One Idea of Pokrovskii

How to Link Economic Problems with Asynchronous Systems?

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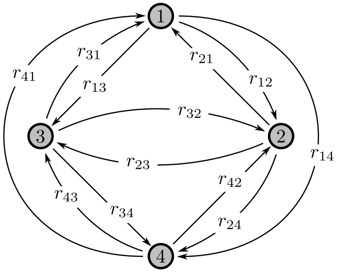
Being in the summer of 2009 in Moscow, Alexei has asked me *to think a bit* about one problem.

He added: *It seems, it is a kind of problems you like.*[1]

Indeed, the formulation of the problem was so simple that I was not able to get rid of it…

[1] Everybody knows that Alexei was a great master in ***Posing the Right Question to the Right Person at the Right Time***.

Problem: consider a full, weighted, oriented graph…



$$r_{ij} = \frac{1}{r_{ji}} > 0$$

Given a triplet $\omega = (i, j, k)$ with $i \neq j$, $k \neq i, j$, let us update the weights in accordance with the following rule:

$$r_{ij}^{\text{new}} = \max\{r_{ij},\ r_{ik} \cdot r_{kj}\}, \qquad r_{ji}^{\text{new}} = 1/r_{ij}^{\text{new}}.$$

Conjecture

For any sequence of triplets $\{\omega_n\}$, the updated wights ***converge to an equilibrium***.

Economic support for transform rules

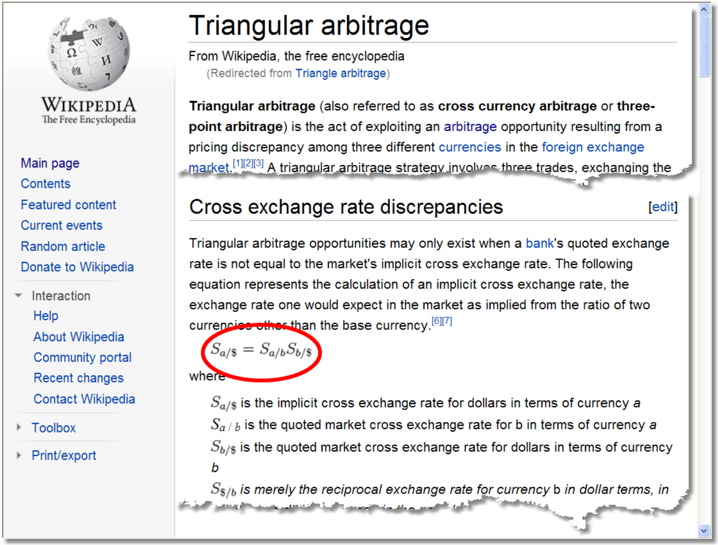


Figure: Transform rules $r_{ij}^{\text{new}} = \max\{r_{ij},\ r_{ik} \cdot r_{kj}\}$ are motivated by economic reasons.

Example of a realistic triangular arbitrage scenario

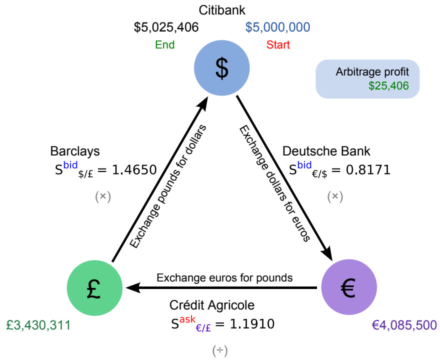
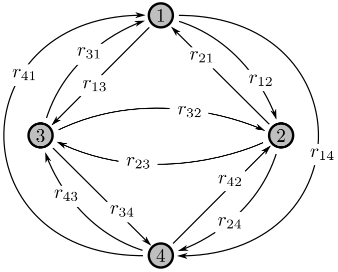


Figure: A visual representation of a realistic triangular arbitrage scenario, using sample bid and ask prices quoted by international banks

Figure: Arbitrage has the effect of ***causing prices to converge***

Additive Reformulation

We have the graph



$$r_{ij} = \frac{1}{r_{ji}} > 0$$

and the set of multiplicative updating rules:

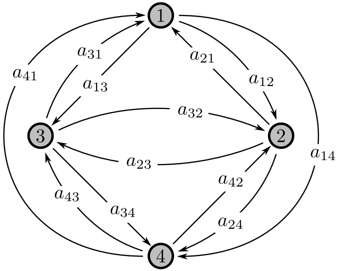
$$r_{ij}^{\text{new}} = \max\{r_{ij},\ r_{ik}\cdot r_{kj}\}, \qquad r_{ji}^{\text{new}} = 1/r_{ij}^{\text{new}}.$$

To simplify Problem, let us set

$$a_{ij} := \log r_{ij} \quad \forall i,j.$$

Additive Reformulation

We obtain the graph



$$a_{ij} = -a_{ji}$$

and the set of additive updating rules:

$$a_{ij}^{\text{new}} = \max\{a_{ij},\ a_{ik} + a_{kj}\}, \qquad a_{ji}^{\text{new}} = -a_{ij}^{\text{new}}.$$

‘Linearization’ of Problem

Each updating rule

$$a_{ij}^{\text{new}} = \max\{a_{ij},\ a_{ik}+a_{kj}\}, \qquad a_{ji}^{\text{new}} = -a_{ij}^{\text{new}}.$$

means the following ‘timing’ operations:

1. given indices i and j we first update a_{ij} to a_{ij}^{new};
2. then, knowing a_{ij}^{new} we update a_{ji} to a_{ji}^{new};
3. as a result, we obtain the updated pair $\left(a_{ij}^{\text{new}}, a_{ji}^{\text{new}}\right)$.

Question

How will look updating rules if we start updating from the pair of indices (j,i) ?

‘Linearization’ of Problem (cont.)

$$a_{ji}^{\text{new}} = \max\left\{a_{ji},\ a_{jk} + a_{ki}\right\}$$

$$\Downarrow$$

$$-a_{ij}^{\text{new}} = \max\left\{-a_{ij},\ -a_{kj} - a_{ik}\right\}$$

$$\Downarrow$$

$$-a_{ij}^{\text{new}} = -\min\left\{a_{ij},\ a_{ik} + a_{kj}\right\}$$

$$\Downarrow$$

$$a_{ij}^{\text{new}} = \min\left\{a_{ij},\ a_{ik} + a_{kj}\right\}$$

If we don’t care which of the weights a_{ij} or a_{ji} is updated first, then we obtain that ***there are valid both of the following updating rules:***

$$a_{ij}^{\text{new}} = \max\left\{a_{ij},\ a_{ik} + a_{kj}\right\}, \qquad a_{ji}^{\text{new}} = -a_{ij}^{\text{new}}.$$

or

$$a_{ij}^{\text{new}} = \min\left\{a_{ij},\ a_{ik} + a_{kj}\right\}, \qquad a_{ji}^{\text{new}} = -a_{ij}^{\text{new}}.$$

Conclusion

max and **min** in the above updating rules are irrelevant and ***may be discarded.***

‘Linearization’ of Problem (cont.)

The rule of updating may be rewritten as follows:

Either a_{ij} is not changed during update or it is changed and then it is updated as follows:

$$a_{ij}^{\text{new}} = a_{ik} + a_{kj},$$

Wow!

Updating rules became **linear**!

‘Linearization’ of Problem (cont.)

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Dimensionality Reduction

Remark

Weights a_{ji} 'mimic' a_{ij} in the skew-symmetric way.

Then instead of the full set of weights $\{a_{ij}\}$ it suffices to consider the set of weights $\{a_{ij}\}$ with $i<j$.

As a result, the updating rules $a_{ij}^{\text{new}} = a_{ik} + a_{kj}$ take the following form:

$$a_{ij}^{\text{new}} = \begin{cases} -a_{ki} + a_{kj} & \text{if} \quad k<i<j, \\ a_{ik} + a_{kj} & \text{if} \quad i<k<j, \\ a_{ik} - a_{jk} & \text{if} \quad i<j<k. \end{cases}$$

It is convenient to represent these last relations in matrix form.

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It is convenient to represent these last relations in matrix form.

By introducing the column-vector

$$\vec{a} = \{a_{12}, a_{23}, a_{34}, a_{13}, a_{24}, a_{14}\}^T$$

the update rules take the 'matrix' form:

$$\vec{a}^{\,\text{new}} = A_{(ijk)}\,\vec{a}, \qquad i<j,\ k\neq i,j,$$

where twelve (6×6)-matrices $A_{(ijk)}$ are as follows:

$$A_{(123)} = \begin{pmatrix} 0 & -1 & 0 & 1 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad A_{(124)} = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 1\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$A_{(231)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0\\ -1 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad A_{(234)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & -1 & 0 & 1 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

etc.

Matrix Representation (cont.)

Now the problem of investigation of the *weights dynamics* may be posed in the following form:

Given an initial vector $\overrightarrow{a}(0) = \overrightarrow{a}_0$ *and a sequence* $\{\omega_n\}$ *of triplets* $\omega_n = (i_n j_n k_n)$ *such that* $i_n < j_n$, $k_n \neq i_n, j_n$, *we need to study the dynamics of the sequence*

$$\overrightarrow{a}(n+1) = A_{\omega_n} \overrightarrow{a}(n)$$

or, what is the same, behavior of the vectors

$$\overrightarrow{a}(n+1) = A_{\omega_n} A_{\omega_{n-1}} \cdots A_{\omega_0} \overrightarrow{a}_0$$

Conjecture (Matrix Reformulation)

For any sequence of triplets $\{\omega_n\}$, the sequence of matrix products $A_{\omega_n} A_{\omega_{n-1}} \cdots A_{\omega_0}$ **is convergent.**

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Further Dimensionality Reduction

Observation

All the matrices $\{A_{(ijk)}\}$ have a **common invariant subspace of fixed points** determined by the relations

$$\begin{cases} a_{13} = a_{12} + a_{23}, \\ a_{14} = a_{13} + a_{34}, \\ a_{24} = a_{23} + a_{34}. \end{cases}$$

Corollary

There exists a change of variables Q such that each of the matrices $Q^{-1}A_{(ijk)}Q$ takes the block-triangle form:

$$B_{(ijk)} := Q^{-1}A_{(ijk)}Q = \left\| \begin{array}{cc} I & C_{(ijk)} \\ 0 & D_{(ijk)} \end{array} \right\|,$$

where $C_{(ijk)}$ and $D_{(ijk)}$ are (3×3)-matrices.

Final Implications

Observation

Each matrix product $B_{\omega_n}B_{\omega_{n-1}}\cdots B_{\omega_0}$ has the following form

$$B_{\omega_n}B_{\omega_{n-1}}\cdots B_{\omega_0} = \left\| \begin{matrix} I & * \\ 0 & D_{\omega_n}D_{\omega_{n-1}}\cdots D_{\omega_0} \end{matrix} \right\|.$$

Corollary

The matrix product $B_{\omega_n}B_{\omega_{n-1}}\cdots B_{\omega_0}$ is convergent ***only if*** *the matrix product $D_{\omega_n}D_{\omega_{n-1}}\cdots D_{\omega_0}$ is convergent.*

$\Downarrow$

We need to investigate the convergence of infinite products of (3×3)-matrices $\{D_{(ijk)}\}$.

Final Implications

Observation

Each matrix product $B_{\omega_n}B_{\omega_{n-1}}\cdots B_{\omega_0}$ has the following form

$$B_{\omega_n}B_{\omega_{n-1}}\cdots B_{\omega_0}=\begin{Vmatrix} I & * \\ 0 & D_{\omega_n}D_{\omega_{n-1}}\cdots D_{\omega_0}\end{Vmatrix}.$$

Corollary

*The matrix product $B_{\omega_n}B_{\omega_{n-1}}\cdots B_{\omega_0}$ is convergent **only if** the matrix product $D_{\omega_n}D_{\omega_{n-1}}\cdots D_{\omega_0}$ is convergent.*

$$\Downarrow$$

We need to investigate the convergence of infinite products of (3×3)-matrices $\{D_{(ijk)}\}$.

Final Implications (cont.)

Matrices $\{D_{(ijk)}\}$ are of the form:

$$D_{(123)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}, \quad D_{(124)} = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad D_{(132)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$D_{(134)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad D_{(142)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad D_{(143)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$D_{(231)} = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}, \quad D_{(234)} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{pmatrix}, \quad D_{(241)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

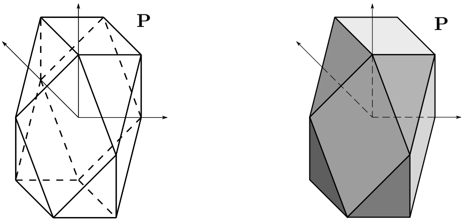
$$D_{(243)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad D_{(341)} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \\ 1 & 0 & 0 \end{pmatrix}, \quad D_{(342)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{pmatrix},$$

Final Implications (cont.)

Observation

All the matrices $\{D_{(ijk)}\}$

- have a **common invariant symmetric body set $\mathbf{P}$** (an elongated *cubeoctahedron*),
- **transform the vertices of $\mathbf{P}$ either to other vertices of $\mathbf{P}$ or to the origin.**



Disproof of Conjecture

Observation

Let $\{\omega_n\}$ be the 12-periodic sequence such that

$$\begin{array}{llll} \omega_0 = (1,4,3), & \omega_1 = (3,4,1), & \omega_2 = (3,4,2), & \omega_3 = (1,4,2), \\ \omega_4 = (1,2,4), & \omega_5 = (2,3,1), & \omega_6 = (1,3,2), & \omega_6 = (2,4,3), \\ \omega_8 = (1,3,4), & \omega_9 = (2,4,1), & \omega_{10} = (1,2,3), & \omega_{11} = (2,3,4), \end{array}$$

then the sequence of matrix products

$$D_{\omega_n} D_{\omega_{n-1}} \cdots D_{\omega_0}, \qquad n = 0, 1, \ldots,$$

is 12-periodic while the sequence of matrix products

$$A_{\omega_n} A_{\omega_{n-1}} \cdots A_{\omega_0}, \qquad n = 0, 1, \ldots,$$

is divergent!

Conjecture of Pokrovskii is false !

Observation

Let $\{\omega_n\}$ be the 12-periodic sequence such that

$$\begin{array}{llll}
\omega_0=(1,4,3), & \omega_1=(3,4,1), & \omega_2=(3,4,2), & \omega_3=(1,4,2),\\
\omega_4=(1,2,4), & \omega_5=(2,3,1), & \omega_6=(1,3,2), & \omega_6=(2,4,3),\\
\omega_8=(1,3,4), & \omega_9=(2,4,1), & \omega_{10}=(1,2,3), & \omega_{11}=(2,3,4),
\end{array}$$

then the sequence of matrix products

$$D_{\omega_n}D_{\omega_{n-1}}\cdots D_{\omega_0},\qquad n=0,1,\ldots,$$

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$$A_{\omega_n}A_{\omega_{n-1}}\cdots A_{\omega_0},\qquad n=0,1,\ldots,$$

is divergent!

Conjecture of Pokrovskii is false !

Observation 2

Let $\{\omega_n\}$ be the 16-periodic sequence such that

$$\begin{array}{llll} \omega_0=(1,4,2), & \omega_1=(1,2,3), & \omega_2=(3,4,1), & \omega_3=(1,4,2),\\ \omega_4=(1,3,4), & \omega_5=(2,4,3), & \omega_6=(2,3,1), & \omega_7=(3,4,2),\\ \omega_8=(2,4,1), & \omega_9=(1,3,4), & \omega_{10}=(3,4,2), & \omega_{11}=(1,4,3),\\ \omega_{12}=(2,3,4), & \omega_{13}=(1,3,2), & \omega_{14}=(1,2,4), & \omega_{15}=(1,4,3), \end{array}$$

then both sequences of matrix products

$$D_{\omega_n}D_{\omega_{n-1}}\cdots D_{\omega_0},\qquad n=0,1,\ldots,$$

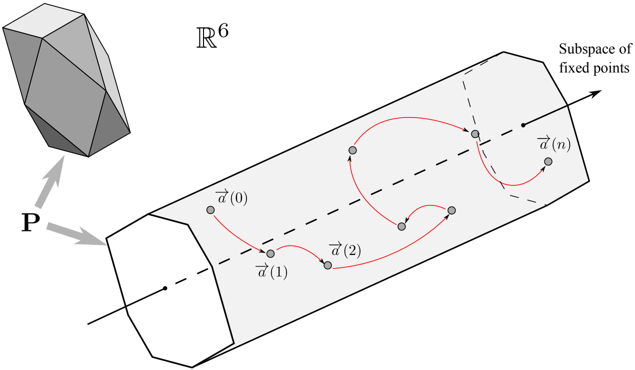
and

$$A_{\omega_n}A_{\omega_{n-1}}\cdots A_{\omega_0},\qquad n=0,1,\ldots,$$

are 16-periodic!

Question

Any trajectory $\{\vec{a}(n)\}$ belongs to a ***tube*** around the space of common fixed points of the matrices $A_{(ijk)}$.



Does this make any economic sense ?

On returning to Cork in 2009, Alexei brought to this work Rod Cross, Brian O'Callaghan and Alexey Pokrovskiy...

- Kozyakin V., A. Pokrovskii, and B. O'Callaghan, *Sequences of Arbitrages*, ArXiv.org e-Print archive, 1004.0561, Apr. 2010, 18p.
- Cross R., V. Kozyakin, B. O'Callaghan, A. Pokrovskii, and A. Pokrovskiy, *Periodic Sequences of Arbitrage: A Tale of Four Currencies*, University of Strathclyde Business School, Department of Economics in its series Working Papers, 2010, No. 10–19 (to be published in Metroeconomica).
- Cross R., V. Kozyakin, D. Lang, B. O'Callaghan, A. Pokrovskii, and A. Pokrovskiy, *Arbitrage sequences and Leijonhufvud's corridor hypothesis*, Centre d'Economie de l'Universite Paris Nord, March 2011 (submitted to Cambridge Journal of Economics)