

# Finiteness Property of a Bounded Set of Matrices with Uniformly Sub-Peripheral Spectrum<sup>1</sup>

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**Abstract**—In the paper, a simple condition guaranteeing the finiteness property, for a bounded set  $\mathcal{S} = \{S_k\}_{k \in K}$  of real or complex  $d \times d$  matrices, is presented. It is shown that existence of a sequence of matrix products  $S_{\sigma(n_l)}$  of length  $n_l$  for  $\mathcal{S}$  with  $n_l \rightarrow \infty$  such that the spectrum of each matrix  $S_{\sigma(n_l)}$  is uniformly sub-

peripheral and  $\rho(\mathcal{S}) := \sup_{n \geq 1} \sup_{i_1, \dots, i_n \in K} \sqrt[n]{\rho(S_{i_1} \dots S_{i_n})} = \lim_{l \rightarrow +\infty} \sqrt[n_l]{\rho(S_{\sigma(n_l)})}$ , guarantees the spectral finiteness property for  $\mathcal{S}$ .

**Keywords:** Joint/generalized spectral radius, finiteness property, peripheral spectrum.

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## 1. INTRODUCTION

In this paper, we prove the finite-step realizability of the joint/generalized spectral radius for a bounded set of matrices with the so-called uniformly sub-peripheral spectrum.

### 1.1. Joint and Generalized Spectral Radii

Throughout this paper, we let

$$\mathcal{S} = \{S_k\}_{k \in K}, \quad \text{card}(K) \geq 2,$$

be a bounded set of  $d \times d$  matrices over the field  $\mathbb{F} = \mathbb{R}, \mathbb{C}$  indexed by elements from some set  $K$ . Let also  $\|\cdot\|$  be a row-vector norm on  $\mathbb{F}^{1 \times d}$  and also the induced matrix-norm on  $\mathbb{F}^{d \times d}$ . Associate with any finite-length word

$$\sigma = \{i_1, \dots, i_n\} \in K^n := \overbrace{K \times \dots \times K}^{n\text{-time}}$$

the matrix  $S_\sigma = S_{i_1} \dots S_{i_n}$ , and define for any integer  $n \geq 1$  two quantities

$$\hat{\rho}_n(\mathcal{S}) = \sup_{\sigma \in K^n} \|S_\sigma\| \quad \text{and} \quad \rho_n(\mathcal{S}) = \sup_{\sigma \in K^n} \rho(S_\sigma).$$

Here  $\rho(A)$  stands for the usual spectral radius for an arbitrary matrix  $A \in \mathbb{F}^{d \times d}$ . Then by the sub-multiplica-

tive property  $\|AB\| \leq \|A\| \cdot \|B\|$  for all  $A, B \in \mathbb{F}^{d \times d}$  there exists the limit

$$\begin{aligned} \hat{\rho}(\mathcal{S}) &= \lim_{n \rightarrow +\infty} \sup \sqrt[n]{\hat{\rho}_n(\mathcal{S})} \\ &= \lim_{n \rightarrow +\infty} \sqrt[n]{\rho_n(\mathcal{S})} = \inf_{n \geq 1} \sqrt[n]{\hat{\rho}_n(\mathcal{S})}, \end{aligned}$$

which does not depend on the choice of the norm  $\|\cdot\|$ . This limit was called by Rota and Strang [1] *the joint spectral radius* of the matrix set  $\mathcal{S}$ . Analogously, there exists the limit

$$\rho(\mathcal{S}) = \lim_{n \rightarrow +\infty} \sup \sqrt[n]{\rho_n(\mathcal{S})} \quad (= \sup_{n \geq 1} \sqrt[n]{\rho_n(\mathcal{S})}),$$

which was called by Daubechies and Lagarias [2] *the generalized spectral radius* of the matrix set  $\mathcal{S}$ . As is shown in [3], for finite matrix sets  $\mathcal{S}$  the quantities  $\hat{\rho}(\mathcal{S})$  and  $\rho(\mathcal{S})$  coincide with each other, and for any  $n$  the following inequalities hold

$$\sqrt[n]{\rho_n(\mathcal{S})} \leq \rho(\mathcal{S}) = \hat{\rho}(\mathcal{S}) \leq \sqrt[n]{\hat{\rho}_n(\mathcal{S})}, \quad (1)$$

which are useful for numerical computation of the joint spectral radius  $\hat{\rho}(\mathcal{S})$ .

### 1.2. Spectral Finiteness Property

In [4] Lagarias and Wang conjectured that for finite sets  $\mathcal{S}$  the value  $\rho(\mathcal{S})$  in fact coincides with  $\sqrt[n]{\rho(S_\sigma)}$  for some  $n$  and  $\sigma \in K^n$ ; that is to say,  $\mathcal{S}$  has the spectral

<sup>1</sup> The article was translated by the authors.

*finiteness property.* If this *Finiteness conjecture* is true, then the problem of determining whether  $\rho(\mathcal{S}) < 1$  is decidable. This is because if  $\rho(\mathcal{S}) < 1$ , then there exists  $n$  such that  $\rho_n(\mathcal{S}) < 1$ , whereas if  $\hat{\rho}(\mathcal{S}) \geq 1$ , the Finiteness conjecture implies that there exists  $n$  such that  $\hat{\rho}_n(\mathcal{S}) \geq 1$ . By checking both conditions for increasing values of  $n$ , one of them will be eventually satisfied and a decision will be made after a finite amount of computation. Note that for a single matrix the problem is decidable. So, the Finiteness conjecture has strong implications on the computation of the joint/generalized spectral radius.

A simplest example of matrix sets having the finiteness property are bounded sets of matrices consisting of upper (or lower) triangular matrices. Another trivial example deliver bounded matrix sets  $\mathcal{S}$  consisting of matrices  $S \in \mathcal{S}$  “isometric to a scalar factor” in some row-vector norm  $\|\cdot\|$  on  $\mathbb{F}^{1 \times d}$ , i.e., such that for any  $x \in \mathbb{F}^{1 \times d}$ ,  $\|xS\| = \lambda_S \|x\|$  with some constant  $\lambda_S$ . One more example was given by Plischke and Wirth [5] who proved that irreducible<sup>2</sup> bounded “symmetric” matrix sets<sup>3</sup> possess the finiteness property. Less trivial examples were constructed by Omladić and Radjavi in [6], where they showed that the finiteness property holds for matrix sets  $\mathcal{S}$  for which the semigroup  $\mathcal{S}^+$  of all the products of matrices from  $\mathcal{S}$  possesses the so-called “sub-multiplicative spectral radius property”, i.e.,  $\rho(FH) \leq \rho(F) \cdot \rho(H)$  for all  $F, H \in \mathcal{S}^+$ .

In [7] Gurvits showed that, for real matrix sets  $\mathcal{S}$ , the Finiteness conjecture holds if there is a real polytope extremal norm<sup>4</sup>. In [4], Lagarias and Wang proved a more general result that Finiteness conjecture holds if there is a piecewise real analytic extremal norm. At last, as showed Guglielmi et al. [8], for complex matrix sets  $\mathcal{S}$ , the Finiteness conjecture holds if there is a complex polytope extremal norm. However, to make use of these results, one needs to know whether a set  $\mathcal{S}$  admits an extremal norm or not. It was shown, e.g., in [9–11] that bounded irreducible sets of matrices always admit extremal norms, yet nothing proves that polytope or piecewise analytic extremal norms are always possible, see, e.g., [12] and references therein. In [8] Guglielmi et al. conjectured that every non-defective<sup>5</sup> finite family of complex matrices that possesses the finiteness property has a complex

polytope extremal norm. Unfortunately, later on this conjecture was disproved by Jungers and Protasov [13].

Despite of the above examples in which the Finiteness conjecture holds, the Finiteness conjecture is turned to be false in general. The first counterexample to the Finiteness conjecture was given by Bousch and Mairesse in [14], and the corresponding proof was essentially based on the analysis of the so-called topological maps and Sturmian measures. Later on in [15, 16] Blondel, Theys and Vladimirov proposed another proof of the counterexample to the Finiteness Conjecture, which extensively exploited combinatorial properties of permutations of products of positive matrices. In the control theory, as well as in the general theory of dynamical systems, the notion of generalized spectral radius is used basically to describe the rate of growth or decrease of the trajectories generated by matrix products. In connection with this, Kozyakin in [17, 18] presented one more proof of the counterexample to the Finiteness conjecture fulfilled in the spirit of the theory of dynamical systems. In this proof, the method of Barabanov norms [9] was the key instrument in disproving the Finiteness conjecture. The related constructions were essentially based on the study of the geometrical properties of the unit balls of some specific Barabanov norms and properties of discontinuous orientation preserving circle maps.

To appreciate the merits of the above mentioned disproofs of the Finiteness conjecture let us point out that the key ideas underlying all the proofs in [14, 16–18] were based on the frequency properties of the Sturmian sequences. In [14] such properties were formulated and investigated in terms of the so-called Sturmian ergodic invariant measures on the spaces of binary sequences. In [16–18], the ergodic theory formally was not mentioned. However, the usage of combinatorial properties of Sturmian sequences in [16] or of the fact that Sturmian sequences naturally arise in symbolic description of trajectories of (discontinuous) orientation preserving circle rotation maps in [17, 18] were essentially motivated namely by ergodic properties of Sturmian sequences.

Unfortunately, all the disproofs [14, 16–18] of the Finiteness conjecture were pure “existence” (or, sooner, “non-existence”) unconstructive results. Only recently, in [19] Hare et al. combined the approaches developed in [14, 16–18] with some rapidly-converging lower bounds for the joint spectral radius based on the multiplicative ergodic theory obtained by Morris [20], which allowed them to build explicitly the set of matrices for which the Finiteness conjecture fails. Namely, for the matrix set

$$S(\alpha) := \left\{ S_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, S_2 = \alpha \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \right\}$$

they computed an explicit value of

$$\alpha_* \approx 0.74932654633036755794396194809\dots$$

<sup>2</sup> A matrix set  $\mathcal{S}$  is called *irreducible*, if the matrices from this set have no common invariant subspaces except  $\{0\}$  and  $\mathbb{F}^{1 \times d}$ . We notice that this is completely different with the “irreducibility” of a Markov transition matrix in probability theory.

<sup>3</sup> A matrix set  $\mathcal{S}$  is called *symmetric* if  $S \in \mathcal{S}$  implies that  $S^* \in \mathcal{S}$ , where  $S^*$  is a matrix conjugate to  $S$ .

<sup>4</sup> A norm  $\|\cdot\|$  is called *extremal* for the matrix set  $\mathcal{S}$  if  $\|S\| \leq \rho(\mathcal{S})$  for all  $S \in \mathcal{S}$ .

<sup>5</sup> A matrix set  $\mathcal{S}$  is called *non-defective* if the semigroup generated by the set  $\rho(\mathcal{S})^{-1}\mathcal{S}$  is bounded.

such that  $\mathcal{S}(\alpha_*)$  does not satisfy the finiteness property. It is still unknown whether  $\alpha_*$  is rational or not.

So, ideas of ergodic theory are proved to be fruitful in disproving the Finiteness conjecture [14, 16–20]. Further development of the ergodic theory approach to investigation of the properties of the joint spectral radius was done by Dai et al. in [21, 22]. Based on the classic multiplicative ergodic theorem and the semi-uniform subadditive ergodic theorem, they showed in particular that there always exists at least one ergodic Borel probability measure on the one-sided symbolic space  $\Sigma_K^+$  of all one-sided infinite sequences  $i: \mathbb{N} \rightarrow K$  such that the joint spectral radius of a finite set of square matrices  $\mathcal{S}$  can be realized almost everywhere with respect to this measure [21].

Since the Finiteness conjecture was proved to be invalid generally, serious efforts were undertaken by some investigators to find less general classes of matrices for which the Finiteness conjecture still might be valid. One of the most interesting such classes constitute matrices with rational entries. In [23, 24], Jungers and Blondel showed that the finiteness property holds for nonnegative rational matrices if and only if it holds for pairs of binary matrices, i.e., matrices with the entries  $\{0, 1\}$ . So they conjectured that pairs of binary matrices always have the finiteness property. In support to this conjecture they proved that the finiteness property holds for pairs of  $2 \times 2$  binary matrices. They gave also a similar result for matrices with negative entries. Namely, they proved that the finiteness property holds for (general) rational matrices if and only if it holds for pairs of sign-matrices, i.e., matrices with entries  $\{-1, 0, 1\}$ . More recently, Cicone et al. in [25] proved that the finiteness property holds for pairs of  $2 \times 2$  sign-matrices; and Dai et al. in [26] proved that for any pair  $\mathcal{S} = \{S_1, S_2\} \subset \mathbb{R}^{d \times d}$ , if one of  $S_1, S_2$  has the rank 1, then  $\mathcal{S}$  possesses the finiteness property.

The aim of this paper is to present yet another sufficient condition enabling the finiteness property of a set of matrices.

## 2. PURE PERIPHERAL SPECTRUM AND MAIN STATEMENT

Recall that an eigenvalue  $\lambda$  of a matrix  $A \in \mathbb{F}^{d \times d}$  is said to belong to the *peripheral spectrum* of  $A$  if  $|\lambda| = \rho(A)$ . If  $|\lambda| = \rho(A)$  for all eigenvalues  $\lambda$  of  $A$ , then we say that  $A$  has the *pure peripheral spectrum*. For example, any unitary matrix has the pure peripheral spectrum. Let us say that a family of matrices  $\mathcal{S}_\sigma$  has the *uniformly sub-peripheral spectrum* if there exists a constant  $\kappa$  with  $0 < \kappa < 1$  such that each eigenvalue  $\lambda$  of  $\mathcal{S}_\sigma$  satisfies  $\kappa\rho(\mathcal{S}_\sigma) \leq |\lambda| \leq \rho(\mathcal{S}_\sigma)$ . Clearly, if the spectrum of every matrix  $\mathcal{S}_\sigma$  is pure peripheral then the whole family of matrices  $\mathcal{S}_\sigma$  has a uniformly sub-peripheral spectrum.

Now our main statement may be formulated as follows:

**Theorem 1.** *Let  $\mathcal{S} = \{S_k\}_{k \in K} \subset \mathbb{F}^{d \times d}$  be a bounded set of matrices. If there exists a sequence of matrix products  $\mathcal{S}_{\sigma(n_l)}$  for  $\mathcal{S}$ , where  $\sigma(n_l) \in K^{n_l}$  and  $n_l \rightarrow +\infty$ , such that its spectrum is uniformly sub-peripheral and*

$$\rho(\mathcal{S}) = \lim_{l \rightarrow +\infty} \sqrt[n_l]{\rho(\mathcal{S}_{\sigma(n_l)})}, \tag{2}$$

*then  $\mathcal{S}$  possesses the spectral finiteness property with  $\rho(\mathcal{S}) = \sup_{k \in K} \rho(S_k)$ .*

In light of the counterexample of Hare et al. [19] where  $\mathcal{S} = \{S_1, S_2\}$  and the spectra of both  $S_1$  and  $S_2$  are pure peripheral, with  $\rho(S_1) = 1$  and  $\rho(S_2) = \alpha_*$ , our assumption that the matrix sequence  $\{\mathcal{S}_{\sigma(n_l)}\}_{l=1}^{+\infty}$  has the uniformly sub-peripheral spectrum is essential for the statement of Theorem 1.

Let us present one example in which the claim of Theorem 1 is evident. If each element of the multiplicative semigroup  $\mathcal{S}^+ \subset \mathbb{F}^{d \times d}$ , generated by  $\mathcal{S}$ , has the pure peripheral spectrum then

$$\begin{aligned} \rho(AB) &= \sqrt[d]{\det(AB)} = \sqrt[d]{\det(A)} \cdot \sqrt[d]{\det(B)} \\ &= \rho(A) \cdot \rho(B), \quad \forall (A, B) \in \mathcal{S}^+. \end{aligned}$$

This implies that

$$\rho(\mathcal{S}) = \sup_{k \in K} \rho(S_k),$$

and so in this case the set of matrices  $\mathcal{S}$  has the spectral finiteness property.

Matrix multiplicative semigroups satisfying  $\rho(AB) = \rho(A) \cdot \rho(B)$  for any their members  $A$  and  $B$  are called *semigroups with multiplicative spectral radius*, see, e.g. [6]. As is shown in [6, Theorem 2.5], for any such irreducible semigroup of matrices there exists a (vector) norm  $\|\cdot\|$  in which each matrix from the semigroup is a direct sum of isometry (in the norm  $\|\cdot\|$ ) and a nilpotent matrix. Nontrivial examples of semigroups with multiplicative spectral radius can be found in [6].

Let us remark that under the conditions of Theorem 1 the set of matrices  $\{\mathcal{S}_{\sigma(n_l)}\}_{l=1}^{+\infty}$  does not need to be a semigroup and moreover this set in general lacks the multiplicative spectral radius property. Still, Theorem 1 is valid in this more restrictive, comparing with Theorem 2.5 from [6], situation, too.

**Proof.** Let  $\{\mathcal{S}_{\sigma(n_l)}\}_{l=1}^{+\infty}$  be a sequence of matrix products for  $\mathcal{S}$  specified by the condition of Theorem 1. Then, since the family of matrix products

$\{\mathcal{S}_{\sigma(n_l)}\}_{l=1}^{+\infty}$  has the uniformly sub-peripheral spectrum, we have

$$\begin{aligned} \kappa \rho(\mathcal{S}_{\sigma(n_l)}) &\leq \sqrt[l]{\det(\mathcal{S}_{i_1(n_l)} \cdots \mathcal{S}_{i_{n_l}(n_l)})} \\ &= \sqrt[l]{\det(\mathcal{S}_{i_1(n_l)}) \cdots \det(\mathcal{S}_{i_{n_l}(n_l)})} \\ &= \sqrt[l]{\det(\mathcal{S}_{i_1(n_l)})} \cdots \sqrt[l]{\det(\mathcal{S}_{i_{n_l}(n_l)})} \leq \rho(\mathcal{S}_{i_1(n_l)}) \cdots \rho(\mathcal{S}_{i_{n_l}(n_l)}) \\ &\leq (\sup_{k \in K} \rho(\mathcal{S}_k))^{n_l}, \end{aligned}$$

where  $\sigma(n_l) = (i_1(n_l), \dots, i_{n_l}(n_l))$ , for some constant  $0 < \kappa < 1$ . Together with (2), these latter inequalities imply that

$$\begin{aligned} \rho(\mathcal{S}) &= \lim_{l \rightarrow +\infty} \sqrt[l]{\rho(\mathcal{S}_{\sigma(n_l)})} \leq \lim_{l \rightarrow +\infty} \sqrt[l]{\kappa^{-1} (\sup_{k \in K} \rho(\mathcal{S}_k))^{n_l}} \quad (3) \\ &= \sup_{k \in K} \rho(\mathcal{S}_k). \end{aligned}$$

On the other hand, by (1) we have

$$\rho(\mathcal{S}) \geq \sup_{k \in K} \rho(\mathcal{S}_k),$$

which together with (3) implies

$$\rho(\mathcal{S}) = \sup_{k \in K} \rho(\mathcal{S}_k).$$

Theorem 1 is thus proved.

**Remark 1.** From Theorem 1 it follows that if a matrix set  $\mathcal{S}$  does not possess the finiteness property then for any sequence  $\{\mathcal{S}_{\sigma(n_l)}\}$  satisfying condition (2) the minimal absolute value of the eigenvalues of  $\mathcal{S}_{\sigma(n_l)}$  divided by  $\rho(\mathcal{S}_{\sigma(n_l)})$  tends to zero as  $l \rightarrow +\infty$ .

Let us recall now from [27] that the limit semigroup  $\mathcal{S}_\infty$  generated by the set of matrices  $\mathcal{S}$  is defined to be the set of all limit points for the matrix sequences  $\{\rho(\mathcal{S})^{-n_l} \mathcal{S}_{\sigma(n_l)}\}_{l=1}^{+\infty}$ , where  $\sigma(n_l) \in K^{n_l}$  and  $n_l \rightarrow \infty$ . As is known [27], the limit semigroup is nonempty bounded when the set of matrices  $\mathcal{S}$  is irreducible.

As a consequence of Theorem 1, we can obtain a sufficient condition for the finiteness property for an irreducible  $\mathcal{S}$ .

**Theorem 2.** Let an irreducible bounded set of matrices  $\mathcal{S} = \{\mathcal{S}_k\}_{k \in K} \subset \mathbb{F}^{d \times d}$  do not possess the finiteness property. Then any matrix  $A \in \mathcal{S}_\infty$  is degenerate, that is,  $\det A = 0$ .

**Proof.** Since the bounded set of matrices  $\mathcal{S}$  is irreducible then  $0 < \rho(\mathcal{S}) < +\infty$ , see, e.g., [27]. So, without loss of generality, it may be assumed that  $\rho(\mathcal{S}) = 1$ . Fix

an arbitrary matrix  $A \in \mathcal{S}_\infty$ . Then there exists a sequence of finite-length words  $\{\sigma(n_l)\}_{l=1}^{+\infty}$  with  $\sigma(n_l) \in K^{n_l}$  and  $n_l \rightarrow \infty$  as  $l \rightarrow +\infty$ , such that

$$A = \lim_{l \rightarrow +\infty} \mathcal{S}_{\sigma(n_l)}.$$

If  $A$  would be singular, then we could stop our proof here. Next, we assume  $\det A \neq 0$  and then  $\rho(A) > 0$ . Since  $\rho(\mathcal{S}_{\sigma(n_l)})$  converges to  $\rho(A)$  as  $l \rightarrow +\infty$ , it follows

$$\sqrt[l]{\rho(\mathcal{S}_{\sigma(n_l)})} \rightarrow 1 = \rho(\mathcal{S}).$$

So, condition (2) holds for the sequence  $\{\mathcal{S}_{\sigma(n_l)}\}$ . Then denoting by  $\lambda_l$  an eigenvalue of  $\mathcal{S}_{\sigma(n_l)}$  with the smallest absolute value we get by Remark 1 that

$$\lambda_l \rightarrow 0 \text{ as } l \rightarrow \infty. \quad (4)$$

Now, by (1) all the other eigenvalues of the matrix  $\mathcal{S}_{\sigma(n_l)}$  have the absolute values do not exceeding 1. So,

$$|\det \mathcal{S}_{\sigma(n_l)}| \leq |\lambda_l|$$

and by (4) we obtain

$$\det A = \lim_{l \rightarrow +\infty} \det \mathcal{S}_{\sigma(n_l)} = 0,$$

which is a contradiction to the assumption.

Due to arbitrariness of the matrix  $A \in \mathcal{S}_\infty$ , the theorem is thus proved.

It is interesting to formulate Theorem 2 equivalently as follows:

**Theorem 3.** Let a bounded set of matrices  $\mathcal{S} = \{\mathcal{S}_k\}_{k \in K} \subset \mathbb{F}^{d \times d}$  be irreducible. If there exists a nonsingular  $A \in \mathcal{S}_\infty$ , then  $\mathcal{S}$  has the finiteness property.

Recall from [28] that an irreducible set of matrices  $\mathcal{S}$  is said to have rank one property if every nonzero element of  $\mathcal{S}_\infty$  has rank one. Then from [28, Corollary 1.6] it follows that for each card  $(K)$ ,  $d \geq 2$  there exists an irreducible finite set of matrices  $\mathcal{S} = \{\mathcal{S}_k\}_{k \in K} \subset \mathbb{F}^{d \times d}$  which satisfies both the finiteness and the rank one properties; for example,

$$\mathcal{S} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & \lambda \end{bmatrix}, \begin{bmatrix} 0 & \lambda \\ \lambda & 0 \end{bmatrix} \right\}, \quad \text{where } 0 < |\lambda| < 1$$

has the finiteness and rank-one properties [28, Example 2]. Therefore, the condition presented in Theorem 3 does not need to be necessary for the finiteness property. However, there always exist open subsets of irreducible pairs  $\mathcal{S} = \{\mathcal{S}_1, \mathcal{S}_2\} \subset \mathbb{R}^{2 \times 2}$  in which the rank one property does not hold [28] and hence the finiteness

property holds from Theorem 3. In addition, the counterexample  $\mathcal{S}(\alpha_*)$  of Hare et al. [19], mentioned in Section 1.2, has the rank one property from Theorem 3.

### 3. A STABILITY CRITERION FROM PERIODICALLY SWITCHED STABILITY

Let us recall that a finite set  $\mathcal{S} = \{S_k\}_{k \in K} \subset \mathbb{F}^{d \times d}$  is called *periodically switched stable* if  $\rho(S_\sigma) < 1$  for all  $\sigma \in K^n$  and all  $n \geq 1$ ; see, e.g., [22, 29]. The following question of substantial importance was posed by E. S. Pyatniskii in 1980s: when does periodically switched stability imply the absolute stability for  $\mathcal{S}$ ?

Since the spectral finiteness property is equivalent to the absolute stability of some periodically switched stable system, then from Theorem 1 it follows immediately the following stability criterion:

**Theorem 4.** *Let  $\mathcal{S} = \{S_k\}_{k \in K} \subset \mathbb{F}^{d \times d}$  be periodically switched stable. If there exists a sequence of words  $\{\sigma(n_l)\}_{l=1}^{+\infty}$  satisfying the requirements of Theorem 1, then  $\mathcal{S}$  is absolutely stable; that is,  $\|S_{i_1} \cdots S_{i_n}\| \rightarrow 0$  as  $n \rightarrow +\infty$  for all one-sided infinite switching sequences  $i: \mathbb{N} \rightarrow K$ .*

So, in the situation of Theorem 1, the stability of  $\mathcal{S}$  is algorithmically decidable.

### 4. CONCLUDING REMARKS

In this paper, we have presented a short survey on the spectral finiteness property for a finite set of  $d \times d$  matrices. We have proved also that if a bounded set  $\mathcal{S}$  of  $d \times d$  matrices satisfies the so-called uniformly sub-peripheral spectrum condition and an approximation property of Lyapunov exponents, then  $\mathcal{S}$  possesses the spectral finiteness property. This result has direct implication for the stability problem for finite sets of matrices.

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