity to that near the origin. Hence all our results can be reformulated in terms of the local behavior of the transformed system near its zero equilibrium. The evolution operator of the latter system does not have the form (3).

2. Main results.

2.1. Assumptions on the map.
Consider system (1) with the map U of the form (2). Assume that the function $\Phi = \Phi(x; \lambda)$ is positively homogeneous of order zero in x for each $\lambda \in D$, and the term $\xi = \xi(x; \lambda)$ vanishes at infinity, i.e.

$$\Phi(\theta x; \lambda) = \Phi(x; \lambda) \quad \text{for all} \quad |x| = 1, \ \theta > 0, \ \lambda \in D; \quad \xi(x, \lambda) \to 0 \ \text{as} \ x \to \infty. \quad (4)$$

For the sake of simplicity it is assumed that λ and $\bar{\lambda}$ are the eigenvalues of the matrix $A(\lambda)$, i.e. a simple complex eigenvalue λ of $A = A(\lambda)$ is used as a parameter of the system. Throughout the paper it is supposed that λ ranges over some neighborhood $D \in \mathbb{C}$ of a closed subset Λ of the unit circle $\{\lambda \in \mathbb{C} : |\lambda| = 1\}$ and that all the eigenvalues $\sigma_j = \sigma_j(\lambda)$ of $A(\lambda)$ different from λ and $\bar{\lambda}$ satisfy

$$||\sigma_j(\lambda)| - 1| \geq \delta > 0 \qquad \text{for all} \qquad \lambda \in D, \ j = 1, \ldots, N - 2. \qquad (5)$$

All the three terms in the right-hand side of (2) are supposed to be continuous with respect to the set of their arguments x and λ. Non-constant homogeneous functions Φ satisfying (4) have discontinuity at the origin. We consider our system for $x \neq 0$.

2.2. Main theorem.
Denote by E_λ the proper plane of the matrix $A(\lambda)$ corresponding to the pair of its eigenvalues $\lambda, \bar{\lambda}$ and by E'_λ the proper $(N-2)$-dimensional subspace of $A(\lambda)$, which is complementary to E_λ. Denote by P_λ the linear projector onto the plane E_λ along the subspace E'_λ; P_λ commutes with $A(\lambda)$. Choose a basis $\{e_1^\lambda, e_2^\lambda\}$ in E_λ depending continuously on λ such that $A(\lambda)$ has the form

$$\begin{pmatrix} \Re e \, \lambda & -\Im m \, \lambda \\ \Im m \, \lambda & \Re e \, \lambda \end{pmatrix}$$

on E_λ. Let Q_λ be a linear invertible map of the complex plane $\mathbb{C}$ onto E_λ defined by $Q_\lambda 1 = e_1^\lambda, Q_\lambda i = e_2^\lambda$. Hence Q_λ satisfies

$$Q_\lambda^{-1} A(\lambda) Q_\lambda \, z = \lambda z, \qquad z \in \mathbb{C}. \qquad (6)$$

Formulations below are independent of the particular choice of the map Q_λ satisfying (6) and the norm $|\cdot|$ in $\mathbb{R}^N$. The map Q_λ defines the complex structure in E_λ, which simplifies the notation below.

Consider the complex-valued 2π-periodic function

$$\Psi_\lambda(\varphi) = Q_\lambda^{-1} P_\lambda \Phi(Q_\lambda e^{i\varphi}; \lambda), \qquad \varphi \in \mathbb{R}, \qquad (7)$$

and its Fourier series

$$\Psi_\lambda(\varphi) = \sum_{k=-\infty}^{\infty} \psi_k^\lambda e^{ik\varphi}, \qquad \psi_k^\lambda \in \mathbb{C}. \qquad (8)$$

For any rational $q = m/n > 0$ (here and henceforth m and n are coprime integers) denote $\lambda_q = e^{2\pi q i}$ and define the continuous 2π-periodic function $\Psi_q^{res} : \mathbb{R} \to \mathbb{C}$ by

$$\Psi_q^{res}(\varphi) = -\sum_{k=-\infty}^{\infty} \psi_{kn+1}^{\lambda_q} e^{ik\varphi} \equiv -\frac{e^{-i\varphi/n}}{n} \sum_{j=0}^{n-1} \Psi_{\lambda_q}\left(\frac{\varphi + 2j\pi}{n}\right) e^{-2j\pi i/n}. \qquad (9)$$

Suppose Ψ_q^{res} is not identically zero and consider the real function $\arg \Psi_q^{res}(\varphi)$ on the open domain $\mathfrak{D}_q = \{\varphi \in \mathbb{R} : \Psi_q^{res}(\varphi) \neq 0\}$. This domain is a unification of disjoint intervals $d^\mu = (\varphi_1^\mu, \varphi_2^\mu)$ (here $\mu \in \mathcal{M}$ is an index, the set $\mathcal{M}$ may be finite or countable). Fix some continuous branch of the complex argument so that the function $\arg \Psi_q^{res}(\varphi)$ is continuous on every interval $d^\mu \in \mathfrak{D}_q$. Denote by $\mathfrak{B}_q$ the range of the function $\arg \Psi_q^{res} : \mathfrak{D}_q \to \mathbb{R}$. Our main assumption is that the interior $\operatorname{Int} \mathfrak{B}_q$ of the set $\mathfrak{B}_q$ is non-empty:

$$\operatorname{Int} \mathfrak{B}_q \neq \varnothing. \tag{10}$$

If $\Psi_q^{res}(\varphi) \neq 0$ for all $\varphi \in \mathbb{R}$, then $\operatorname{Int} \mathfrak{B}_q$ is an open interval whenever it is non-empty. For example, this is always the case if $\psi_1^{\lambda_q} \neq 0$ and n is sufficiently large. If the function Ψ_q^{res} has zeros, then the set $\operatorname{Int} \mathfrak{B}_q$ may be a union of disjoint intervals.

Denote by $\mathfrak{R}^\mu$ the range of the function $\arg \Psi_q^{res}$ on the interval d_μ. If (10) holds, then at least one interval $\mathfrak{R}^\mu$ has interior points, therefore the set $\mathfrak{B}^* = \bigcup_{\mu \in \mathcal{M}} \operatorname{Int} \mathfrak{R}^\mu$ is nonempty. Generically, $\mathfrak{B}^* = \operatorname{Int} \mathfrak{B}_q$, counterexamples are cumbersome and degenerate.

Theorem 1. *Let (10) hold for some rational positive $q = m/n$. Let $\mathfrak{B} \subset \mathfrak{B}^*$ be a closed non-empty set. Then there are positive numbers $\varepsilon = \varepsilon(q, \mathfrak{B})$ and $r_j = r_j(q, \mathfrak{B})$, $j = 1, 2$ such that for every λ from the set*

$$S_q = S_q(\mathfrak{B}) = \{\lambda \in D \subset \mathbb{C} : \ 0 < |\lambda - \lambda_q| \leq \varepsilon, \ \arg(\lambda - \lambda_q) \in \mathfrak{B}\}$$

system (1) has at least one periodic point x_λ with the minimal period n and a norm $|x_\lambda| \in \left(r_1|\lambda - \lambda_q|^{-1}, r_2|\lambda - \lambda_q|^{-1}\right)$.

Assume that we let λ vary along the ray $\arg(\lambda - \lambda_q) = \nu$ for some fixed $\nu \in \mathfrak{B}^*$. Consider the points φ_ν satisfying $\arg \Psi_q^{res}(\varphi_\nu) = \nu$. Let us call φ_λ robust if the function $\arg \Psi_q^{res}$ is strictly monotone in a vicinity of φ_λ. Generically, all φ_λ are robust and there exists a finite number of such points. Each robust point φ_ν defines the n-periodic orbit $(x_1, \ldots, x_n)$, which satisfies asymptotically

$$x_k = |\lambda - \lambda_q|^{-1} |\Psi_q^{res}(\varphi_\nu)| Q_{\lambda_q}(e^{i(\varphi_\nu^j + 2\pi k)/n}) + o(|\lambda - \lambda_q|^{-1})$$

as $|\lambda - \lambda_q| \to 0$. Thus, the number of periodic orbits for a fixed ν is generically equal to the number of robust points φ_ν if these points are identified modulo 2π.

Let an open set $\mathfrak{B}'$ contain the closure $\overline{\mathfrak{B}}_q$ of the set $\mathfrak{B}_q$. It follows from the proof of Theorem 1 that system (1) does not have n-periodic points with a sufficiently large norm for $\arg(\lambda - \lambda_q) \notin \mathfrak{B}'$ if $|\lambda - \lambda_q|$ is sufficiently small. Hence Theorem 1 describes pretty accurately the Arnold tongue near its vertex λ_q.

2.3. **Geometric interpretation of Theorem 1.** In two dimensions, system (1) can be rewritten as $z_{k+1} = U(z_k; \lambda)$ (everything is complexified) where

$$U(z; \lambda) = \lambda z + \Phi(e^{i\varphi}; \lambda) + o(1), \quad z = re^{i\varphi} \in \mathbb{C}. \tag{11}$$

We consider values of the parameter λ from a small neighborhood of the point $\lambda_q = e^{2\pi q i} \in \mathbb{C}$ of the unit circle $S \in \mathbb{C}$, $q = m/n$. Formula (11) implies the representation $U^n(z; \lambda) = \lambda^n z - n\lambda_q^{-1} e^{i\varphi} \Psi_q^{res}(n\varphi) + o(1)$ for the n-th iteration U^n of the operator U, where the last term vanishes as $z \to \infty$, $\lambda \to \lambda_q$ and Ψ_q^{res} is defined by (9):

$$\Psi_q^{res}(\varphi) = -\frac{e^{-i\varphi/n}}{n} \sum_{j=0}^{n-1} \Phi(e^{(\varphi+2j\pi)i/n}; \lambda_q) e^{-2j\pi i/n}.$$

Consequently, using the new small parameter $\xi = \lambda - \lambda_q$ and the approximation $\lambda^n = 1 + \lambda_q^{-1} n\xi + o(\xi)$, $\xi \to 0$, we can write the equation $z - U^n(z;\lambda) = 0$ for the n-periodic points of U in the form $z(n\lambda_q^{-1}\xi + o(\xi)) = n\lambda_q^{-1}e^{i\varphi}\Psi_q^{res}(n\varphi) + o(1)$. Now, neglecting the small terms we arrive at the natural approximation

$$\xi r = \Psi_q^{res}(n\varphi) \tag{12}$$

to the equation $z = U^n(z;\lambda)$ with $r = |z|$, $\varphi = \arg z$. In the proof of Theorem 1 we show that solutions of the equations $z = U^n(z;\lambda)$ and (12) are close for small ξ.

FIGURE 2. Arnold tongues for $\Phi(e^{i\varphi};\lambda) = 3|\sin\varphi| + i\cos\varphi$ and $n = 5$.

The solutions $z = re^{i\varphi}$ of equation (12) and the set of parameter values ξ for which these solutions exist are described straightforwardly in terms of the curve $\Gamma_q = \{\Psi_q^{res} : \mathbb{R} \to \mathbb{C}\}$ and the set $\mathfrak{S}_q = \{\tilde{\xi} \in \mathbb{C} : \tilde{\xi} = \theta\Psi_q^{res}(\varphi),\ \theta \geq 0, \varphi \in \mathbb{R}\}$, which is generically an angle, but may have a more complex structure (see the right picture of Fig. 3). Indeed, (12) has a non-zero solution z for $\xi \neq 0$ iff $\xi \in \mathfrak{S}_q$. Moreover, each transversal intersection of the ray $L_\xi = \{\tilde{\xi} \in \mathbb{C} : \tilde{\xi} = \theta\xi,\ \theta > 0\} \subset \mathfrak{S}_q$ with the curve Γ_q defines a different solution $z_\xi^j = r_\xi^j e^{i\varphi_\xi^j}$ with $0 \leq n\varphi_\xi^j < 2\pi$. Because the function Ψ_q^{res} is 2π-periodic, the n points $z_\xi^{j,s} = r_\xi^j e^{i\varphi_\xi^j + 2\pi s/n}$, $s = 0, ..., n-1$, are solutions of (12) together with each point $z_\xi^{j,0} = z_\xi^j$: these n points approach (in appropriate sense) a n-periodic orbit of the system $z_{k+1} = U(z_k;\lambda)$ as $\xi \to 0$ under the conditions of Theorem 1. Thus, the number of large n-periodic orbits generically equals the number of intersections of L_ξ and Γ_q for small ξ.

Figs. 2 and 3 show various curves $\Gamma = \Gamma_q$ and the corresponding sets $\mathfrak{S} = \mathfrak{S}_q$ (shadowed). The right picture of Fig. 2 represents the simplest (also, 'the most generic') situation. Here the origin lies outside the region bounded by a simple curve Γ_q and each ray $L_\xi \subset \mathfrak{S}_q$, except for the two rays forming the boundary of the angle $\mathfrak{S}_q$, crosses Γ_q at two points. Another generic case is shown on the left picture of Fig. 3: a simple curve Γ_q surrounds the origin and the set $\mathfrak{S}_q$ coincides with $\mathbb{C}$; on this picture, each ray L_ξ crosses Γ_q at one point. However, if n is sufficiently large then a necessary condition for $\mathfrak{S}_q = \mathbb{C}$ is $\int_0^{2\pi} \Psi_q^{res}(\varphi)\, d\varphi = 0$. Two other pictures of Fig. 3 show exotic cases of co-dimensions 1 and 2.

The non-degeneracy condition (10) means that the angle $\mathfrak{S}_q$ has a non-empty interior $\text{Int}\,\mathfrak{S}_q$. Consider any closed angle $\tilde{\mathfrak{S}} \subset \text{Int}\,\mathfrak{S}_q \cup \{0\}$. Theorem 1 states that the intersection Ξ_ε of the angle $\tilde{\mathfrak{S}}$ with the punctured circle $0 < |\lambda - \lambda_q| \leq \varepsilon$ of a sufficiently small radius ε becomes a part of the Arnold tongue $\mathfrak{A}_q$ after shifting by λ_q (see, Fig. 2), i.e. the map (11) has at least one n-periodic orbit whenever

FIGURE 3. Various curves $\Gamma = \Gamma_q$ and sets $\mathfrak{S} = \mathfrak{S}_q$ (shadowed).

$\lambda - \lambda_q \in \Xi_\varepsilon$ and the norms of the points of this orbit are of order $|\lambda - \lambda_q|^{-1}$. Here ε depends on the angle distance between the angle $\tilde{\mathfrak{S}}$ and the boundary of $\mathfrak{S}_q$. The Arnold tongue $\mathfrak{A}_q$ with an acute angle at its vertex λ_q is directed inside the unit circle if $\Re e(\lambda_q^{-1}\Psi_q^{res}(\varphi)) < 0$ for all φ, outside the unit circle if $\Re e(\lambda_q^{-1}\Psi_q^{res}(\varphi)) > 0$ for all φ, and covers a part of the unit circle if the function $\Re e(\lambda_q^{-1}\Psi_q^{res}(\varphi))$ takes values of the both signs.

2.4. **Poincare maps of continuous time systems.** A natural class of discrete time systems (1) is generated by Poincare maps U of ODEs $x' = g(t,x)$, $x \in \mathbb{R}^N$, with T-periodic in t smooth right-hand sides. If $g(t,x) = Bx + G(t,x) + o(1)$, $|x| \to \infty$, where B is a $N \times N$ matrix and $G(t,x) \equiv G(t,x/|x|)$ for $x \neq 0$, then the Poincare map has the form $U(x) = Ax + \Phi(x) + o(1)$, $|x| \to \infty$ (which is (2) without a parameter), where $A = e^{TB}$ and the bounded term

$$\Phi(x) = e^{TB}\int_0^T e^{-tB}G(t, e^{TB}x)\,dt, \quad x \in \mathbb{R}^N,$$

is positively homogeneous of order zero: $\Phi(x) \equiv \Phi(x/|x|)$. Consequently, the above results may be reformulated for such ODEs depending on a parameter.

Consider the simplest two-dimensional example written in the complex form:

$$z' = \nu z + a(t)f(z) + b(t), \qquad z \in \mathbb{C}, \tag{13}$$

where $a = a(t)$, $b = b(t)$ are complex valued 2π-periodic functions and ν is the complex parameter. Assume that $f : \mathbb{C} \to \mathbb{C}$ has a radial limit F at infinity:

$$\lim_{\theta \to \infty} \sup_{z \in S} |f(\theta z) - F(z)| = 0, \qquad F(z) = F(z/|z|). \tag{14}$$

Then the Poincare map of equation (13) equals $U(z;\nu) = \lambda z + \Phi(z;\nu) + o(1)$ in a vicinity of infinity, where $\lambda = e^{2\pi\nu}$ is our standard parameter, which lies on the complex unit circle if ν is on the imaginary axis; the positively homogeneous term Φ coincides with the function (7) and equals

$$\Phi(z;\nu) = \Psi_\lambda(\varphi) = \int_0^{2\pi} e^{\nu(2\pi - t)}\big(a(t)F(e^{\nu t + i\varphi}) + b(t)\big)\,dt,$$

where $\varphi = \arg z$. Now, it is easy to see that given a $\nu = iq$ and the corresponding $\lambda_q = e^{2\pi\nu} = e^{2\pi q i}$ with any rational $q = m/n$, function (9) is defined by

$$\Psi_q^{res}(n\varphi) = -\frac{e^{i(2\pi q - \varphi)}}{n}\int_0^{2n\pi} e^{-iqt}a(t)F(e^{i(qt+\varphi)})\,dt$$

(the integral of $e^{-iqt}b(t)$ over the segment $[0, 2n\pi]$ is zero for each q). Equivalently, in terms of the Fourier coefficients of the functions

$$a(t) = \sum_{k=-\infty}^{\infty} \alpha_k e^{ikt}, \qquad F(e^{i\varphi}) = \sum_{k=-\infty}^{\infty} \phi_k e^{ik\varphi},$$

one obtains

$$\Psi_q^{res}(\varphi) = -2\pi e^{2\pi qi} \sum_{k=-\infty}^{\infty} \alpha_{-mk} \phi_{nk+1} e^{ik\varphi}.$$

If $a(t) \equiv const$, then Theorem 1 is not applicable, since $\Psi_q^{res} \equiv const$ in this case.

The radial limit F in (14) is discontinuous in some natural applications. A simple example is the real scalar equation $x'' + \nu_1 x' + \nu_2 x = b(t) + a(t)f(x)$, where f is a nonlinearity with saturation: $f(x) \to f_\pm$ as $x \to \pm\infty$ with $f_+ \neq f_-$. For small ν_1 and positive ν_2 this equation has the complex form (13) with ν close to the imaginary axis. The corresponding limit function $F(z) = F(e^{i\varphi})$ is discontinuous; more precisely, it is piecewise constant with jumps at the points $\varphi = \varphi_0 + \pi k$, $k \in \mathbb{Z}$. However, the Poincare map is continuous and Theorem 1 can be applicable to the discrete time system generated by this map.

3. Proofs.

3.1. Reduction to planar case and the main lemma.

We denote by E_λ^+ the proper subspace of the matrix $A(\lambda)$ corresponding to all its eigenvalues $\sigma_j(\lambda)$ satisfying $|\sigma_j(\lambda)| > 1$ and by E_λ^- the proper subspace of $A(\lambda)$ corresponding to its eigenvalues $\sigma_j(\lambda)$ satisfying $|\sigma_j(\lambda)| < 1$. Consequently, $E_\lambda^+ \oplus E_\lambda^- = E_\lambda'$ and $E_\lambda^+ \oplus E_\lambda^- \oplus E_\lambda = \mathbb{R}^N$. Let P_λ^+ be the linear projector onto the subspace E_λ^+ along the complementary proper subspace $E_\lambda^- \oplus E_\lambda$ of $A(\lambda)$ and let P_λ^- be the linear projector onto E_λ^- along the complementary proper subspace $E_\lambda^+ \oplus E_\lambda$ of $A(\lambda)$ so that $P_\lambda^+ + P_\lambda^- + P_\lambda = I$. By these definitions, there is a norm $|\cdot|_\lambda$ in $\mathbb{R}^N$ depending continuously on λ and a $\kappa < 1$ such that for all $\lambda \in D$

$$|A(\lambda)x|_\lambda \leq \kappa|x|_\lambda, \quad x \in E_\lambda^-; \qquad |A(\lambda)x|_\lambda \geq \kappa^{-1}|x|_\lambda, \quad x \in E_\lambda^+. \qquad (15)$$

From (15) it follows the existence of a $\beta > 0$ such that

$$|P_\lambda^- U(x; \lambda)|_\lambda < \beta \qquad \text{if} \qquad |P_\lambda^- x|_\lambda \leq \beta, \qquad (16)$$

$$|P_\lambda^- U(x; \lambda)|_\lambda < |P_\lambda^- x|_\lambda \qquad \text{if} \qquad |P_\lambda^- x|_\lambda \geq \beta, \qquad (17)$$

$$|\eta P_\lambda^+ U(x; \lambda) + (1 - \eta)P_\lambda^+ A(\lambda)x|_\lambda > |P_\lambda^+ x|_\lambda \qquad \text{if} \qquad |P_\lambda^+ x|_\lambda \geq \beta \qquad (18)$$

for every $\lambda \in D$ and every $\eta \in [0, 1]$. Fix β up to the end of the proofs. Estimates (16), (17), and (18) (the latter with $\eta = 1$) imply that all the periodic points of the map $U(\cdot; \lambda)$ lie in the interior of the set

$$\Omega_\lambda = \{x \in \mathbb{R}^N : |P_\lambda^- x|_\lambda \leq \beta, |P_\lambda^+ x|_\lambda \leq \beta\}.$$

Consider the nonlinear projector onto the set $\{x \in \mathbb{R}^N : |P_\lambda^+ x|_\lambda \leq \beta\}$ defined by

$$W_\lambda(x) = x + \left(|P_\lambda^+ x|_\lambda^{-1} \min\{\beta, |P_\lambda^+ x|_\lambda\} - 1\right)P_\lambda^+ x$$

and define the map

$$V_\lambda(x) = U(W_\lambda(x); \lambda), \qquad x \in \mathbb{R}^N, \lambda \in D.$$

If all the eigenvalues of $A(\lambda)$ different from $\lambda, \bar{\lambda}$ have module less than 1, then $V_\lambda(x) = U(x; \lambda)$ for all $x \in \mathbb{R}^N$; if not, then $V_\lambda(x) = U(x; \lambda)$ on the set $\{x \in \mathbb{R}^N :$

$|P_\lambda^+ x|_\lambda \le \beta\} \supset \Omega_\lambda$. Therefore in any case each periodic point of the map $U(\cdot; \lambda)$ is a periodic point of the map V_λ too. But relations (16) and $P_\lambda^- W_\lambda(x) = P_\lambda^- x$ imply

$$|P_\lambda^- V_\lambda(x)|_\lambda < \beta \qquad \text{if} \qquad |P_\lambda^- x|_\lambda \le \beta, \tag{19}$$

while relations (18) (with $\eta = 1$) and $|P_\lambda^+ W_\lambda(x)|_\lambda = \beta$ if $|P_\lambda^+ x|_\lambda \ge \beta$ imply

$$|P_\lambda^+ V_\lambda(x)|_\lambda > \beta \qquad \text{if} \qquad |P_\lambda^+ x|_\lambda \ge \beta. \tag{20}$$

Combining estimates (19) and (20), one concludes that for any periodic point $x \in \Omega_\lambda$ of the map V_λ all the iterations $V_\lambda^k(x)$, $k \in \mathbb{N}$, also belong to the set Ω_λ, where V_λ coincides with $U(\cdot; \lambda)$. Consequently, the periodic points of $U(\cdot; \lambda)$ coincide with the periodic points of V_λ lying in Ω_λ. The map V_λ is defined in such a way that the images of the set Ω_λ under the iterated maps V_λ^k have uniformly bounded projections onto the subspace E_λ' along the subspace E_λ of $\mathbb{R}^N$ for all k, i.e.

$$|(I - P_\lambda)V_\lambda^k(x)|_\lambda \le c_0, \qquad x \in \Omega_\lambda, \ k \in \mathbb{N}. \tag{21}$$

Lemma 1. *For any $\kappa_2 > \kappa_1 > 0$ the n-th iteration of the operator V_λ satisfies*

$$\left| Q_\lambda^{-1} P_\lambda V_\lambda^n(x) - z - n\lambda_q^{-1}((\lambda - \lambda_q)z - e^{i\varphi}\Psi_q^{res}(n\varphi)) \right| \to 0 \tag{22}$$

with $z = Q_\lambda^{-1} P_\lambda x$, $\varphi = \arg z$ as $\lambda \to \lambda_q$ for all x from the set

$$\Omega_\lambda^{q,\kappa_1,\kappa_2} = \{x \in \Omega_\lambda : \ \kappa_1 \le |\lambda - \lambda_q||Q_\lambda^{-1} P_\lambda x| \le \kappa_2\}.$$

This lemma will be proved in the last subsection.

3.2. **Proof of Theorem 1.** Let us cover the set S_q by a finite number of sectors

$$S_J = \{\lambda \in D \subset \mathbb{C} : \ 0 < |\lambda - \lambda_q| \le \varepsilon, \ \arg(\lambda - \lambda_q) \in J\},$$

where each closed interval J belongs to $\text{Int}\,\mathfrak{R}^\mu$ for some μ (the intervals $\mathfrak{R}^\mu$ are defined prior to the formulation of the theorem). Now it suffices to prove the conclusion of the theorem for the part of S_q contained in one sector S_J. The inclusion $J \subset \text{Int}\,\mathfrak{R}^\mu$ allows us to choose a segment $\mathcal{F} = [\varphi_-, \varphi_+] \subset d^\mu$ on the φ axis such that the image of $\mathcal{F}$ under the mapping $\arg \Psi_q^{res}$ contains J in its interior. By construction, there are positive κ_1, κ_2 such that $0 < \kappa_1 \le |\Psi_q^{res}(\varphi)| \le \kappa_2$, $\varphi \in \mathcal{F}$.

Let $z = re^{\varphi i} \in \mathbb{C}$, $h \in E_\lambda'$, $x = Q_\lambda z + h \in \mathbb{R}^N$, $\mathbb{H}_\lambda(x) = h - (I - P_\lambda)V_\lambda^n(x) \in E_\lambda'$. In the space $\mathbb{E}_\lambda = \mathbb{C} \times E_\lambda'$ of pairs (z, h) consider the two vector fields

$$\mathbb{W}_\lambda(z, h) = \left(n^{-1}\lambda_q e^{-i\varphi} Q_\lambda^{-1} P_\lambda(V_\lambda^n(x) - x), \mathbb{H}_\lambda(x) \right),$$

$$\mathbb{V}_\lambda(z, h) = \left((\lambda - \lambda_q)r - \Psi_q^{res}(n\varphi), \mathbb{H}_\lambda(x) \right)$$

on the set $G_\lambda = \{(z, h) : \ r \in \mathcal{R}_\lambda, \varphi \in \mathcal{F}, \|h\|_\lambda \le \beta\} \subset \mathbb{E}_\lambda$, where $\|h\|_\lambda = \max\{|P_\lambda^- h|_\lambda, |P_\lambda^+ h|_\lambda\}$, $\mathcal{R}_\lambda = [.5\kappa_1|\lambda - \lambda_q|^{-1}, 2\kappa_2|\lambda - \lambda_q|^{-1}]$.

The equation $x = V_\lambda^n(x)$ is equivalent to $\mathbb{W}_\lambda(z, h) = 0$. Hence, to prove the existence of a fixed point x_* of V_λ^n we shall show that for any $\lambda \in S_J$ sufficiently close to λ_q the vector field $\mathbb{V}_\lambda$ is non-singular on the boundary ∂G_λ of the set G_λ and its rotation $\gamma(\mathbb{V}_\lambda, \partial G_\lambda)$ on ∂G_λ is not equal to 0. Due to Lemma 1 and Rouche's Theorem, this implies that $\mathbb{V}_\lambda$ and $\mathbb{W}_\lambda$ are homotopic on ∂G_λ, hence $\gamma(\mathbb{V}_\lambda, \partial G_\lambda) = \gamma(\mathbb{W}_\lambda, \partial G_\lambda) \ne 0$, which ensures that $\mathbb{W}_\lambda$ has a zero $(z_*, h_*) \in G_\lambda$.

The vector field $\mathbb{V}_\lambda$ is non-singular on the boundary ∂G_λ by construction: its first component is non-singular on the boundary of $G_\lambda^z = \{z \in \mathbb{C} : |z| \in \mathcal{R}_\lambda, \arg z \in \mathcal{F}\}$, its second component is non-singular on the sphere $G_\lambda^h = \{h \in E_\lambda' : \|h\|_\lambda = \beta\}$. From the Index Product Formula [5] it follows that $\gamma(\mathbb{V}_\lambda, \partial G_\lambda) = \gamma_1 \gamma_2$, where γ_1 is the rotation of the planar vector field $(\lambda - \lambda_q)r - \Psi_q^{res}(n\varphi)$ on the boundary of

the set G_λ^z and γ_2 is the rotation of the vector field $\mathbb{H}_\lambda(Q_\lambda z + h)$ on the sphere G_λ^h (the latter rotation is the same for each fixed $z \in G_\lambda^z$). From the definition of the constants κ_j and the intervals $\mathcal{F}$, $\mathcal{R}_\lambda$ it follows in a standard way that $|\gamma_1| = 1$. Relations (16) – (18) can be used to show that the vector field $\mathbb{H}_\lambda(x)$ is homotopic to the non-degenerate linear vector field $P_\lambda^- h + A^n(\lambda) P_\lambda^+ h$ on the sphere G_λ^h, hence $|\gamma_2| = 1$ too, which proves $|\gamma(\mathbb{V}_\lambda, \partial G_\lambda)| = 1$.

The rest part of the proof is to show that n is the least period of the periodic point $x_* = Q_\lambda z_* + h_*$ of U if λ is close to λ_q and hence $|x_*|$ is sufficiently large. Let $k < n$ be a positive integer. Fix a vicinity O_q of the point λ_q that does not contain points $e^{2p\pi i}$ with rational $p = s/k$. For any $\lambda \in O_q$ the matrix $I - A^k(\lambda)$ is invertible and the norm of its inverse $\left(I - A^k(\lambda)\right)^{-1}$ is uniformly bounded on O_q. Hence the set of all fixed points of U^k is *a priori* bounded. $\square$

3.3. **Proof of Lemma 1.** We use the denomination $o(1)$ for the terms that vanish as $\lambda \to \lambda_q$ and the denominations c, c_1 etc. for constants where their exact value does not play any role. Let $x \in \Omega_\lambda^{q, \kappa_1, \kappa_2}$. Define $x_k = V_\lambda^k(x)$, $z_k = Q_\lambda^{-1} P_\lambda x_k$, and $z = Q_\lambda^{-1} P_\lambda x$. Let us first show that for any integer $k > 0$

$$\left| z_k - \lambda^k z - \sum_{j=0}^{k-1} \lambda^{k-j-1} Q_\lambda^{-1} P_\lambda \Phi(Q_\lambda(\lambda^j z); \lambda) \right| \to 0, \qquad \lambda \to \lambda_q. \qquad (23)$$

Indeed, relations $z_1 = Q_\lambda^{-1} P_\lambda V_\lambda(x) = \lambda z + Q_\lambda^{-1} P_\lambda \Phi(x; \lambda) + Q_\lambda^{-1} P_\lambda \xi(x; \lambda)$ imply

$$|z_1 - \lambda z - Q_\lambda^{-1} P_\lambda \Phi(Q_\lambda z; \lambda)| \le |Q_\lambda^{-1} P_\lambda (\Phi(x; \lambda) - \Phi(Q_\lambda z; \lambda) + \xi(x; \lambda))|,$$

where $\Phi(x; \lambda) - \Phi(Q_\lambda z; \lambda) = \Phi(x/|x|; \lambda) - \Phi(Q_\lambda z/|Q_\lambda z|; \lambda) = o(1)$, $\xi(x; \lambda) = o(1)$ due to (4) and continuity of Φ (we take into account that $x - Q_\lambda z = x - P_\lambda x$ is bounded for all $x \in \Omega_\lambda$ and $|x| \to \infty$ as $\lambda \to \lambda_q$ for $x \in \Omega_\lambda^{q, \kappa_1, \kappa_2}$, hence $x/|x| - Q_\lambda z/|Q_\lambda z| = o(1)$). Consequently (23) holds for $k = 1$. From the equality $z_{k+1} = \lambda z_k + Q_\lambda^{-1} P_\lambda \Phi(W_\lambda(x_k); \lambda) + Q_\lambda^{-1} P_\lambda \xi(W_\lambda(x_k); \lambda)$ and relations $\xi(W_\lambda(x_k); \lambda) = o(1)$ and (23) it follows that

$$z_{k+1} - \lambda^{k+1} z - \sum_{j=0}^{k-1} \lambda^{k-j} Q_\lambda^{-1} P_\lambda \Phi(Q_\lambda(\lambda^j z); \lambda) - Q_\lambda^{-1} P_\lambda \Phi(W_\lambda(x_k); \lambda) = o(1). \quad (24)$$

We assume that $|\lambda - \lambda_q|$ is sufficiently small, therefore (23) implies $|z_k - \lambda^k z| \le c$, $k \le n$. Combining the estimate $|P_\lambda^- x_k|_\lambda \le \beta$ (which follows from (19) for all k) with the relations $(I - P_\lambda^+) W_\lambda(\tilde{x}) = (I - P_\lambda^+)\tilde{x}$, $|P_\lambda^+ W_\lambda(\tilde{x})|_\lambda \le \beta$ valid for all $\tilde{x} \in \mathbb{R}^N$, we obtain $|W_\lambda(x_k) - P_\lambda x_k|_\lambda \le 2\beta$ and consequently

$$|W_\lambda(x_k) - Q_\lambda(\lambda^k z)| \le |W_\lambda(x_k) - P_\lambda x_k| + |Q_\lambda(z_k - \lambda^k z)| \le c_1 < \infty,$$

i.e. the difference $W_\lambda(x_k) - Q_\lambda(\lambda^k z)$ is bounded for $x \in \Omega_\lambda^{q, \kappa_1, \kappa_2}$. Because of the continuity and homogeneity (4) of Φ, the estimate $|W_\lambda(x_k) - Q_\lambda(\lambda^k z)| \le c_1$ and the relation $|Q_\lambda(\lambda^k z)| \to \infty$ as $\lambda \to \lambda_q$ (both valid for $x \in \Omega_\lambda^{q, \kappa_1, \kappa_2}$, $k \le n$) imply that $\Phi(W_\lambda(x_k); \lambda) - \Phi(Q_\lambda(\lambda^k z); \lambda) = o(1)$. Therefore the last term in the l.h.s. of (24) can be replaced with the term $Q_\lambda^{-1} P_\lambda \Phi(Q_\lambda(\lambda^k z); \lambda)$, which transforms (24) to

$$z_{k+1} - \lambda^{k+1} z - \sum_{j=0}^{k} \lambda^{k-j} Q_\lambda^{-1} P_\lambda \Phi(Q_\lambda(\lambda^j z); \lambda) = o(1). \qquad (25)$$

Note that one obtains (25) from (23) by replacing k with $k + 1$. By induction we conclude that (23) is valid for all $1 \le k \le n$.

Because the function Φ and the projectors are continuous,

$$\sum_{j=0}^{n-1}\Big(\lambda^{n-j-1}Q_\lambda^{-1}P_\lambda\Phi(Q_\lambda(\lambda^j z);\lambda) - \lambda_q^{n-j-1}Q_{\lambda_q}^{-1}P_{\lambda_q}\Phi(Q_{\lambda_q}(\lambda_q^j z);\lambda_q)\Big) = o(1). \quad (26)$$

Relation $\lambda_q^n = 1$ implies

$$\lambda^n = \Big(1 + \frac{\lambda - \lambda_q}{\lambda_q}\Big)^n = 1 + n\lambda_q^{-1}(\lambda - \lambda_q) + O((\lambda - \lambda_q)^2), \qquad \lambda \to \lambda_q,$$

and consequently $|\lambda^n z - z - n\lambda_q^{-1}(\lambda - \lambda_q)z| \le c_2|\lambda - \lambda_q|$, where the estimate $|z| \le \kappa_2|\lambda - \lambda_q|$ is used. Combining the last estimate with (23) for $k = n$ and (26), we arrive at

$$z_n - z - n\lambda_q^{-1}(\lambda - \lambda_q)z - \sum_{j=0}^{n-1}\lambda_q^{n-j-1}Q_{\lambda_q}^{-1}P_{\lambda_q}\Phi(Q_{\lambda_q}(\lambda_q^j z);\lambda_q) = o(1). \quad (27)$$

Here

$$\sum_{j=0}^{n-1}\lambda_q^{n-j-1}Q_{\lambda_q}^{-1}P_{\lambda_q}\Phi(Q_{\lambda_q}(\lambda_q^j z);\lambda_q) = -n\lambda_q^{-1}e^{i\varphi}\Psi_q^{res}(n\varphi)$$

with $\varphi = \arg z$, which follows from the relations

$$\sum_{j=0}^{n-1}\lambda_q^{n-j-1}Q_{\lambda_q}^{-1}P_{\lambda_q}\Phi(Q_{\lambda_q}(\lambda_q^j z);\lambda_q) = \sum_{j=0}^{n-1}e^{2\pi q(n-j-1)i}\Psi_{\lambda_q}(2\pi qj + \varphi)$$

$$= \sum_{j=0}^{n-1}e^{-2\pi q(j+1)i}\sum_{k=-\infty}^{\infty}\psi_k^{\lambda_q}e^{ik(2\pi qj+\varphi)} = e^{-2\pi qi}\sum_{k=-\infty}^{\infty}\psi_k^{\lambda_q}e^{ik\varphi}\sum_{j=0}^{n-1}e^{2\pi qj(k-1)i}$$

and

$$\sum_{j=0}^{n-1}e^{2\pi qj(k-1)i} = \begin{cases} n & \text{if } k-1 \text{ is a multiple of n;} \\ 0 & \text{otherwise.} \end{cases}$$

Therefore (27) coincides with (22). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

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