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MATHEMATICS

Strong Trajectory Attractor for a Dissipative Reaction-Diffusion System

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The method of trajectory attractors makes it possible to effectively investigate the behavior as $t \to +\infty$ of solutions to evolution partial differential equations for which either the corresponding initial boundary value problem is ill-posed, or the well-posedness of this problem is not proved, in particular, to the 3D Navier–Stokes system. So far, for instructive examples, the attraction of families of trajectories to an attractor has been proved, as rule, only in a suitable "maximal" weak topology of the problem under consideration. In this paper, for the first time, it is proved for a very general dissipative reaction-diffusion system that its solutions converge to a trajectory attractor in the "maximal" strong topology on the corresponding phase space, in which solutions to this system are constructed. The reaction-diffusion system under consideration is ill-posed, because it does not satisfies conditions for the uniqueness solvability of the corresponding Cauchy problem.

The proof of the theorem about the strong attraction to a trajectory attractor uses the energy identity method, which was previously used in [1, 2] to construct global attractors of noncompact semigroups for various dissipative evolution equations in unbounded domains.

1. A DISSIPATIVE REACTION-DIFFUSION SYSTEM AND ITS WEAK SOLUTIONS

We consider the system of partial differential equations of the form

$$\partial_t u = \mathbf{d}\Delta u - f(u) + g(x), \quad x \in \Omega \Subset \mathbb{R}^n, \quad t \geq 0 \quad (1)$$

with respect to the unknown vector function $u = (u^1(x, t), u^2(x, t), \ldots, u^N(x, t))$. The function $f(u) = (f^1(u), f^2(u), \ldots, f^N(u))$ describes a nonlinear interaction. The inhomogeneous external action $g(x) = (g^1(x), g^2(x), \ldots, g^N(x))$ is assumed to be known. The $N \times N$ matrix $\mathbf{d}$ has positive symmetric part $\frac{\mathbf{d} + \mathbf{d}^*}{2} \geq \beta \mathbf{I}$, where $\beta > 0$. On the boundary $\partial\Omega$ of the domain, the Dirichlet condition

$$u|_{\partial\Omega} = 0 \quad (2)$$

is set. (Other boundary conditions, such as the Neumann conditions, are considered in a similar way.)

Systems of the form (1) describe complex composite chemical reaction in an inhomogeneous media, in which diffusion and cross-diffusion of chemical components may occur (the matrix $\mathbf{d}$ is nondiagonal). The inhomogeneous external action $g(x)$ can also be interpreted as a flux of light, radiation, etc. (see real-life examples of reaction-diffusion systems in survey [3]). Note that the complex Ginzburg–Landau equation can also be written in the form (1) (see [4]).

We introduced the spaces $\mathbf{H} := [L_2(\Omega)]^N$ and $\mathbf{V} := [H_0^1(\Omega)]^N$. The norms on these spaces are denoted, respectively, by

$$|v|^2 := \int_\Omega \sum_{i=1}^N |v^i(x)|^2 dx \quad \text{and} \quad \|v\|^2 := \int_\Omega \sum_{i=1}^N |\nabla v^i(x)|^2 dx.$$

By $\mathbf{V}' = [H^{-1}(\Omega)]^N$ we denote the space dual to $\mathbf{V}$.

It is assumed that $g(\cdot) \in \mathbf{V}'$, $f(\cdot) \in C(\mathbb{R}^N; \mathbb{R}^N)$, and

$$\sum_{i=1}^N |f^i(v)|^{\frac{p_i}{p_i - 1}} \leq C_0 \left(\sum_{i=1}^N |v^i|^{p_i} + 1 \right), \quad (3)$$

$$\sum_{i=1}^N \gamma_i |v^i|^{p_i} - C_1 \leq \sum_{i=1}^N f^i(v) v^i, \quad \forall v \in \mathbb{R}^N, \quad (4)$$

where $\gamma_i > 0$ for $i = 1, 2, \ldots, N$. For definiteness, we assume that $2 \leq p_1 \leq \ldots \leq p_N$. Inequality (3) is related to the fact that, in real-life reaction-diffusion systems, the functions $f^i(u)$ are polynomials. Inequality (4) is called the dissipativity condition for the reaction-diffusion system (1).

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Let $q_i = \dfrac{p_i}{p_i - 1}$, $\dfrac{1}{p_i} + \dfrac{1}{q_i} = 1$, $1 < q_i \le 2$, $i = 1, 2, \ldots, N$.
We use the vector notations $\mathbf{p} = (p_1, p_2, \ldots, p_N)$ and $\mathbf{q} = (q_1, q_2, \ldots, q_N)$ and the spaces

$$\mathbf{L}_\mathbf{p}(\Omega) := L_{p_1}(\Omega) \times \ldots \times L_{p_N}(\Omega),$$
$$\mathbf{L}_\mathbf{p}(0, M; \mathbf{L}_\mathbf{p}(\Omega)) := L_{p_1}(0, M; L_{p_1}(\Omega)) \times \ldots$$
$$\ldots \times L_{p_N}(0, M; L_{p_N}(\Omega)).$$

We also consider the Banach space $\mathcal{L}(0, M) := \mathbf{L}_2(0, M; \mathbf{V}') + \mathbf{L}_\mathbf{q}(0, M; \mathbf{L}_\mathbf{q}(\Omega))$.

If $u(x, t) \in \mathbf{L}_\mathbf{p}(0, M; \mathbf{L}_\mathbf{p}(\Omega))$, then it follows from condition (3) that $f(u(x, t)) \in \mathbf{L}_\mathbf{q}(0, M; \mathbf{L}_\mathbf{q}(\Omega))$, and

$$\sum_{i=1}^{N} \| f_i(u(\cdot))\|^{q_i}_{L_{q_i}(0, M; L_{q_i})} \le C_0\left(\sum_{i=1}^{N} \|u(\cdot)\|^{p_i}_{L_{p_i}(0, M; L_{p_i})} + |\Omega| M\right). \tag{5}$$

Moreover, if $u(x, t) \in \mathbf{L}_2(0, M; \mathbf{V})$, then $\mathbf{d}\Delta u(x, t) + g(x) \in \mathbf{L}_2(0, M; \mathbf{V}')$.

A function $u(x, t)$, where $x \in \Omega$, and $t \ge 0$, is called a (global) weak solution of (1) if $u(x, t) \in \mathbf{L}_\mathbf{p}(0, M; \mathbf{L}_\mathbf{p}(\Omega)) \cap \mathbf{L}_2(0, M; \mathbf{V})$ for each $M > 0$ and, for any function $\varphi(x) \in \mathbf{L}_\mathbf{p}(\Omega) \cap \mathbf{V}$, the following identity holds in a generalized sense:

$$\frac{d}{dt}\int_\Omega u(x, t) \cdot \varphi(x) dx + \int_\Omega \{\mathbf{d}\nabla u(x, t) \cdot \nabla\varphi(x) + f(u(x, t)) \cdot \varphi(x)\} dx = \langle g, \varphi\rangle.$$

It follows from Eq. (1) that $\partial_t u(x, t) \in \mathcal{L}(0, M)$. By the Sobolev embedding theorem, we have $\mathcal{L}(0, M) \subset \mathbf{L}_\mathbf{q}(0, M; \mathbf{H}^{-\mathbf{r}}(\Omega))$, where $\mathbf{H}^{-\mathbf{r}}(\Omega) = H^{-r_1}(\Omega) \times \ldots \times H^{-r_N}(\Omega)$, $\mathbf{r} = (r_1, r_2, \ldots, r_N)$, and $r_i \equiv \max\left\{1, n\left(\dfrac{1}{q_i} - \dfrac{1}{2}\right)\right\}$ for $i = 1, 2, \ldots, N$. Therefore, for any weak solution of (1), we have $\partial_t u(x, t) \in \mathbf{L}_\mathbf{q}(0, M; \mathbf{H}^{-\mathbf{r}}(\Omega))$.

For simplicity, we shall write sometimes $u(t)$ instead of $u(x, t)$.

If $u(t)$ is a weak solution of system (1), then $u(\cdot) \in \mathbf{C}([0, M]; \mathbf{H}^{-\mathbf{r}}(\Omega))$. Moreover, if $u(\cdot) \in \mathbf{L}_\infty(0, M; \mathbf{H})$, then, according to the classical Lions–Magenes lemma (see [5, 6]), we have $u(\cdot) \in \mathbf{C}_w([0, M]; \mathbf{H})$. Thus, on the equations of system (1), we can impose the initial conditions

$$u|_{t=0} = u_0, \quad \text{where} \quad u_0 \in \mathbf{H}. \tag{6}$$

The Galerkin method (see [6–9]) proves the following assertion.

Statement 1. *For any* $u_0 \in \mathbf{H}$, *system* (1) *has a global weak solution*

$$u(t) \in \mathbf{L}_\infty(\mathbb{R}_+; \mathbf{H}) \cap \mathbf{L}_2^{\text{loc}}(\mathbb{R}_+; \mathbf{V}) \cap \mathbf{L}_\mathbf{p}^{\text{loc}}(\mathbb{R}_+; \mathbf{L}_\mathbf{p}(\Omega)),$$

satisfying the initial condition (6).

Remark 1. If only conditions (3) and (4) hold, then the weak solution of problem (1), (6) may be nonunique, because the function $f(u)$ is not assumed to be Lipschitz. The uniqueness theorem is valid under, e.g., the additional assumption $f(u) \in C^1(\mathbb{R}^N; \mathbb{R}^N)$; moreover, the Jacobian $\mathbf{J}(u) = \dfrac{\partial f(u)}{\partial u}$ must satisfy the inequality $\mathbf{J}(u) + \alpha\mathbf{I} \ge 0$ for all $u \in \mathbb{R}^N$ at some $\alpha > 0$ (see [7–9]). However, this condition is very restrictive, and we do not impose it in constructing a trajectory attractor of system (1).

In [9], it was proved that any weak solution $u(t)$ $(t \ge 0)$ of the reaction-diffusion-system (1) is a strongly continuous function belonging to the space $\mathbf{C}(\mathbb{R}_+; \mathbf{H})$; moreover, the real function $|u(t)|^2$, where $t \ge 0$, is absolutely continuous, and the differential relation

$$\frac{1}{2}\frac{d}{dt}|u(t)|^2 + (\mathbf{d}\nabla u(t), \nabla u(t)) + (f(u(t)), u(t)) = \langle g, u(t)\rangle, \quad t \ge 0 \tag{7}$$

holds. This relation is called the energy identity for the reaction-diffusion system.

It follows from (7) that any weak solution $u(t)$ of system (1) satisfies the inequalities

$$|u(t)|^2 \le |u(0)|^2 e^{-\lambda_1\beta t} + R_1^2, \tag{8}$$

$$\beta\int_t^{t+1} \|u(s)\|^2 ds + 2\sum_{i=1}^{N}\gamma_i\int_t^{t+1} \|u_i(s)\|^{p_i}_{L_{p_i}} ds \le |u(t)|^2 + R_2^2, \tag{9}$$

where λ_1 is the first eigenvalue of the scalar operator $-\Delta$ with Dirichlet boundary conditions. The quantities R_1 and R_2 depend on $\|g\|_{\mathbf{V}'}$ and do not depend on $u(0)$.

To construct a trajectory attractor for system (1), we must describe the trajectory space (see [9, 10]).

Definition 1. The trajectory space $\mathcal{K}^+$ of the reaction-diffusion system (1) is defined as the set of all weak solutions $u(t)$ of this system in the class

$$u(\cdot) \in \mathbf{L}_\infty(\mathbb{R}_+; \mathbf{H}) \cap \mathbf{L}_2^{\text{loc}}(\mathbb{R}_+; \mathbf{V}) \cap \mathbf{L}_\mathbf{p}^{\text{loc}}(\mathbb{R}_+; \mathbf{L}_\mathbf{p}(\Omega)).$$

It follows from Statement 1 that, for any $u_0 \in \mathbf{H}$, there exists a trajectory $u(\cdot) \in \mathcal{K}^+$ for which $u(0) = u_0$. Therefore, system (1) has sufficiently large trajectory space.

2. CONSTRUCTION OF A WEAK TRAJECTORY ATTRACTOR

Consider the linear space $\mathcal{F}_+^{\mathrm{loc}}$ consisting of functions $v(x, t)$, where $x \in \Omega$ and $t \geq 0$, satisfying the conditions

$$v(\cdot) \in L_\infty(0, M; \mathbf{H}) \cap \mathbf{L}_2(0, M; \mathbf{V}) \cap \mathbf{L_p}(0, M; \mathbf{L_p}(\Omega)), \tag{10}$$

$$\partial_t v(\cdot) \in \mathcal{L}(0, M) = \mathbf{L}_2(0, M; \mathbf{V}') + \mathbf{L_q}(0, M; \mathbf{L_q}(\Omega)), \quad \forall M > 0. \tag{11}$$

We endow the space $\mathcal{F}_+^{\mathrm{loc}}$ with the topology $\Theta_+^{s,\,\mathrm{loc}}$ of strong local convergence of the sequences $\{v_m(\cdot)\}$ and $\{\partial_t v_m(\cdot)\}$ as $m \to \infty$ in the norms on the Banach spaces (10) and (11), respectively, for each $M > 0$. It is easy to show that the topology $\Theta_+^{s,\,\mathrm{loc}}$ is metrizable, and the corresponding metric space $\mathcal{F}_+^{s,\,\mathrm{loc}} = \mathcal{F}_+^{\mathrm{loc}} \cap \Theta_+^{s,\,\mathrm{loc}}$ is complete.

On the space $\mathcal{F}_+^{\mathrm{loc}}$, we also consider the topology $\Theta_+^{w,\,\mathrm{loc}}$ of weak local convergence, which is determined by the weak and the $*$-weak convergence of the sequences $\{v_m(\cdot)\}$ and $\{\partial_t v_m(\cdot)\}$ in the same spaces for each $M > 0$. The space $\mathcal{F}_+^{w,\,\mathrm{loc}} = \mathcal{F}_+^{\mathrm{loc}} \cap \Theta_+^{w,\,\mathrm{loc}}$ is not metrizable, but it is a Hausdorff Fréchet–Urysohn space with countable topological base. Obviously, the topology $\Theta_+^{s,\,\mathrm{loc}}$ is stronger than $\Theta_+^{w,\,\mathrm{loc}}$.

In $\mathcal{F}_+^{\mathrm{loc}}$, we consider the linear subspace $\mathcal{F}_+^{\mathrm{b}}$ of elements $v \in \mathcal{F}_+^{\mathrm{loc}}$ for which $v \in \mathbf{L}_\infty(\mathbb{R}_+; \mathbf{H}) \cap \mathbf{L}_2^{\mathrm{b}}(\mathbb{R}_+; \mathbf{V}) \cap \mathbf{L}_\mathbf{p}^{\mathrm{b}}(\mathbb{R}_+; \mathbf{L_p}(\Omega))$ and $\partial_t v \in \mathcal{L}^b(\mathbb{R}_+) := \mathbf{L}_2^{\mathrm{b}}(\mathbb{R}_+; \mathbf{V}') + \mathbf{L}_2^{\mathrm{b}}(\mathbb{R}_+; \mathbf{L_q}(\Omega))$ with finite norm

$$\|v\|_{\mathcal{F}_+^{\mathrm{b}}} := \|v\|_{\mathbf{L}_\infty(\mathbb{R}_+;\, \mathbf{H})} + \|v\|_{\mathbf{L}_2^{\mathrm{b}}(\mathbb{R}_+;\, \mathbf{V})} + \|v\|_{\mathbf{L}_\mathbf{p}^{\mathrm{b}}(\mathbb{R}_+;\, \mathbf{L_p}(\Omega))} + \|\partial_t v\|_{\mathcal{L}^b(\mathbb{R}_+)}. \tag{12}$$

Recall that the norm on the space $L_p^{\mathrm{b}}(\mathbb{R}_+; X)$, where X is a Banach space and $p \geq 1$, is defined by

$$\|\chi\|_{L_p^{\mathrm{b}}(\mathbb{R}_+;\, X)} := \left[\sup_{t \in \mathbb{R}_+} \int_t^{t+1} \|\chi(s)\|_X^p ds\right]^{1/p}.$$

The space $\mathcal{F}_+^{\mathrm{b}}$ is Banach; we use its norm (12) to define bounded families of trajectories in $\mathcal{K}^+$.

Note that any ball $\mathcal{B}_r := \{v \in \mathcal{F}_+^{\mathrm{b}} \mid \|v\|_{\mathcal{F}_+^{\mathrm{b}}} \leq r\}$ in the space $\mathcal{F}_+^{\mathrm{b}}$ is compact in the weak topology $\Theta_+^{w,\,\mathrm{loc}}$ (see [13]). In [9], the following assertion was proved.

Statement 2. *The trajectory space $\mathcal{K}^+$ of system* (1) *is closed in the topology $\Theta_+^{w,\,\mathrm{loc}}$ (and in the topology $\Theta_+^{s,\,\mathrm{loc}}$).*

Consider the translations semigroup $\{T(\tau)\} := \{T(\tau), \tau \geq 0\}$, which acts on the space $\mathcal{F}_+^{\mathrm{loc}}$ as

$$T(\tau)v(t) = v(\tau + t), \quad t \geq 0.$$

Note that $T(\tau)\colon \mathcal{K}^+ \mapsto \mathcal{K}^+$ for any $\tau \geq 0$. The definition of the local topology implies the continuity of the semigroup $\{T(\tau)\}$.

Statement 3. *The semigroup $\{T(\tau)\}$ is continuous in the topology $\Theta_+^{w,\,\mathrm{loc}}$ (and in $\Theta_+^{s,\,\mathrm{loc}}$).*

Inequalities (8), (9), and (5) imply the following property.

Statement 4. *The space $\mathcal{K}^+$ belongs to $\mathcal{F}_+^{\mathrm{b}}$, and for any function $u(\cdot) \in \mathcal{K}^+$,*

$$\|T(\tau)u(\cdot)\|_{\mathcal{F}_+^{\mathrm{b}}}^2 \leq C|u(0)|^2 e^{-\sigma\tau} + R_0^2, \quad \forall \tau \geq 0, \tag{13}$$

where $\sigma = \beta\lambda_1$ and R_0 depends on λ_1 and on $\|g\|_{V'}$.

It follows from inequality (13) that the set

$$\mathcal{P}_0 = \left\{u \in \mathcal{K}^+ \,\middle|\, \|u\|_{\mathcal{F}_+^{\mathrm{b}}} \leq 2R_0\right\} \tag{14}$$

is absorbing for the semigroup $\{T(\tau)\}$ on $\mathcal{K}^+$; i.e., for any set $\mathfrak{B} \subset \mathcal{K}^+$ bounded in $\mathcal{F}_+^{\mathrm{b}}$, there exists a number $\tau_1 = \tau_1(\mathfrak{B}) \geq 0$ such that $T(\tau)\mathfrak{B} \subseteq \mathcal{P}_0$ for all $\tau \geq \tau_1$. Moreover, this set is invariant with respect to $\{T(\tau)\}$, that is,

$$T(\tau)\mathcal{P}_0 \subseteq \mathcal{P}_0, \quad \forall \tau \geq 0. \tag{15}$$

Let us construct a weak trajectory attractor for the translation semigroup $\{T(\tau)\}$ acting on the trajectory space $\mathcal{K}^+$. Recall the definition of a (weakly) attracting set. A set $\mathcal{P} \subseteq \mathcal{K}^+$ is said to be weakly attracting (in the topology $\Theta_+^{w,\,\mathrm{loc}}$) for the semigroup $\{T(\tau)\}$ if any neighborhood $\Theta_+^{w,\,\mathrm{loc}}$ of the set $\mathcal{P}$ in the (weak) topology $\mathcal{O}(\mathcal{P})$ is an absorbing set, i.e., for any bounded set $\mathfrak{B} \subset \mathcal{K}^+$, there exists a $\tau_1 = \tau_1(\mathfrak{B}, \mathcal{O}) \geq 0$ such that $T(\tau)\mathfrak{B} \subseteq \mathcal{O}(\mathcal{P})$ for all $\tau \geq \tau_1$. Below, we recall the definition of a (weak) trajectory attractor (see [9–11]).

Definition 2. A set $\mathfrak{A} \subseteq \mathcal{K}^+$ is called a (weak) trajectory attractor for the translation semigroup $\{T(\tau)\}$ on $\mathcal{K}^+$ if it bounded in the norm of $\mathcal{F}_+^{\mathrm{b}}$, compact in the topology $\Theta_+^{w,\,\mathrm{loc}}$, strictly invariant with respect to $\{T(\tau)\}$ (i.e., $T(\tau)\mathfrak{A} = \mathfrak{A}$ for all $\tau \geq 0$), and is a weakly attracting set for the semigroup $\{T(\tau)\}$ on $\mathcal{K}^+$ in the topology $\Theta_+^{w,\,\mathrm{loc}}$.

In [9], a weak trajectory attractor for the reaction-diffusion system (1) was constructed (see also [12]).

Theorem 1. *The set*

$$\mathfrak{A} = \bigcap_{\tau \geq 0} T(\tau)\mathcal{P}_0 \qquad (16)$$

is a weak trajectory attractor for the semigroup $\{T(\tau)\}$ *acting on the trajectory space* $\mathcal{K}^+$.

3. A STRONG TRAJECTORY ATTRACTOR

The definition of a strong trajectory attractor is obtained by replacing the weak attraction condition in Definition 2 by the condition of strong attraction in the topology $\Theta_+^{s,\,\text{loc}}$.

Below, we state the main result of this paper.

Theorem 2. *The weak trajectory attractor* $\mathfrak{A}$ *specified in Theorem* 1 *is a strong trajectory attractor for the semigroup* $\{T(\tau)\}$ *acting on the trajectory space* $\mathcal{K}^+$ *of the reaction-diffusion system* (1).

Proof. It is sufficient to show that the set $T(1)\mathcal{P}_0$, where $\mathcal{P}_0$ is the absorbing set for the semigroup $\{T(\tau)\}$ defined by (14), is compact in the strong topology of the space $\mathbf{L}_2(0, M; \mathbf{V}) \cap \mathbf{L}_\mathbf{p}(\Omega \times [0, M])$ for any $M > 0$. (Note that $\mathbf{L}_\mathbf{p}(\Omega \times [0, M]) = \mathbf{L}_\mathbf{p}(0, M; \mathbf{L}_\mathbf{p}(\Omega))$.) Without loss of generality, we can assume that $M = 1$.

Thus, let us verify that any sequence of trajectories $\{u_m(\cdot)\} \subset \mathcal{P}_0$ has a subsequence strongly convergent in the space $\Phi := \mathbf{L}_2(1, 2; \mathbf{V}) \cap \mathbf{L}_\mathbf{p}(\Omega \times [1, 2])$. For simplicity, we consider only the case of a homogeneous system (1) for $g \equiv 0$. (The general case is handled in a similar way.)

The set $\mathcal{P}_0$ is compact in the topology $\Theta_+^{w,\,\text{loc}}$. Therefore, without loss of generality, we can assume that $u_m(\cdot) \rightharpoonup \hat{u}(\cdot)$ as $m \to \infty$ weakly in the spaces $\mathbf{L}_2(0, 2; \mathbf{V})$ and $\mathbf{L}_\mathbf{p}(\Omega \times [0, 2])$. Here, $\hat{u}(\cdot)$ is a solution of the reaction-diffusion system (1) belonging to $\mathcal{P}_0$. It follows from Eqs. (1) and the Lions–Magenes lemma that $u_m(t) \rightharpoonup \hat{u}(t)$ as $m \to \infty$ weakly in $\mathbf{H}$ for each $t \in [0, 2]$. According to standard embedding theorems, $u_m(\cdot) \to \hat{u}(\cdot)$ strongly in the space $\mathbf{L}_2(\Omega \times [0, 2])$, and $u_m(x, t) \to \hat{u}(x, t)$ for almost all $(x, t) \in \Omega \times [0, 2]$.

Note that if a Banach space X contains a sequence $\chi_m \rightharpoonup \hat{\chi}$ weakly in X as $m \to \infty$, then $\|\hat{\chi}\|_X \leq \liminf_{m \to \infty} \|\chi_m\|_X$ (see, e.g., [14]). Therefore, if a sequence $\{u_m(\cdot)\}$ weakly converges, then the norms in the corresponding Banach spaces satisfy the inequalities

$$|\hat{u}(2)|^2 \leq \liminf_{m \to \infty} |u_m(2)|^2, \qquad (17)$$

$$\int_0^2 t(\mathbf{d}\nabla\hat{u}(t), \nabla\hat{u}(t))dt \leq \liminf_{m \to \infty} \int_0^2 t(\mathbf{d}\nabla u_m(t), \nabla u_m(t))dt, \qquad (18)$$

$$\int_0^2 t\|\hat{u}^i(t)\|^{p_i}_{L_{p_i}(\Omega \times [0, 2])}dt \leq \liminf_{m \to \infty} \int_0^2 t\|u^i_m(t)\|^{p_i}_{L_{p_i}(\Omega \times [0, 2])}dt, \quad i = 1, 2, \ldots, N. \qquad (19)$$

The norms in (18) and (19) correspond to the weight spaces $\mathbf{L}_{2,\,t}(0, 2; \mathbf{H})$ and $\mathbf{L}_{\mathbf{p},\,t}(0, 2; \mathbf{L}_\mathbf{p}(\Omega))$ with weight t. Obviously, the weak convergence $u_m(t) \rightharpoonup \hat{u}(t)$ as $m \to \infty$ holds in the weight spaces too.

Consider the scalar function $F(v) = \sum_{i=1}^N f^i(v)v^i - \sum_{i=1}^N \gamma_i |v^i|^{p_i}$, where $v \in \mathbb{R}^N$. We have

$$\int_0^2\int_\Omega tF(\hat{u}(x, t))dxdt \leq \liminf_{m \to \infty} \int_0^2\int_\Omega tF(u_m(x, t))dxdt, \qquad (20)$$

the proof of this inequality uses the inequality $F(u_m(x, t)) + C_1 \geq 0$, which follows from (4), the convergence of $u_m(x, t)$ for almost all $(x, t) \in \Omega \times [0, 2]$, and Fatou's theorem on bounds for integrals of convergent sequences of nonnegative functions (see, e.g., [13]).

Recall that weak solutions $u_m(\cdot)$ and $\hat{u}(\cdot)$ of system (1) satisfy the energy identity (7). Multiplying this identity by t, integrating the result over $[0, 2]$, and using the definition of the function $F(u)$, we obtain

$$\frac{1}{2}|u_m(2)|^2 + \int_0^2 t(\mathbf{d}\nabla u_m(t), \nabla u_m(t))dt + \int_0^2 t\sum_{i=1}^N \gamma_i\|u^i_m(t)\|^{p_i}_{L_{p_i}(\Omega \times [0, 2])}dt + \int_0^2\int_\Omega tF(u_m(x, t))dxdt = \frac{1}{2}\int_0^2 |u_m(t)|^2 dt, \qquad (21)$$

$$\frac{1}{2}|\hat{u}(2)|^2 + \int_0^2 t(\mathbf{d}\nabla \hat{u}(t), \nabla \hat{u}(t))dt + \int_0^2 t\sum_{i=1}^N \gamma_i\|\hat{u}^i(t)\|^{p_i}_{L_{p_i}(\Omega \times [0, 2])}dt + \int_0^2\int_\Omega tF(\hat{u}(x, t))dxdt = \frac{1}{2}\int_0^2 |\hat{u}(t)|^2 dt. \qquad (22)$$

Recall that $u_m(\cdot) \to \hat{u}(\cdot)$ strongly in the space $\mathbf{L}_2([0, 2]; \mathbf{H})$ as $m \to \infty$; therefore, the right-hand side of Eq. (21) tends to that of Eq. (22). Hence, the left-hand sides of these equations converge as $m \to \infty$ as well. It follows from inequalities (17)–(20) that the number sequences on the right-hand sides have limits as $m \to \infty$, and these limits coincide with the quantities on the left-hand sides of (17)–(20), respectively. In particular, we have

$$\lim_{m \to \infty} \int_0^2 t(\mathbf{d}\nabla u_m(t), \nabla u_m(t))dt = \int_0^2 t(\mathbf{d}\nabla \hat{u}(t), \nabla \hat{u}(t))dt,$$

$$\lim_{m \to \infty} \int_0^2 t\|u_m^i(t)\|^{p_i}_{L_{p_i}(\Omega \times [0, 2])} dt = \int_0^2 t\|\hat{u}^i(t)\|^{p_i}_{L_{p_i}(\Omega \times [0, 2])} dt,$$

$$i = 1, 2, \dots, N.$$

As is known, in a uniformly convex Banach space X, the weak convergence $\chi_m \rightharpoonup \hat{\chi}$ of vectors and $\|\chi_m\|_X \to \|\hat{\chi}\|_X$ of their norms implies the strong convergence $\|\chi_m - \hat{\chi}\|_X \to 0$ as $m \to \infty$ (this follows from Mazur's theorem; see [14]). The weight spaces $\mathbf{L}_{2,t}(0, 2; \mathbf{H})$ and $\mathbf{L}_{\mathbf{p},t}(0, 2; \mathbf{L}_\mathbf{p}(\Omega))$ are uniformly convex. Therefore, the weak convergence of the sequence of functions $u_m(\cdot)$ and their norms in the space $\mathbf{L}_{2,t}(0, 2; \mathbf{H}) \cap \mathbf{L}_{\mathbf{p},t}(0, 2; \mathbf{L}_\mathbf{p}(\Omega))$ implies the strong convergence $u_m(\cdot) \to \hat{u}(\cdot)$ as $m \to \infty$ in this space and in the space $\mathbf{L}_{2,t}(1, 2; \mathbf{H}) \cap \mathbf{L}_{\mathbf{p},t}(1, 2; \mathbf{L}_\mathbf{p}(\Omega))$, which is, obviously, equivalent to the space $\Phi := \mathbf{L}_2(1, 2; \mathbf{V}) \cap \mathbf{L}_\mathbf{p}(\Omega \times [1, 2])$.

We have proved the strong compactness of the set $T(1)\mathcal{P}_0$ in the strong topology of the space $\mathbf{L}_2^{\text{loc}}(\mathbb{R}_+; \mathbf{V}) \cap \mathbf{L}_\mathbf{p}^{\text{loc}}(\mathbb{R}_+; \mathbf{L}_\mathbf{p}(\Omega))$. The compactness of the corresponding set of derivatives $\partial_t u$ in the strong topology of $\mathbf{L}_2^{\text{loc}}(\mathbb{R}_+; \mathbf{V}') + \mathbf{L}_\mathbf{q}^{\text{loc}}(\mathbb{R}_+; \mathbf{L}_\mathbf{q}(\Omega))$ follows directly from the equation and from the continuity of the Nemytskii operator $u \mapsto f(u)$, which, by virtue of (5), acts from $\mathbf{L}_\mathbf{p}$ to $\mathbf{L}_\mathbf{q}$ (see [15]). It remains to note that the compactness of the set $T(1)\mathcal{P}_0$ in the space $C^{\text{loc}}(\mathbb{R}_+; \mathbf{H})$ follows from the continuity of the embedding

$$\{\mathbf{L}_2(0, 1; \mathbf{V}) \cap \mathbf{L}_\mathbf{p}(0, 1; \mathbf{L}_\mathbf{p}(\Omega))\}$$
$$\cap \{\partial_t v \in \mathbf{L}_2(0, 1; \mathbf{V}') + \mathbf{L}_\mathbf{q}(0, 1; \mathbf{L}_\mathbf{q}(\Omega))\}$$
$$\subset C(0, 1; \mathbf{H}),$$

which was proved in, e.g., [9].

Corollary 1. *The complex Ginzburg–Landau equation* (*see* [4]) *has a strong trajectory attractor in the corresponding trajectory space.*

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REFERENCES

1. J.-M. Ghidaglia, J. Differ. Equations **110** (2), 356–359 (1994).
2. I. Moise, R. Rosa, and X. Wang, Nonlinearity **11** (5), 1369–1393 (1998).
3. V. K. Vanag, *Dissipative Structures in Reaction-Diffusion Systems* (Izhevsk, 2008) [in Russian].
4. M. I. Vishik and V. V. Chepyzhov, Mat. Sb. **196** (6), 17–42 (2005).
5. J.-L. Lions and E. Magenes, *Problemes aux limites non homogénes et applications* (Mir, Moscow, 1971; Dunod, Paris, 1968).
6. J.-L. Lions, *Quelques methodes de résolution des problémes aux limites nonlinéaires* (Paris, Dunod, 1969; Moscow, Mir, 1972).
7. R. Temam, *Infinite-Dimensional Dynamical Systems in Mechanics and Physics* (Springer, New York, 1988).
8. A. V. Babin and M. I. Vishik, *Attractors of Evolution Equations* (Nauka, Moscow, 1989) [in Russian].
9. V. V. Chepyzhov and M. I. Vishik, *Attractors for Equations of Mathematical Physics* (Am. Math. Soc., Providence, R.I., 2002).
10. V. V. Chepyzhov and M. I. Vishik, J. Math. Pures Appl. **76**, 913–964 (1997).
11. M. I. Vishik and V. V. Chepyzhov, Mat. Zametki **71** (2), 194–213 (2002).
12. V. V. Chepyzhov and M. I. Vishik, Topol. Meth. Nonlin. Anal. J. Juliusz Schauder Center **7** (1), 49–76 (1996).
13. A. N. Kolmogorov and S. V. Fomin, *Elements of Function Theory and Functional Analysis* (Nauka, Moscow, 1981) [in Russian].
14. K. Yoshida, *Functional Analysis* (Academic, New York, 1965; Mir, Moscow, 1967).
15. M. A. Krasnosel'skii, *Topological Methods in the Theory of Nonlinear Integral Equations* (GITTL, Moscow, 1956) [in Russian].