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TRAJECTORY ATTRACTOR FOR REACTION-DIFFUSION SYSTEM WITH DIFFUSION COEFFICIENT VANISHING IN TIME

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Abstract. We consider a non-autonomous reaction-diffusion system of two equations having in one equation a diffusion coefficient depending on time ($\delta = \delta(t) \geq 0, t \geq 0$) such that $\delta(t) \to 0$ as $t \to +\infty$. The corresponding Cauchy problem has global weak solutions, however these solutions are not necessarily unique. We also study the corresponding "limit" autonomous system for $\delta = 0$. This reaction-diffusion system is partly dissipative. We construct the trajectory attractor $\mathfrak{A}$ for the limit system. We prove that global weak solutions of the original non-autonomous system converge as $t \to +\infty$ to the set $\mathfrak{A}$ in a weak sense. Consequently, $\mathfrak{A}$ is also as the trajectory attractor of the original non-autonomous reaction-diffusions system.

1. **Introduction.** Global attractors for autonomous and non-autonomous reaction-diffusion systems have been constructed in many papers for the case when the corresponding initial boundary value problem has a unique global solution in the corresponding function space (see, for instance, the books [1]– [5] and the references therein).

In the present paper, we study a system of two reaction-diffusion equations where one equation has a time-dependent diffusion coefficient ($\delta = \delta(t) \geq 0, t \geq 0$) such that $\delta(t) \to 0$ as $t \to +\infty$. For this system, the corresponding Cauchy problem has a global weak solution but this solution is not necessarily unique, so that the conventional theory of global attractor is not directly applicable to this system. We study the long-time behavior of this system using the trajectory attractor method. This approach is very effective in the study of weak solutions of various dissipative equations of mathematical physics for which the uniqueness theorem of the corresponding Cauchy problem fails or is not proved yet (for instance, the 3D Navier-Stokes system and other equations and system; see, e.g., [6]– [12]).

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We study the following non-autonomous reaction-diffusion system:

$$\partial_t u = \Delta u - f(u,v) + g_1(x), \quad u|_{\partial\Omega} = 0, \tag{1}$$

$$\partial_t v = \delta(t)\Delta v - h(u,v) + g_2(x), \quad v|_{\partial\Omega} = 0, \tag{2}$$

$$x \in \Omega \Subset \mathbb{R}^3, \quad t \geq 0,$$

where $g_1 \in L_2(\Omega)$ and $g_2 \in H_0^1(\Omega)$. The scalar functions $u = u(x,t)$ and $v = v(x,t)$ are the unknowns. The (continuous) nonlinear functions $f(u,v)$ and $h(u,v)$ satisfy some appropriate conditions (see Section 2).

We assume that the diffusion coefficient $\delta(\cdot) \in L_\infty(\mathbb{R}_+;\mathbb{R}_+)$ and satisfies

$$\int_t^{t+1} \delta(s)ds \to 0 \quad \text{as} \quad t \to +\infty.$$

Consider the following initial data for system (1)–(2):

$$u|_{t=0} = u_0 \in L_2(\Omega), \quad v|_{t=0} = v_0 \in H_0^1(\Omega). \tag{3}$$

Note that the conditions for the nonlinear functions f and h do not ensure the unique solvability of the Cauchy problem (1)–(3).

In Section 2, we study weak solutions to the reaction-diffusion system (1)–(2). A pair of functions $(u(x,t), v(x,t)), x \in \Omega, t \geq 0$, is called a (global) weak solution to system (1)–(2) if, for every $M > 0$,

$$u(\cdot) \in L_{p_1}(0,M;L_{p_1}(\Omega)) \cap L_2(0,M;H_0^1(\Omega)),$$

$$v(\cdot) \in L_{p_2}(0,M;L_{p_2}(\Omega)) \cap L_\infty(0,M;H_0^1(\Omega)),$$

and the functions $u(x,t)$ and $v(x,t)$ satisfy equations (1) and (2) in the sense of distributions. Here, $(p_1 - 1, p_2 - 1)$ are the degrees of polynomial growth of the functions (f,h) with respect to (u,v) (see Section 2). We prove the existence of a global weak solution to problem (1)–(3).

In Section 3, we consider the limit (as $t \to +\infty$) reaction-diffusion system for the system (1)–(2). This *autonomous* system reads

$$\partial_t u = \Delta u - f(u,v) + g_1(x), \quad u|_{\partial\Omega} = 0, \tag{4}$$

$$\partial_t v = -h(u,v) + g_2(x), \quad v|_{\partial\Omega} = 0. \tag{5}$$

This system is called *partly dissipative* since the diffusion coefficient in equation (5) equals to zero. The global attractor for this system has been constructed in [13] under some additional conditions that guarantee the unique solvability of the Cauchy problem (4)–(5), (3). Without this conditions, the uniqueness fails, and in [12] the trajectory attractor $\mathfrak{A}$ for the system (4)–(5) has been constructed. The definition and the properties of the set $\mathfrak{A}$ are given in Section 3.

In Section 4, the main result of this paper is established. We prove that weak solutions $(u(x,t), v(x,t))$ of the non-autonomous system (1)–(2) converge as $t \to +\infty$ to the trajectory attractor $\mathfrak{A}$ of the autonomous system (4)–(5) in the corresponding weak sense. Consequently, the set $\mathfrak{A}$ is the trajectory attractor of the original non-autonomous reaction-diffusion system (1)–(2)

2. **Reaction-diffusion system with time dependent diffusion coefficient.**

In a bounded domain $\Omega \Subset \mathbb{R}^3$, we consider the following reaction-diffusion system:

$$\partial_t u = \Delta u - f(u,v) + g_1(x), \tag{6}$$

$$\partial_t v = \delta(t)\Delta v - h(u,v) + g_2(x), \tag{7}$$

where the unknowns are scalar functions $u = u(x,t)$ and $v = v(x,t),\ x \in \Omega, t \geq 0$. In the equations (6)–(7), Δ is the Laplace operator acting in the domain Ω. At the boundary $\partial\Omega$, we impose the Dirichlet conditions

$$u|_{\partial\Omega} = 0, \quad v|_{\partial\Omega} = 0. \tag{8}$$

In equation (7), the diffusion coefficient δ depends on time and $\delta(t) \geq 0$ for $t \geq 0$. We assume that $\delta(\cdot) \in L_\infty(\mathbb{R}_+)$ and

$$\int_t^{t+1} \delta(s)ds \to 0 \quad \text{as} \quad t \to +\infty. \tag{9}$$

The diffusion coefficient in equation (6) equals to one.

We assume that the nonlinear functions $f, h : \mathbb{R}^2 \to \mathbb{R}$ are continuous in $\mathbb{R}^2$ and satisfy the following inequalities:

$$\sigma_1\left(|u|^{p_1} + |v|^{p_2}\right) - C \leq f(u,v)u + h(u,v)v \leq C_0(|u|^{p_1} + |v|^{p_2} + 1), \tag{10}$$

$$|f(u,v)|^{q_1} + |h(u,v)|^{q_2} \leq C_0(|u|^{p_1} + |v|^{p_2} + 1), \quad \forall u, v \in \mathbb{R}, \tag{11}$$

where $\sigma_1, C, C_0, p_1, p_2$ are some positive constants, $p_1, p_2 \geq 2$, and $q_i = p_i/(p_i - 1)$, for $i = 1, 2$. We also assume that

$$h \in C^1(\mathbb{R}^2), \quad h(0,0) = 0, \tag{12}$$

and the following inequalities hold

$$\frac{\partial h}{\partial v}(u,v) \geq \sigma_2 > 0, \tag{13}$$

$$\left|\frac{\partial h}{\partial u}(u,v)\right| \leq D, \quad \forall u, v \in \mathbb{R}. \tag{14}$$

We note that the positive quantities σ_1 and σ_2 which bound the expressions in (10) and (13) mean the dissipativity of the system. The numbers σ_1 and σ_2 can be arbitrary small. For definiteness, we set

$$\sigma_1 = \sigma_2 =: \sigma.$$

We do not assume that the function f in equation (6) is differentiable so the Cauchy problem for system (6)–(8) can have more than one solution in the corresponding function space.

The functions $g_1(x)$ and $g_2(x)$ in equations (6) and (7) satisfy the conditions

$$g_1 \in L_2(\Omega), \quad g_2 \in H_0^1(\Omega). \tag{15}$$

An example of a system of this type is the following non-autonomous FitzHugh-Nagumo system:

$$\begin{cases} \partial_t u = \Delta u - u(u-\beta)(u-1) - v, \\ \partial_t v = \delta(t)\Delta v + \alpha u - \gamma v, \end{cases}$$

where $f(u,v) = u(u-\beta)(u-1) + v,\ h(u,v) = \gamma v - \alpha u$, and α, β, γ are positive coefficients. Then $\sigma = \min\{1, \gamma\}$. In this example, $p_1 = 4, p_2 = 2$, and $g_1 \equiv 0$, $g_2 \equiv 0$ (see [1], [14], and [15]).

For simplicity of notations, we set

$$H := L_2(\Omega) \quad \text{and} \quad V := H_0^1(\Omega).$$

We denote by $\|\cdot\|_X$ the norm in a Banach space X and we denote for brevity by $\|\cdot\|$ and $\|\cdot\|_1$ the norms in the spaces H and V, respectively. Recall that the Poincaré inequality implies that the norm of a function w in $V = H_0^1(\Omega)$ can be defined by the formula

$$\|w\|_1 := \|\nabla w\| = \left(\int_\Omega \sum_{i=1}^{3} |\partial_{x_i} w(x)|^2 dx \right)^{1/2}.$$

For arbitrary functions $u \in L_{p_1}(0, M; L_{p_1}(\Omega))$ and $v \in L_{p_2}(0, M; L_{p_2}(\Omega))$, it follows from (10) and (11) that

$$f(u, v) \in L_{q_1}(0, M; L_{q_1}(\Omega)), \quad h(u, v) \in L_{q_2}(0, M; L_{q_2}(\Omega)), \tag{16}$$

and

$$\|f(u, v)\|^{q_1}_{L_{q_1}(0,M;L_{q_1})} + \|h(u, v)\|^{q_2}_{L_{q_2}(0,M;L_{q_2})} \leq C_1 \left(\|u\|^{p_1}_{L_{p_1}(0,M;L_{p_1})} + \|v\|^{p_2}_{L_{p_2}(0,M;L_{p_2})} + 1 \right). \tag{17}$$

In addition, if $u, v \in L_2(0, M; V)$, then

$$\Delta u + g_1, \ \delta\Delta v + g_2 \in L_2(0, M; H^{-1}(\Omega)). \tag{18}$$

The Sobolev embedding theorem implies that $H_0^{r_i}(\Omega) \subset L_{p_i}(\Omega)$ for $r_i \geq 3(1/2 - 1/p_i)$, $i = 1, 2$, and hence, for the dual spaces, we have the embedding

$$L_{q_i}(\Omega) \subset H^{-r_i}(\Omega).$$

Therefore, if $r_i \geq 1$, then due to (16) and (18), the right-hand sides of equations (6) and (7) belong to the spaces $L_{q_1}(0, M; H^{-r_1}(\Omega))$ and $L_{q_2}(0, M; H^{-r_2}(\Omega))$, respectively. We set

$$r_i = \max\{1, 3(1/2 - 1/p_i)\}, \quad i = 1, 2.$$

Now, we can look for solutions of equations (6) and (7) in the spaces of distributions $\mathcal{D}'(0, M; H^{-r_1}(\Omega))$ and $\mathcal{D}'(0, M; H^{-r_2}(\Omega))$, respectively (see [16]), and moreover

$$\partial_t u \in L_{q_1}(0, M; H^{-r_1}(\Omega)), \quad \partial_t v \in L_{q_2}(0, M; H^{-r_2}(\Omega)).$$

Definition 2.1. A pair of functions $(u(x, t), v(x, t))$ is called a *weak solution* to the system (6)–(7) in the domain $\Omega \times \mathbb{R}_+$, if, for every $M > 0$,

$$u \in L_{p_1}(0, M; L_{p_1}(\Omega)) \cap L_2(0, M; V),$$
$$v \in L_{p_2}(0, M; L_{p_2}(\Omega)) \cap L_2(0, M; V),$$

and the functions $u(x, t)$ and $v(x, t)$ satisfy equations (6)–(7) in the sense of distributions in the space $\mathcal{D}'(0, M; H^{-r_1}(\Omega) \times H^{-r_2}(\Omega))$ (see [2], [3], and [16]).

From equations (6)–(8), we see that

$$u(\cdot) \in L_\infty(0, M; H) \quad \text{and} \quad v(\cdot) \in L_\infty(0, M; H);$$

using the Lions-Magenes lemma (see [17]), we obtain

$$u(\cdot) \in C_w([0, M]; H), \quad v(\cdot) \in C_w([0, M]; H), \quad \forall M > 0.$$

Consequently, for every $t \geq 0$, $u(t)$ and $v(t)$ are well-defined and, in particular, the initial conditions of the form

$$u|_{t=0} = u_0 \in H, \quad v|_{t=0} = v_0 \in H \tag{19}$$

make sense. For brevity, we will often omit the space variable x in the arguments of the functions u and v.

In [3], we proved that any weak solution $(u(\cdot), v(\cdot))$ to a system of the form (6)–(7) has the following properties:

(i): $u(\cdot) \in C(\mathbb{R}_+; H), v(\cdot) \in C(\mathbb{R}_+; H)$;

(ii): the real function $\|u(t)\|^2 + \|v(t)\|^2$ is absolutely continuous for $t \geq 0$ and satisfies the energy identity

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left\{\|u(t)\|^2 + \|v(t)\|^2\right\} + \|\nabla u(t)\|^2 + \delta(t)\|\nabla v(t)\|^2 \\ + \int_\Omega f(u(x,t), v(x,t))u(x,t)dx + \int_\Omega h(u(x,t), v(x,t))v(x,t)dx \\ = \langle g_1, u(t)\rangle + \langle g_2, v(t)\rangle\,. \quad (20)\end{aligned}$$

Here, $\langle\cdot,\cdot\rangle$ denotes the scalar product of functions in H. Formally to obtain (20), we take the scalar product in H of equation (6) and $u(t)$, and the scalar product of equation (7) and $v(t)$, and add the results.

Using inequality (10) and identity (20), we prove the following result (see [12]).

Proposition 1. *For any weak solution $(u(t), v(t))$ to the problem (6)–(7), (19), the following inequalities hold:*

$$\begin{aligned}&\|u(t)\|^2 + \|v(t)\|^2 + 2\int_0^t \left(\|u(s)\|_1^2 + \delta(s)\|v(s)\|_1^2\right) \mathrm{e}^{-\sigma(t-s)}ds \\ &\leq \left(\|u_0\|^2 + \|v_0\|^2\right)e^{-\sigma t} + R_1^2, \quad (21)\\ &2\int_t^{t+1} \left(\|u(s)\|_1^2 + \delta(s)\|v(s)\|_1^2\right)ds + \sigma\int_t^{t+1}\left(\|u(s)\|_{L_{p_1}}^{p_1} + \|v(s)\|_{L_{p_2}}^{p_2}\right)ds \\ &\leq \left(\|u_0\|^2 + \|v_0\|^2\right)e^{-\sigma t} + R_2^2, \quad \forall t \geq 0. \quad (22)\end{aligned}$$

Here the quantities R_1 and R_2 depend on σ, $\|g_1\|$, and $\|g_2\|$.

We now assume that

$$v|_{t=0} = v_0 \in V. \quad (23)$$

The existence of a weak solution to system (6)–(7) with initial data (19) for arbitrary $u_0 \in H, v_0 \in V$, is established using the Galerkin method with basis consisting of the eigenfunctions of the Laplace operator with zero boundary conditions (see, e.g., [2], [3]). Using the standard approach, we firstly establish a priori estimates for the Galerkin approximations $(u_m(t), v_m(t))$ of order $m \in \mathbb{N}$. Then we prove the existence of a subsequence with indices $\{m'\} \subset \{m\}$, that converges to a weak solution $(u(t), v(t))$ to the problem (6)–(7), (19) as $m' \to \infty$ in the sense of distributions (see [12]).

Proposition 2. *Under assumption (23), problem (6)–(7), (19) has a weak solution $(u(t), v(t))$ such that $v(\cdot) \in L_\infty(\mathbb{R}_+; V)$ and the following inequality holds:*

$$\|v(t)\|_1^2 \leq \|v_0\|_1^2\mathrm{e}^{-\sigma t} + C_5\left(\|u_0\|^2 + \|v_0\|^2\right)e^{-\sigma t} + R^2, \quad \forall t \geq 0, \quad (24)$$

where $R^2 = \sigma^{-1}D^2R_1^2 + 2\sigma^{-2}\|g_2\|_1^2$, $C_5 = \sigma^{-1}D^2$ and R_1 is defined in (21).

The Lions–Magenes lemma we recalled above implies that the weak solution so obtained satisfies $v(\cdot) \in C_w(\mathbb{R}_+; V)$, that is, for every $t \geq 0$, the value $v(t) \in V$ is well-defined and the inequality (24) hold for $t \geq 0$.

We now define the linear space $\mathcal{F}_+^{\mathrm{loc}}$. By definition,

$$\mathcal{F}_+^{\mathrm{loc}} = \left\{\begin{array}{c}(y(x,t), z(x,t)), x \in \Omega, t \geq 0 \mid \\ y \in L_\infty^{\mathrm{loc}}(\mathbb{R}_+; H) \cap L_2^{\mathrm{loc}}(\mathbb{R}_+; V) \cap L_{p_1}^{\mathrm{loc}}(\mathbb{R}_+; L_{p_1}(\Omega)), \\ z \in L_\infty^{\mathrm{loc}}(\mathbb{R}_+; V) \cap L_{p_2}^{\mathrm{loc}}(\mathbb{R}_+; L_{p_2}(\Omega)), \\ \partial_t y \in L_{q_1}^{\mathrm{loc}}(\mathbb{R}_+; H^{-r_1}(\Omega)), \partial_t z \in L_{q_2}^{\mathrm{loc}}(\mathbb{R}_+; H^{-r_2}(\Omega))\end{array}\right\}. \quad (25)$$

We consider the linear subspace $\mathcal{F}_+^{\mathrm{b}} \subset \mathcal{F}_+^{\mathrm{loc}}$ of functions (y,z) with finite norm

$$\|(y,z)\|_{\mathcal{F}_+^{\mathrm{b}}} := \|y\|_{L_\infty(\mathbb{R}_+;H)} + \|y\|_{L_2^{\mathrm{b}}(\mathbb{R}_+;V)} + \|y\|_{L_{p_1}^{\mathrm{b}}(\mathbb{R}_+;L_{p_1})} + \|\partial_t y\|_{L_{q_1}^{\mathrm{b}}(\mathbb{R}_+;H^{-r_1})} + \|z\|_{L_\infty(\mathbb{R}_+;V)} + \|z\|_{L_{p_2}^{\mathrm{b}}(\mathbb{R}_+;L_{p_2})} + \|\partial_t z\|_{L_{q_2}^{\mathrm{b}}(\mathbb{R}_+;H^{-r_2})}. \tag{26}$$

Recall that the norm in the space $L_p^{\mathrm{b}}(\mathbb{R}_+;X),\ p \geq 1$, where X is a Banach space, is defined by the formula

$$\|y\|^p_{L_p^{\mathrm{b}}(\mathbb{R}_+;X)} := \sup_{t\geq 0} \int_t^{t+1} \|y(s)\|_X^p ds.$$

Clearly the space $\mathcal{F}_+^{\mathrm{b}}$ with norm (26) is a Banach space.

We now define the space $\mathcal{K}_+^\delta(N)$ of solutions (trajectories) to system (6)–(7) which depends on $N > 0$. Here, $\delta = \delta(t)$ denotes the diffusion coefficient in (7).

Definition 2.2. The space $\mathcal{K}_+^\delta(N)$ consists of functions $(u(\cdot),v(\cdot)) \in \mathcal{F}_+^{\mathrm{loc}}$ such that

(i): the pair $(u(t),v(t)), t \geq 0$, is a weak solution to (6)–(7);

(ii): the function $v(t)$ satisfies the inequality

$$\|v(t)\|_1^2 \leq N\mathrm{e}^{-\sigma t} + R^2, \quad \forall t \geq 0, \tag{27}$$

with σ and R from (24).

Recall that if $(u(\cdot),v(\cdot)) \in \mathcal{F}_+^{\mathrm{loc}}$ and $(u(t),v(t))$ is a weak solution to system (6)–(7), then, by the Lions-Magenes lemma, $v \in C_w(\mathbb{R}_+;V)$ and the inequality (27) is meaningful for all $t \geq 0$.

We note that the trajectory space $\mathcal{K}_+^\delta(N)$ is non-empty. Indeed, solving the Cauchy problem (6)–(7), (19) with $u_0 \in H$, $v_0 \in V$, using the Galerkin method, we find a weak solution $(u(t),v(t))$ to this problem, which satisfies inequality (24) (see Proposition 2). Therefore, if the norms $\|u_0\|, \|v_0\|_1$ are sufficiently small and satisfy the condition $\|v_0\|_1^2 + C_5\left(\|u_0\|^2 + \|v_0\|^2\right) \leq N$, then inequality (27) holds for the weak solution we have constructed, that is, it belongs to $\mathcal{K}_+^\delta(N)$.

Consider the translation operators $T(\tau), \tau \geq 0$, acting on the space $\mathcal{F}_+^{\mathrm{loc}}$ by the formula

$$T(\tau)(y(t),z(t)) = (y(t+\tau), z(t+\tau)), \quad t \geq 0. \tag{28}$$

Proposition 3. *The trajectory space $\mathcal{K}_+^\delta(N)$ lies in $\mathcal{F}_+^{\mathrm{b}}$ and the following inequality holds:*

$$\|T(\tau)(u,v)\|_{\mathcal{F}_+^{\mathrm{b}}} \leq C_6\left(\|u(0)\|^2 + N\right) e^{-\rho\tau} + R_3^2, \quad \forall \tau \geq 0, \tag{29}$$

where $C_6 \geq 0$ and $\rho > 0$ are independent of g_1, g_2, while $R_3 = R_3(\|g_1\|, \|g_2\|)$.

Proof. We apply inequalities (21), (22), (27) and obtain

$$\|T(\tau)u\|^2_{L_\infty(\mathbb{R}_+;H)} + \|T(\tau)u\|^{p_1}_{L_2^{\mathrm{b}}(\mathbb{R}_+;V)} + \|T(\tau)u\|^2_{L_{p_1}^{\mathrm{b}}(\mathbb{R}_+;L_{p_1})} + \|T(\tau)v\|^2_{L_\infty(\mathbb{R}_+;V)} + \|T(\tau)v\|^{p_2}_{L_{p_2}^{\mathrm{b}}(\mathbb{R}_+;L_{p_2})} \leq C_7(\|u(0)\|^2 + N)e^{-\rho_1\tau} + R_4^2, \tag{30}$$

for the corresponding ρ_1 and R_4. It remains to check analogous estimates for the norms

$$\|T(\tau)\partial_t u\|_{L_{q_1}^{\mathrm{b}}(\mathbb{R}_+;H^{-r_1})} \text{ and } \quad \|T(\tau)\partial_t v\|_{L_{q_2}^{\mathrm{b}}(\mathbb{R}_+;H^{-r_2})}.$$

We use equation (6) and obtain the following estimates for $\partial_t u$:

$$\begin{aligned}&\left[\int_t^{t+1}\|\partial_t u(s)\|_{H^{-r_1}}^{q_1}ds\right]^{1/q_1}\\ \le&\left[\int_t^{t+1}\|\Delta u(s)\|_{H^{-r_1}}^{q_1}ds\right]^{1/q_1}+\left[\int_t^{t+1}\|f(u(s),v(s))\|_{H^{-r_1}}^{q_1}ds\right]^{1/q_1}+\|g_1\|_{H^{-r_1}}\\ \le&C_8\left[\int_t^{t+1}\|u(s)\|_1^2ds\right]^{\frac{1}{2}}+C_9\left[\left(\int_t^{t+1}\|u(s)\|_{L_{p_1}}^{p_1}+\|v(s))\|_{L_{p_2}}^{p_2}ds\right)+1\right]^{\frac{1}{q_1}}+\|g_1\|\\ \le&C_{10}(\|u(0)\|^2+N)e^{-\rho_2\tau}+R_5^2,\ \forall t\ge\tau,\end{aligned}$$

where we again use estimate (22), the choice of the number r_1, and the inequality $q_1\le 2$. Consequently,

$$\|T(\tau)\partial_t u\|_{L_{q_1}^{\mathrm{b}}(\mathbb{R}_+;H^{-r_1})}\le C_{11}(\|u(0)\|^2+N)e^{-\rho_3\tau}+R_6^2. \tag{31}$$

Similarly from equation (7), we derive the estimate

$$\|T(\tau)\partial_t v\|_{L_{q_2}^{\mathrm{b}}(\mathbb{R}_+;H^{-r_2})}\le C_{12}(\|u(0)\|^2+N)e^{-\rho_4\tau}+R_7^2, \tag{32}$$

where we use the inequalities

$$\begin{aligned}\left[\int_t^{t+1}\|\delta(s)v(s)\|_{H^{-r_2}}^{q_2}ds\right]^{1/q_2}&\le C_8'\left[\int_t^{t+1}\|\delta(s)v(s)\|_1^2ds\right]^{1/2}\\&\le C_9'(\|u(0)\|^2+N)e^{-\rho_2\tau}+R_5^2,\end{aligned}$$

since $\delta(\cdot)\in L_\infty(\mathbb{R}_+)$. Combining inequalities (30), (31), and (32), we obtain estimate (29), which in particular implies that $\mathcal{K}_+^\delta(N)\subset\mathcal{F}_+^{\mathrm{b}}$. □

3. The limit reaction-diffusion system with zero diffusion coefficient and its trajectory attractor.

Consider the "limit" system for equations (6)–(7) with diffusion coefficient $\delta\equiv 0$:

$$\partial_t u=\Delta u-f(u,v)+g_1(x), \tag{33}$$

$$\partial_t v=-h(u,v)+g_2(x). \tag{34}$$

This system is autonomous. As before, at the boundary $\partial\Omega$, we set the Dirichlet conditions

$$u|_{\partial\Omega}=0,\quad v|_{\partial\Omega}=0. \tag{35}$$

We keep the same notations as in § 2. In particular, functions f and h satisfy conditions (10)–(14), and the functions g_1,g_2 satisfy (15).

A *weak solution* to system (33)–(34) in the domain $\Omega\times\mathbb{R}_+$ is a pair of functions $(u(x,t),v(x,t))$

$$u\in L_{p_1}^{\mathrm{loc}}(\mathbb{R}_+;L_{p_1}(\Omega))\cap L_2^{\mathrm{loc}}(\mathbb{R}_+;V), \tag{36}$$

$$v\in L_{p_2}^{\mathrm{loc}}(\mathbb{R}_+;L_{p_2}(\Omega))\cap L_\infty^{\mathrm{loc}}(\mathbb{R}_+;V), \tag{37}$$

that satisfy (33)–(34) in the distribution space $\mathcal{D}'(\mathbb{R}_+;H^{-r_1}(\Omega)\times H^{-r_2}(\Omega))$ (the exponents r_1 and r_2 are defined in § 2).

As was shown in § 2, (36) and (37) give that

$$\partial_t u\in L_{q_1}^{\mathrm{loc}}(\mathbb{R}_+;H^{-r_1}(\Omega))\quad\text{and}\quad\partial_t v\in L_{q_2}^{\mathrm{loc}}(\mathbb{R}_+;H^{-r_2}(\Omega)).$$

It follows from Lions-Magenes lemma that $u(\cdot) \in C_w(\mathbb{R}_+; H)$ and $v(\cdot) \in C_w(\mathbb{R}_+; V)$; therefore the values $u(t)$ and $v(t)$ are well-defined for all $t \geq 0$ and for equations (33)–(34) the following initial conditions are meaningful:

$$u|_{t=0} = u_0 \in H, \tag{38}$$

$$v|_{t=0} = v_0 \in V. \tag{39}$$

The existence of weak solutions to problem (33)–(35), (38), (39) is proved using the Galerkin method. Recall that any weak solution to system (33)–(34) satisfy the following energy identity:

$$\frac{1}{2}\frac{d}{dt}\left\{\|u(t)\|^2 + \|v(t)\|^2\right\} + \|\nabla u(t)\|^2 + \int_\Omega f(u(x,t),v(x,t))u(x,t)dx$$
$$+ \int_\Omega h(u(x,t),v(x,t))v(x,t)dx = \langle g_1, u(t)\rangle + \langle g_2, v(t)\rangle .$$

which is proved similarly to (20). This equality implies the following estimates.

Proposition 4. *Any weak solution $(u(t),v(t)), t \geq 0$, to problem (33)–(35), (38), (39) satisfies the inequalities*

$$\|u(t)\|^2 + \|v(t)\|^2 + 2\int_0^t \|u(s)\|_1^2 \mathrm{e}^{-\sigma(t-s)}ds \leq \left(\|u_0\|^2 + \|v_0\|^2\right)e^{-\sigma t} + R_1^2, \tag{40}$$

$$2\int_t^{t+1} \|u(s)\|_1^2 ds + \sigma\int_t^{t+1}\left\{\|u(s)\|_{L_{p_1}}^{p_1} + \|v(s)\|_{L_{p_2}}^{p_2}\right\}ds \tag{41}$$
$$\leq \left(\|u_0\|^2 + \|v_0\|^2\right)e^{-\sigma t} + R_2^2,\ \forall t \geq 0.$$

The values R_1 and R_2 are the same as in Proposition 1.

The proof is similar to the proof of Proposition 1 (see [12]).

Note that to construct a weak solution to problem (33)–(35), (38), (39) using the Galerkin method, it is not sufficient to have estimates of the form (40) and (41)! This construction is possible if we prove a special estimate. We have to take into account the smoothness of the initial data (39) and to use the identity

$$\frac{1}{2}\frac{d}{dt}\left(\|v_m(t)\|_1^2\right) - \int_\Omega h(u_m(x,t),v_m(x,t))\Delta v_m(x,t)dx = \langle \nabla g_2, \nabla v_m(t)\rangle,\ t \geq 0, \tag{42}$$

that is valid for any Galerkin approximation $(u_m(x,t),v_m(x,t))$ of problem (33)–(35), (38), (39). Using identity (42) similar to Proposition 2, we prove

Proposition 5. *Problem (33)–(35), (38), (39) has a weak solution $(u(t),v(t))$ in the function classes (36)–(37) that satisfies the inequality*

$$\|v(t)\|_1^2 \leq \|v_0\|_1^2 \mathrm{e}^{-\sigma t} + C_5\left(\|u_0\|^2 + \|v_0\|^2\right)e^{-\sigma t} + R^2, \quad \forall t \geq 0, \tag{43}$$

where the value R and the constant C_5 are the same as in Proposition 2 (see [12]).

We now construct the trajectory attractor for system (33)–(34). We shall use the spaces $\mathcal{F}_+^{\mathrm{loc}}$ and $\mathcal{F}_+^{\mathrm{b}}$ introduced in § 2.

In the space $\mathcal{F}_+^{\mathrm{loc}}$, we consider the following topology that, for simplicity, we define in terms of the weak convergence of the corresponding sequences. By definition, a sequence of functions $\{(y_m(\cdot), z_m(\cdot))\} \subset \mathcal{F}_+^{\mathrm{loc}}$ converges as $m \to \infty$ to a function

$(y(\cdot), z(\cdot)) \in \mathcal{F}_+^{\mathrm{loc}}$ in the topology Θ_+^{loc}, if for each $M > 0$

1) $y_m(\cdot) \rightharpoonup y(\cdot)$ weak-$*$ in $L_\infty(0, M; H)$, weakly in $L_2(0, M; V)$, and weakly in $L_{p_1}(0, M; L_{p_1}(\Omega))$,
2) $z_m(\cdot) \rightharpoonup z(\cdot)$ weak-$*$ in $L_\infty(0, M; V)$ and weakly in $L_{p_2}(0, M; L_{p_2}(\Omega))$,
3) $\partial_t y_m(\cdot) \rightharpoonup \partial_t y(\cdot)$ weakly in $L_{q_1}(0, M; H^{-r_1}(\Omega))$, and
4) $\partial_t z_m(\cdot) \rightharpoonup \partial_t z(\cdot)$ weakly in $L_{q_2}(0, M; H^{-r_2}(\Omega))$

as $m \to \infty$. The space $\mathcal{F}_+^{\mathrm{loc}}$ with topology Θ_+^{loc} is a linear Hausdorff space and a Frechét-Urysohn space with countable topology base (see [3]).

Remark 1. Any ball

$$\mathcal{B}_r = \left\{ \|(y, z)\|_{\mathcal{F}_+^{\mathrm{b}}} \leq r \right\}$$

in the space $\mathcal{F}_+^{\mathrm{b}}$ is a compact set in the topology Θ_+^{loc}. The general theorems on compact topology spaces imply that the set B_r with topology induced from Θ_+^{loc} is a metrizable space (see, e.g., [18]).

The trajectory space $\mathcal{K}_+^0(N)$ for system (33)–(34) is defined similarly to spaces $\mathcal{K}_+^\delta(N)$ of system (6)–(7) (see Definition 2.1).

Definition 3.1. The space $\mathcal{K}_+^0(N)$ consists of functions $(u(\cdot), v(\cdot)) \in \mathcal{F}_+^{\mathrm{loc}}$ such that

(i): $(u(t), v(t)), t \geq 0$, is a weak solution to system (33)–(34);
(ii): the function $v(t)$ satisfies the inequality

$$\|v(t)\|_1^2 \leq N\mathrm{e}^{-\sigma t} + R^2, \quad \forall t \geq 0, \tag{44}$$

where the values σ and R are taken from inequality (43).

Using Proposition 5, we prove that the space $\mathcal{K}_+^0(N)$ is non-empty and sufficiently large for any $N > 0$.

Consider the translation operators $T(\tau), \tau \geq 0$, acting on the space $\mathcal{F}_+^{\mathrm{loc}}$ by the formula (28). This operators clearly form a semigroup $\{T(\tau)\} := \{T(\tau), \tau \geq 0\}$ called the *translation semigroup*. Note that the semigroup $\{T(\tau)\}$ takes the trajectory space $\mathcal{K}_+^0(N)$ to itself, that is,

$$T(\tau) : \mathcal{K}_+^0(N) \to \mathcal{K}_+^0(N), \quad \forall \tau \geq 0. \tag{45}$$

Indeed, if $(u, v) \in \mathcal{K}_+^0(N)$, then the function $T(\tau)(u(t), v(t)) = (u(t+\tau), v(t+\tau))$ is also a weak solution to the autonomous system (33)–(34). Besides, since $v(t)$ satisfies (44), we have

$$\|v(t+\tau)\|_1^2 \leq N\mathrm{e}^{-\sigma(t+\tau)} + R^2 \leq N\mathrm{e}^{-\sigma t} + R^2,$$

and, hence, $T(\tau)v(t) = v(t+\tau)$ also satisfies (44), i.e., $T(\tau)(u, v) \in \mathcal{K}_+^0(N)$ for $\tau \geq 0$.

Proposition 3 holds for the space $\mathcal{K}_+^0(N)$, since, the system (33)–(34) is a particular case of system (6)–(7) for $\delta \equiv 0$. We recall this assertion.

Proposition 6. *The trajectory space $\mathcal{K}_+^0(N)$ belongs to $\mathcal{F}_+^{\mathrm{b}}$ and the following inequality holds:*

$$\|T(\tau)(u, v)\|_{\mathcal{F}_+^{\mathrm{b}}} \leq C_6 \left(\|u(0)\|^2 + N\right) e^{-\rho\tau} + R_3^2, \quad \forall \tau \geq 0, \tag{46}$$

where $C_6 \geq 0$ and $\rho > 0$ are independent of g_1, g_2, and $R_3 = R_3(\|g_1\|, \|g_2\|)$.

Proposition 7. *The space $\mathcal{K}_+^0(N)$ is closed in Θ_+^{loc} for each $N \geq 0$.*

Proof. Consider an arbitrary sequence $w_m = (u_m(t), v_m(t)) =: (u_m, v_m) \in \mathcal{K}_+^0(N)$, $m = 1, 2, \ldots$, that converges in the topology Θ_+^{loc} as $m \to \infty$ to a function $w = (u(t), v(t)) =: (u, v) \in \mathcal{F}_+^{\mathrm{loc}}$. We claim that $w \in \mathcal{K}_+^0(N)$. The convergence stated above means that, on any interval $[0, M]$, we have

$$\left.\begin{array}{lll} u_m(\cdot) \rightharpoonup u(\cdot) & \text{weak-}* \text{ in } L_\infty(0, M; H) \\ u_m(\cdot) \rightharpoonup u(\cdot) & \text{weakly in } L_2(0, M; V) \\ v_m(\cdot) \rightharpoonup v(\cdot) & \text{weak-}* \text{ in } L_\infty(0, M; V) \end{array}\right\}, \tag{47}$$

$$\left.\begin{array}{ll} u_m(\cdot) \rightharpoonup u(\cdot) & \text{weakly in } L_{p_1}(0, M; L_{p_1}(\Omega)) \\ v_m(\cdot) \rightharpoonup v(\cdot) & \text{weakly in } L_{p_2}(0, M; L_{p_2}(\Omega)) \end{array}\right\}, \tag{48}$$

$$\left.\begin{array}{ll} \partial_t u_m(\cdot) \rightharpoonup \partial_t u(\cdot) & \text{weakly in } L_{q_1}(0, M; H^{-r_1}(\Omega)) \\ \partial_t v_m(\cdot) \rightharpoonup \partial_t v(\cdot) & \text{weakly in } L_{q_2}(0, M; H^{-r_2}(\Omega)) \end{array}\right\} \tag{49}$$

as $m \to \infty$. Then, clearly, the sequence $\{u_m\}$ is bounded in the spaces $L_\infty(0, M; H)$, $L_2(0, M; V)$, and $L_{p_1}(0, M; L_{p_1}(\Omega))$, the sequence of functions $\{v_m\}$ is bounded in $L_\infty(0, M; V)$ and $L_{p_2}(0, M; L_{p_2}(\Omega))$, while the derivatives $\{\partial_t u_m\}$ and $\{\partial_t v_m\}$ are bounded in the spaces $L_{q_1}(0, M; H^{-r_1}(\Omega))$ and $L_{q_2}(0, M; H^{-r_2}(\Omega))$, respectively. Therefore, owing to inequality (17) $\{f(u_m, v_m)\}$ and $\{h(u_m, v_m)\}$ are bounded in the spaces $L_{q_1}(0, M; L_{q_1}(\Omega))$ and $L_{q_2}(0, M; L_{q_2}(\Omega))$, respectively. Then passing, if necessarily, to a subsequence $\{m'\} \subset \{m\}$ but retaining the same notation, we may assume that

$$\left.\begin{array}{ll} f(u_m, v_m) \rightharpoonup \varphi(\cdot) & \text{weakly in } L_{q_1}(0, M; L_{q_1}(\Omega)), \\ h(u_m, v_m) \rightharpoonup \chi(\cdot) & \text{weakly in } L_{q_2}(0, M; L_{q_2}(\Omega)) \end{array}\right\} \tag{50}$$

as $m \to \infty$, where $\varphi = \varphi(x, t)$ and $\chi = \chi(x, t)$ are some functions from the spaces $L_{q_1}(0, M; L_{q_1}(\Omega))$ and $L_{q_2}(0, M; L_{q_2}(\Omega))$, respectively.

Since the pair $(u_m(t), v_m(t))$ is a weak solution, these functions satisfy (33)–(34) in the sense of distributions:

$$\begin{aligned} \partial_t u_m &= \Delta u_m - f(u_m, v_m) + g_1(x), \\ \partial_t v_m &= -h(u_m, v_m) + g_2(x). \end{aligned}$$

Then, properties (47), (49), and (50) implies that, as distribution, the pair of functions $(u(t), v(t))$ satisfy the equations

$$\begin{aligned} \partial_t u &= \Delta u - \varphi(x, t) + g_1(x), \\ \partial_t v &= -\chi(x, t) + g_2(x), \quad 0 \le t \le M. \end{aligned}$$

Recall that the sequence of functions $\{u_m(t)\}$ is bounded in $L_2(0, M; V)$, and the sequence of derivatives $\{\partial_t u_m(t)\}$ is bounded in $L_{q_1}(0, M; H^{-r_1}(\Omega))$. Besides, the embedding $V \Subset H \equiv L_2(\Omega)$ is compact. Therefore, by the Aubin theorem (see [19] and [20]), the sequence $\{u_m(t)\}$ is precompact in the space $L_2(0, M; L_2(\Omega))$. Consequently,

$$u_m(\cdot) \rightharpoonup u(\cdot) \ \text{ as } m \to \infty \quad \text{strongly in } L_2(\Omega \times]0, M[).$$

Passing to a subsequence again and keeping the notation, we obtain that

$$u_m(x, t) \to u(x, t) \ \text{ as } m \to \infty \quad \text{for almost all } (x, t) \in \Omega \times]0, M[.$$

Similarly, we prove that

$$v_m(x, t) \to v(x, t) \ \text{ as } m \to \infty \quad \text{for almost all } (x, t) \in \Omega \times]0, M[.$$

Using the continuity of functions f and h, we obtain that

$$f(u_m(x,t), v_m(x,t)) \to f(u(x,t), v(x,t)),$$
$$h(u_m(x,t), v_m(x,t)) \to h(u(x,t), v(x,t)) \text{ as } m \to \infty$$

for almost all $(x,t) \in \Omega\times]0,M[$. The sequences $\{f(u_m,v_m)\}$ and $\{h(u_m,v_m)\}$ are bounded in $L_{q_1}(0,M;L_{q_1}(\Omega))$ and $L_{q_2}(0,M;L_{q_2}(\Omega))$, respectively. We now apply the well-known Lions lemma on weak convergence (see [16, Ch. 1, Lemma 1.3]), which implies that

$$f(u_m,v_m) \rightharpoonup f(u,v) \text{ weakly in } L_{q_1}(0,M;L_{q_1}(\Omega)),$$
$$h(u_m,v_m) \rightharpoonup h(u,v) \text{ weakly in } L_{q_2}(0,M;L_{q_2}(\Omega)).$$

as $m \to \infty$. Hence from (50), we conclude that

$$\varphi(x,t) \equiv f(u(x,t),v(x,t)) \quad \text{and} \quad \chi(x,t) \equiv h(u(x,t),v(x,t))$$

almost everywhere in $\Omega\times]0,M[$. That is, the pair of functions $(u(x,t),v(x,t))$ is a weak solution to system (33)–(34). It remains to check inequality (44) for the function $v(x,t)$. Indeed, the functions $v_m(x,t)$ satisfy (44). Therefore, for every $\theta \geq 0$

$$\operatorname{ess\,sup}\left\{\|v_m(\theta)\|_1^2 \mid t \leq \theta \leq t+1\right\} \leq N\mathrm{e}^{-\sigma t} + R^2, \quad \forall m \in \mathbb{N}.$$

Recall that

$$v_m(\cdot) \rightharpoonup v(\cdot) \text{ weak-}* \text{ in } L_\infty(0,M;V) \text{ as } m \to \infty, \quad \forall M > 0.$$

Hence, for every fixed $t \geq 0$

$$\begin{aligned} &\operatorname{ess\,sup}\left\{\|v(\theta)\|_1^2 \mid t \leq \theta \leq t+1\right\} \\ &\leq \liminf_{m\to\infty} \left\{\|v_m(\theta)\|_1^2 \mid t \leq \theta \leq t+1\right\} \leq N\mathrm{e}^{-\sigma t} + R^2. \end{aligned} \tag{51}$$

The Lions-Magenes lemma implies that $v(\cdot) \in C_w(\mathbb{R}_+;V)$, and the real function $\|v(t)\|_1$ for $t \geq 0$, is lower semicontinuous, that is, in particular,

$$\|v(t)\|_1 \leq \liminf_{\theta\to t^+} \|v(\theta)\|_1$$

(see, e.g., [3]). Applying this relation together with (51), we find that

$$\|v(t)\|_1 \leq N\mathrm{e}^{-\sigma t} + R^2, \ \forall t \geq 0.$$

We have established inequality (44) for the function $v(x,t)$, and so $w = (u,v) \in \mathcal{K}_+^0(N)$. We have proved that $\mathcal{K}_+^0(N)$ is closed in the topology Θ_+^{loc}. □

We now study the translation semigroup $\{T(\tau)\}$ acting no the trajectory space $\mathcal{K}_+^0(N)$.

Definition 3.2. A set $\mathcal{P} \subseteq \mathcal{K}_+^0(N)$ is called *absorbing* for the semigroup $\{T(\tau)\}$, if for every set $B \subset \mathcal{K}_+^0(N)$ bounded in $\mathcal{F}_+^{\mathrm{b}}$, there is $\tau_1 = \tau_1(B) \geq 0$ such that $T(\tau)B \subseteq \mathcal{P}$ for all $\tau \geq \tau_1$.

Definition 3.3. A set $\mathcal{P} \subseteq \mathcal{K}_+^0(N)$ is called *attracting* for the semigroup $\{T(\tau)\}$, if any neighbourhood $\mathcal{O}(\mathcal{P})$ of the set $\mathcal{P}$ in the topology Θ_+^{loc} is an absorbing set, i.e., for every set $B \subset \mathcal{K}_+^0(N)$ bounded in $\mathcal{F}_+^{\mathrm{b}}$, there is $\tau_1 = \tau_1(B,\mathcal{O}) \geq 0$ such that $T(\tau)B \subseteq \mathcal{O}(\mathcal{P})$ for all $\tau \geq \tau_1$.

It is clear that any absorbing set is attracting. We now give the main definition of the trajectory attractor for the semigroup $\{T(\tau)\}$.

Definition 3.4. A set $\mathfrak{A} \subset \mathcal{K}_+^0(N)$ is called the *trajectory attractor* of the semigroup $\{T(\tau)\}$ on $\mathcal{K}_+^0(N)$, if it is bounded in $\mathcal{F}_+^{\mathrm{b}}$, compact in Θ_+^{loc}, strictly invariant with respect to $\{T(\tau)\}$, i.e.,

$$T(\tau)\mathfrak{A} = \mathfrak{A}, \quad \forall \tau \geq 0,$$

and $\mathfrak{A}$ is an attracting set of $\{T(\tau)\}$ on $\mathcal{K}_+^0(N)$ in the topology Θ_+^{loc}.

It is easy to see that the trajectory attractor of the translation semigroup is unique. Our aim is to construct the trajectory attractor $\mathfrak{A}$ of the semigroup $\{T(\tau)\}$ on $\mathcal{K}_+^0(N)$.

Inequality (46) implies that the set

$$P = \left\{ (u, v) \in \mathcal{K}_+^0(N) \mid \|(u, v)\|_{\mathcal{F}_+^{\mathrm{b}}} \leq 2R_3^2 \right\}$$

is absorbing for the semigroup $\{T(\tau)\}$ on $\mathcal{K}_+^0(N)$. The set P is bounded in $\mathcal{F}_+^{\mathrm{b}}$. Consider the topology on P induced by Θ_+^{loc}. This topological space is compact and metrizable. Using (45) and the obvious inequality

$$\|T(\tau)(u, v)\|_{\mathcal{F}_+^{\mathrm{b}}} \leq \|(u, v)\|_{\mathcal{F}_+^{\mathrm{b}}}, \quad \forall \tau \geq 0,$$

we find that the semigroup $\{T(\tau)\}$ maps P into itself:

$$T(\tau)P \subseteq P, \quad \forall \tau \geq 0.$$

It follows easily that the translation semigroup $\{T(\tau)\}$ is continuous on $\mathcal{K}_+^0(N)$ in the topology Θ_+^{loc}. Thus, we have a continuous semigroup $\{T(\tau)\}$ acting on the compact metric space P. Then the general theorem on the existence of a global attractor is applicable (see, e.g., [1], [2], and [4]). The global attractor $\mathfrak{A}(N) \subseteq P$ of the semigroup $\{T(\tau)\}$ is constructed by the standard formula

$$\mathfrak{A}(N) := \bigcap_{\tau \geq 0} \left[\bigcup_{\theta \geq \tau} T(\theta)P \right]_{\Theta_+^{\mathrm{loc}}}. \tag{52}$$

The set $\mathfrak{A}(N)$ has the following properties: it is bounded in $\mathcal{F}_+^{\mathrm{b}}$, compact in Θ_+^{loc}, strictly invariant, that is,

$$T(\tau)\mathfrak{A}(N) = \mathfrak{A}(N), \quad \forall \tau \geq 0, \tag{53}$$

and, as it is a global attractor, $\mathfrak{A}(N)$ attracts any set $B \subseteq P$. However, P is an absorbing set of the semigroup $\{T(\tau)\}$, therefore, $\mathfrak{A}(N)$ attracts any bounded set $B \subset \mathcal{K}_+^0(N)$. Consequently, $\mathfrak{A}(N)$ is the trajectory attractor of $\{T(\tau)\}$ on $\mathcal{K}_+^0(N)$.

Proposition 8. *The trajectory attractor constructed above is independent of* N : $\mathfrak{A}(N) = \mathfrak{A}$. *In particular,* $\mathfrak{A} = \mathfrak{A}(0)$, *i.e.,*

$$\sup \left\{ \|v(t)\|_1^2 \mid t \geq 0 \right\} \leq R^2, \quad \forall (u, v) \in \mathfrak{A}. \tag{54}$$

Proof. Assume that $N > 0$. It follows from the definition of the space $\mathcal{K}_+^0(N)$ that $\mathcal{K}_+^0(N) \subseteq \mathcal{K}_+^0(N_1)$ for $N_1 \geq N$. Hence $\mathfrak{A}(N) \subseteq \mathfrak{A}(N_1)$ for $N_1 \geq N$. From (44) it follows that $T(\tau)\mathcal{K}_+^0(N_1) \subseteq \mathcal{K}_+^0(N)$ for $\tau \geq \sigma^{-1} \ln (N_1/N)$ and, in particular, $T(\tau)\mathfrak{A}(N_1) \subseteq \mathcal{K}_+^0(N)$. Since the attractor is strictly invariant, $\mathfrak{A}(N_1) = T(\tau)\mathfrak{A}(N_1)$, i.e., $\mathfrak{A}(N_1) \subseteq \mathcal{K}_+^0(N)$. But then, again because of the strict invariance of the attractors, we have that $\mathfrak{A}(N_1) \subseteq \mathfrak{A}(N)$ for all $N_1 \geq N$. Thus, $\mathfrak{A}(N_1) = \mathfrak{A}(N)$ for $N_1 \geq N$. We have proved that $\mathfrak{A} = \mathfrak{A}(N)$ is independent of N for $N > 0$, i.e., for every $(u, v) \in \mathfrak{A}$ we have the inequality

$$\sup \left\{ \|v(t)\|_1^2 \mid t \geq 0 \right\} \leq N + R^2, \quad \forall N > 0,$$

consequently, we obtain (54). □

It follows from (52) and (53) that

$$\mathfrak{A} = \bigcap_{\tau \geq 0} T(\tau)P.$$

In conclusion, we describe the structure of the trajectory attractor $\mathfrak{A}$ using complete trajectories of system (33)–(34), that is, weak solutions to this system defined on the entire time axis.

We define the spaces $\mathcal{F}^{\mathrm{loc}}$, its subspace $\mathcal{F}^{\mathrm{b}}$, and the topology Θ^{loc} on $\mathcal{F}^{\mathrm{loc}}$ similarly to $\mathcal{F}_+^{\mathrm{loc}}, \mathcal{F}_+^{\mathrm{b}}$, and Θ_+^{loc} replacing $\mathbb{R}_+$ $(t \geq 0)$ in their definitions by the entire real axis $\mathbb{R}$ $(-\infty < t < \infty)$. In particular,

$$\mathcal{F}^{\mathrm{loc}} = \left\{ \begin{array}{c} (y(x,t), z(x,t)), x \in \Omega, t \in \mathbb{R} \mid \\ y \in L_\infty^{\mathrm{loc}}(\mathbb{R}; H) \cap L_2^{\mathrm{loc}}(\mathbb{R}; V) \cap L_{p_1}^{\mathrm{loc}}(\mathbb{R}; L_{p_1}(\Omega)), \\ z \in L_\infty^{\mathrm{loc}}(\mathbb{R}; V) \cap L_{p_2}^{\mathrm{loc}}(\mathbb{R}; L_{p_2}(\Omega)), \\ \partial_t y \in L_{q_1}^{\mathrm{loc}}(\mathbb{R}; H^{-r_1}(\Omega)), \partial_t z \in L_{q_2}^{\mathrm{loc}}(\mathbb{R}; H^{-r_2}(\Omega)) \end{array} \right\},$$

(see (25)) and the norm in the space $\mathcal{F}^{\mathrm{b}}$ is defined by the formula (cf. (26))

$$\begin{aligned} \|(y,z)\|_{\mathcal{F}^{\mathrm{b}}} := {} & \|y\|_{L_\infty(\mathbb{R};H)} + \|y\|_{L_2^{\mathrm{b}}(\mathbb{R};V)} + \|y\|_{L_{p_1}^{\mathrm{b}}(\mathbb{R};L_{p_1})} + \|\partial_t y\|_{L_{q_1}^{\mathrm{b}}(\mathbb{R};H^{-r_1})} \\ & + \|z\|_{L_\infty(\mathbb{R};V)} + \|z\|_{L_{p_2}^{\mathrm{b}}(\mathbb{R};L_{p_2})} + \|\partial_t z\|_{L_{q_2}^{\mathrm{b}}(\mathbb{R};H^{-r_2})}. \end{aligned} \tag{55}$$

We consider weak solutions $\{u(t), v(t)\}, t \in \mathbb{R}$, to system (33)–(34) that belong to the space $\mathcal{F}^{\mathrm{loc}}$ and satisfy equations (33) and (34) in the distribution sense.

Definition 3.5. The kernel $\mathcal{K}^0$ of the system of equations (33)–(34) in the space $\mathcal{F}^{\mathrm{b}}$ is called the set of weak solutions $\{u(t), v(t)\}, t \in \mathbb{R}$, of this system from $\mathcal{F}^{\mathrm{loc}}$ that belong to $\mathcal{F}^{\mathrm{b}}$ (i.e., that have the finite norm $\|(u,v)\|_{\mathcal{F}^{\mathrm{b}}}$ defined in (55)) and which satisfy the following inequality

$$\sup \{\|v(t)\|_1 \mid t \in \mathbb{R}\} \leq R, \tag{56}$$

where the value R is taken from inequality (44).

We denote by Π_+ the operator restricting functions on $\mathbb{R}$ to $\mathbb{R}_+$. This operator maps a function $\{\varphi(t), t \in \mathbb{R}\}$ into the function $\{\Pi_+\varphi(t), t \geq 0\}$, where $\Pi_+\varphi(t) \equiv \varphi(t)$ for $t \geq 0$.

We have the following result.

Theorem 3.6. *The kernel $\mathcal{K}^0$ of system (33)–(34) is bounded in the space $\mathcal{F}^{\mathrm{b}}$ and compact in the topology Θ^{loc}. The trajectory attractor $\mathfrak{A}$ of system (33)–(34) coincides with the restriction to $\mathbb{R}_+$ of the kernel $\mathcal{K}^0$:*

$$\mathfrak{A} = \Pi_+\mathcal{K}^0. \tag{57}$$

Proof. Suppose that $\zeta = (y,z) \in \mathcal{K}^0$. We claim that $\Pi_+\zeta \in \mathfrak{A}$. Consider the set $B_\zeta = \{\Pi_+\zeta(\eta + t) \mid \eta \in \mathbb{R}\}$, which contains ζ. Every pair of functions $\Pi_+\zeta(\eta+t) = (y(\eta+t), z(\eta+t)), t \geq 0$, is a weak solution to system (33)–(34). Also, by (56) we obtain $B_\zeta \subseteq \mathcal{K}_+^0(0)$. Moreover, the set B_ζ is bounded in $\mathcal{F}_+^{\mathrm{b}}$ and is strictly invariant with respect to the semigroup $\{S(\tau)\}$ because clearly $S(\tau)B_\zeta = B_\zeta$ for all $\tau \geq 0$. Recall that $\mathfrak{A}(0)$ attracts $S(\tau)B_\zeta$ as $\tau \to +\infty$. Hence,

$$B_\zeta \subseteq \mathfrak{A}(0),$$

and, in particular, for all $\zeta \in \mathcal{K}$, the function $\Pi_+\zeta \in \mathfrak{A}(0) \equiv \mathfrak{A}$ (see Proposition 8). We have shown that

$$\Pi_+\mathcal{K}^0 \subseteq \mathfrak{A}.$$

Let us establish the reverse inclusion. We must prove that every weak solution (u, v) to system (33)–(34) belonging to $\mathfrak{A} = \mathfrak{A}(0)$ admits an extension to the negative semiaxis as a weak solution preserving inequality (56) for all $t \in \mathbb{R}$.

Indeed, the strict invariance property of the set $\mathfrak{A}$ implies that, for any $(u, v) \in \mathfrak{A}$, there is a weak solution $(u_1(t), v_1(t)), t \geq 0$, such that $(u_1, v_1) \in \mathfrak{A} \subseteq \mathcal{K}_+^0(0)$, and $(u_1(t+1), v_1(t+1)) = (u(t), v(t))$ for all $t \geq 0$. We set $(\tilde{u}(t), \tilde{v}(t)) = (u_1(t+1), v_1(t+1))$ for $t \geq -1$. It is clear that $(\tilde{u}(t), \tilde{v}(t))$ is a weak solution for $t \geq -1$. This pair coincides with $(u(t), v(t))$ for $t \geq 0$, and

$$\sup\{\|v(t)\|_1 \mid t \geq -1\} \leq R.$$

Repeating this process to (u_1, v_1) in place of (u, v), we extend the original solution to the semiaxis $\{t \geq -2\}$ keeping the necessary properties, then to $\{t \geq -3\}$, an so on. As a result, we construct the complete weak solution $(\tilde{u}(t), \tilde{v}(t)), t \in \mathbb{R}$, to system (33)–(34), that satisfies the inequality

$$\sup\{\|v(t)\|_1 \mid t \in \mathbb{R}\} \leq R,$$

and belongs to $\mathcal{F}^{\mathrm{b}}$, since $\mathfrak{A}$ is bounded in $\mathcal{F}_+^{\mathrm{b}}$. Then $(\tilde{u}, \tilde{v}) \in \mathcal{K}^0$, and furthermore, $\Pi_+(\tilde{u}, \tilde{v}) = (u, v)$. Consequently, $(u, v) \in \Pi_+\mathcal{K}^0$ for all $(u, v) \in \mathfrak{A}$, that is,

$$\mathfrak{A} \subseteq \Pi_+\mathcal{K}^0,$$

and identity (57) is established. □

4. **Convergence of solutions of the original reaction-diffusion system to the trajectory attractor $\mathfrak{A}$ of the limit reaction-diffusion system.** Recall that the diffusion coefficient δ depends on time, $\delta(\cdot) \in L_\infty(\mathbb{R}_+)$ and

$$\int_t^{t+1} \delta(s)ds \to 0 \quad \text{as} \quad t \to +\infty. \tag{58}$$

Let $B = \{(u(x,t), v(x,t)),\ x \in \Omega, t \geq 0\}$ be a family of weak solutions of the original non-autonomous reaction-diffusion system (6)–(7) belonging to $\mathcal{K}_+^\delta(N)$ and bounded in $\mathcal{F}_+^{\mathrm{b}}$. Recall that a pair $(u(\cdot), v(\cdot))$ belongs to $\mathcal{K}_+^\delta(N)$ if $(u(x,t), v(x,t))$ is a weak solution to (6)–(7) and $(u(\cdot), v(\cdot))$ satisfies the inequality

$$\|v(t)\|_1^2 \leq N\mathrm{e}^{-\sigma t} + R^2, \quad \forall t \geq 0. \tag{59}$$

We study the behaviour of the family $T(\tau)B$ in the topology Θ_+^{loc} as $\tau \to +\infty$. The central result of the paper is the following theorem on stabilization.

Theorem 4.1. *For every $N > 0$ and for any set of trajectories $B = \{(u(t), v(t))\} \in \mathcal{K}_+^\delta(N)$, which is bounded in $\mathcal{F}_+^{\mathrm{b}}$, the family $T(\tau)B = \{(u(t+\tau), v(t+\tau))\}$ converges in the topology Θ_+^{loc} as $\tau \to +\infty$ to the trajectory attractor $\mathfrak{A}$ of the limit autonomous system (33)–(34):*

$$T(\tau)B \to \mathfrak{A}\ (\tau \to +\infty) \text{ in } \Theta_+^{\mathrm{loc}}. \tag{60}$$

Proof. Assume that (60) fails for some bounded (in $\mathcal{F}_+^{\mathrm{b}}$) set $B \subset \mathcal{K}_+^\delta(N)$. Then there is a neighbourhood $\mathcal{O}(\mathfrak{A})$ in Θ_+^{loc}, a sequence $\tau_n \to +\infty$, and a sequence

$$(u_n(t), v_n(t)) =: w_n \in B \subset \mathcal{K}_+^\delta(N),\ n = 1, 2, \ldots$$

such that

$$T(\tau_n)w_n \notin \mathcal{O}(\mathfrak{A}) \text{ for } n = 1, 2, \ldots \tag{61}$$

The sequence $\{w_n\}$ is bounded in $\mathcal{F}_+^{\mathrm{b}}$, that is, for some r,

$$\|w_n(\cdot)\|_{\mathcal{F}_+^{\mathrm{b}}} \leq r, \quad \forall n. \tag{62}$$

Since $w_n \in \mathcal{K}_+^{\delta}(N)$, it follows from (59) that

$$\|v_n(t+\tau_n)\|_1^2 \leq N\mathrm{e}^{-\sigma(t+\tau_n)} + R^2, \quad \forall t \geq 0. \tag{63}$$

We set

$$\begin{aligned} &U_n(t) := u(t+\tau_n),\ V_n(t) := v(t+\tau_n), \\ &W_n(t) := (U_n(t), V_n(t)) \qquad \text{for } t \geq -\tau_n, . \end{aligned}$$

Then clearly

$$W_n = T(\tau_n) w_n.$$

From (61) and (63) we conclude that

$$W_n \notin \mathcal{O}(\mathfrak{A}) \quad \text{ for } n = 1, 2, \dots \tag{64}$$

$$\|V_n(t)\|_1^2 \leq N\mathrm{e}^{-\sigma(t+\tau_n)} + R^2, \quad \forall t \geq -\tau_n. \tag{65}$$

We denote by $\mathcal{F}_\ell^{\mathrm{loc}}$ and $\mathcal{F}_\ell^{\mathrm{b}}$ for $\ell \in \mathbb{R}$ the spaces similar to $\mathcal{F}_0^{\mathrm{loc}} := \mathcal{F}_+^{\mathrm{loc}}$ and $\mathcal{F}_0^{\mathrm{b}} := \mathcal{F}_+^{\mathrm{b}}$, which consist of functions on the semiaxis $]\ell, +\infty[$ (see (25)), and the norm in $\mathcal{F}_\ell^{\mathrm{b}}$ is defined by the formula (26) with $\mathbb{R}_+ =]0, +\infty[$ replaced by $]\ell, +\infty[$. By analogy to Θ_+^{loc}, in the spaces $\mathcal{F}_\ell^{\mathrm{loc}}$ and $\mathcal{F}_\ell^{\mathrm{b}}$, we introduce the topology $\Theta_\ell^{\mathrm{loc}}$.

From (62) we have that

$$\|W_n(\cdot)\|_{\mathcal{F}_{-\tau_n}^{\mathrm{b}}} \leq r, \quad \forall n. \tag{66}$$

Thus, for every fixed $M > 0$, the sequence $\{W_n\}_{\tau_n \geq M}$ is bounded in $\mathcal{F}_{-M}^{\mathrm{b}}$ and hence this sequence is precompact in the space $\Theta_{-M}^{\mathrm{loc}}$. Therefore, for every $M > 0$, there is a subsequence $\{n'\} \subset \{n\}$, $n' = n'(M)$, such that $\{W_{n'}\}$ is convergent in the topology $\Theta_{-M}^{\mathrm{loc}}$. Then, using the well-known Cantor diagonal construction, we can find a subsequence of indices $\{n''\} \subset \{n\}$ and a function $W = (U(t), V(t)), t \in \mathbb{R}$, such that

$$W_{n''}(\cdot) \to W \text{ as } n'' \to \infty \quad \text{in } \Theta_{-M}^{\mathrm{loc}}, \quad \text{for every } M > 0, \tag{67}$$

and in view of (66)

$$\|W(\cdot)\|_{\mathcal{F}^{\mathrm{b}}} = \sup_{M>0} \|W(\cdot)\|_{\mathcal{F}_{-M}^{\mathrm{b}}} \leq r, \tag{68}$$

that is $W \in \mathcal{F}^{\mathrm{b}}$. From (65) we find that

$$\sup \{\|V(t)\|_1 \mid t \in \mathbb{R}\} \leq R. \tag{69}$$

We claim that the function $W(t) = (U(t), V(t)), t \in \mathbb{R}$, belong to the kernel $\mathcal{K}^0$ of the limit equation (33)–(34). Indeed, it follows from (68) that, for every $M > 0$,

$$\begin{aligned} &U(\cdot) \in L_\infty(-M, M; H) \cap L_2(-M, M; V) \cap L_{p_1}(-M, M; L_{p_1}(\Omega)), \\ &V(\cdot) \in L_\infty(-M, M; V) \cap L_{p_2}(-M, M; L_{p_2}(\Omega)). \end{aligned}$$

We must prove that the pair $(U(t), V(t))$ satisfies the system (33)–(34) in the sense of distributions in the space $\mathcal{D}'(-M, M; H^{-r_1}(\Omega) \times H^{-r_2}(\Omega))$ for every $M > 0$.

We note that, for large $n = n(M)$, the pair $(U_n(t), V_n(t))$ satisfies the system

$$\partial_t U_n = \Delta U_n - f(U_n, V_n) + g_1(x), \tag{70}$$

$$\partial_t V_n = \delta(t+\tau_n)\Delta V_n - h(U_n, V_n) + g_2(x), \tag{71}$$

in the space $\mathcal{D}'(-M, M; H^{-r_1}(\Omega) \times H^{-r_2}(\Omega))$.

From (67) we conclude that

$$(U_{n''}(t), V_{n''}(t)) \to (U(t), V(t)) \quad \text{as } n'' \to \infty \quad \text{in } \Theta_{-M,M} \tag{72}$$

for every fixed $M > 0$. Applying the arguments from the proof of Proposition 7, we obtain that,

$$f(U_{n''}, V_{n''}) \rightharpoonup f(U, V) \text{ weakly in } L_{q_1}(-M, M; L_{q_1}(\Omega)), \tag{73}$$

$$h(U_{n''}, V_{n''}) \rightharpoonup h(U, V) \text{ weakly in } L_{q_2}(-M, M; L_{q_2}(\Omega)) \tag{74}$$

as $n'' \to \infty$.

Consider the behaviour of the term $\delta(t + \tau_n)\Delta V_n$ as $n \to \infty$ in details. From (58), we learn that

$$\int_{-M}^{M} \delta(s + \tau_n) ds \to 0 \quad \text{as } n \to \infty.$$

Inequality (65) implies that ΔV_n is bounded in $L_\infty(-M, M; H^{-1})$. Therefore,

$$\delta(t + \tau_n)\Delta V_n(t) \to 0 \quad \text{weak-}* \text{ in } L_\infty(-M, M; H^{-1}(\Omega)) \quad \text{as } n \to \infty. \tag{75}$$

The convergencies (72)–(74) and (75) are stronger than the convergence of distributions in the space $\mathcal{D}'(-M, M; H^{-r_1}(\Omega) \times H^{-r_2}(\Omega))$. Passing to the limit in system (70)–(71) as $n'' \to \infty$, we obtain that the functions $(U(t), V(t)), t \in \mathbb{R}$, satisfy the equations

$$\partial_t U = \Delta U - f(U, V) + g_1(x),$$
$$\partial_t V = -h(U, V) + g_2(x)$$

in the sense of distributions. Moreover, inequality (69) implies that $W(\cdot) \in \mathcal{K}^0$. Then, owing to Theorem 3.6,

$$\Pi_+ W(\cdot) \in \mathfrak{A}.$$

From (67) we also conclude that

$$\Pi_+ W_{n''}(\cdot) \to \Pi_+ W(\cdot) \quad \text{as } n'' \to \infty \quad \text{in } \Theta_0^{\text{loc}}$$

and hence, if n'' is sufficiently large, then

$$\Pi_+ W_{n''}(\cdot) \in \mathcal{O}(\Pi_+ W) \subset \mathcal{O}(\mathfrak{A}),$$

which contradicts (64). Consequently, property (60) is established. □

In conclusion, we summarize the main result of this note: the non-autonomous reaction-diffusion system (6)–(7) with diffusion coefficient $\delta(t)$ that vanishes in time has the trajectory attractor which coincides with the trajectory attractor $\mathfrak{A}$ of the corresponding "limit" autonomous reaction-diffusion system (33)–(34) with zero diffusion coefficient, $\delta \equiv 0$.

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