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On isotopic weavings

By

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Abstract. In this paper we consider configurations of straight lines in general position in a plane with all intersection points marked to show which of the two lines is "above" the other. We prove that there exist two isotopic configurations such that one of them can be obtained as a projection of a collection of straight lines in 3-space, and the other not. We investigate some isotopism class of configurations of six lines and find a necessary and sufficient condition for configurations from this class to be a projection of a collection of lines in 3-space.

Consider a finite set of pairwise intersecting straight lines in a plane such that no three lines intersect in a point. Such a set is called *a weaving* if all the intersection points are marked to show which of the two lines is locally "above" the other. A weaving is called *realizable* if there is a collection of straight lines in 3-space projected onto this weaving guaranteeing the local situations prescribed at the intersection points. Without loss of generality we can assume that this projection is orthogonal. So in this paper projection is understood to be an orthogonal projection.

Two weavings are said to be *isotopic* if there is an isotopy of the plane moving one of them to the other such that the lines of the weaving remain straight lines during this isotopy.

Some problems concerning realizability of weavings were solved in [2], [3] and [5]. Some results on the classification of finite sets of lines in space were obtained in [1] and [4].

The aim of this paper is to prove the following theorem.

Theorem 1. *There exist two isotopic weavings one of which is realizable and the other is not.*

The statement of this theorem is due to G. A. Galperin and A. B. Skopenkov.

An example of such two weavings is given in Fig. 1 and Fig. 2. Let W be the weaving shown in Fig. 1. Theorem 2 gives a necessary and sufficient condition for the weaving isotopic to W to be realizable.

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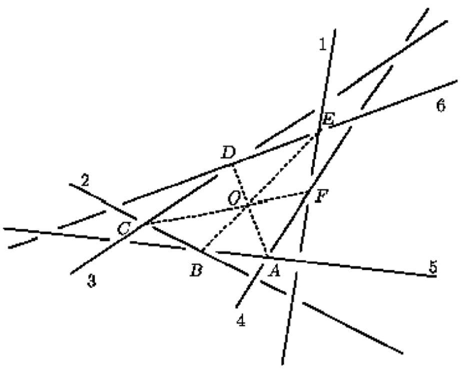


Figure 1. This weaving is realizable

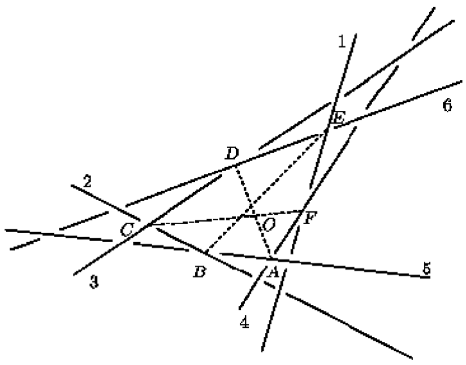


Figure 2. This weaving is not realizable

Theorem 1 is just an evident consequence of Lemmas 1 and 2.

Lemma 1. *A weaving isotopic to W is not realizable if the diagonals AD, BE and CF of the hexagon $ABCDEF$ intersect in a point.*

Proof. Suppose there exist six lines $1', 2', \dots, 6'$ in space such that their projections on the horizontal plane form the given weaving (in such a way that line j' projects onto line j for any $j = 1, \dots, 6$).

Rotate line $2'$ in the vertical plane about the point, which belongs to line $2'$ and projects on the intersection point of lines 2 and 6, until line $2'$ crosses line $5'$.

Similarly, rotate lines $4'$ and $6'$ until they intersect lines $1'$ and $3'$, respectively.

Rotate line $5'$ in the vertical plane about the intersection point of lines $5'$ and $2'$ until line $5'$ crosses line $6'$.

Rotate line $4'$ in the vertical plane about the intersection point of lines $4'$ and $1'$ until line $4'$ crosses line $5'$. Then rotate line $3'$ in the vertical plane about the intersection point of lines $3'$ and $6'$ until line $3'$ crosses line $4'$.

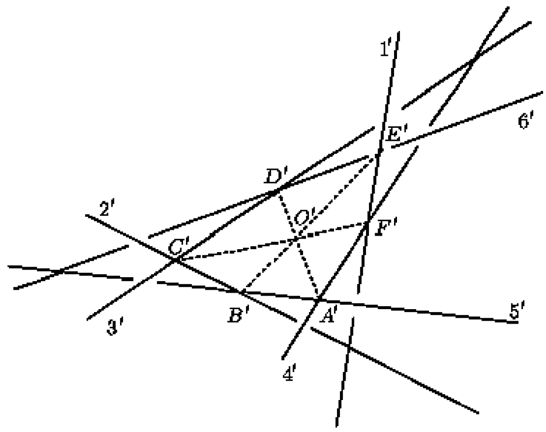
Similarly, rotating lines $2'$ and $1'$ we obtain that line $2'$ intersects lines $3'$ and $1'$.

The obtained configuration of lines is shown in Fig. 3.

Let A', B', C', D', F' be the intersection points of the lines of the obtained configuration projected on the points A, B, C, D, F, respectively. Let E' be a point on line $6'$ projected on the point E. Let O' be the intersection point of segments $A'D'$ and $C'F'$. Let π_1 be a plane containing the lines $1'$ and $2'$. Let π_3 be a plane containing the lines $5'$ and $6'$. Then O' belongs to π_1 and π_3. Hence the line $B'O'$ intersects the lines $1'$ and $6'$. Therefore the lines $1'$ and $6'$ intersect in the point E'. But the lines $1'$ and $6'$ are skew. This contradiction proves the lemma. □

Lemma 2. *There exists a realizable weaving isotopic to W.*

Proof. Clearly, after deleting line 2, the weaving in Fig. 1 becomes realizable. Consider five lines $1', 3', 4', 5', 6'$ in space such that their projections on the horizontal plane form



The configuration obtained in the proof of Lemma 1

Fig. 1 with line 2 deleted (in such a way that line j' projects onto line j for any $j = 1, \dots, 6, j \neq 2$). Let $2'$ be a line which crosses the lines $3'$ and $5'$ and is projected onto the line 2.

Move line 2 in the horizontal plane towards the intersection point of the lines 3 and 5 so that it remains parallel to itself. Simultaneously, move line $2'$ so that it remains crossing the lines $3'$ and $5'$ and is projected onto the line 2.

After some time line $2'$ will lie above line $6'$ and below the lines $1'$ and $4'$.

Then rotate line $2'$ in the vertical plane by a very small angle in such a way that it will lie below line $3'$ and above line $5'$.

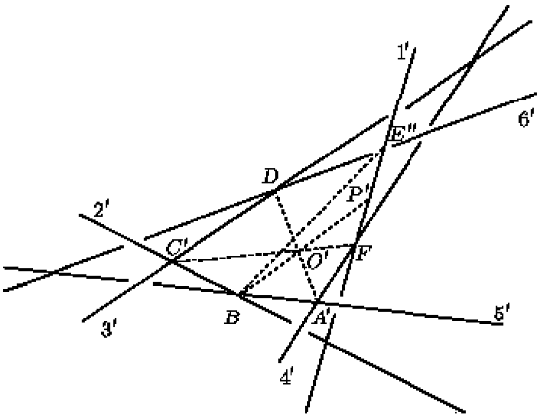
Consider the obtained collection of lines $1', 2', \dots, 6'$. Its projection on the horizontal plane is the weaving isotopic to W. □

Consider a weaving isotopic to W. Orient each side of the hexagon *ABCDEF* from the vertex of the hexagon, where this side is "above" the other, to the vertex, where this side is "below" the other. Orient the diagonals *AD*, *BE* and *CF* of the hexagon *ABCDEF* from the points D, B and F to the points A, E and C, respectively. Assume that the diagonals *AD*, *BE* and *CF* of the hexagon *ABCDEF* do not intersect in a point. Then they form a non-degenerate triangle Δ and their orientations induce orientations on the sides of this triangle.

Theorem 2. *The weaving isotopic to W is realizable iff the diagonals AD, BE and CF of the hexagon ABCDEF do not intersect in a point and the triangle* Δ *is cooriented with the hexagon ABCDEF.*

P r o o f. The proof of the fact that the weaving is not realizable if the triangle Δ is contraoriented with the hexagon $ABCDEF$ is analogous to the proof of Lemma 1.

Suppose the triangle Δ is cooriented with the hexagon $ABCDEF$ (see Fig. 2). Without loss of generality we can assume that the weaving belongs to a horizontal plane π.



The configuration obtained in the proof of Theorem 2

Consider configurations of six lines $1', 2', \ldots, 6'$ in space such that the following conditions hold:

1) The line j' projects onto the line j for any $j = 1, 2, \ldots, 6$.
2) The lines $2'$ and $5'$ intersect in the point B. The lines $3'$ and $6'$ intersect in the point D. The lines $1'$ and $4'$ intersect in the point F.
3) For each pair $(i, j) \in \{(1, 2), (3, 4), (5, 6), (2, 3), (4, 5)\}$ the lines i' and j' intersect.

Define points A', C', O' and planes π_1 and π_3 as in the proof of Lemma 1. Let π_2 be a plane containing the lines $3'$ and $4'$. For any $\alpha \in (-\pi/2, \pi/2)$ there is the only configuration of lines satisfiing conditions 1, 2 and 3 such that the angle between π_2 and π is equal to α (this angle is suggested to be positive if A' is above π and negative if A' is below π). Evidently we can choose α in such a way that the line i' will lie above the line j' for each pair $(i, j) \in \{(2, 6), (6, 4), (4, 2), (1, 3), (3, 5), (5, 1)\}$.

Let P' be the intersection point of the lines BO' and $1'$. Let E'' be a point on the line $1'$ projected on the point E. Then P' belongs to the plane π_3 and to the open interval with endpoints E'' and F'. Hence this interval intersects the plane π_3. The point F' is below π_3 because the point C' is above π_3 and $O' \in \pi_3$. Therefore the point E'' is above the plane π_3. Hence the line $1'$ lies above the line $6'$.

The obtained configuration of lines is shown in Fig. 4.

With this configuration, let us do all transformations as in the proof of Lemma 1 in the inverse order. Thus we obtain a configuration which is a realization of the given weaving. □

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References

[1] V. F. MAZUROVSKIJ, Configurations of six skew lines. Zap. Nauchn. Sem. Leningad. Otdel. Mat. Inst. Steklova **167**, 121–134 (1988); English transl.: J. Soviet. Math. **52**(1), 2825–2832 (1990).

[2] J. PACH, R. POLLACK and E. WELZL, Weaving patterns of lines and line segments in space. Algorithmica **9**, 561–571 (1993).

[3] D. REPOVŠ, A. B. SKOPENKOV and F. SPAGGIARI, An infinite sequence of non-realizable weavings. Submitted 2001.

[4] O. YA. VIRO, Topological problems concerning lines and points of three-dimensional space. Dokl. Akad. Nauk SSSR **284**, 5, 1049–1052 (1985); English transl.: Soviet. Math. Dokl. **32**, 528–531 (1985).

[5] W. WHITELEY, Weavings, sections and projections of spherical polyhedra. Discrete. Appl. Math. **32**(3), 275–294 (1991).

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