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Linear codes with covering radius 3

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Abstract The shortest possible length of a q-ary linear code of covering radius R and codimension r is called the length function and is denoted by $\ell_q(r, R)$. Constructions of codes with covering radius 3 are here developed, which improve best known upper bounds on $\ell_q(r, 3)$. General constructions are given and upper bounds on $\ell_q(r, 3)$ for $q = 3, 4, 5, 7$ and $r \leq 24$ are tabulated.

Keywords Concatenation · Covering radius · Projective geometry · Saturating set

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1 Introduction

Let $\mathbb{F}_q$ be the Galois field of order q, and let $\mathbb{F}_q^* = \mathbb{F}_q \setminus \{0\}$. Only nonbinary cases are considered in this work, that is, we assume that $q \geq 3$. We denote by $\mathbb{F}_q^n$ the n-dimensional space of q-ary vectors of length n. A linear code in $\mathbb{F}_q^n$ with codimension r, minimum distance d, and covering radius R is said to be an $[n, n-r, d]_q R$ code. In this notation we can omit d and R. All words in $\mathbb{F}_q^r$ can be obtained as a linear combination of at most R columns of a parity check matrix of an $[n, n-r]_q R$ code. The minimum length n such that an $[n, n-r]_q R$ code

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exists is called the *length function* and is denoted by $\ell_q(r, R)$ [3]. For surveys of covering codes, see [2,5].

Related to any $[n, n-r]_q R$ code is the concept of *density*,

$$\mu_q(n, R, r) = q^{-r} \sum_{i=0}^{R} (q-1)^i \binom{n}{i},$$

which necessarily is greater than or equal to 1, and generally one may study the asymptotic density

$$\liminf_{r\to\infty} \mu_q(\ell_q(r, R), R, r).$$

for some given q and R. Similarly, the asymptotic density of a specific family of $[n(r), n(r)-r]_q R$ codes, where $n(r)$ is some increasing function of r, is equal to

$$\liminf_{r\to\infty} \mu_q(n(r), R, r).$$

Let $\mathrm{PG}(v, q)$ be the v-dimensional projective space over $\mathbb{F}_q$. For an introduction to such spaces and the geometrical objects therein, see [17,18]. We say that a set of points $S \subseteq \mathrm{PG}(v, q)$ is *ρ-saturating* if for any point $x \in \mathrm{PG}(v, q)$ there exist $\rho + 1$ points in S generating a subspace of $\mathrm{PG}(v, q)$ in which x lies and ρ is the smallest integer for which this property holds [6, Definition 1.1]. Saturating sets have also been called *spanning sets* [3]. The points of a ρ-saturating n-set in $\mathrm{PG}(r-1, q)$ can thus be considered as r-dimensional columns of a parity check matrix of an $[n, n-r]_q R$ code with $R = \rho + 1$.

This work is devoted to constructing infinite families of codes with covering radius $R = 3$. A common technique for constructing codes is that of taking a code with a given covering radius and small codimension as a starting code and producing an infinite family of codes with the same covering radius and with almost the same density. Different families are here obtained for the three cases of $r \pmod 3$. Starting codes for such families are obtained by first applying a q^m-concatenating construction to "short" codes, including $[n, n-4]_q 3$ and $[n, n-5]_q 3$ codes. The codes constructed in this manner are further used as starting codes in other q^m-concatenating constructions.

The paper is organized as follows. In Sect. 2, we develop and investigate a general q^m-concatenating construction and six of its variants for covering radius $R = 3$. In these constructions, certain partitions of the column sets of the involved parity check matrices are of central importance. The constructions are utilized in Sect. 3 to get particular codes as well as infinite families of codes, leading to new upper bounds on $\ell_q(r, 3)$, in other words, on the smallest size of 2-saturating sets in $\mathrm{PG}(r-1, q)$. The paper is concluded with tables of upper bounds on $\ell_q(r, 3)$ for $q = 3, 4, 5, 7$ and $r \le 24$.

2 Constructions

The constructions in the sequel are based on an approach of using an $[n_0, n_0 - r_0]_q R$ starting code and concatenating constructions to obtain specific codes as well as infinite families of $[n, n-(r_0 + Rm)]_q R$ codes with length $n = q^m n_0 + N_m$ for some $N_m \le R\theta_{m,q}$, where

$$\theta_{m,q} = \frac{q^m - 1}{q - 1}. \tag{1}$$

In the q^m-concatenating constructions, every column of the parity check matrix of the starting code is repeated q^m times with Rm additional entries, and N_m further columns are added to complete the new parity check matrix. This approach has earlier been used in [6–13,15].

The following conventions are used throughout this paper. All matrices are q-ary. An element of $\mathbb{F}_{q^m}$ written in a q-ary matrix denotes an m-dimensional column that is a q-ary representation of this element, and vice versa, a q-ary column with m entries can be treated as an element of $\mathbb{F}_{q^m}$. Note that in linear combinations of q-ary columns, all coefficients belong to $\mathbb{F}_q^*$.

Throughout the paper, we denote the elements of $\mathbb{F}_{q^m}$ by $\{\xi_1, \xi_2, \ldots, \xi_{q^m}\}$ so that $\xi_1 = 0$, $\xi_2 = 1$, and $\{\xi_2, \xi_3, \ldots, \xi_{\theta_{m,q}+1}\}$ form the columns of a parity check matrix $\mathbf{H}_m$ of the q-ary Hamming code of codimension m; note that the length of such a code is given by (1).

An $s \times t$ zero matrix is denoted by $\mathbf{0}_s$ for clarity, as the number of columns is clear from the context in all places. The index s may also be omitted if it is irrelevant or obvious from the context.

2.1 Basic construction and matrix partitions

The following definition [6, Definition 2.4] plays a central role in the constructions to be discussed.

Definition 1 Let $0 \le \ell \le R$. A partition of the column set of a parity check matrix of an $[n, n-r]_q R$ code into nonempty subsets is called an (R, ℓ)*-partition* if every column of $\mathbb{F}_q^r$ (including the zero column) can be obtained as a linear combination with *nonzero q-ary coefficients* of at least ℓ and at most R columns belonging to *distinct* subsets of the partition.

If $\ell = 0$, then we get the all-zero column as the linear combination of 0 columns. Obviously, an (R, ℓ)-partition is an (R, ℓ')-partition for any nonnegative $\ell' < \ell$. A refinement of an (R, ℓ)-partition is also an (R, ℓ)-partition, and a partition into one-element subsets—which obviously cannot be refined any further—is called *trivial*. An $[n, n-r, d]_q R$ code that admits an (R, ℓ)-partition is said to be an $[n, n-r, d]_q R, \ell$ code, where the minimum distance d may be omitted. By considering the trivial partition it is clear that any $[n, n-r]_q R$ code is an $[n, n-r]_q R, 0$ code.

When studying and applying (R, ℓ)-partitions, it is advantageous if ℓ is as large as possible and, simultaneously, the number of subsets in the partition is as small as possible.

We shall now outline a general construction. The details of the construction will be specified in the subsequent discussion. However, even if some details are left open, we will still be able to prove certain properties of the obtained codes.

Construction A. We start from an $[n_0, n_0 - r_0]_q 3, \ell_0$ code C_0 with an $r_0 \times n_0$ parity check matrix $\mathbf{H}_0 = [\mathbf{h}_1\ \mathbf{h}_2\ \cdots\ \mathbf{h}_{n_0}]$ that has columns $\mathbf{h}_i \in \mathbb{F}_q^{r_0}$. Moreover, consider some $(3, \ell_0)$-partition $\mathcal{P}_0$ with p_0 subsets, and let $m \ge 1$ be an integer parameter.

To every column $\mathbf{h}_i$ we assign a value $\beta_i \in \mathbb{F}_{q^m} \cup \{*, \#\}$ so that $\beta_i \ne \beta_j$ whenever the columns $\mathbf{h}_i$ and $\mathbf{h}_j$ belong to distinct subsets of the partition $\mathcal{P}_0$. Consequently, if $\mathbf{h}_i$ and $\mathbf{h}_j$ belong to the same subset it is possible to put either $\beta_i = \beta_j$ or $\beta_i \ne \beta_j$.

Let $\mathbf{A}$ be some $(r_0 + 3m) \times N_m$ matrix with $N_m \le (3 - \ell_0)\theta_{m,q}$, the parameters and details of which are left open at this moment. Notice, however, that $\mathbf{A}$ has no columns if $\ell_0 = 3$. We now construct an $[n, n - (r_0 + 3m)]_q R, \ell$ code with length $n = q^m n_0 + N_m$ and parity check matrix

$$\mathbf{H} = \left[\mathbf{A}\ \mathbf{B}(\mathbf{h}_1, \beta_1)\ \mathbf{B}(\mathbf{h}_2, \beta_2)\ \cdots\ \mathbf{B}(\mathbf{h}_{n_0}, \beta_{n_0})\right], \tag{2}$$

where

$$\mathbf{B}(\mathbf{h}_i, \beta_i) = \begin{bmatrix} \mathbf{h}_i & \mathbf{h}_i & \cdots & \mathbf{h}_i \\ \xi_1 & \xi_2 & \cdots & \xi_{q^m} \\ \beta_i\xi_1 & \beta_i\xi_2 & \cdots & \beta_i\xi_{q^m} \\ \beta_i^2\xi_1 & \beta_i^2\xi_2 & \cdots & \beta_i^2\xi_{q^m} \end{bmatrix} \quad \text{if} \quad \beta_i \in \mathbb{F}_{q^m},$$

$$\mathbf{B}(\mathbf{h}_i, *) = \begin{bmatrix} \mathbf{h}_i & \mathbf{h}_i & \cdots & \mathbf{h}_i \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \xi_1 & \xi_2 & \cdots & \xi_{q^m} \end{bmatrix}, \quad \mathbf{B}(\mathbf{h}_i, \#) = \begin{bmatrix} \mathbf{h}_i & \mathbf{h}_i & \cdots & \mathbf{h}_i \\ 0 & 0 & \cdots & 0 \\ \xi_1 & \xi_2 & \cdots & \xi_{q^m} \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Let $\mathbf{b}_{i,j}$ be the jth column of $\mathbf{B}(\mathbf{h}_i, \beta_i)$. For typographical reasons, we consider columns as tuples in the proofs of this paper. For example, $\mathbf{b}_{i,j} = (\mathbf{h}_i, \xi_j, \beta_i\xi_j, \beta_i^2\xi_j)$ if $\beta_i \in \mathbb{F}_{q^m}$.

Lemma 1 *The code obtained by Construction A has minimum distance* 3 *for any* $q \geq 3$.

Proof For odd $q \geq 3$, we let $\xi_u = -1$ to get $2\mathbf{b}_{i,1} - \mathbf{b}_{i,2} - \mathbf{b}_{i,u} = \mathbf{0}$. For even $q \geq 4$, we let $\xi_u \in \mathbb{F}_q \setminus \{0, 1\}$ and get $(\xi_u^{-1} + 1)\mathbf{b}_{i,1} + \mathbf{b}_{i,2} + \xi_u^{-1}\mathbf{b}_{i,u} = \mathbf{0}$. □

We continue by studying (R, ℓ)-partitions—parameters of R and ℓ and numbers of subsets in (R, ℓ)-partitions—of codes obtained in Construction A. We define partitions $\mathcal{P}_i$, $1 \leq i \leq 3$, of the columns of the matrix $\left[\mathbf{B}(\mathbf{h}_1, \beta_1)\ \mathbf{B}(\mathbf{h}_2, \beta_2)\ \cdots\ \mathbf{B}(\mathbf{h}_{n_0}, \beta_{n_0})\right]$ into ip_0 subsets as follows.

The partition $\mathcal{P}_1$ is an obvious extension from $\mathcal{P}_0$. In $\mathcal{P}_1$ two columns, $\mathbf{b}_{i,j}$ and $\mathbf{b}_{i',j'}$, belong to the same subset iff $\mathbf{h}_i$ and $\mathbf{h}_{i'}$ belong to the same subset of $\mathcal{P}_0$ (which is obviously the case if $i = i'$).

The partitions $\mathcal{P}_2$ and $\mathcal{P}_3$ are refinements of $\mathcal{P}_1$ based on a partition of the columns of each matrix $\mathbf{B}(\mathbf{h}_i, \beta_i)$ in the following way:

$$\mathbf{B}(\mathbf{h}_i, \beta_i) = \{\mathbf{b}_{i,1}, \mathbf{b}_{i,2}\} \cup \{\mathbf{b}_{i,3}, \ldots, \mathbf{b}_{i,q^m}\} \quad \text{for } \mathcal{P}_2,$$
$$\mathbf{B}(\mathbf{h}_i, \beta_i) = \{\mathbf{b}_{i,1}\} \cup \{\mathbf{b}_{i,2}, \ldots, \mathbf{b}_{i,\theta_{m,q}+1}\} \cup \{\mathbf{b}_{i,\theta_{m,q}+2}, \ldots, \mathbf{b}_{i,q^m}\} \quad \text{for } \mathcal{P}_3.$$

Lemma 2 *Let* $q \geq 4$. *A column of* $\left[\mathbf{B}(\mathbf{h}_1, \beta_1)\ \mathbf{B}(\mathbf{h}_2, \beta_2)\ \cdots\ \mathbf{B}(\mathbf{h}_{n_0}, \beta_{n_0})\right]$ *can be obtained as a linear combination (with nonzero q-ary coefficients) of two columns of distinct subsets of the partition* $\mathcal{P}_2$ *and of two as well as three columns of distinct subsets of* $\mathcal{P}_3$.

Proof For $\mathcal{P}_2$, consider some $\xi_u \in \mathbb{F}_{q^m} \setminus \{0, 1\}$ (that is $u \geq 3$). Then for arbitrary $i \in \{1, 2, \ldots, n\}$ and $\xi_v \in \mathbb{F}_q \setminus \{0, 1, \xi_u\}$, $\mathbf{b}_{i,u} = (1 - \xi_v)\mathbf{b}_{i,1} + \xi_v\mathbf{b}_{i,w}$ where $\xi_w = \xi_u\xi_v^{-1}$. Moreover, $\mathbf{b}_{i,1} = \xi_v(\xi_v - 1)^{-1}\mathbf{b}_{i,2} - (\xi_v - 1)^{-1}\mathbf{b}_{i,v}$ takes care of $\mathbf{b}_{i,1}$ as well as $\mathbf{b}_{i,2}$.

For $\mathcal{P}_3$ and the case of linear combinations of two columns, the results for $\mathcal{P}_2$ in the first part of the proof can be applied as long as ξ_v is chosen for the last equality so that $\mathbf{b}_{i,2}$ and $\mathbf{b}_{i,v}$ are from different subsets.

For $\mathcal{P}_3$ and combinations of three columns, for $\xi_u \in \{\xi_2, \xi_3, \ldots, \xi_{\theta_{m,q}+1}\}$ we take $\mathbf{b}_{i,u} = (\xi_w - 1)\xi_w\mathbf{b}_{i,1} - \xi_w^2\mathbf{b}_{i,v} + (\xi_w + 1)\mathbf{b}_{i,u}$, where $\xi_u = \xi_w\xi_v$, $\xi_w \in \mathbb{F}_q \setminus \{0, 1\}$. (Note that, as ξ_u stems from the parity check matrix of a Hamming code, $\mathbf{b}_{i,u}$ and $\mathbf{b}_{i,v}$ are indeed in different sets of $\mathcal{P}_3$.) For any such ξ_u and ξ_v, we also get $\mathbf{b}_{i,1} = \xi_w\mathbf{b}_{i,1} - \xi_w\mathbf{b}_{i,v} + \mathbf{b}_{i,u}$. Finally, for $\xi_u \in \{\xi_{\theta_{m,q}+2}, \ldots, \xi_{q^m}\}$, take $\mathbf{b}_{i,u} = (\xi_w^{-1} - 1)\mathbf{b}_{i,1} - (\xi_w^{-1} - 1)\mathbf{b}_{i,u} + \mathbf{b}_{i,v}$, where $\xi_u = \xi_w\xi_v$ and $\xi_v \in \{\xi_2, \xi_3, \ldots, \xi_{\theta_{m,q}+1}\}$ (which exists by the argument above). □

Finally, we define partitions of the set $\{\xi_2, \xi_3, \ldots, \xi_{\theta_{m,q}+1}\}$. Treating elements of this set (stemming from a parity check matrix of a Hamming code) as points of the projective geometry PG$(m-1, q)$, consider an arbitrary point p in this geometry and the $\theta_{m-1,q}$ lines $L_i, i = 1, 2, \ldots$ that pass through p. The points of L_i are p and $p_{i,1}, p_{i,2}, \ldots, p_{i,q}$. The partition $\mathcal{P}_2'$ is then

$$\{p, p_{1,1}, p_{2,1}, \ldots, p_{\theta_{m-1,q},1}\} \cup \{p_{i,j} : 1 \le i \le \theta_{m-1,q},\ 2 \le j \le q\},$$

and $\mathcal{P}_3'$ is a refinement of $\mathcal{P}_2'$ obtained by letting $\{p\}$ form a new subset. The following lemma is of the same flavor as Lemma 2.

Lemma 3 *Let $m \ge 2$. Each element of $\{\xi_2, \xi_3, \ldots, \xi_{\theta_{m,q}+1}\}$ can be obtained as a linear combination (with nonzero q-ary coefficients) of two elements of distinct subsets of the partition $\mathcal{P}_2'$ and of two as well as three elements of distinct subsets of $\mathcal{P}_3'$.*

Proof The result follows from the basic fact that any point on a line is a linear combination of any other two points on the same line. Such a linear combination may further be combined with a third point to get combinations of three points. As all this can be done in an arbitrary manner, the points can be picked from different subsets of the partitions. □

2.2 Specific constructions

In a sequence of theorems in this section, we present Constructions A1 to A6 as specific instances of Construction A. Recall that the unspecified details in Construction A are the choices of $\mathbf{A}$, β_i, and m.

To prove that (an instance of) Construction A gives codes with covering radius 3, one must show that any column $(a, c_1, c_2, c_3) \in \mathbb{F}_{q^{r_0}}\mathbb{F}_{q^m}\mathbb{F}_{q^m}\mathbb{F}_{q^m}$ can be obtained as a linear combination of at most three columns of the matrix $\mathbf{H}$ in (2). By definition, as the initial code C_0 has an (R, ℓ_0)-partition, we can obtain

$$a = \sum_{i=1}^{k} s_i \mathbf{h}_{j_i},\ \ s_i \in \mathbb{F}_q^* \tag{3}$$

for $\ell_0 \le k \le 3$ with all columns in the combination from different subsets of the partition $\mathcal{P}_0$ (the special case with $a = 0$, $\ell_0 = 0$ will be treated separately). As the columns belong to different subsets, the values of β_{j_i} are distinct, which in turn provides a nonzero determinant (guaranteeing a solution) in a system of k linear equation and k variables $x_1, x_2, \ldots, x_k \in \mathbb{F}_q^m$,

$$\sum_{i=1}^{k} f_{t_u}(\beta_{j_i}) s_i x_i = c_{t_u},\ \ u = 1, 2, \ldots, k \tag{4}$$

with $t_u \in \{1, 2, 3\}$ and where $f_j(y) = y^{j-1}$ if $y \in \mathbb{F}_{q^m}$ and $f_j(y) \in \{0, 1\}$ otherwise; cf. the matrices related to (2). A solution $x_1, x_2, \ldots, x_k$ explicitly defines the columns in a desired linear combination. These columns have the form $\mathbf{b}_{j_i, v_i}, i = 1, \ldots, k$, where v_i satisfies $x_i = \xi_{v_i}$. If $k < 3$, not all c_i are included in the equation system (4); then we further use at most $3 - k$ columns from the matrix $\mathbf{A}$ in the linear combination.

Throughout this section we assume that the values of a, s_i, j_i, and k be taken from (3). The basic parameters of the codes obtained, such as length, follow directly from the constructions and are therefore not addressed in the proofs. The linear combinations used in the proofs to show that the covering radius of the constructed codes is 3 are chosen so that they simultaneously prove that the code has a certain $(3, \ell)$-partition. There are commonly several possible ways of obtaining desired linear combinations; some of the proofs (of the covering radius) can be made slightly more elegant if one does not care about the existence of $(3, \ell)$-partitions.

As for proving existence of $(3, \ell)$-partition—in particular for the all-zero vector—Lemmata 2 and 3 are needed. However, as it is anyway not possible to include all technical details for the proof that the described partitions indeed are $(3, \ell)$-partitions, the application of these lemmata is not further emphasized.

2.2.1 Construction A1

In *Construction A1*, for $\ell_0 = 0$ and $q^m - 1 \geq p_0$, consider $\beta_i \in \mathbb{F}_{q^m}^*$, and let

$$\mathbf{A} = [\mathbf{A}_1\ \mathbf{A}_2],\ \mathbf{A}_1 = \begin{bmatrix} \mathbf{0}_{r_0} \\ \mathbf{H}_m \\ \mathbf{0}_{2m} \end{bmatrix},\ \mathbf{A}_2 = \begin{bmatrix} \mathbf{0}_{r_0} \\ \mathbf{0}_m \\ \mathbf{D}_{2m} \end{bmatrix},$$

where $\mathbf{D}_{2m}$ is a parity check matrix of an $[n', n' - 2m]_q 2, 0$ code D with a $(2, 0)$-partition into p' subsets.

Theorem 1 *Construction A1 gives an* $[n, n - r, 3]_q 3, 1$ *code with* $n = q^m n_0 + (q^m - 1)/(q - 1) + n'$, $r = r_0 + 3m$ *that has a* $(3, 1)$*-partition with* $p_0 + 3 + p'$ *subsets. Moreover, if* $m \geq 2$ *and every linear combination of one and two columns of* $\mathbf{D}_{2m}$ *can be represented as a linear combination (with nonzero* q*-ary coefficients) of three different columns of* $\mathbf{D}_{2m}$*, then an* $[n, n - r, 3]_q 3, 3$ *code is obtained that has a* $(3, 3)$*-partition with* $3p_0 + 3 + n'$ *subsets.*

Proof To prove that the covering radius is 3, we first consider the case $a \neq 0$. Then for (4) we use the form

$$\sum_{i=1}^{k} s_i \beta_{j_i}^{v-1} x_i = c_v \tag{5}$$

where $v = 1$ if $k = 1$; $v \in \{2, 3\}$ if $k = 2$; and $v \in \{1, 2, 3\}$ if $k = 3$. For $k = 3$ the solution of (5) directly gives a linear combination of three columns $\mathbf{b}_{i,j}$, so we only need to consider the cases $k = 1$ and $k = 2$ separately.

For $k = 1$, we get (a, c_1, c_2', c_3') (for some c_2', c_3') as a linear combination with just one column $\mathbf{b}_{i,j}$. We can then get $(a, c_1, c_2, c_3) - (a, c_1, c_2', c_3') = (0, 0, c_2 - c_2', c_3 - c_3')$ as a linear combination of two columns of $\mathbf{A}_2$ (as the code D has covering radius 2).

For $k = 2$, we get (a, c_1', c_2, c_3) (for some c_1') as a linear combination of two columns $\mathbf{b}_{i,j}$. We can then get $(a, c_1, c_2, c_3) - (a, c_1', c_2, c_3) = (0, c_1 - c_1', 0, 0)$ as linear combination of one column of $\mathbf{A}_1$ (as H_m has covering radius 1).

Finally, for $a = 0$, we get any column $(0, c_1, c_2, c_3)$ as a linear combination of at most three columns of $\mathbf{A}$; note that the last $3m$ rows of $\mathbf{A}$ form a parity check matrix of a code that is the direct sum of codes with covering radius 1 (from $\mathbf{A}_1$) and 2 (from $\mathbf{A}_2$).

A (3,1)-partition with the desired number of subsets is obtained by partitioning $[\mathbf{B}(\mathbf{h}_1, \beta_1)\ \mathbf{B}(\mathbf{h}_2, \beta_2)\ \cdots\ \mathbf{B}(\mathbf{h}_{n_0}, \beta_{n_0})]$ as $\mathcal{P}_1$, then refining $\mathbf{B}(\mathbf{h}_1, \beta_1)$ into three subsets $\{\mathbf{b}_{1,1}\}, \{\mathbf{b}_{1,2}\}, \{\mathbf{b}_{1,3}, \ldots, \mathbf{b}_{1,q^m}\}$ (to be able to represent the all-zero column by a linear combination of three columns from distinct subsets), and finally partitioning $\mathbf{A}$ into $\mathbf{A}_1$ and p' subsets of $\mathbf{A}_2$ (from its (2,0)-partition).

To prove the last part of the result, partition $[\mathbf{B}(\mathbf{h}_1, \beta_1)\ \mathbf{B}(\mathbf{h}_2, \beta_2)\ \cdots\ \mathbf{B}(\mathbf{h}_{n_0}, \beta_{n_0})]$ as $\mathcal{P}_3$, A_1 (via its part H_m) as $\mathcal{P}_3'$, and use the trivial partition for A_2. (Lemma 3 is here useful in considering the various cases.) □

Example 1 For $m = 2$, $q \geq 5$, we use the parity check matrix of the $[2q + 1, 2q - 3]_q 2$ code shown in [15, Eq. 1] as $\mathbf{D}_4$ in Theorem 1. With arguments and approaches similar to those in the proofs of Lemma 3 and results in [15], it can be shown that every linear combination of one or two columns of that matrix is a linear combination of three distinct columns of it. Then the assumption in the last part of Theorem 1 holds, and we get a code with a (3,3)-partition with $3p_0 + 2q + 4 \leq 3n_0 + 2q + 4$ subsets.

2.2.2 Construction A2

In *Construction A2*, for $\ell_0 = 1$ and $q^m \geq p_0$, consider $\beta_i \in \mathbb{F}_{q^m}$, and let

$$\mathbf{A} = [\mathbf{A}_1\ \mathbf{A}_2], \quad \mathbf{A}_1 = \begin{bmatrix} \mathbf{0}_{r_0+m} \\ \mathbf{H}_m \\ \mathbf{0}_m \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} \mathbf{0}_{r_0+m} \\ \mathbf{0}_m \\ \mathbf{H}_m \end{bmatrix}. \tag{6}$$

Theorem 2 *Construction A2 gives an* $[n, n-r, 3]_q 3, 3$ *code with* $n = q^m n_0 + 2(q^m - 1)/(q-1)$, $r = r_0 + 3m$ *that has a* $(3, i)$*-partition with* $ip_0 + 2$ *subsets for* $1 \leq i \leq 3$.

Proof The proof that the covering radius is 3 is similar to that of Theorem 1, so we only point out the main differences here. For (4) we again use the form (5), but now $v \in \{1, 3\}$ if $k = 2$; the other cases coincide. A column of $\mathbf{A}_1$ is used to get $(a, c_1, c_2, c_3) - (a, c_1, c_2', c_3) = (0, 0, c_2 - c_2', 0)$ for $k = 2$, and columns of $\mathbf{A}_1$ and $\mathbf{A}_2$ are combined to get $(a, c_1, c_2, c_3) - (a, c_1, c_2', c_3') = (0, 0, c_2 - c_2', c_3 - c_3')$ for $k = 1$.

Desired $(3, i)$-partitions for $1 \leq i \leq 3$ are obtained by taking the partitions $\mathcal{P}_i$ and the two subsets formed by the columns of $\mathbf{A}_1$ and $\mathbf{A}_2$. □

2.2.3 Construction A3

In *Construction A3*, for $\ell_0 = 2$ consider $\beta_i \in \mathbb{F}_{q^m}$ if q is odd and $\beta_i \in \mathbb{F}_{q^m} \cup \{\#\}$ if q is even—so $q^m \geq p_0$ and $q^m \geq p_0 - 1$, respectively—and let

$$\mathbf{A} = \begin{bmatrix} \mathbf{0}_{r_0+2m} \\ \mathbf{H}_m \end{bmatrix}.$$

Theorem 3 *Construction A3 gives an* $[n, n-r, 3]_q 3, 3$ *code with* $n = q^m n_0 + (q^m - 1)/(q-1)$, $r = r_0 + 3m$ *that has a* $(3, i)$*-partition with* $(i-1)p_0 + 1$ *subsets for* $i \in \{2, 3\}$.

Proof The proof that the covering radius is 3 is similar to those of Theorems 1 and 2. In particular, if all β_i belong to F_{q^m}, then the system of equations takes the form of (5). If q is even, $k = 3$, and $\beta_i = \#$ for some i, then (4) becomes

$$\begin{bmatrix} s_1 & s_2 & 0 \\ s_1\beta_{j_1} & s_2\beta_{j_2} & s_3 \\ s_1\beta_{j_1}^2 & s_2\beta_{j_2}^2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}, \tag{7}$$

where the determinant of the 3×3 matrix is $s_1 s_2 s_3 (\beta_{j_1}^2 - \beta_{j_2}^2)$, which is nonzero when q is even.

The case $k = 2$ can be considered via the first two rows of (7) and by using an appropriate column from $\mathbf{A}$ as the third column in the linear combination.

Desired $(3, i)$-partitions for $2 \leq i \leq 3$ are obtained by taking the partitions $\mathcal{P}_{i-1}$ and one set with the columns of $\mathbf{A}$. □

2.2.4 Construction A4

In *Construction A4*, for $\ell_0 = 3$ consider $\beta_i \in \mathbb{F}_{q^m} \cup \{*\}$ if q is odd and $\beta_i \in \mathbb{F}_{q^m} \cup \{*, \#\}$ if q is even—so $q^m \geq p_0 - 1$ and $q^m \geq p_0 - 2$, respectively—and let $\mathbf{A}$ be empty.

Theorem 4 *Construction A4 gives an* $[n, n-r, 3]_q 3, 3$ *code with* $n = q^m n_0,\ r = r_0 + 3m$ *that has a* $(3, 3)$*-partition with* p_0 *subsets.*

Proof The proof coincides with the proofs of Theorems 1, 2, and 3 if $\beta_i \in \mathbb{F}_{q^m} \cup \{\#\}$ for all i. If $\beta_i \in \mathbb{F}_{q^m} \cup \{*\}$ and some $\beta_i = *$, then we solve

$$\begin{bmatrix} s_1 & s_2 & 0 \\ s_1\beta_{j_1} & s_2\beta_{j_2} & 0 \\ s_1\beta_{j_1}^2 & s_2\beta_{j_2}^2 & s_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}, \tag{8}$$

and if $\beta_i = *, \beta_j = \#$ for some i, j—this can occur only when q is even—then we solve

$$\begin{bmatrix} s_1 & 0 & 0 \\ s_1\beta_{j_1} & s_2 & 0 \\ s_1\beta_{j_1}^2 & 0 & s_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}. \tag{9}$$

The solution of (9) is straightforward: $x_i = c_i s_i^{-1}$ for $i \in \{1, 2, 3\}$. Finally, we get a (3,3)-partition by taking the partition $\mathcal{P}_1$. □

2.2.5 Construction A5

In the last two constructions to be considered we have $\beta_i \in \mathbb{F}_{q^m} \cup \{*\}$ and the further requirement that $\cup_i \beta_i = \mathbb{F}_{q^m} \cup \{*\}$. Consequently, we must have $n_0 \geq q^m + 1$.

In *Construction A5*, for $\ell_0 = 0$, q odd, and $n_0 \geq q^m + 1 \geq p_0$, consider $\beta_i \in \mathbb{F}_{q^m} \cup \{*\}$ so that $\cup_i \beta_i = \mathbb{F}_{q^m} \cup \{*\}$, and let $\mathbf{A}$ be as in (6).

Theorem 5 *Construction A5 gives an* $[n, n-r, 3]_q 3, \ell$ *code with* $n = q^m n_0 + 2(q^m - 1)/(q-1)$, $r = r_0 + 3m$, *and with* $\ell = 1$ *if* $m = 1$, $\ell = 3$ *if* $m \geq 2$. *The code has a* $(3, 1)$*-partition with* $p_0 + 4$ *subsets and a* $(3, i)$*-partition with* $i(p_0 + 2)$ *subsets for* $i \in \{2, 3\}$, $m \geq 2$.

Proof We first consider the case $a \neq 0$. The proof that the covering radius is 3 coincides with that of Theorem 2 whenever all $\beta_{j_i} \in \mathbb{F}_{q^m}$ with j_i taken from the solution of (3), and parts of the proof of Theorem 4 takes care of the cases with $k = 3$.

If $k = 2$ and some $\beta_{j_i} = *$, then we can use the first and third rows of (8) with $s_2 = 0$ (that is, ignoring the second column of the 3×3 matrix). We then get any linear combination (a, c_1, c_2', c_3) by using two columns; a column of $\mathbf{A}_1$ takes care of $(a, c_1, c_2, c_3) - (a, c_1, c_2', c_3) = (0, 0, c_2 - c_2', 0)$, if $c_2 - c_2' \neq 0$.

The final case for $a \neq 0$ is $k = 1, \beta_{j_1} = *$. If $c_1 = 0$, then the solution of $s_1 x_1 = c_3$ gives a column that in a linear combination with a column of $\mathbf{A}_1$ gives a desired combination using two columns. If $c_1 \neq 0$, then determine y from $\beta_y = c_2/c_1$ (which necessarily has a solution as β_i takes all values of $\mathbb{F}_q^m$). By letting

$$\xi_z = c_1,\ \xi_u = s_1^{-1}(c_3 - \beta_y^2 \xi_z) = s_1^{-1}(c_3 - c_2^2/c_1), \tag{10}$$

we get the following linear combination of three columns: $\mathbf{b}_{y,z} - \mathbf{b}_{y,1} + s_1 \mathbf{b}_{j_1,u} = (a, c_1, c_2, c_3)$.

Next consider $a = 0$. The cases with $c_1 = 0$ are handled with (at most) two columns, one from $\mathbf{A}_1$ and one from $\mathbf{A}_2$. If $c_1 \neq 0$, then we take z as in (10) to get $\mathbf{b}_{y,z} - \mathbf{b}_{y,1} = (a, c_1, c_2, c_3')$ and finally a third column in the linear combination can be taken from $\mathbf{A}_2$ if $c_3' \neq c_3$.

A (3,1)-partition with the desired number of subsets is obtained by partitioning $\left[\mathbf{B}(\mathbf{h}_1, \beta_1)\ \mathbf{B}(\mathbf{h}_2, \beta_2)\ \cdots\ \mathbf{B}(\mathbf{h}_{n_0}, \beta_{n_0})\right]$ as $\mathcal{P}_1$, by further refining $\mathbf{B}(\mathbf{h}_1, \beta_1)$ into three subsets as in the proof of Theorem 1, and finally taking the two subsets formed by the columns of $\mathbf{A}_1$ and $\mathbf{A}_2$.

For $m \geq 2$, a $(3, 2)$-partition and a $(3, 3)$-partition are obtained via $\mathcal{P}_2$ and $\mathcal{P}_2'$, and $\mathcal{P}_3$ and $\mathcal{P}_3'$, respectively ($\mathcal{P}_i'$ applied to $\mathbf{A}_1$ and $\mathbf{A}_2$). □

Note that the requirement that $\cup_i \beta_i = \mathbb{F}_{q^m} \cup \{*\}$ means that if $q^m + 1 > p_0$ we must have $\beta_i \neq \beta_j$ in some cases when $\mathbf{h}_i$ and $\mathbf{h}_j$ belong to the same subset of the partition $\mathcal{P}_0$. The same comment applies to the following construction.

2.2.6 *Construction A6*

In *Construction A6*, for $\ell_0 = 0$, q even, and $n_0 \geq q^m + 1 \geq p_0$, consider $\beta_i \in \mathbb{F}_{q^m} \cup \{*\}$ so that $\cup_i \beta_i = \mathbb{F}_{q^m} \cup \{*\}$, and let

$$\mathbf{A} = \begin{bmatrix} \mathbf{0}_{r_0+m} \\ \mathbf{H}_m \\ \mathbf{0}_m \end{bmatrix}.$$

Theorem 6 *Construction A6 gives an* $[n, n-r, 3]_q 3, \ell$ *code with* $n = q^m n_0 + (q^m - 1)/(q-1)$, $r = r_0 + 3m$, *and with* $\ell = 1$ *if* $m = 1$, $\ell = 3$ *if* $m \geq 2$. *The code has a* $(3, 1)$*-partition with* $p_0 + 3$ *subsets and a* $(3, i)$*-partition with* $i(p_0 + 1)$ *subsets for* $i \in \{2, 3\}$, $m \geq 2$.

Proof We first consider the case $a \neq 0$. The proof that the covering radius is 3 coincides with that of Theorem 2 for all solutions of (3) with $k = 2, 3$.

For $k = 1$, $\beta_{j_1} = *$, we can use the same arguments as in the proof of Theorem 5 for the same values of these parameters. The case $k = 1$, $\beta_{j_1} \in \mathbb{F}_{q^m}$, splits into four subcases.

First, if $c_2 \neq c_1\beta_{j_1}$ and $c_3 \neq c_1\beta_{j_1}^2$, we put

$$\beta_y = \frac{c_3 + c_2\beta_{j_1}}{c_2 + c_1\beta_{j_1}}, \quad \xi_{x_1} = \frac{c_1\beta_y + c_2}{(\beta_{j_1} + \beta_y)s_1}, \quad \xi_z = \frac{c_1\beta_{j_1} + c_2}{\beta_{j_1} + \beta_y}.$$

As $c_3 \neq c_1\beta_{j_1}^2$, we have $\beta_y \neq \beta_{j_1}$. As $c_2 \neq c_1\beta_{j_1}$ it holds that $\xi_z \neq 0$ and therefore $\mathbf{b}_{y,z} \neq \mathbf{b}_{y,1}$. The desired linear combination is $s_1\mathbf{b}_{j_1,x_1} + \mathbf{b}_{y,z} + \mathbf{b}_{y,1}$.

Throughout this paper, detailed evaluation of linear combinations have been omitted. However, as the expressions involved are more complex in this particular case, and also to give an explicit example of calculations, we here give all details:

$$\begin{aligned} s_1\mathbf{b}_{j_1,x_1} + \mathbf{b}_{y,z} + \mathbf{b}_{y,1} &= s_1(\mathbf{h}_{j_1}, \xi_{x_1}, \beta_{j_1}\xi_{x_1}, \beta_{j_1}^2\xi_{x_1}) + (\mathbf{h}_y, \xi_z, \beta_y\xi_z, \beta_y^2\xi_z) + (\mathbf{h}_y, 0, 0, 0) \\ &= (s_1\mathbf{h}_{j_1},\ s_1\xi_{x_1} + \xi_z,\ s_1\beta_{j_1}\xi_{x_1} + \beta_y\xi_z,\ s_1\beta_{j_1}^2\xi_{x_1} + \beta_y^2\xi_z) \end{aligned}$$

with

$$\begin{aligned}
s_1\mathbf{h}_{j_1} &= a,\\
s_1\xi_{x_1}+\xi_z &= \frac{c_1\beta_y+c_2}{\beta_{j_1}+\beta_y}+\frac{c_1\beta_{j_1}+c_2}{\beta_{j_1}+\beta_y}=\frac{c_1\beta_y+c_2+c_1\beta_{j_1}+c_2}{\beta_{j_1}+\beta_y}=c_1,\\
s_1\beta_{j_1}\xi_{x_1}+\beta_y\xi_z &= \beta_{j_1}\frac{c_1\beta_y+c_2}{\beta_{j_1}+\beta_y}+\beta_y\frac{c_1\beta_{j_1}+c_2}{\beta_{j_1}+\beta_y}\\
&= \frac{\beta_{j_1}c_1\beta_y+\beta_{j_1}c_2+\beta_y c_1\beta_{j_1}+\beta_y c_2}{\beta_{j_1}+\beta_y}=c_2,\\
s_1\beta_{j_1}^2\xi_{x_1}+\beta_y^2\xi_z &= \beta_{j_1}^2\frac{c_1\beta_y+c_2}{\beta_{j_1}+\beta_y}+\beta_y^2\frac{c_1\beta_{j_1}+c_2}{\beta_{j_1}+\beta_y}\\
&= \frac{\beta_{j_1}^2c_1\beta_y+\beta_{j_1}^2c_2+\beta_y^2c_1\beta_{j_1}+\beta_y^2c_2}{\beta_{j_1}+\beta_y}\\
&= \frac{\beta_{j_1}c_1\beta_y(\beta_{j_1}+\beta_y)+c_2(\beta_{j_1}^2+\beta_y^2)}{\beta_{j_1}+\beta_y}\\
&= \beta_{j_1}c_1\beta_y+c_2(\beta_{j_1}+\beta_y)=\beta_y(c_2+\beta_{j_1}c_1)+c_2\beta_{j_1}=c_3,
\end{aligned}$$

where we in several places make use of the fact that q is even.

Second, if $c_2=c_1\beta_{j_1}$ and $c_3\neq c_1\beta_{j_1}^2$, we put $\beta_y=*$, $\xi_{x_1}=c_1/s_1$, $\xi_z=c_3+c_1\beta_{j_1}^2$. The linear combination is the same as in the previous case.

Third, if $c_3=c_1\beta_{j_1}^2$, we put $\xi_{x_1}=c_1/s_1$ and obtain $s_1\mathbf{b}_{j_1,x_1}=(a,c_1,c_2,c_3)$ if $c_2=c_1\beta_{j_1}$. Fourth, if $c_2\neq c_1\beta_{j_1}$ we further use an appropriate column of $\mathbf{A}$ in the linear combination.

Next consider the case $a=0$. If $c_1\neq 0$, let β_y be the solution of $\beta_y^2=c_3/c_1$ (which always has a solution in $\mathbb{F}_{q^m}$ when q is even). Moreover, let $\xi_z=c_1$. If $c_1=0$ and $c_3\neq 0$, we put $\beta_y=*$, $\xi_z=c_3$. In both these cases, the linear combination is obtained as $(0,c_1,0,c_3)=\mathbf{b}_{y,z}+\mathbf{b}_{y,1}$ if $c_2=0$, and further taking a column of $\mathbf{A}$ if $c_2\neq 0$.

A (3,1)-partition with the desired number of subsets is obtained by partitioning $\left[\mathbf{B}(\mathbf{h}_1,\beta_1)\ \mathbf{B}(\mathbf{h}_2,\beta_2)\ \cdots\ \mathbf{B}(\mathbf{h}_{n_0},\beta_{n_0})\right]$ as $\mathcal{P}_1$, by further refining $\mathbf{B}(\mathbf{h}_1,\beta_1)$ into three subsets as in the proof of Theorem 1, and finally taking the subset formed by the columns of $\mathbf{A}$.

For $m\geq 2$, a $(3,2)$-partition and a $(3,3)$-partition are obtained via $\mathcal{P}_2$ and $\mathcal{P}_2'$, and $\mathcal{P}_3$ and $\mathcal{P}_3'$, respectively ($\mathcal{P}_i'$ applied to $\mathbf{A}$). □

3 Infinite families of codes

In this section, code families obtained by applying the different versions of Construction A are presented. This discussion is divided into three parts, one for each possible value of the codimension modulo 3. In addition to Construction A, the well-known direct sum construction is useful.

3.1 Direct sum construction

The *direct sum* of an $[n_1,n_1-r_1]_qR_1$ code and an $[n_2,n_2-r_2]_qR_2$ gives an $[n_1+n_2,n_1+n_2-(r_1+r_2)]_qR_1+R_2$ code [5, Theorem 3.2.1].

Example 2 The direct sum of the $[40,33]_32$ code of [4, Theorem 2] and the $[13,10]_31$ Hamming code gives a $[53,23]_33$ code. A $[40,32]_43$ code can be obtained as the direct sum of the $[19,14]_42$ code of [13, Theorem 1] and the $[21,18]_41$ Hamming code. The direct sum of the $[37, 21]_4 2$ code from [7, Table I] and the $[21, 18]_4 1$ Hamming code gives a $[58, 39]_4 3$ code. The direct sum of the $[28, 23]_5 2$ code of [12, Table 1] and the $[31, 28]_5 1$ Hamming code forms a $[59, 51]_5 3$ code. The direct sum of the $[44, 39]_7 2$ code of [7, Table I] and the $[57, 54]_7 1$ Hamming code forms a $[101, 93]_7 3$ code. The direct sum of the $[15141, 15130]_7 2$ code from [7, Table I] and the $[2801, 2796]_7 1$ Hamming code forms a $[17942, 17926]_7 3$ code.

Example 3 For $q = 5, 9$, the direct sum of the $[n_1 = 2q^2 + q + 1, n_1 - 6]_q 2$ code of [7, Example 5] and the $[n_2 = (q^3 - 1)/(q - 1), n_2 - 3]_q 1$ Hamming code forms an $[n = 3q^2 + 2q + 2, n - 9]_q 3$ code. For $q = 7, 8, 11, 13, 17, 19$, the direct sum of the $[n_1 = 2q^2 + q, n_1 - 6]_q 2$ code of [15, Theorem 9] and the $[n_2 = (q^3 - 1)/(q - 1), n_2 - 3]_q 1$ Hamming code gives an $[n = 3q^2 + 2q + 1, n - 9]_q 3$ code.

Theorem 7 *There is an infinite family of* $[n, n - (3t + 1)]_5 3$ *codes with*

$$n = \begin{cases} (5^{t+1} - 1)/4 + \ell_5(t, 2) & \text{if } t \equiv 1, 3 \pmod 4, \\ (5^{t+1} + 5^{t/2} - 2)/4 + \ell_5(t/2, 2) \cdot 5^{t/2} & \text{if } t \equiv 2 \pmod 4, \\ (5^{t+1} + 2 \cdot 5^{(t+2)/2} - 3)/4 & \\ \quad + \ell_5((t-2)/2, 2) \cdot 5^{(t+2)/2} & \text{if } t \equiv 0 \pmod 4 \end{cases} \tag{11}$$

for $t = 3$ *and* $t \geq 5$.

Proof By [12, Theorems 5,10], for $t = 3$ and $t \geq 5$, there is an infinite family of $[n_1, n_1 - (2t + 1)]_5 2$ codes with

$$n_1 = \begin{cases} 5^t + \ell_5(t, 2) & \text{if } t \equiv 1, 3 \pmod 4, \\ 5^t + (5^{t/2} - 1)/4 + \ell_5(t/2, 2) \cdot 5^{t/2} & \text{if } t \equiv 2 \pmod 4, \\ 5^t + (5^{(t+2)/2} - 1)/2 & \\ \quad + \ell_5((t-2)/2, 2) \cdot 5^{(t+2)/2} & \text{if } t \equiv 0 \pmod 4. \end{cases}$$

Take the direct sum of these codes and the $[n_2 = (5^t - 1)/4, n_2 - t]_5 1$ Hamming code. □

The asymptotic density of the code family (11) is approximately 4.167; the values of $\ell_5(r, 2)$ are not known in general, but this contribution to the length turns out to be insignificant when calculating the asymptotic behavior. Note that upper bounds on $\ell_5(r, 2)$ can be taken from [7,12]; in particular, bounds for $r \leq 24$ are summarized in [12, Table 1].

Theorem 8 *There is an infinite family of* $[n, n - (3t + 1)]_3 3$ *codes with*

$$n = \begin{cases} (7 \cdot 3^t - 3)/4 & \text{if } t \equiv 0, 2 \pmod 4, \\ (7 \cdot 3^t + 3^{(t+3)/2} - 6)/4 & \text{if } t \equiv 1 \pmod 4, \\ (7 \cdot 3^t + 3^{(t+5)/2} - 6)/4 & \text{if } t \equiv 3 \pmod 4 \end{cases} \tag{12}$$

for $t = 2$ *and* $t \geq 4$.

Proof Take the direct sum of the $[n_1, n_1 - (2t + 1)]_3 2$ codes, coming from [4, Theorem 1] and [12, Theorem 9], and the $[n_2 = (3^t - 1)/2, n_2 - t]_3 1$ Hamming code. □

The asymptotic density of the code family (12) is approximately 2.382.

Theorem 9 *There is an infinite family of* $[n, n - (3t + 2)]_3 3$ *codes with*

$$n = \begin{cases} (11 \cdot 3^t - 3)/4 & \text{if } t \equiv 0, 2 \pmod 4, \\ (11 \cdot 3^t + 3^{(t+3)/2} - 6)/4 & \text{if } t \equiv 1 \pmod 4, \\ (11 \cdot 3^t + 3^{(t+5)/2} - 6)/4 & \text{if } t \equiv 3 \pmod 4 \end{cases} \tag{13}$$

for $t = 2$ *and* $t \geq 4$.

Proof Take the direct sum of the $[n_1, n_1 - (2t + 1)]_3 2$ codes of [4, Theorem 1], [12, Theorem 9] and the $[n_2 = (3^{t+1} - 1)/2, n_2 - (t + 1)]_3 1$ Hamming code. □

The asymptotic density of the code family (13) is approximately 3.082.

3.2 Codes with codimension $r = 3t$

In [15, Theorem 12] an infinite family of codes with the following parameters is obtained:

$$[n = 3q^{t-1} + q^{t-2}, n - 3t]_q 3,\ t \geq 2,\ q \geq 5,$$

where $t \neq 3$ for $q \in \{5, 7, 8, 9, 11, 13, 17, 19\}$. The asymptotic density of this family of codes is bounded from above by

$$\frac{9}{2} - \frac{9}{q} + \frac{3}{2q^2} + \frac{14}{3q^3} - \frac{1}{2q^4}.$$

Theorem 10 *There is an infinite family of*

$$[n = 958 \cdot 3^{t-6}, n - 3t]_3 3 \tag{14}$$

codes for $t = 6$ *and for* $t \geq 10$. *Moreover, there exist* $[35, 26]_3 3$, $[2875, 2854]_3 3$, $[8626, 8602]_3 3$*, and* $[25879, 25852]_3 3$ *codes.*

Proof The direct sum of the $[22, 16]_3 2$ code from [12, Table 3] and the $[13, 10]_3 1$ Hamming code gives a $[35, 26]_3 3$ code C_0. We checked by computer that C_0 is a $[35, 26]_3 3, 2$ code that has a (3,2)-partition with 25 subsets. As $25 < 3^3$, we can use C_0 as a starting code in Construction A3 for any $m \geq 3$. By Theorem 3, $[n = 35 \cdot 3^m + (3^m - 1)/2, n - (3m + 9)]_3 3, 3$ codes C with a (3,3)-partition with $2 \cdot 25 + 1 = 51$ subsets are then obtained. In particular, for $3 \leq m \leq 6$, $[958, 940]_3 3$, $[2875, 2854]_3 3$, $[8626, 8602]_3 3$, and $[25879, 25852]_3 3$ codes are obtained.

Since $51 < 3^4$, the $[958, 940]_3 3, 3$ code obtained can be used as the starting code for Construction A4 for any $m \geq 4$; the parameters of (14) then follow by Theorem 4. □

The asymptotic density of the code family (14) is approximately 3.026. In the next theorem, families of quaternary codes are built up using the same approach as in the proof of Theorem 10.

Theorem 11 *There is an infinite family of*

$$[n = 213 \cdot 4^{t-4}, n - 3t]_4 3 \tag{15}$$

codes for $t = 4$ *and* $t \geq 7$. *Moreover, there are* $[853, 838]_4 3$*, and* $[3413, 3395]_4 3$ *codes.*

Proof The construction of [14, Theorem 7] gives a $[13, 7]_4 3$ code C_0. We checked by computer that C_0 is a $[13, 7]_4 3, 2$ code, which has a (3,2)-partition with 11 subsets. As $11 < 4^2$, we can use C_0 as the starting code in Construction A3 for any $m \geq 2$. By Theorem 3, $[n = 13 \cdot 4^m + (4^m - 1)/3, n - (3m + 6)]_4 3, 3$ codes C with a (3,3)-partition with $2 \cdot 11 + 1 = 23$ subsets are then obtained. In particular, for $2 \leq m \leq 4$, $[213, 201]_4 3$, $[853, 838]_4 3$, and $[3413, 3395]_4 3$ codes are obtained.

Since $23 < 4^3$, the $[213, 201]_4 3, 3$ code obtained can be used as the starting code for Construction A4 for any $m \geq 3$; the parameters of (15) then follow by Theorem 4. □

The asymptotic density of the code family (15) is approximately 2.592.

3.3 Codes with codimension $r = 3t + 1$

Theorem 12 *There is an infinite family of*

$$[n = 341 \cdot 4^{t-4}, n - (3t + 1)]_4 3 \tag{16}$$

codes for $t = 4$ *and* $t \geq 7$. *Moreover, there are* $[21, 14]_4 3$, $[94, 84]_4 3$, $[1386, 1370]_4 3$, *and* $[5546, 5527]_4 3$ *codes.*

Proof By using the $[5, 1]_4 3, 0$ Reed-Solomon code, a $[9, 5]_4 2$ code D from [3, p. 104], and $m = 2$ in Construction A1, one gets a $[94, 84]_4 3$ code (Theorem 1).

By Theorem 6, the $[5, 1]_4 3, 0$ Reed-Solomon code—with a trivial (3,0)-partition into $p_0 = 5$ subsets—used in Construction A4 with $m = 1$ leads to a $[21, 14]_4 3, 1$ code C with a (3,1)-partition into $5 + 3 = 8$ subsets, which can now—as $8 < 4^2$—be used as a starting code in Construction A2 with $m = 3, 4$ to form $[1386, 1370]_4 3$ and $[5546, 5527]_4 3$ codes (Theorem 2).

The code C used in Construction A6 with $m = 2$ gives a $[341, 328]_4 3, 3$ code that has a (3,3)-partition with $3(8+1) = 27$ subsets (Theorem 6), which can be used as a starting code for Construction A4 with $m \geq 3$. The assertion now follows from Theorem 4. □

The asymptotic density of the code family (16) is approximately 2.659.

Theorem 13 *There are* $[32, 25]_5 3$ *and* $[812, 799]_5 3$ *codes.*

Proof By using the $[6, 2]_5 3, 0$ Reed-Solomon code and its trivial (3,0)-partition—that is, $p_0 = 6$—in Construction A5 with $m = 1$, a $[32, 25]_5 3$ code C is obtained that has a (3,0)-partition with $6 + 4 = 10$ subsets (Theorem 5). By further taking C as the starting code for Construction A5 with $m = 2$, a $[812, 799]_5 3$ code is obtained. □

For codimension $r = 3t + 1$ and alphabets of size $q \geq 7$, $[n_0, n_0 - 4]_q 3$ codes can be used as seeds. The best known upper bounds on $\ell_q(4, 3)$ are therefore collected in Table 1. Entries that give the exact value are indicated by a dot. An integer in the column d tells that there exists a code attaining the best known upper bound that has that particular minimum distance. The information in the table has been collected from [8, Tables III–V], [10, Theorem 3.5, Corollary 3.8], [11, Tables II, III], and [14, Table 1]. Also, a few new computer results obtained in this work are included.

The main importance of the minimum distances in Table 1 is the information whether codes with minimum distance $d \leq R$ exist, as this tells something about (R, ℓ)-partitions.

Lemma 4 *An* $[n, n - r, d]_q R$ *code is an* $[n, n - r, d]_q R, 1$ *code iff* $d \leq R$.

Proof The all-zero column of $\mathbb{F}_q^r$ can be obtained as the linear combination of 0 columns or (at least) d columns of the parity check matrix of the code. If $d > R$, linear combinations of the latter type are not possible. □

Any good covering code with $R = 3$ has $d \geq 3$, so the interesting question is whether codes with $d = R = 3$ exist. Note that for the distances $d = 4$ and $d = 5$, we have an incomplete cap and a complete arc, respectively, in the projective space $PG(3, q)$. The bound in the following lemma is not best possible, but is still sufficient for our needs.

Lemma 5 $\ell_q(4, 3) \leq q - 1$ *for* $q \geq 8$.

Table 1 Upper bounds on $\ell_q(4,3)$ for $q \le 563$

q	Bound	d	q	Bound	d	q	Bound	d	q	Bound	d
2	5.	3,4	81	17	4	227	27	3,5	379	33	3,5
3	5.	4,5	83	17	4	229	27	3,5	383	33	3,5
4	5.	5	89	18	3,5	233	27	3,5	389	33	4
5	6.	3,4,5	97	19	3,5	239	27	3,5	397	34	3,5
7	7.	3,4	101	19	3,5	241	28	3,5	401	34	3,5
8	7.	3,4,5	103	19	3,5	243	28	3,5	409	34	3,5
9	7.	4	107	19	4	251	28	3,5	419	34	3
11	8.	3,4,5	109	20	3,5	256	28	3,5	421	34	3
13	8	4,5	113	20	3,5	257	28	3,5	431	35	3,5
16	9	3,4,5	121	20	4	263	28	3,5	433	35	3
17	9	3,4,5	125	21	3,5	269	29	3,5	439	35	3,5
19	9	4,5	127	21	3,5	271	29	3,5	443	35	3,5
23	10	3,4,5	128	21	3,5	277	29	3,5	449	35	3,5
25	11	3,4,5	131	21	3,5	281	29	3,5	457	35	4
27	11	3,4,5	137	22	3,5	283	29	3,5	461	36	3,5
29	11	3,4,5	139	22	3,5	289	29	4	463	36	3
31	11	4	149	22	3,5	293	29	4	467	36	3
32	12	3,4,5	151	22	4	307	30	3,5	479	36	3
37	12	4,5	157	23	3,5	311	30	4	487	36	3,5
41	13	3,4,5	163	23	5	313	30	4	491	36	4
43	13	4,5	167	24	3,5	317	30	3,4	499	37	3,5
47	14	3,4,5	169	24	3,5	331	31	3,5	503	37	3,5
49	14	3,4,5	173	24	3,5	337	31	3	509	37	3,5
53	15	3,4,5	179	24	3,5	343	31	4	512	36	3
59	15	3,4,5	181	24	4	347	32	3,5	521	37	4
61	15	4	191	25	3,5	349	32	3,5	523	38	3
64	16	3,4,5	193	25	3,5	353	32	3,5	529	38	5
67	16	3,4,5	197	25	3,5	359	32	3,5	541	38	5
71	16	4,5	199	25	5	361	32	3	547	38	4
73	16	4	211	26	3,5	367	32	4	557	39	5
79	17	3,5	223	27	3,5	373	33	3,5	563	39	5

Proof A complete n-arc in the projective plane $\mathrm{PG}(2,q)$ corresponds to an $[n, n-3]_q2$ code [18, Sect. 1.3]. There are complete n-arcs with $n = (q+4)/2$ for even $q \ge 8$ and $n = (q+3)/2$ for odd $q \ge 9$ [16, Eqs. 12, 13]. So, $\ell_q(3,2) \le q/2 + 2$ if $q \ge 8$. The direct sum construction then gives that $\ell_q(4,3) \le \ell_q(3,2) + 1 \le q/2 + 3$, which is smaller than q for $q \ge 8$. □

Irrespective of the minimum distance of the starting code we get the following result.

Theorem 14 *For $q \ge 8$ there is an infinite family of*

$$[n = q^{t-1}\ell_q(4,3) + 3\lfloor q^{t-2}\rfloor + 2\lfloor q^{t-3}\rfloor, n - (3t+1)]_q3$$

codes for $t = 1, 2, 3$ *and* $t \geq 5$. *Moreover, there are* $[n = q^3 \ell_q(4,3) + (q^3 - 1)/(q-1) + \ell_q(6,2), n - 13]_q 3$ *codes.*

Proof Take an $[n_0 = \ell_q(4,3), n_0 - 4]_q 3$ code and its trivial (3, 0)-partition as a starting code for Construction A1 with $m \geq 1$ (which is possible due to Lemma 5) to get, by Theorem 1, $[n = q^m \ell_q(4,3) + (q^m - 1)/(q-1) + \ell_q(2m, 2), n - (3m+4)]_q 3$ codes. In particular, for $m = 1$ we obtain—since $\ell_q(2,2) = 2$—an $[n = q\ell_q(4,3) + 3, n - 7]_q 3$ code, and for $m = 2$—using an $[2q+1, 2q-3]_q 2$ code [15, Theorem 5]—we get an $[n = q^2 \ell_q(4,3) + 3q + 2, n - 10]_q 3, 3$ code C, which has a (3,3)-partition with $3\ell_q(4,3) + 2q + 4$ subsets; see Example 1.

As $3\ell_q(4,3) + 2q + 4 \leq 5q + 1 < q^2$ for $q \geq 8$ ($\ell_q(4,3) \leq q - 1$ by Lemma 5), we can take C as the starting code for Construction A4 with $m \geq 2$; the lengths n of the infinite code family obtained follow from Theorem 4. □

Whenever a starting code with minimum distance 3 can be used, we get a stronger result.

Theorem 15 *For* $q \geq 7$, *if there is an* $[n_0, n_0 - 4, 3]_q 3$ *code with* $n_0 \leq q$, *then there is an infinite family of*

$$[n = q^{t-1} n_0 + 2\lfloor q^{t-2} \rfloor, n - (3t+1)]_q 3$$

codes for $t = 1, 2$ *and* $t \geq 4$. *Moreover there are —irrespective whether the inequality* $n_0 \leq q$ *is satisfied or not—*$[n = q^2 n_0 + 2q + 2, n - 10]_q 3$ *codes.*

Proof By Lemma 4, an $[n_0, n_0 - 4, 3]_q 3$ code is an $[n_0, n_0 - 4]_q 3, 1$ code and can be used as a starting code for Construction A2 with $m \geq 1$ (under the assumption that $n_0 \leq q$) to get, by Theorem 2, $[n = q^m n_0 + 2(q^m - 1)/(q-1), n - (3m+4)]_q 3, 3$ codes that have a (3,3)-partition with $3n_0 + 2$ subsets. In particular, for $m = 1$ we get an $[n = qn_0 + 2, n - 7]_q 3, 3$ code C and for $m = 2$ we get an $[n = q^2 n_0 + 2q + 2, n - 10]_q 3, 3$ code. Finally, as $n_0 \leq q$, we have $3n_0 + 2 \leq 3q + 2 < q^2$ whenever $q \geq 7$, so the code C can be used as a starting code for Construction A4 with $m \geq 2$; the lengths n of the infinite code family obtained follow from Theorem 4. □

Note that all codes from Table 1 with $q \leq 7$ have cardinality at most q, which is required in Theorem 15.

The asymptotic density of the families obtained by Theorems 14 and 15 can be calculated for any desired prime power $q \geq 7$. To get a rough impression of these densities, calculations reveal that they are bounded from above by 10.7 for $q \leq 83$, by 15.2 for $q \leq 343$, and by 20.9 for $q \leq 563$.

3.4 Codes with codimension $r = 3t + 2$

Theorem 16 *There is an infinite family of*

$$[n = 154 \cdot 4^{t-3}, n - (3t+2)]_4 3 \tag{17}$$

codes for $t = 3$ *and* $t \geq 6$. *Moreover, there are* $[618, 604]_4 3$ *and* $[2474, 2457]_4 3$ *codes.*

Proof The $[9, 4, 3]_4 3$ code from [8, Table IV] is a $[9, 4]_4 3, 1$ code by Lemma 4. By using that code in Construction A2 with $m \geq 2$, one gets an $[n = 9 \cdot 4^m + 2(4^m - 1)/3, n - (3m+5)]_4 3, 3$ code C that has a (3, 3)-partition with $3 \cdot 9 + 2 = 29$ subsets (Theorem 2). In particular, for $2 \leq m \leq 4$, we obtain $[154, 143]_4 3$, $[618, 604]_4 3$, and $[2474, 2457]_4 3$ codes.

As the $[154, 143]_4 3, 3$ code has a (3,3)-partition with $29 < 4^3$ subsets, we can use it as the starting code in Construction A4 for $m \geq 3$ (Theorem 4) to get the code family of (17). □

The asymptotic density of the code family (17) is approximately 3.919.

Theorem 17 *There is an infinite family of*

$$[n = 267 \cdot 5^{t-3}, n - (3t + 2)]_5 3 \tag{18}$$

codes for $t = 3$ *and* $t \geq 6$. *Moreover, there are* $[1337, 1323]_5 3$ *and* $[6687, 6670]_5 3$ *codes.*

Proof By using the $[10, 5]_5 3$ code from [14, Table 1] in Construction A1 with $m \geq 2$ and taking D as a code attaining $\ell_5(2m, 2)$, one gets an $[n = 10 \cdot 5^m + (5^m - 1)/4 + \ell_5(2m, 2), n - (3m + 5)]_5 3$ code (Theorem 1). However, since $\ell_5(r, 2)$ is not known in general, we take as D codes attaining the best known upper bound on $\ell_5(2m, 2)$. In particular, for $m = 2, 3, 4$, we let D be $[11, 7]_5 2$ [12], $[56, 50]_5 2$ [7], and $[281, 273]_5 2$ [12] codes, respectively.

In this manner, we get $[267, 256]_5 3$, $[1337, 1323]_5 3$ and $[6687, 6670]_5 3$ codes. By Theorem 1 and Example 1—with $\mathbf{D}_4$ from [15, Eq. 1]—there is a $[267, 256]_5 3, 3$ code C that has a (3,3)-partition with $3 \cdot 10 + 10 + 4 = 44$ subsets. As $44 < 5^3$, we can take C as the starting code for Construction A4 for $m \leq 3$, and the infinite family (18) follows by Theorem 4. □

The asymptotic density of the code family (18) is approximately 4.159.

For codimension $r = 3t + 2$ and alphabet sizes $q \geq 7$, an approach similar to that in Sect. 3.3 can be taken—now with $[n_0, n_0 - 5]_q 3$ codes as seeds. Therefore, the best known upper bounds on $\ell_q(5, 3)$ are collected in Table 2, with exact entries indicated by a dot. This data is gathered from [8, Tables III, IV], [9, Table 1], and [14, Table 1]; for $37 \leq q \leq 43$ we take the direct sum of codes with codimension 3 and covering radius 2 from [8, Table I] and the $[q + 1, q - 1]_q 1$ Hamming code.

Indeed, the question whether there are good codes with minimum distance $d = 3$ is here also of central importance, so a minimum distance column d is included, like for Table 1. Distances $d = 4$ and $d = 6$ correspond to incomplete caps and arcs, respectively, in PG$(4, q)$.

A rough upper bound on $\ell_q(5, 3)$ will later be needed and is provided by the following lemma.

Lemma 6 $\ell_q(5, 3) \leq 2q - 1$ *for* $q \geq 7$.

Proof By the direct sum construction with a code attaining $\ell_q(3, 2)$ and a $[q + 1, q - 1]_q 1$ Hamming code, we get (see the proof of Lemma 5) $\ell_q(5, 3) \leq \ell_q(3, 2) + (q + 1) \leq (q/2 + 2) + (q + 1) \leq 2q - 1$ for $q \geq 8$. The case $q = 7$ follows from Table 2. □

Theorems 18 and 19 are analogous to Theorems 14 and 15. First, we present the more general result.

Table 2 Upper bounds on $\ell_q(5, 3)$ for $q \leq 43$

q	Bound	d	q	Bound	d	q	Bound	d
2	6.	5,6	11	18	3,4	27	36	3,4
3	8.	3,4	13	21	3,4	29	38	3,4
4	9.	3,4	16	24	3,4	31	40	3,4
5	10.	4	17	25	3,4	32	41	3,4
7	13	3,4	19	27	4	37	54	3
8	14	3,4	23	32	3,4	41	58	3
9	16	3,4	25	34	3,4	43	60	3

Theorem 18 *For $q \geq 8$ there is an infinite family of*

$$[n = q^{t-1}\ell_q(5,3) + 3\lfloor q^{t-2}\rfloor + 2\lfloor q^{t-3}\rfloor, n - (3t+2)]_q 3$$

codes for $t = 1, 3$ and $t \geq 5$. Moreover, there are $[n = q^3\ell_q(5,3) + (q^3-1)/(q-1) + \ell_q(6,2), n-14]_q 3$ codes.

Proof Take an $[n_0 = \ell_q(5,3), n_0 - 5]_q 3$ code and its trivial $(3,0)$-partition as a starting code for Construction A1 with $m \geq 2$ (by Lemma 6, $n_0 \leq 2q - 1 < q^2$ as we have $q \geq 8$) to get, by Theorem 1, an $[n = q^m\ell_q(5,3) + (q^m - 1)/(q-1) + \ell_q(2m,2), n - (3m+5)]_q 3$ code. Moreover, by taking the $[2q+1, 2q-3]_q 2$ code from [15, Eq. 1] as D in the construction, we get by Theorem 1 and Example 1 that the resulting $[n = q^2\ell_q(5,3)+3q+2, n-11]_q 3$, 3 code C has a (3,3)-partition with $3\ell_q(5,3) + 2q + 4$ subsets.

As $3\ell_q(5,3) + 2q + 4 \leq 8q + 1 < q^2$ (utilizing Lemma 6) for $q \geq 9$ and $3\ell_8(4,3) + 2 \cdot 8 + 4 = 3 \cdot 14 + 2 \cdot 8 + 4 = 62 < 8^2$ (see Table 2), we can take C as the starting code for Construction A4 with $m \geq 2$ whenever $q \geq 8$; the lengths n of the infinite code family obtained follow from Theorem 4. □

By using a starting code that has minimum distance 3, we get the following, stronger result.

Table 3 Upper bounds on $\ell_q(r,3)$ for $q = 3, 4, r \leq 24$

r	$q=3$	Reference	$q=4$	Reference
4	5**.**	[1,6]	5**.**	[14]
5	8**.**	[1,6]	9**.**	[8,14]
6	11**.**	[6,19]	13	[14, Theorem 7]
7	14	[1]	21	Theorem 12
8	24	Theorem 9	40	Example 2
9	35	Theorem 10	58	Example 2
10	53	Theorem 8	94	Theorem 12
11	76	[6]	154	Theorem 16
12	107	[6]	213	Theorem 11
13	141	Theorem 8	341	Theorem 12
14	222	Theorem 9	618	Theorem 16
15	323	[6]	853	Theorem 11
16	431	[6]	1386	Theorem 12
17	674	[6]	2474	Theorem 16
18	958	Theorem 10	3413	Theorem 11
19	1275	Theorem 8	5546	Theorem 12
20	2004	Theorem 9	9856	Theorem 16
21	2875	Theorem 10	13632	Theorem 11
22	3887	[6]	21824	Theorem 12
23	6074	[6]	39424	Theorem 16
24	8626	Theorem 10	54528	Theorem 11

Theorem 19 *For $q \geq 7$, if there is an $[n_0, n_0 - 5, 3]_q 3$ code with $3n_0 + 1 \leq q^2$, then there is an infinite family of*

$$[n = q^{t-1} n_0 + 2\lfloor q^{t-2} \rfloor + 2\lfloor q^{t-3} \rfloor, n - (3t+2)]_q 3$$

codes for $t = 1, 3$ and $t \geq 5$. Moreover there are—irrespective whether the inequality $3n_0 + 1 \leq q^2$ is satisfied or not—$[n = q^3 n_0 + 2q^2 + 2q + 2, n - 14]_q 3$ codes.

Proof By Lemma 4, an $[n_0, n_0 - 5, 3]_q 3$ code is an $[n_0, n_0 - 5]_q 3, 1$ code and can be used as a starting code for Construction A2 with $m \geq 2$ ($3n_0 + 1 \leq q^2$ implies that $n_0 \leq q^2$) to get, by Theorem 2, $[n = q^m n_0 + 2(q^m - 1)/(q-1), n - (3m+5)]_q 3, 3$ codes that have a (3,3)-partition with $3n_0 + 2$ subsets. In particular, for $m = 2$ we get an $[n = q^2 n_0 + 2q + 2, n - 11]_q 3, 3$ code C and for $m = 3$ we get an $[n = q^3 n_0 + 2q^2 + 2q + 2, n - 14]_q 3, 3$ code. Under the assumptions that $3n_0 + 1 \leq q^2$ and $q \geq 7$, we get that $3n_0 + 2 \leq q^2 + 1$, and the code C can be used as a starting code for Construction A4 with $m \geq 2$; the lengths n of the infinite code family obtained follow from Theorem 4. □

The asymptotic density of the families obtained by Theorems 18 and 19 can be calculated for any desired prime power $q \geq 7$; calculations reveal that they are bounded from above by 10.7 for $q \leq 27$, by 12.4 for $q \leq 32$, and by 20.9 for $q \leq 43$.

Table 4 Upper bounds on $\ell_q(r, 3)$ for $q = 5, 7, r \leq 24$

r	$q = 5$	Reference	$q = 7$	Reference
4	6.	[14]	7.	[14]
5	10.	[8,14]	13	[14]
6	16	[14, Theorem 7]	22	[14, Theorem 7]
7	32	Theorem 13	51	Theorem 15
8	59	Example 2	101	Example 2
9	87	Example 3	162	Example 3
10	162	Theorem 7	359	Theorem 15
11	267	Theorem 17	653	Theorem 19
12	400	[15, Theorem 12]	1078	[15, Theorem 12]
13	812	Theorem 13	2499	Theorem 15
14	1337	Theorem 17	4573	Theorem 19
15	2000	[15, Theorem 12]	7546	[15, Theorem 12]
16	3934	Theorem 7	17942	Example 2
17	6687	Theorem 17	31997	Theorem 19
18	10000	[15, Theorem 12]	52822	[15, Theorem 12]
19	20312	Theorem 7	122451	Theorem 15
20	33375	Theorem 17	223979	Theorem 19
21	50000	[15, Theorem 12]	369754	[15, Theorem 12]
22	97787	Theorem 7	857157	Theorem 15
23	166875	Theorem 17	1567853	Theorem 19
24	250000	[15, Theorem 12]	2588278	[15, Theorem 12]

4 Tables of bounds

We summarize the results of this paper by presenting the best known upper bounds on $\ell_q(r, 3)$ for $3 \leq q \leq 7$ and $r \leq 24$ in Tables 3 and 4. The results in the table follow from the present work and earlier results published in [1,6,8,11,14,19]. Proper references are given for all entries. Once again we indicate the exact values by a dot. The codes providing these exact values are classified in [8,11], except for $[10, 5]_5 3$ codes. Note that in [6] an $[11, 5]_3 3$ code is obtained, while in [19] it is proved that the smallest covering radius of a $[10, 4]_3 R$ code is 4. These facts give the equality $\ell_3(6, 3) = 11$.

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