[redacted]

New Inductive Constructions of Complete Caps in $PG(N,q)$, q even

Alexander A. Davydov,[1] Massimo Giulietti,[2] Stefano Marcugini,[2] Fernanda Pambianco[2]

[1] *Institute for Information Transmission Problems, Russian Academy of Sciences, Bol'shoi Karetnyi per. 19, GSP-4, Moscow 127994, Russian Federation, E-mail: adav@iitp.ru*

[2] *Dipartimento di Matematica e Informatica, Università degli Studi di Perugia, Via Vanvitelli 1, Perugia 06123, Italy, E-mail: giuliet@dipmat.unipg.it; gino@dipmat.unipg.it; fernanda@dipmat.unipg.it*

Received January 20, 2009; revised April 17, 2009

Published online in Wiley InterScience (www.interscience.wiley.com).
DOI 10.1002/jcd.20230

Abstract: **Some new families of small complete caps in $PG(N,q)$, q even, are described. By using inductive arguments, the problem of the construction of small complete caps in projective spaces of arbitrary dimensions is reduced to the same problem in the plane. The caps constructed in this article provide an improvement on the currently known upper bounds on the size of the smallest complete cap in $PG(N,q)$, $N \geq 4$, for all $q \geq 2^3$. In particular, substantial improvements are obtained for infinite values of q square, including $q = 2^{2Cm}$, $C \geq 5$, $m \geq 3$; for $q = 2^{Cm}$, $C \geq 5$, $m \geq 9$, with C, m odd; and for all $q \leq 2^{18}$.** © 2009 Wiley Periodicals, Inc. J Combin Designs

Keywords: *projective space; complete cap; complete arc*

1. INTRODUCTION

A cap in $PG(N,q)$, the projective N-dimensional space over the finite field with q elements $\mathbb{F}_q$, is a set of points no three of which are collinear. A cap of size k is denoted as a k-cap. When $N=2$, a cap is also called an arc in $PG(2,q)$.

Contract grant sponsor: Italian Ministry MIUR.

A cap is said to be complete if it is not contained in a larger cap. The most important problem on caps is to determine the spectrum of possible values of k for which there exists a complete k-cap in $PG(N,q)$; for the known results, see [14, 2, 4], and the references therein. The smallest and the largest sizes of a complete cap are of particular interest. This work is mainly devoted to the construction of small complete caps that provide upper bounds on the smallest possible size of a complete cap. New values of k in the spectrum are also obtained.

Interestingly, the problem of determining the possible sizes of complete caps is related to Coding Theory. In fact, complete k-caps in $PG(N,q)$ with $k>N+1$ and linear quasi-perfect $[k,k-N-1,4]_q2$ -codes over $\mathbb{F}_q$ with covering radius 2 are equivalent objects (with the exceptions of the complete 5-cap in $PG(3,2)$ giving rise to a binary $[5,1,5]_22$-code and the complete 11-cap in $PG(4,3)$ corresponding to the Golay $[11,6,5]_32$-code over $\mathbb{F}_3$), see e.g. [8, 9, 11, 13].

Classical examples of complete caps are nonsingular conics for $N=2$ and elliptic quadrics for $N=3$; also, for both $N=2$ and $N=3$, most of the known explicit constructions of complete caps are based on subsets of points of a quadric. For $N\geq 4$ no such natural model for complete caps exists, a consequence of that being the rarity of constructions of complete caps.

In this article we describe new infinite families of complete caps in $PG(N,q)$ for $N\geq 4$ and even q, which arise as a result of some inductive procedures based on complete arcs in $PG(2,q)$. Our main construction is described in Theorems 3.6 and 4.4; see also Theorem 1.1 below. It should be noted that an arbitrary complete plane arc can be taken as the starting point for this construction; then, all known (and future) results on the spectra of sizes of complete plane arcs (see e.g. [1–3, 7, 10–16, 19, 22]) provide results in higher dimension via Theorem 1.1.

Theorem 1.1. *Let $q>8$, q even. Assume that there exists a complete k-arc in $PG(2,q)$ with $k<q-5$. Then there exists a complete n-cap in $PG(N,q)$ with*

$$n=\begin{cases}\dfrac{k}{q}q^{N/2}+3(q^{(N-2)/2}+q^{(N-4)/2}+\cdots+q)-N+3, & N\geq 4 \text{ even}.\\ \left(2+\dfrac{k}{q}\right)q^{(N-1)/2}+3(q^{(N-3)/2}+q^{(N-5)/2}+\cdots+q)-N+4, & N\geq 5 \text{ odd}.\end{cases}$$

Other inductive constructions presented in this article (see Theorems 3.10, 3.14, 4.6) allow to obtain smaller complete caps in $PG(N,q)$ from complete k-arcs having some special properties, which are connected with a new concept of "sum-points" for a k-arc (see Section 2). Significantly, it turns out that these properties are possessed by the smallest known complete arcs in $PG(2,q)$ for any even $q\leq 2^{17}$ (see Table II), by the complete k-arcs of [10] with $k\leq (q+4)/2$ and by the Abatangelo complete $(q+8)/3$-arcs of [1] (see Lemmas 2.12 and 2.13).

As a consequence of our results, substantial improvements on the known bounds on the size $t_2(N,q)$ of the smallest complete cap in $PG(N,q)$ (or, equivalently, the minimal length k for which there exists an $[k,k-N-1,4]_q2$-code) are obtained for $q\geq 8$. In [20], with using results of [21], it was proved that

$$t_2(N,q)\leq\begin{cases}q^{N/2}+s_{N,q}, & N \text{ even},\\ (2+1)q^{(N-1)/2}+s_{N,q}, & N \text{ odd},\end{cases}\tag{1.1}$$

where

$$s_{N,q}=3(q^{\lfloor (N-2)/2\rfloor}+q^{\lfloor (N-2)/2\rfloor -1}+\cdots+q)+2.$$

For about a decade, (1.1) was the best known upper bound on $t_2(N,q)$, $q>2$, $N>3$, with the exceptions of few small values of both N and q, see [14, 2, 4, 19]. Inequality (1.1) was improved in [10]:

$$t_2(N,q)\le \begin{cases} \frac{1}{2}q^{N/2}+s_{N,q}, & N \text{ even},\\ (2+\frac{1}{2})q^{(N-1)/2}+s_{N,q}, & N \text{ odd},\end{cases} \quad q\ge 32. \tag{1.2}$$

Better upper bounds were obtained for specific values of $q\le 2^{15}$, see also [11]. Also, in [11] it was proved that

$$t_2(N,q)\le \tfrac{1}{3}q^{N/2}+\tfrac{5}{3}q^{(N-2)/2}+s_{N,q}-2, \quad q\ge 2^8 \text{ square}, \quad N\ge 4 \text{ even}. \tag{1.3}$$

All the above upper bounds on $t_2(N,q)$ are improved in this article. Theorems 3.15 and 4.7 yield the following inequality.

- For $q>8$, q even,

$$t_2(N,q)\le \begin{cases} \dfrac{t_2(2,q)}{q}q^{N/2}+s_{N,q}-N+1, & N \text{ even.}\\ \left(2+\dfrac{t_2(2,q)}{q}\right)q^{(N-1)/2}+s_{N,q}-N+2, & N \text{ odd.}\end{cases} \tag{1.4}$$

Then, each upper bound on $t_2(2,q)$ for q even gives rise to an upper bound on $t_2(N,q)$. For instance, from [22, Remark 2] the following inequalities are obtained:

- For $q=2^{2Cm}\ge 2^{30}$, $m\ge 2$, $C\ge 5$,

$$t_2(N,q)\le \begin{cases} \dfrac{1}{2^{m-1}}q^{N/2}+s_{N,q}-N+1, & N \text{ even.}\\ \left(2+\dfrac{1}{2^{m-1}}\right)q^{(N-1)/2}+s_{N,q}-N+2, & N \text{ odd.}\end{cases} \tag{1.5}$$

- For $q=2^{Cm}\ge 2^{30}$, $m\ge 2$, $C\ge 5$,

$$t_2(N,q)\le \begin{cases} \dfrac{1}{2^{(m/3)-1}}q^{N/2}+s_{N,q}-N+1, & N \text{ even.}\\ \left(2+\dfrac{1}{2^{(m/3)-1}}\right)q^{(N-1)/2}+s_{N,q}-N+2, & N \text{ odd.}\end{cases} \tag{1.6}$$

From [16], taking into account that 7 divides $2^{3t}-1$ and putting $s=7, u=3$ in the relation $u(q-1)/s+3$ for the size of a complete arc, we have the inequality

- For $q=2^{3t}\ge 2^{15}$,

$$t_2(N,q)\le \begin{cases} \frac{3}{7}q^{N/2}+\frac{18}{7}q^{(N-2)/2}+s_{N,q}-N+1, & N \text{ even.}\\ (2+\frac{3}{7})q^{(N-1)/2}+\frac{18}{7}q^{(N-3)/2}+s_{N,q}-N+2, & N \text{ odd.}\end{cases} \tag{1.7}$$

For $q=2^{14},2^{18}$ we can use the bounds $t_2(2,q)\le 6(\sqrt{q}-1)$, see [3].

Other results arise by Theorems 3.16, 3.17, and 4.8, together with Table II and Lemmas 2.12 and 2.13.

- For $q \geq 32$,

$$t_2(N,q) \leq \begin{cases} \frac{1}{2}q^{N/2} + \frac{5}{6}s_{N,q} + \frac{1}{3}, & N \geq 4 \text{ even.} \\ (2+\frac{1}{2})q^{(N-1)/2} + \frac{5}{6}s_{N,q} + \frac{1}{3}, & N \geq 5 \text{ odd.} \end{cases} \quad (1.8)$$

- For $q \geq 2^6$ square,

$$t_2(N,q) \leq \begin{cases} \frac{1}{3}q^{N/2} + s_{N,q} + \frac{2}{3}, & N \geq 4 \text{ even.} \\ (2+\frac{1}{3})q^{(N-1)/2} + s_{N,q} + \frac{2}{3}, & N \geq 5 \text{ odd.} \end{cases} \quad (1.9)$$

- For $q \leq 2^{15}$,

$$t_2(N,q) \leq \begin{cases} \frac{t_q}{q}q^{N/2} + \frac{t_q}{3q}(s_{N,q}-2), & N \geq 4 \text{ even,} \\ \left(2+\frac{t_q}{q}\right)q^{(N-1)/2} + \frac{t_q}{3q}(s_{N,q}-2), & N \geq 5 \text{ odd,} \end{cases}$$

where t_q is as in the following table:

$\log_2 q$	3	4	5	6	7	8	9	10	11	12	13	14	15
t_q	6	9	14	22	34	55	86	124	201	307	461	665	993

It is easy to see that these new upper bounds improve the known bounds on $t_2(N,q)$ for any $q \geq 8$ and any dimension $N \geq 4$. In order to assess the above improvements, we introduce and discuss two parameters, $\Delta_{N,q}$ and $R_{N,q}$. Define $\Delta_{N,q}$ as the difference between the best known upper bounds on $t_2(N,q)$ and the new bounds obtained in this work. By $R_{N,q}$ we denote the ratio between the coefficients of the main term $q^{\lfloor N/2 \rfloor}$ in our upper bounds and in the best known ones.

If q is not a square, then $\Delta_{N,q} \gtrsim \frac{1}{6}s_{N,q}$; if in addition $q = 2^{Cm}$, $m \geq 9$, $C \geq 5$, then $\Delta_{N,q} \gtrsim (\frac{1}{2} - 1/(2^{(m/3)-1}))q^{\lfloor N/2 \rfloor}$; and if $q = 2^{3t}$, $t \geq 5$ odd, then $\Delta_{N,q} \gtrsim \frac{1}{14}q^{\lfloor N/2 \rfloor}$. If q is a square, then $\Delta_{N,q} \gtrsim \frac{5}{3}q^{(N-2)/2}$ for N even and $\Delta_{N,q} \gtrsim \frac{1}{6}q^{(N-1)/2}$ for N odd. Furthermore, if $q = 2^{2Cm}$, $m \geq 3$, $C \geq 5$, then $\Delta_{N,q} \gtrsim (\frac{1}{3} - 1/(2^{m-1}))q^{N/2} + \frac{5}{3}q^{(N-2)/2}$ for N even and $\Delta_{N,q} \gtrsim (\frac{1}{2} - 1/(2^{m-1}))q^{(N-1)/2}$ for N odd.

If q is a square then $R_{N,q} \approx \frac{14}{15}$ for N odd; moreover, if $q = 2^{2Cm}$, $m \geq 3$, $C \geq 5$, then $R_{N,q} \approx 3/(2^{m-1})$ for N even and $R_{N,q} \approx (2^m+1)/(5 \cdot 2^{m-2})$ for N odd. If $q = 2^{Cm}$, $m \geq 9$, $C \geq 5$, m, C odd, then $R_{N,q} \approx 1/(2^{(m/3)-2})$ for N even and $R_{N,q} \approx (2^{(m/3)}+1)/(5 \cdot 2^{(m/3)-2})$ for N odd. Finally, if $q = 2^{3t}$, $t \geq 5$ odd, then $R_{N,q} \approx \frac{6}{7}$ for N even, $R_{N,q} \approx \frac{34}{35}$ for N odd.

For $2^3 \leq q \leq 2^{18}$, in Table I we list only some of the values of the new upper bounds on $t_2(N,q)$ obtained in this work, and those of the corresponding $\Delta_{N,q}$. In each entry $\Delta_{N,q}$ of the table, we cite the article where the best previously known bound was proved. Some comparisons are also given after Theorems 3.14 and 4.6.

TABLE I. New Upper Bounds on $t_2(N,q)$ for Small N and q.

N	$t_2(N,2^3)$	$\Delta_{N,2^3}$	$t_2(N,2^4)$	$\Delta_{N,2^4}$	$t_2(N,2^5)$	$\Delta_{N,2^5}$	$t_2(N,2^6)$	$\Delta_{N,2^6}$
4	≤ 54	18 [19]	≤ 153	25 [11]	≤ 462	148 [10]	≤ 1430	812 [10]
5	≤ 182	36 [20]	≤ 665	153 [20]	≤ 2510	148 [10]	≤ 9622	812 [10]
6	≤ 438	292 [20]	≤ 2457	409 [11]	≤ 14798	4756 [10]	≤ 91542	52012 [10]

The article is organized as follows. Section 2 contains some preliminary ideas and results on k-arcs in $PG(2,q)$. The concept of sum-points for a k-arc is introduced, and k-arcs with only one sum-point are investigated. These arcs will be the base for some of the inductive constructions of complete caps that are described in Section 3 for N even and Section 4 for N odd. Most of these caps are such that the intersection with an M-dimensional subspace of $PG(N,q)$ is a cap of the same type $K^{(M)}_{m_1,m_2}$, see (3.2). Being quite technical, the investigation of caps of type $K^{(M)}_{m_1,m_2}$ is postponed to the Appendix.

2. SUM-POINTS FOR PLANE ARCS

Throughout the article, q is a power of 2. Let $\mathbb{F}_q$ denote the finite field with q elements, and let $\mathbb{F}_q^* = \mathbb{F}_q \setminus \{0\}$. Let X_0, X_1, X_2 denote homogeneous coordinates for points of $PG(2,q)$.

In this section we prove some preliminary results on plane arcs. Let l_∞ be the line of $PG(2,q)$ of equation $X_0 = 0$. The points of an arc K not lying on l_∞ are the *affine points* of K, and the subset of affine points of K is the *affine part* of K. An arc is said to be *affine* if it coincides with its affine part. An *affinely complete arc* is an affine arc whose secants cover all the points in $PG(2,q) \setminus l_\infty$. We recall that an R-secant of K is a line l such that $|l \cap K| = R$. As usual, we say that a point is written in his normalized form if the first nonzero coordinate is equal to 1.

Let K be a complete arc in $PG(2,q)$, and let Q be a point in $PG(2,q) \setminus K$ written in its normalized form. For every secant l of K through Q, let $c_1^{(l)}, c_2^{(l)}$ be the elements in $\mathbb{F}_q^*$ such that

$$Q = c_1^{(l)} P_1 + c_2^{(l)} P_2$$

where P_1 and P_2 are the points on $l \cap K$ written in their normalized form.

Definition 2.1. The point Q is said to be a *sum-point* for K if $c_1^{(l)} = c_2^{(l)}$ for every secant l of K through Q.

Remark 2.2. In general, collineations do not preserve the number of sum-points for an arc. In this sense the concept of "sum-points" is not geometrical.

We denote by $\beta(K)$ the number of sum-points for a complete arc K. When $\beta(K) = 1$, we denote by $p(K)$ the number of secants of K passing through the only sum-point.

Lemma 2.3. *Let K be a complete arc in $PG(2,q)$. Then $\beta(K) \geq 1$.*

TABLE II. Parameters for $q \leq 2^{18}$.

q	$\bar{t}_2^A(2,q)$	$\bar{t}_2(2,q)$	$\bar{t}_2^*(2,q)$	$\bar{p}(2,q)$	$(\bar{t}_2^*(2,q)-2)\bar{p}(2,q)$	References
2^3	6•	6•	6•	1•	$4 < 2^3-1$	[2]
2^4	9•	9•	9•	1•	$7 < 2^4-1$	[2]
2^5	14	14	14	1	$12 < 2^5-1$	[2]
2^6	22	22	22	1	$20 < 2^6-1$	[2]
2^7	32	34	34	1	$32 < 2^7-1$	[10, Section 5, Lemma 4.3]
2^8	55	55	55	2	$106 < 2^8-1$	⋆, [7]
2^9	86	86	86	3	$252 < 2^9-1$	[2]
2^{10}	124	124	124	1	$122 < 2^{10}-1$	⋆, [7], Lemma 2.14
2^{11}	201	201	201	4	$796 < 2^{11}-1$	⋆
2^{12}	307	307	307	5	$1525 < 2^{12}-1$	⋆, [7]
2^{13}	461	461	461	6	$2754 < 2^{13}-1$	⋆
2^{14}	665	665	665	11	$7293 < 2^{14}-1$	⋆, [7], Lemma 2.14
2^{15}	993	993	993	14	$13874 < 2^{15}-1$	⋆
2^{16}	$\frac{2^{16}+8}{3}$	$\frac{2^{16}+8}{3}$	$\frac{2^{16}+8}{3}$	1	$\frac{2^{16}+2}{3} < 2^{16}-1$	[1], Lemma 2.13
2^{17}	$2^{16} \leq$	$2^{16}+2 \leq$	$2^{16}+2 \leq$	1	$2^{16} < 2^{17}-1$	[10], Lemma 2.12
2^{18}	3066	3066	$2^{17}+2$	1	$2^{17} < 2^{18}-1$	[3, 10], Lemma 2.12

Proof. Let K be a complete arc in $PG(2,q)$. Note that, as q is even,

$$(0,m,m') = c_1(1,a,b) + c_2(1,a',b'), \quad (1,a,b) \neq (1,a',b')$$

yields $c_1 = c_2$. Therefore, if l_∞ is either a 0-secant or a 1-secant of K, then every point in $l_\infty \setminus K$ is a sum-point. Assume then that l_∞ is a 2-secant, and let $K \cap l_\infty = \{(0, X_1, X_2), (0, 1, f)\}$, where X_1 is either 0 or 1. Then the point of coordinates $(0, X_1 + 1, X_2 + f)$ is clearly a sum-point, which proves the assertion. □

Lemma 2.4. *Let K be a complete arc in $PG(2,q)$ such that l_∞ is a secant of K. If the affine part of K is affinely complete then $\beta(K) = 1$.*

Proof. We first prove that any point $P = (1, x, y)$ is not a sum-point for K. Two distinct affine points of K, say $(1,a,b)$ and $(1,a',b')$, are collinear with P, that is,

$$(1,x,y) = c_1(1,a,b) + c_2(1,a',b'),$$

for some $c_1, c_2 \in \mathbb{F}_q^*$. As q is even, it is not possible that $c_1 = c_2$. Now let $K \cap l_\infty = \{(0, X_1, X_2), (0, 1, f)\}$, with $X_1 \in \{0, 1\}$. Then it is straightforward that the point of coordinates $(0, X_1 + 1, X_2 + f)$ is the only sum-point for K on l_∞. □

Remark 2.5. The converse of Lemma 2.4 does not hold, as it can be shown that there exist arcs K in $PG(2,q)$, $q = 8, 16$, such that $\beta(K) = 1$ but the secants of the affine part of K do not cover all the affine points of $PG(2,q)$, see Table II.

For a k-arc K in $PG(2,q)$, let

$$Cov_\infty(K) = \{m \mid (0,1,m) \text{ is covered by the secants of } K\},$$

$$S_\infty(K) = \{X_2 + Y_2 \mid (X_0, X_1, X_2), (Y_0, Y_1, Y_2) \in K,\ X_0 = Y_0, X_1 = Y_1, X_2 \neq Y_2\}.$$

Also, for any element $m \in \mathbb{F}_q$, let

$$S_m(K) = \left\{ X_1+Y_1 \mid (X_0,X_1,X_2),(Y_0,Y_1,Y_2) \in K,\ X_0=Y_0, X_1 \neq Y_1, \frac{X_2+Y_2}{X_1+Y_1}=m \right\}.$$

Note that the size of $S_m(K)$ is at most $k/2$, as there are at most $k/2$ pairs of points of K collinear with $(0,1,m)$. The size of $S_\infty(K)$ is at most $k/2$ as well, as $S_\infty(K)$ corresponds to pair of points of K collinear with $(0,0,1)$. In particular, when K is complete and $\beta(K)=1$ and the only sum-point for K is $(0,0,1)$, we have

$$1 \leq |S_\infty(K)| \leq p(K).$$

Note also that $0 \notin S_\infty(K)$. Similarly, if $\beta(K)=1$ and the only sum-point for K is $(0,1,m)$ we have

$$1 \leq |S_m(K)| \leq p(K).$$

As a matter of terminology, we say that a projectivity ψ of $PG(2,q)$ is *integral for K* if it can be represented by a matrix $A \in GL(3,q)$ with the following property: for each point P of K written in its normalized form, $A \cdot P$ is the normalized form for the point $\psi(P)$.

Lemma 2.6. *Let K be a complete arc in $PG(2,q)$. Let ψ be any projectivity of $PG(2,q)$ which is integral for K. Then a point $Q \in PG(2,q) \setminus K$ is a sum-point for K if and only if $\psi(Q)$ is a sum-point for $\psi(K)$. In particular, $\beta(\psi(K))=\beta(K)$.*

Proof. Assume that Q is not a sum-point for K. Then there exists $c_1 \neq c_2$ such that

$$Q = c_1P_1 + c_2P_2,$$

where P_1 and P_2 are points of K written in their normalized form. Let A be a matrix representing ψ and such that for each point P of K written in its normalized form, $A \cdot P$ is the normalized form for $\psi(P)$. Then

$$\psi(Q) = A \cdot Q = c_1(A \cdot P_1) + c_2(A \cdot P_2),$$

whence $\psi(Q)$ is not a sum-point for K. The converse can be proved in a similar way. □

Remark 2.7. Let K be a complete arc such that $\beta(K)=1$. From the proof of Lemma 2.6 it follows that for any projectivity ψ which is integral for K, the value of $p(\psi(K))$ coincides with $p(K)$.

Lemma 2.8. *For every complete arc K in $PG(2,q)$ with $\beta(K)=1$ there is a projectivity ψ such that $\beta(\psi(K))=1$, $p(\psi(K))=p(K)$, and the only sum-point for $\psi(K)$ is* $(0,0,1)$.

Proof. As $\beta(K)=1$, the line l_∞ is a secant of K. Let $K \cap l_\infty = \{(0,X_1,f),(0,1,g)\}$. If $X_1=1$, the lemma is proved by taking ψ as the identical projectivity. Assume then that $X_1=0$. Then $f=1$. Let $\psi(x,y,z)=(x, y(g+1)+z, z+gy)$. Clearly ψ is integral for K. Also, $\psi(K) \cap l_\infty = \{(0,1,1),(0,1,0)\}$, whence $(0,0,1)$ is a sum-point for $\psi(K)$. By Lemma 2.6, the assertion is proved. □

Lemma 2.9. *For every complete arc K in $PG(2,q)$ with $\beta(K)=1$ there exists a projectivity ψ such that $\beta(\psi(K))=1$, $\psi(K)\cap l_\infty=\{(0,0,1),(0,1,0)\}$, $p(\psi(K))=p(K)$, and the only sum-point for $\psi(K)$ is $(0,1,1)$.*

Proof. As $\beta(K)=1$, the line l_∞ is a secant of K. Let $K\cap l_\infty=\{(0,X_1,f),(0,1,g)\}$. If $X_1=0$ then $f=1$, whence the assertion holds for $\psi(x,y,z)=(x,y,z+gy)$. Assume then that $X_1=1$. Then let $\psi(x,y,z)=(x,(f/(f+g))y+(1/(f+g))z,(g/(f+g))y+(1/(f+g))z)$. By Lemma 2.6, the claim follows. □

Lemma 2.10. *For every complete arc K in $PG(2,q)$ with $\beta(K)=1$ there is a projectivity ψ such that $\beta(\psi(K))=1$, $p(\psi(K))=p(K)$, the only sum-point for $\psi(K)$ is $(0,0,1)$, and $1\notin S_\infty(\psi(K))$.*

Proof. By Lemma 2.8 we can assume that the only sum-point for K is $(0,0,1)$. Then $K\cap l_\infty=\{(0,1,f),(0,1,g)\}$ for some $f,g\in\mathbb{F}_q$. For a pair $P_1=(X_0,X_1,X_2)$, $P_2=(Y_0,Y_1,Y_2)$ of points of K collinear with $(0,0,1)$, let $\Delta_{P_1,P_2}=X_2+Y_2$. As there are at most $k/2$ distinct values of Δ_{P_1,P_2}, there exists an element $w\in\mathbb{F}_q^*$ such that

$$w\notin\{\Delta_{P_1,P_2}\mid P_1,P_2 \text{ collinear with } (0,0,1)\}.$$

Let $\psi(X_0,X_1,X_2)=(X_0,X_1,X_2/w)$. Then $\psi(K)\cap l_\infty=\{(0,1,f/w),(0,1,g/w)\}$. Note that $\Delta_{\psi(P_1),\psi(P_2)}=(1/w)\Delta_{P_1,P_2}\neq 1$, for every pair $\psi(P_1),\psi(P_2)$ of points collinear with $(0,0,1)$, whence $1\notin S_\infty(\psi(K))$. Also, the point $(0,0,1)$ is a sum-point for $\psi(K)$. Finally, Lemma 2.6 ensures that $\beta(\psi(K))=\beta(K)=1$. □

Lemma 2.11. *In $PG(2,q)$ for every complete k-arc K with $\beta(K)=1$ and*

$$(k-2)p(K)<q-1 \tag{2.1}$$

there exists a collineation ψ such that $\psi(K)\cap l_\infty=\{(0,0,1),(0,1,0)\}$, $\beta(\psi(K))=1$, $p(\psi(K))=p(K)$, the only sum-point for $\psi(K)$ is $(0,1,1)$, and with the property that

$$\psi(K)\cap\{(1,a,Aa^2)\mid A\in S_1(\psi(K)),a\in\mathbb{F}_q\}=\emptyset. \tag{2.2}$$

Proof. By Lemma 2.9, we can assume that $K\cap l_\infty=\{(0,0,1),(0,1,0)\}$ and $\beta(K)=1$, the only sum-point for K being $(0,1,1)$. Let $K_w=\phi_w(K)$ where $\phi_w(x,y,z)=(x,wx+y,z)$, $w\in\mathbb{F}_q^*$. Note that $\beta(K_w)=\beta(K)=1$ by Lemma 2.6. As $\phi_w(K)\cap l_\infty=\{(0,0,1),(0,1,0)\}$, it follows that the only sum-point for K_w is $(0,1,1)$. Also, it is straightforward that $S_1(K_w)=S_1(K)$.

Now let $R=\{(1,a,Aa^2)\mid A\in S_1(K),a\in\mathbb{F}_q\}$. Note that for any affine point $P=(1,b,c)$ in $PG(2,q)$, the point $\phi_w(P)$ belongs to R if and only if $w^2=(1/A)(c+Ab^2)$. When P ranges over the affine points of K and A over the set $S_1(K)$, the number of values $(1/A)(c+Ab^2)$ is at most $(k-2)|S_1(K)|$. This proves that $K_w\cap R\neq\emptyset$ for at most $(k-2)|S_1(K)|$ values of w. As $|S_1(K)|\leq p(K)$, from 2.1 it follows that there exists $w_0\in\mathbb{F}_q^*$ such that $K_{w_0}\cap R=\emptyset$. Then the assertion follows for $\psi=\phi_{w_0}$. □

Lemma 2.12. *For any even $q\geq 32$, in $PG(2,q)$ there exists a complete k-arc K with $k\leq(q+4)/2$, $\beta(K)=1$, $p(K)=1$.*

Proof. By [10, Proposition 3.2, Lemmas 4.1–4.3], for any even $q\geq 32$, in $PG(2,q)$ there exists a complete $(q/\alpha+2)$-arc K, $\alpha\geq 2$ integer, which is obtained from an affinely complete q/α-arc K^A by adding two points lying on the line l_∞. Moreover, the number of points on ℓ_∞ covered by K^A is $q/\alpha-1$. This means that there are at least 3 points on

ℓ_∞ uncovered by K^A. Such points can be assumed to be $(0,1,f)$, $(0,1,g)$, and $(0,0,1)$. Then it is straightforward that K is a complete arc with the only sum-point $(0,0,1)$ (see also the proof of Lemma 2.4). As K^A does not cover the sum-point $(0,0,1)$, $p(K)=1$ holds. □

Lemma 2.13. *Let $q=2^h$, $h\geq 6$ even. Then there exists a complete $(q+8)/3$-arc K in $PG(2,q)$ such that $\beta(K)=1$, $p(K)=1$.*

Proof. The following construction comes from [1]. Let g be a primitive element of the field $\mathbb{F}_q$. The points Y_1, Y_2, and X_∞, the point sets C_3, D_3, F_3, and K are defined as follows in [1]: $Y_1=(0,1,g^{-1})$, $Y_2=(0,1,g^{-2})$, $X_\infty=(1,0,0)$, $C_3=\{(1,g^{3r},g^{-3r})\,|\,r=0,1,\ldots,(q-4)/3\}$, $D_3=\{(1,g^{1-3r},g^{-(1-3r)})\,|\,r=0,1,\ldots,(q-4)/3\}$, $F_3=\{(1,g^{2-3r},g^{-(2-3r)})\,|\,r=0,1,\ldots,(q-4)/3\}$, $K=C_3\cup\{Y_1,Y_2,X_\infty\}$. In [1] the following assertions are proved:

(i) every point in $T=PG(2,q)\setminus(C_3\cup D_3\cup F_3\cup l_\infty\cup\{(0,0,1),(0,1,0),(1,0,0)\})$ lies on some bisecant of C_3;
(ii) for every point $D\in D_3$ there is a point $P_r=(1,g^{3r},g^{-3r})\in C_3$ such that the points D, P_r, and Y_1 are collinear; similarly, for every point $F\in F_3$ there exists a point $P_r=(1,g^{3r},g^{-3r})$ of C_3 collinear with F and Y_2;
(iii) K is a complete $((q+8)/3)$-arc.

By [17, Chapter 4, Corollary 28], g can be assumed to have trace equal to 1. Note that l_∞ is a 2-secant of K and $(0,0,1)$ is the only sum-point lying on l_∞, see the proof of Lemma 2.3. Sum-points not belonging to l_∞ can only be of the form $S_{r,i}=P_r+Y_i$, $i\in\{1,2\}$. In fact, sum-points of the form $X_\infty+Y_i$ do not exist as, by (i) and (ii), every point of $PG(2,q)\setminus K$ lies on some bisecant of $K\setminus\{X_\infty\}$. If $S_{r,i}\in T$ then, by (i), $S_{r,i}$ lies on a bisecant of C_3 and then it is not a sum-point, see the proof of Lemma 2.4. If $S_{r,i}\notin T$ then $S_{r,i}\in D_3\cup F_3$. This means that $(1,g^{3r},g^{-3r})+(0,1,g^{-i})=(1,a,a^{-1})$, $a\in\mathbb{F}_q$, whence $(g^{3r})^2+g^{3r}+g^i=0$ for some $i\in\{1,2\}$. But this is impossible as the traces of g and g^2 are both equal to one.

Therefore, the only sum-point is $(0,0,1)$. As it can be obtained only as the sum Y_1+Y_2, $p(K)=1$ holds. □

Let $\overline{t}_2(2,q)$ and $\overline{t}_2^A(2,q)$ be the smallest *known* size of a complete arc and an affinely complete arc in $PG(2,q)$. As every complete arc is projectively equivalent to an affine arc, we have $\overline{t}_2^A(2,q)\leq\overline{t}_2(2,q)\leq\overline{t}_2^A(2,q)+2$. Let $\overline{t}_2^*(2,q)$ be the smallest *known* size of a complete arc K in $PG(2,q)$ with $\beta(K)=1$, and let $\overline{p}(2,q)$ be the smallest *known* value of $p(K)$ for arcs of size $\overline{t}_2^*(2,q)$ with $\beta(K)=1$. For $q\leq 2^{18}$, the values of these parameters, either known in the literature or obtained in this work, are listed in Table II. Any value in the table which is not only the smallest *known*, but also the smallest *possible*, is followed by a dot. It should be noted that for some q we have $\overline{t}_2^A(2,q)<\overline{t}_2(2,q)$. For $q=2^3,2^4,2^5,2^6,2^9$, we use the previously known small complete arcs. For $q=2^7,2^{17}$, by [10, Section 5, Lemma 4.3], complete $\overline{t}_2(2,q)$-arcs K are obtained from affinely complete $\overline{t}_2^A(2,q)$-arcs K^A by adding two points lying on l_∞, as in the proof of Lemma 2.12. By the same arguments, the values of $\overline{t}_2^*(2,2^{18})$ and $\overline{p}(2,2^{18})$ are obtained. The value of $\overline{t}_2(2,2^{18})$ comes from [3], where $6(\sqrt{q}-1)$-arcs in $PG(2,q)$, $q=4^{2h+1}$ are constructed and for $h\leq 4$ it is proved that they are complete. Unfortunately, nothing is known on $\beta(K)$ for these arcs. For $q=2^{16}$ the entries follow from [1] and Lemma 2.13.

For $q=2^8, 2^{10}, 2^{11}, 2^{12}, 2^{13}, 2^{14}, 2^{15}$, the complete $\overline{t}_2(2,q)$-arcs in Table II are new. For $q=2^{11}, 2^{13}, 2^{15}$, they have been obtained by using the randomized greedy algorithms [4, 2] with a random starting set. For $q=2^8, 2^{10}, 2^{12}, 2^{14}$, our arcs are based on the $(4\sqrt{q}-4)$-arcs $\mathcal{K}_w$ in $PG(2,q)$ introduced in [7, p. 115]:

$$\mathcal{K}_w = \{(1, 1/\alpha, \alpha), (1, 1/w\alpha, w\alpha), (1, w^{\sqrt{q}-1}/\alpha, \alpha), (1, 1/w\alpha, w^{\sqrt{q}}\alpha) | \alpha \in \mathbb{F}^*_{\sqrt{q}}\} \quad (2.3)$$

where w is an element of $\mathbb{F}_q \setminus \mathbb{F}_{\sqrt{q}}$ satisfying $w^2+w+d=0$, with $d \in \mathbb{F}_{\sqrt{q}}$. Let γ be a primitive element of $\mathbb{F}_q$. For $q=2^{12}$, we put $d=\gamma^{(q-1)/3}$ and use $\mathcal{K}_w$ as the starting set for the greedy algorithms. For $q=2^8$, the starting set is a subset of $\mathcal{K}_w$. For $q=2^{10}, 2^{14}$, we modify $\mathcal{K}_w$.

Lemma 2.14. *Let $q=4^{2h+1}$, $h \geq 1$. Let $w=\gamma^{(q-1)/3}$. Then $w^2+w+1=0$. Let $\mathcal{K}'_w$ be the point set obtained from the arc $\mathcal{K}_w$ (2.3) by changing every point $(1, 1/w\alpha, w^{\sqrt{q}}\alpha)$ with the point $(1, 1/w^2\alpha, w^{2\sqrt{q}}\alpha)$. Then $\mathcal{K}'_w$ is a $(4\sqrt{q}-4)$-arc in $PG(2,q)$, $q=2^6, 2^{10}, 2^{14}$. For $q=2^6, 2^{10}$, the arc is complete.*

Proof. The assertion about w is trivial. The properties of $\mathcal{K}'_w$ have been checked by computer. □

If $q=2^{14}$ the arc $\mathcal{K}'_w$ turns out not to be complete, and in order to obtain a complete arc in $PG(2, 2^{14})$ we use $\mathcal{K}'_w$ as the starting point for the greedy algorithms.

Finally, it should be noted that for $q=2^3, \ldots, 2^6, 2^8, \ldots, 2^{15}$, the values of $\overline{t}^*_2(2,q)$ and $\overline{p}(2,q)$ have been obtained by acting on complete $\overline{t}_2(2,q)$-arcs with both randomly chosen collineations and projectivities of type $\phi(X_0, X_1, X_2) = (X_0, X_1+wX_2, X_2)$.

On the basis of Table II, together with the results of some computer search, we make the following conjecture.

Conjecture 2.15. *Every complete arc in $PG(2,q)$ is projectively equivalent to an arc with only one sum-point.*

Let K be an affinely complete arc in $PG(2,q)$. Without loss of generality, assume that the point $(0,0,1)$ is covered by the secants of K. The following results will be needed in the sequel.

Lemma 2.16. *For an affinely complete arc K in $PG(2,q)$ such that $(0,0,1)$ is covered by the secants of K, it can be assumed that $1 \notin S_\infty(K)$.*

Proof. See the proof of Lemma 2.10. □

Lemma 2.17. *Let K be an affinely complete k-arc K in $PG(2,q)$ such that $(0,0,1)$ is covered by the secants of K. Assume that $k < q-5$. Then there exist $m_1, m_2 \in \mathbb{F}^*_q$ with $m_1 \neq m_2$, $(m_1+m_2)^3 \neq 1$, $m_i \neq 1$, such that*

$$1 \notin S_{m_1}(K) \cup S_{m_2}(K).$$

Proof. First we prove that the number of values of m for which $1 \in S_m(K)$ is at most k. For $a \in \mathbb{F}_q$, let n_a be the number of pairs $(1,a,b), (1,a+1,d)$ of points in K. Let l_1 be the line of equation $X_1 = aX_0$ and l_2 that of equation $X_1 = (a+1)X_0$. It is straightforward to check that n_a is equal to

- 4, if both lines l_1 and l_2 are secants to K;
- 2, if one of the two lines is a 1-secant to K and the other is a 2-secant;
- 1, if both lines are 1-secant to K;
- 0, if at least one line is a 0-secant to K.

Then n_a is less than or equal to the number of points in K belonging to $l_1 \cup l_2$. When a ranges over $\mathbb{F}_q$, we obtain at most k pairs $(1,a,b)$, $(1,a+1,d)$ of points in K. Then by the assumption $k < q-5$, there are at least 6 values of $m \in \mathbb{F}_q$ such that $1 \notin S_m(K)$. Let θ_1 and θ_2 be the roots of $T^3 = 1$ distinct from 1. Choose $m_1 \notin \{0,1\}$ such that $1 \notin S_{m_1}(K)$. Then there exists $m_2 \notin \{0,1,m_1,\theta_1+m_1,\theta_2+m_1\}$, with $1 \notin S_{m_2}(K)$. This completes the proof. □

3. CAPS IN PROJECTIVE SPACES OF EVEN DIMENSION

Throughout this section the following notation is fixed. Let s be a positive integer. Let $X_0, X_1, \ldots, X_{2s+2}$ be homogeneous coordinates for the points of $PG(2s+2,q)$. For $i = 0, \ldots, 2s+1$, let H_i be the subspace of $PG(2s+2,q)$ of equations $X_0 = \cdots = X_i = 0$. Let $AG(N,q)$ be the N-dimensional affine space over $\mathbb{F}_q$. As usual, a point in $AG(N,q)$ is identified with a vector in $\mathbb{F}_q^N$. For any integer $j \geq 1$, let

$$\mathcal{P}^j = \{(a_1, a_1^2, \ldots, a_j, a_j^2) \mid a_1, \ldots, a_j \in \mathbb{F}_q\} \subset AG(2j,q). \tag{3.1}$$

The set $\mathcal{P}^j$ is a cap in $AG(2j,q)$, as it was first noticed in [20].

The *product construction*, first introduced in [18], is the starting point for our constructions of small complete caps in $PG(2s+2,q)$.

Proposition 3.1 (see [6]). *Let $C_1 \subset \mathbb{F}_q^{N_1+1}$ be a set of representatives of a cap $C = \langle C_1 \rangle \subset PG(N_1,q)$, and let $C_2 \subset AG(N_2,q)$ be a cap. Then the product*

$$(C:C_2) := \{(P,Q) \mid P \in C_1, Q \in C_2\} \subset PG(N_1+N_2,q) \quad \textit{is a cap.}$$

In this section we consider products $(K:\mathcal{P}^s)$, where K is an arc in $PG(2,q)$. Completeness of K in $PG(2,q)$ is not enough to guarantee the completeness of $(K:\mathcal{P}^s)$ in $PG(2s+2,q)$. In order to obtain a complete cap, the following inductive construction of a cap in $PG(2s+1,q)$ will be a key tool.

Let $m_1, m_2 \in \mathbb{F}_q^*$ with $m_1 \neq m_2$, $(m_1+m_2)^3 \neq 1$, $m_i \neq 1$. Let $K^{(1)}_{m_1,m_2}$ be the subset of $PG(1,q)$ consisting of points $\{(1,0),(0,1)\}$. For $i \geq 1$, let

$$K^{(2i+1)}_{m_1,m_2} = A_1^{(2i+1)} \cup A_2^{(2i+1)} \cup A_3^{(2i+1)} \cup \{(1,0,\ldots,0)\} \subset PG(2i+1,q) \tag{3.2}$$

where

$$A_1^{(2i+1)} = \{(1,m_1,a_1,a_1^2,a_2,a_2^2,\ldots,a_i,a_i^2),(1,m_2,a_1,a_1^2,a_2,a_2^2,\ldots,a_i,a_i^2) \mid$$
$$a_1,\ldots,a_i \in \mathbb{F}_q, (a_1,\ldots,a_i) \neq (0,\ldots,0)\},$$

$$A_2^{(2i+1)} = \{(0,1,a_1,a_1^2,a_2,a_2^2,\dots,a_i,a_i^2)\,|\,a_1,\dots,a_i\in\mathbb{F}_q,(a_1,\dots,a_i)\neq(0,\dots,0)\},$$
$$A_3^{(2i+1)} = \{(0,0,b_0,b_1,\dots,b_{2i-1})\,|\,(b_0,b_1,\dots,b_{2i-1})\in K_{m_1,m_2}^{(2i-1)}\}.$$

Let $K_2^*(m_1,m_2)=K_{m_1,m_2}^{(2s+1)}\setminus\{(1,0,\dots,0)\}$.

Proposition 3.2. *If $q>4$, then the set $K_2^*(m_1,m_2)$ is a cap in $PG(2s+1,q)$ which covers all the points in $PG(2s+1,q)$ with the exception of points*

$$(1,m,0,0,\dots,0),\quad m\in\mathbb{F}_q.$$

Being quite technical, the proof of Proposition 3.2 is postponed to the Appendix.

We are now in a position to construct complete caps from products $(K:\mathcal{P}^s)$, K being a suitable arc in $PG(2,q)$. Three cases will be investigated.

(I) K affinely complete.
(II) K complete, $\beta(K)=1$.
(III) K complete, $\beta(K)=1$, $(k-2)p(K)<q-1$.

A. Case (I)

By Lemma 2.16 we can assume without loss of generality that

(Ia) K is affinely complete, $(0,0,1)$ is covered by K, $1\notin S_\infty(K)$.

Lemma 3.3. *The cap $(K:\mathcal{P}^s)$ covers all points in $PG(2s+2,q)\setminus H_0$.*

Proof. Let $Q=(1,\alpha,\beta,c_1,c_1',\dots,c_s,c_s')$ be any point in $PG(2s+2,q)\setminus H_0$. Write

$$(1,\alpha,\beta)=\gamma(1,a,b)+(\gamma+1)(1,c,d)$$

with $(1,a,b)$, $(1,c,d)$ in K. Assume first that $\gamma\notin\{0,1\}$. We look for λ_i, μ_i in $\mathbb{F}_q$ such that

$$Q=\gamma(1,a,b,\lambda_1,\lambda_1^2,\dots,\lambda_s,\lambda_s^2)+(\gamma+1)(1,c,d,\mu_1,\mu_1^2,\dots,\mu_s,\mu_s^2),$$

that is,

$$\begin{cases} c_i^2=\gamma^2\lambda_i^2+(\gamma+1)^2\mu_i^2 \\ c_i'=\gamma\lambda_i^2+(\gamma+1)\mu_i^2 \end{cases}\quad \text{for each } i=1,\dots,s\,.$$

As $\gamma(\gamma+1)\neq 0$, such λ_i,μ_i certainly exist.

We now need to consider the case $\gamma\in\{0,1\}$, that is, $(1,\alpha,\beta)\in K$. Fix any $\delta\notin\{0,1\}$. We look for λ_i, μ_i in $\mathbb{F}_q$ such that

$$Q=\delta(1,\alpha,\beta,\lambda_1,\lambda_1^2,\dots,\lambda_s,\lambda_s^2)+(\delta+1)(1,\alpha,\beta,\mu_1,\mu_1^2,\dots,\mu_s,\mu_s^2),$$

that is,

$$\begin{cases} c_i^2=\delta^2\lambda_i^2+(\delta^2+1)\mu_i^2 \\ c_i'=\delta\lambda_i^2+(\delta+1)\mu_i^2 \end{cases}\quad \text{for each } i=1,\dots,s\,.$$

As $\delta^2(\delta+1)+(\delta^2+1)\delta=\delta(\delta+1)\neq 0$, such λ_i,μ_i certainly exist. □

Lemma 3.4. *The points on H_0 covered by $(K:\mathcal{P}^s)$ are*

$$(0,1,m,a_1,Aa_1^2,\dots,a_s,Aa_s^2),\quad m\in Cov_\infty(K),\quad A\in S_m(K),\quad a_i\in\mathbb{F}_q,$$

$$(0,0,1,a_1,Aa_1^2,\dots,a_s,Aa_s^2),\quad A\in S_\infty(K),\quad a_i\in\mathbb{F}_q,$$

$$(0,0,0,\dots,1,m,a_l,ma_l^2,\dots,a_s,ma_s^2),\quad l\geq 2,\quad m\in\mathbb{F}_q^*,\quad a_i\in\mathbb{F}_q,$$

$$(0,0,\dots,0,1,m),\ m\in\mathbb{F}_q^*.$$

Proof. Let $Q_1=(1,a,b,\lambda_1,\lambda_1^2,\dots,\lambda_s,\lambda_s^2)$ and $Q_2=(1,c,d,\mu_1,\mu_1^2,\dots,\mu_s,\mu_s^2)$ be two points in $(K:\mathcal{P}^s)$. The line through Q_1 and Q_2 meets H_0 in

$$Q=(0,a+c,b+d,(\lambda_1+\mu_1),(\lambda_1+\mu_1)^2,\dots,(\lambda_s+\mu_s),(\lambda_s+\mu_s)^2).$$

Assume that $a+c\neq 0$. Let $m=(b+d)/(a+c)$, $a_i=(\lambda_i+\mu_i)/(a+c)$. Then

$$Q=(0,1,m,a_1,(a+c)a_1^2,\dots,a_s,(a+c)a_s^2).$$

If $a+c=0$, $b+d\neq 0$, let $a_i=(\lambda_i+\mu_i)/(b+d)$. Then

$$Q=(0,0,1,a_1,(b+d)a_1^2,\dots,a_s,(b+d)a_s^2).$$

Finally, assume that $a+b=0$, $c+d=0$. Let l be the minimum integer for which $\lambda_l+\mu_l\neq 0$, and let $m=\lambda_l+\mu_l$. If $l<s$, let $a_i=(\lambda_i+\mu_i)/(\lambda_l+\mu_l)$. Then

$$Q=(0,0,0,\dots,1,m,a_{l+1},ma_{l+1}^2,\dots,a_s,ma_s^2).$$

If $l=s$, then $Q=(0,0,\dots,0,1,m)$. □

Let m_1, m_2 be as in Lemma 2.17. Let $K_2(m_1,m_2)$ be the natural embedding of $K_2^*(m_1,m_2)$ in $H_0\subset PG(2s+2,q)$.

Proposition 3.5. *Assume that $k<q-5$. Then*

$$\mathcal{X}=(K:\mathcal{P}^s)\cup K_2(m_1,m_2)\quad \textit{is a cap.}$$

Proof. Note that $1\notin(S_{m_1}(K)\cup S_{m_2}(K)\cup S_\infty(K))$. Then by Lemma 3.4 no point in $K_2(m_1,m_2)$ is covered by $(K:\mathcal{P}^s)$; the converse is also true as $K_2(m_1,m_2)\subset H_0$ and $(K:\mathcal{P}^s)\cap H_0=\emptyset$. Then no three points in $(K:\mathcal{P}^s)\cup K_2(m_1,m_2)$ are collinear. □

Theorem 3.6. *Let $M=2s+2$, $s\geq 1$, $q>8$. Assume that* (Ia) *holds and $k<q-5$. Let $\mathcal{X}$ be the cap*

$$\mathcal{X}=(K:\mathcal{P}^s)\cup K_2(m_1,m_2)\subset PG(M,q).$$

Then the size of $\mathcal{X}$ is

$$(k+3)\cdot q^{(M-2)/2}+3(q^{(M-4)/2}+q^{(M-6)/2}+\cdots+q)-M+3.$$

Moreover,

- *if* $Cov_\infty(K)=\mathbb{F}_q$, *then* $\mathcal{X}$ *is a complete cap*;
- *if* $\mathbb{F}_q\setminus Cov_\infty(K)=\{m_0\}$, *then*

$$\mathcal{X}'=\mathcal{X}\cup\{(0,1,m_0,0,\ldots,0)\}$$

  is a complete cap;
- *if* $\mathbb{F}_q\setminus Cov_\infty(K)\supseteq\{m_0,m_0'\}$, *then*

$$\mathcal{X}'=\mathcal{X}\cup\{(0,1,m_0,0,\ldots,0),(0,1,m_0',0,\ldots,0)\}$$

  is a complete cap.

Proof. The claim on the size of $\mathcal{X}$ follows from straightforward computation. The points in $PG(M,q)\setminus H_0$ are covered by $(K:\mathcal{P}^s)$ according to Lemma 3.3. Points of H_0 not of type $(0,1,m,0,\ldots,0)$ are covered by $K_2(m_1,m_2)$ according to Proposition 3.2. Points in H_0 of type $(0,1,m,0,\ldots,0)$, $m\in Cov_\infty(K)$, are covered by $(K:\mathcal{P}^s)$ according to Lemma 3.4. Then the claim follows. □

B. Case (II)

By Lemma 2.10 we can assume without loss of generality that
(IIa) K is complete, $\beta(K)=1$, the only sum-point for K is $(0,0,1)$, $1\notin S_\infty(K)$.

Lemma 3.7. *The cap* $(K:\mathcal{P}^s)$ *covers all points in* $PG(2s+2,q)\setminus H_1$.

Proof. Let $Q=(\delta,\alpha,\beta,c_1,c_1',\ldots,c_s,c_s')$ be any point in $PG(2s+2,q)\setminus H_1$, where (δ,α,β) is written in its normalized form. As $(\delta,\alpha,\beta)\neq(0,0,1)$ there exists $\gamma_1\neq\gamma_2$ such that

$$(\delta,\alpha,\beta)=\gamma_1(Y_0,Y_1,Y_2)+\gamma_2(Y_0',Y_1',Y_2')$$

with (Y_0,Y_1,Y_2), (Y_0',Y_1',Y_2') in K. The existence of λ_i, μ_i in $\mathbb{F}_q$ such that

$$Q=\gamma_1(Y_0,Y_1,Y_2,\lambda_1,\lambda_1^2,\ldots,\lambda_s,\lambda_s^2)+\gamma_2(Y_0',Y_1',Y_2',\mu_1,\mu_1^2,\ldots,\mu_s,\mu_s^2)$$

can then be proved as in the proof of Lemma 3.3. □

Lemma 3.8. *The points on* H_1 *covered by* $(K:\mathcal{P}^s)$ *are*

$$(0,0,1,a_1,Aa_1^2,\ldots,a_s,Aa_s^2),\quad A\in S_\infty(K),\ a_i\in\mathbb{F}_q. \tag{3.3}$$

$$(0,0,0,\ldots,1,m,a_l,ma_l^2,\ldots,a_s,ma_s^2),\quad l\geq 2,\ m\in\mathbb{F}_q^*,\ a_i\in\mathbb{F}_q. \tag{3.4}$$

$$(0,0,0\ldots,,0,0,1,m),\quad m\in\mathbb{F}_q^*. \tag{3.5}$$

Proof. The proof is similar to that of Lemma 3.4. □

Let m_1 and m_2 be as in Lemma 2.17. Let $\bar{K}_2^*(m_1,m_2)=K_{m_1,m_2}^{(2s-1)}\setminus\{(1,0,\ldots,0)\}$, and let $\bar{K}_2(m_1,m_2)$ be the natural embedding of $\bar{K}_2^*(m_1,m_2)$ in the subspace H_2 of $PG(2s+2,q)$ of equations $X_0=X_1=X_2=0$.

Proposition 3.9. *Assume that $k<q-5$. Let m_1, m_2 be as in Lemma* 2.17. *Then the set*

$$\mathcal{X} := (K:\mathcal{P}^s)\cup \bar{K}_2(m_1,m_2)\cup\{(0,0,1,a_1,a_1^2,\ldots,a_s,a_s^2)\mid a_i\in\mathbb{F}_q\}$$

is a cap.

Proof. Let $\bar{K}^{(0)}$ denote the cap $\{(0,0,1,a_1,a_1^2,\ldots,a_s,a_s^2)\mid a_i\in\mathbb{F}_q\}$. By Lemma 3.8 $(K:\mathcal{P}^s)\cup\bar{K}^{(0)}$ is a cap since $1\notin S_\infty(K)$. Again by Lemma 3.8, $(K:\mathcal{P}^s)\cup\bar{K}_2(m_1,m_2)$ is a cap as well. Clearly a point in $\bar{K}_2(m_1,m_2)$ cannot be collinear with a point in $(K:\mathcal{P}^s)$ and a point in $\bar{K}^{(0)}$. Then it remains to show that no two points in $\bar{K}^{(0)}$ are collinear with a point in $\bar{K}_2(m_1,m_2)$. But this follows from the fact that points in H_2 covered by $\bar{K}^{(0)}$ are points of type $(0,0,0,\ldots,1,m,a_{l+1},ma_{l+1}^2,\ldots,a_s,ma_s^2)$. □

Theorem 3.10. *Let $M=2s+2$, $s\geq 1$, $q>8$. Assume that* (IIa) *holds and $k<q-5$. Let $\mathcal{X}$ be the cap*

$$\mathcal{X} := (K:\mathcal{P}^s)\cup \bar{K}_2(m_1,m_2)\cup\{(0,0,1,a_1,a_1^2,\ldots,a_s,a_s^2)\mid a_i\in\mathbb{F}_q\}\subset PG(M,q).$$

Then,

- *the size of $\mathcal{X}$ is*

$$(k+1)\cdot q^{(M-2)/2}+3(q^{(M-4)/2}+q^{(M-6)/2}+\cdots+q)-M+5;$$

- *$\mathcal{X}$ is complete.*

Proof. The claim on the size of $\mathcal{X}$ follows by straightforward computation. Let $\bar{K}^{(0)}$ be as in the proof of Proposition 3.9. The points in $PG(M,q)\setminus H_1$ are covered by $(K:\mathcal{P}^s)$ according to Lemma 3.7. It is straightforward to check that $\bar{K}^{(0)}$ covers all the points in $H_1\setminus H_2$, with the exception of $(0,0,1,0,\ldots,0)$, which is covered by $(K:\mathcal{P}^s)$. Points of H_2 not of type $(0,0,0,1,m,0,\ldots,0)$ are covered by $\bar{K}_2(m_1,m_2)$ according to Proposition 3.2. Points in H_2 of type $(0,0,0,1,m,0,\ldots,0)$ are covered by $(K:\mathcal{P}^s)$ according to Lemma 3.8. Then the claim follows. □

C. Case (III)

By Lemma 2.11 we can assume without loss of generality that

(IIIa) K is complete, $\beta(K)=1$, $K\cap l_\infty=\{(0,0,1),(0,1,0)\}$, $(k-2)p(K)<q-1$,

$$K\cap\{(1,a,Aa^2)\mid A\in S_1(K), a\in\mathbb{F}_q\}=\emptyset. \tag{3.6}$$

We first consider the product cap $(K:\mathcal{P}^j)$ in $PG(2j+2,q)$, with $1\leq j\leq s$. Let $Y_0,\ldots,Y_{2j+2}$ be homogeneous coordinates for points in $PG(2j+2,q)$.

Lemma 3.11. *The cap $(K:\mathcal{P}^j)$ covers all points in $PG(2j+2,q)$, but some points on the subspace of equations $Y_0=0$, $Y_1=Y_2$.*

Proof. The proof is similar to that of Lemma 3.7. Note that here the only sum-point of K is $(0,1,1)$. □

Lemma 3.12. *The points on the subspace of* $PG(2j+2,q)$ *of equations* $Y_0=0$, $Y_1=Y_2$ *covered by* $(K:\mathcal{P}^j)$ *are*

$$(0,1,1,a_1,Aa_1^2,\ldots,a_{j-1},Aa_{j-1}^2,a_j,Aa_j^2),\quad A\in S_1(K),\ a_i\in F_q.$$

$$(0,0,0,\ldots,1,m,a_l,ma_l^2,\ldots,a_j,ma_j^2),\quad l\geq 2,\ m\in F_q^*,\ a_i\in F_q.$$

$$(0,0,0\ldots,,0,0,1,m),\quad m\in F_q^*.$$

Proof. The proof is similar to that of Lemma 3.4. □

For $j=0,\ldots,s-1$, let V_{2j+2} be the $(2j+2)$-dimensional subspace of $PG(2s+2,q)$ of equations $X_0=\cdots=X_{2s-2j-2}=0$, $X_{2s-2j-1}=X_{2s-2j}$. Let Φ_j be the following isomorphism between $PG(2j+2,q)$ and V_{2j+2}:

$$\Phi_j(Y_0,Y_1,\ldots,Y_{2j+2})=(0,0,\ldots,0,Y_0,Y_0,Y_1,\ldots,Y_{2j+2}).$$

For $j=1,\ldots,s-1$, let $K^{(j)}$ be the image by Φ_j of $(K:\mathcal{P}^j)\subset PG(2j+2,q)$. Also, let $K^{(0)}$ be the image of K by Φ_0.

Proposition 3.13. *The set*

$$\mathcal{X}:=(K:\mathcal{P}^s)\cup\left(\bigcup_{j=0}^{s-1}K^{(j)}\right)\quad \textit{is a cap.}$$

Proof. Note that the subspace of $PG(2j+2,q)$ of equations $Y_0=0$, $Y_1=Y_2$ is mapped by Φ_j onto V_{2j} for any $j\geq 1$. Then Lemma 3.12, together with (3.6), yields that the subset of points covered by the cap $K^{(j)}$ is disjoint from $\bigcup_{u=0}^{j-1}K^{(u)}$. This proves the assertion. □

Theorem 3.14. *Let* $M=2s+2$, $s\geq 1$. *Assume that* (IIIa) *holds. Then the cap*

$$\mathcal{X}:=(K:\mathcal{P}^s)\cup\left(\bigcup_{j=0}^{s-1}K^{(j)}\right)\subset PG(M,q)$$

is a complete cap of size

$$k\frac{q^{M/2}-1}{q-1}=k(q^{(M-2)/2}+q^{(M-4)/2}+q^{(M-6)/2}+\cdots+q+1).$$

Proof. The claim on the size of $\mathcal{X}$ follows from straightforward computation. Note that $(K:\mathcal{P}^s)$ covers all the points in $PG(M,q)\setminus V_{2s}$, and that for each $j=1,\ldots,s-1$, the cap $K^{(j)}$ covers all the points in $V_{2j+2}\setminus V_{2j}$, see Lemma 3.12. Taking into account that $K^{(0)}$ covers all the points in V_2, the completeness of $\mathcal{X}$ follows. □

By Theorem 3.14, taking into account the values of $\overline{t}_2^*(2,q)$ from Table II, we obtain complete $k_{4,q}$-caps in $PG(4,q)$ with the following sizes (the best known sizes from [10, 11, 20]) are given in parentheses): $k_{4,128}=34q+34$ $(35q+2)$, $k_{4,256}=55q+55$ $(67q+2)$, $k_{4,512}=86q+86$ $(131q+2)$, $k_{4,1024}=124q+124$ $(131q+2)$, $k_{4,2048}=201q+201$ $(259q+2)$, $k_{4,4096}=307q+307$ $(515q+2)$. See also Table I.

D. New Upper Bounds on $t_2(N,q)$ for N and q Even

Theorem 3.15. *Let N and q be even, $N>2$. If $q>8$, then*

$$t_2(N,q) \le (t_2(2,q)+3)\cdot q^{(N-2)/2} + 3(q^{(N-4)/2} + q^{(N-6)/2} + \cdots + q) - N + 3,$$

and

$$t_2(N,q) \le (t_2^A(2,q)+3)\cdot q^{(N-2)/2} + 3(q^{(N-4)/2} + q^{(N-6)/2} + \cdots + q) - N + 5,$$

where $t_2^A(2,q) \le t_2(2,q)$ is the size of the smallest affinely complete arc in $PG(2,q)$.

Proof. The assertion follows from Theorem 3.6. □

Theorem 3.16. *Let N and q be even, $N>2$. If $q>8$, then*

$$t_2(N,q) \le (t_2^S(2,q)+1)\cdot q^{(N-2)/2} + 3(q^{(N-4)/2} + q^{(N-6)/2} + \cdots + q) - N + 5,$$

where $t_2^S(2,q)$ is the size of the smallest complete arc in $PG(2,q)$ with only one sum-point.

Proof. The assertion follows from Theorem 3.10, together with Lemma 2.10. □

Theorem 3.17. *Let N and q be even, $N>2$. Then*

$$t_2(N,q) \le t_2^{S^+}(2,q)(q^{(N-2)/2} + q^{(N-4)/2} + q^{(N-6)/2} + \cdots + q + 1),$$

where $t_2^{S^+}(2,q)$ is the size of the smallest complete k-arc K in $PG(2,q)$ with only one sum-point and with the property that $(k-2)p(K) < q-1$.

Proof. The assertion follows from Theorem 3.14, together with Lemma 2.11. □

4. CAPS IN PROJECTIVE SPACES OF ODD DIMENSION

We keep the notation of the previous section. We consider complete k-arcs K in $PG(2,q)$ such that:

$$(*) K \text{ is affine, } k<q-5,\ 1 \notin S_\infty(K),\ Y_2 \ne Y_1^2 \text{ for each point } (1,Y_1,Y_2) \in K.$$

Lemma 4.1. *Any complete k-arc in $PG(2,q)$ with $k<q-5$ is projectively equivalent to a k-arc satisfying property $(*)$.*

Proof. We can assume that K is affine as every arc has an external line, which can be moved to l_∞. By Lemma 2.16 we can also assume that $1 \notin S_\infty(K)$. When $(1,a,b)$ ranges over K, the number of values of b/a^2 is at most k. Therefore, there exists an element w in $\mathbb{F}_q$ such that $b \ne (wa)^2$ for every point $(1,a,b) \in K$. The projectivity $\psi(X_0,X_1,X_2) = (X_0, wX_1, X_2)$ then maps K onto an arc $\psi(K)$ such that $Y_2 \ne Y_1^2$ for each point $(1,Y_1,Y_2) \in \psi(K)$. It is straightforward that $S_\infty(\psi(K))$ coincides with $S_\infty(K)$, whence the lemma is proved. □

Let $K_0 = \{(1,1),(1,0)\}$ be the trivial complete cap in $PG(1,q)$. We consider the product cap $(K_0 : \mathcal{P}^{s+1}) \subset PG(2s+3,q)$. Let H_0 be the subspace of $PG(2s+3,q)$ of equation $X_0 = 0$.

Lemma 4.2. *The cap* $(K_0 : \mathcal{P}^{s+1}) \subset PG(2s+3, q)$ *covers all the points in* $PG(2s+3, q) \setminus H_0$. *Points in* H_0 *covered by* $(K_0 : \mathcal{P}^{s+1})$ *are precisely*

$$(0, 1, a_1, a_1^2, \ldots, a_{s+1}, a_{s+1}^2), \quad a_i \in \mathbb{F}_q.$$

$$(0, 0, \ldots, 1, m, a_l, ma_l^2, \ldots, a_{s+1}, ma_{s+1}^2), \quad l \geq 2, \ m \in \mathbb{F}_q^*, \ a_i \in \mathbb{F}_q.$$

$$(0, 0 \ldots, 0, 0, 1, m), \quad m \in \mathbb{F}_q^*.$$

Proof. Let $Q = (1, \gamma, c_1, c_1', \ldots, c_s, c_s', c_{s+1}, c_{s+1}')$ be any point in $PG(2s+3, q) \setminus H_0$. Clearly

$$(1, \gamma) = \gamma(1, 1) + (1 + \gamma)(1, 0).$$

By an argument analogous to that of the proof of Lemma 3.3, it can then be proved that when $\gamma \notin \{0, 1\}$ there exist λ_i, μ_i in $\mathbb{F}_q$ such that

$$Q = \gamma(1, 1, \lambda_1, \lambda_1^2, \ldots, \lambda_{s+1}, \lambda_{s+1}^2) + (1+\gamma)(1, 0, \mu_1, \mu_1^2, \ldots, \mu_{s+1}, \mu_{s+1}^2),$$

and that when $\gamma \in \{0, 1\}$ there exist δ, λ_i, μ_i in $\mathbb{F}_q$ such that

$$Q = \delta(1, \gamma, \lambda_1, \lambda_1^2, \ldots, \lambda_{s+1}, \lambda_{s+1}^2) + (1+\delta)(1, \gamma, \mu_1, \mu_1^2, \ldots, \mu_{s+1}, \mu_{s+1}^2).$$

This proves that $(K_0 : \mathcal{P}^{s+1})$ covers all the points in $PG(2s+3, q) \setminus H_0$. The proof of the second statement of the Lemma is analogous to that of Lemma 3.4. □

Now, let $\mathcal{X} \subset PG(2s+2, q)$ be as in Proposition 3.5. Let $\bar{\mathcal{X}}$ be the natural embedding of $\mathcal{X}$ in the hyperplane H_0 of $PG(2s+3, q)$.

Proposition 4.3. *The set* $(K_0 : \mathcal{P}^{s+1}) \cup \bar{\mathcal{X}}$ *is a cap in* $PG(2s+3, q)$.

Proof. By Lemma 4.2 together with property (*) it follows that $\bar{\mathcal{X}}$ is disjoint from the set of points in H_0 covered by $(K_0 : \mathcal{P}^{s+1})$. This proves the assertion. □

Theorem 4.4. *Let* $M = 2s+3$, $s \geq 1$, $q > 8$. *Assume that* (*) *holds. Then the set* $(K_0 : \mathcal{P}^{s+1}) \cup \bar{\mathcal{X}}$ *is a complete cap in* $PG(M, q)$ *of size*

$$2q^{(M-1)/2} + (k+3) \cdot q^{(M-3)/2} + 3(q^{(M-5)/2} + q^{(M-7)/2} + \cdots + q) - M + 4.$$

Proof. By Proposition 4.3 the set $(K_0 : \mathcal{P}^{s+1}) \cup \bar{\mathcal{X}}$ is a cap, which is complete by Lemma 4.2 and Theorem 3.6. □

From now we assume that K is a k-arc in $PG(2, q)$ satisfying property (IIIa) of the previous section, and that $\mathcal{X} \subset PG(2s+2, q)$ is as in Proposition 3.13. Let $\bar{\mathcal{X}}'$ be the natural embedding of $\mathcal{X}$ in the hyperplane H_0 of $PG(2s+3, q)$.

Proposition 4.5. *The set* $(K_0 : \mathcal{P}^{s+1}) \cup \bar{\mathcal{X}}'$ *is a cap in* $PG(2s+3, q)$.

Proof. Note that the plane arc K is disjoint from points $\{(1, a, a^2) \mid a \in \mathbb{F}_q\}$. Then from Lemma 4.2 it follows that the cap $\bar{\mathcal{X}}'$ is disjoint from the set of points in H_0 covered by $(K_0 : \mathcal{P}^{s+1})$. This proves the assertion. □

Theorem 4.6. *Let* $M=2s+3$, $s\geq 1$. *Assume that* (IIIa) *holds. Then the set* $(K_0:\mathcal{P}^{s+1})\cup\bar{\mathcal{X}}'$ *is a complete cap in* $PG(2s+3,q)$ *of size*

$$2q^{(M-1)/2}+k(q^{(M-3)/2}+q^{(M-5)/2}+q^{(M-7)/2}+\cdots+q+1).$$

Proof. The claim follows from Lemma 4.2 together with Theorem 3.14. □

By Theorem 4.6, taking into account the values of $\overline{t}_2^*(2,q)$ from Table II, we obtain complete $k_{5,q}$-caps in $PG(5,q)$ with the following sizes (the best known sizes from [10, 11, 20] are given in parentheses): $k_{5,128}=2q^2+34q+34$ $(2q^2+35q+2)$, $k_{5,256}=2q^2+55q+55$ $(2q^2+67q+2)$, $k_{5,512}=2q^2+86q+86$ $(2q^2+131q+2)$, $k_{4,1024}=2q^2+124q+124$ $(2q^2+131q+2)$, $k_{4,2048}=2q^2+201q+201$ $(2q^2+259q+2)$, $k_{4,4096}=2q^2+307q+307$ $(2q^2+515q+2)$. See also Table I.

A. New Upper Bounds on $t_2(N,q)$ for N Odd, q Even

Theorem 4.7. *Let* N *be odd*, $N>3$, *and let* q *be even. If* $q>8$, *then*

$$t_2(N,q)\leq 2q^{(N-1)/2}+(t_2(2,q)+3)\cdot q^{(N-3)/2}+3(q^{(N-5)/2}+q^{(N-7)/2}+\cdots+q)-N+4.$$

Proof. The assertion follows from Theorem 4.4 together with Lemma 4.1. □

Theorem 4.8. *Let* N *be odd*, $N>3$, *and let* q *be even. Then*

$$t_2(N,q)\leq 2q^{(N-1)/2}+t_2^{S^+}(2,q)(q^{(N-3)/2}+q^{(N-5)/2}+q^{(N-7)/2}+\cdots+q+1),$$

where $t_2^{S^+}(2,q)$ *is the size of the smallest complete* k*-arc* K *in* $PG(2,q)$ *with only one sum-point and with the property that* $(k-2)p(K)<q-1$.

Proof. The assertion follows from Theorem 4.6, together with Lemma 2.11. □

ACKNOWLEDGMENTS

This research was performed within the activity of GNSAGA of the Italian INDAM, with the financial support of the Italian Ministry MIUR, project *Strutture Geometriche*, *Combinatoria e loro Applicazioni*, PRIN 2006-2007.

REFERENCES

[1] V. Abatangelo, A class of complete $[(q+8)/3]$-arcs of $PG(2,q)$, with $q=2^h$ and h (≥ 6) even, Ars Combin 16 (1983), 103–111.

[2] A. A. Davydov, G. Faina, S. Marcugini, and F. Pambianco, Computer search in projective planes for the sizes of complete arcs, J Geom 82 (2005), 50–62.

[3] A. A. Davydov, M. Giulietti, S. Marcugini, and F. Pambianco, On sharply transitive sets in $PG(2,q)$, Innovations Incidence Geom 6–7 (2009), 139–151.

[4] A. A. Davydov, S. Marcugini, and F. Pambianco, Complete caps in projective spaces $PG(n,q)$, J Geom 80 (2004), 23–30.

[5] A. A. Davydov and P. R. J. Östergård, Recursive constructions of complete caps, J Statist Planning Infer 95 (2001), 167–173.

[6] Y. Edel, Extensions of generalized product caps, Des Codes Cryptogr 31 (2004), 5–14.

[7] G. Faina and M. Giulietti, On small dense arcs in Galois planes of square order, Discrete Math 267 (2003), 113–125 (Gaeta, 2000).

[8] E. M. Gabidulin, A. A. Davydov, and L. M. Tombak, Linear codes with covering radius 2 and other new covering codes, IEEE Trans Inform Theory 37 (1991), 219–224.

[9] M. Giulietti, Small complete caps in Galois affine spaces, J Algebraic Comb 25 (2007), 149–168.

[10] M. Giulietti, Small complete caps in $PG(N,q)$, q even, J Comb Des 15 (2007), 420–436.

[11] M. Giulietti and F. Pasticci, Quasi-perfect linear codes with minimum distance 4, IEEE Trans Inform Theory 53 (2007), 1928–1935.

[12] J. W. P. Hirschfeld, Projective Geometries Over Finite Fields, 2nd edn, Oxford University Press, Oxford, UK, 1998.

[13] J. W. P. Hirschfeld and L. Storme, The packing problem in statistics, coding theory and finite projective spaces, J Statist Planning Infer 72 (1998), 355–380.

[14] J. W. P. Hirschfeld and L. Storme, The packing problem in statistics, coding theory and finite projective spaces: Update 2001, Finite geometries, In: Proceedings of Fourth Isle of Thorns Conference, A. Blokhuis, J. W. P. Hirschfeld, D. Jungnickel, and J. A. Thas (Editors), Kluwer, Dordrecht, Brighton, UK, April 2000. Developments in Math 3 (2001), 201–246.

[15] J. H. Kim and V. Vu, Small complete arcs in projective planes, Combinatorica 23 (2003), 311–363.

[16] G. Korchmáros, New examples of complete k-arcs in $PG(2,q)$, Europ J Combin 4 (1983), 329–334.

[17] F. J. MacWilliams and N. J. A. Sloane, The Theory of Error-Correcting Codes, North-Holland, Amsterdam, Netherlands, 1977.

[18] A. C. Mukhopadhyay, Lower bounds on $m_t(r,s)$, J Combin Theory A 25 (1978), 1–13.

[19] P. R. J. Östergård, Computer search for small complete caps, J Geom 69 (2000), 172–179.

[20] F. Pambianco and L. Storme, Small complete caps in spaces of even characteristic, J Combin Theory A 75 (1996), 70–84.

[21] B. Segre, On complete caps and ovaloids in three-dimensional Galois spaces of characteristic two, Acta Arith 5 (1959), 315–332.

[22] T. Szőnyi, Small complete arcs in Galois planes, Geom Dedicata 18 (1985), 161–172.

APPENDIX A: PROOF OF PROPOSITION 3.2

To prove Proposition 3.2, some ideas and results on translation caps from [10] will be useful. A *translation cap* in an affine space $AG(M,q)$ is a cap corresponding to a coset of an additive subgroup of $\mathbb{F}_q^M$. The prototype of a translation cap in $AG(2,q)$ is the parabola $\mathcal{P}=\{(a,a^2)\,|\,a\in\mathbb{F}_q\}$. The cartesian product of translation caps is still a translation cap, see [10, Lemma 2.7], whence the cap $\mathcal{P}^i$ defined as in (3.1) is a translation cap.

Let $m_1,m_2\in\mathbb{F}_q^*$, with $m_1\neq m_2$, $(m_1+m_2)^3\neq 1$, $m_i\neq 1$. Let $K^{(2s+1)}_{m_1,m_2}$, $A_1^{(2s+1)}$, $A_2^{(2s+1)}$, $A_3^{(2s+1)}$ be as in Section 3.

Throughout this section, let $(X_1,X_2,\ldots,X_{2s+2})$ denote homogenous coordinates for the points in $PG(2s+1,q)$. Also, let L_1 be the hyperplane of equation $X_1=0$, and L_2 be the $(2s-1)$-dimensional subspace of equations $X_1=X_2=0$. Denote $U=(1,0,\ldots,0)$.

Lemma A1 ([10, Proposition 2.5]). *Let $q > 2$. Then through every point in $AG(2i, q) \setminus \mathcal{P}^i$ there pass exactly $(q-2)/2$ secants of $\mathcal{P}^i$.*

Lemma A2. *Let h_1, h_2 be any distinct elements in $\mathbb{F}_q$. Let $C^{(i)}_{h_1,h_2} \subset AG(2i+1, q)$ be the cartesian product of $\{h_1, h_2\}$ by $\mathcal{P}^i$. Then $C^{(i)}_{h_1,h_2}$ is a cap. Moreover, through every point in $AG(2i+1, q) \setminus C^{(i)}_{h_1,h_2}$ there pass at least one secant of $C^{(i)}_{h_1,h_2}$.*

Proof. By [10, Proposition 2.9] the assertion holds for $h_1 = 0$, $h_2 = 1$. As clearly any $C^{(i)}_{h_1,h_2}$ is affinely equivalent to $C^{(i)}_{0,1}$, the claim follows. □

The set $PG(2s+1, q) \setminus L_1$ can be viewed as affine space $AG(2s+1, q)$, and similarly $L_1 \setminus L_2$ as an affine space $AG(2s, q)$. Note that for $s > 0$,

$$\begin{aligned} &A_1^{(2s+1)} \cup \{(1, m_1, 0\ldots, 0), (1, m_2, 0\ldots, 0)\} = C^{(s)}_{m_1,m_2},\\ &A_2^{(2s+1)} \cup \{(0, 1, 0\ldots, 0)\} = \mathcal{P}^s. \end{aligned} \tag{A1}$$

We are now in a position to prove that $K^{(2s+1)}_{m_1,m_2}$ is a cap.

Lemma A3. *The set $K^{(2s+1)}_{m_1,m_2}$ is a cap.*

Proof. We prove the claim by induction on s. The case $s = 0$ is trivial. Assume that $s > 0$. Throughout this proof, let $K = K^{(2s+1)}_{m_1,m_2}$ and $A_j = A_j^{(2s+1)}$ for $j = 1, 2, 3$.

Note that A_3 is a cap by inductive hypothesis. Lemma A2 together with (A1) yields that A_1 is a cap. Similarly, A_2 is a cap by (A1).

Assume that K is not a cap. Let P_1, P_2, and P_3 be collinear points in K. Assume first that no P_i coincides with U. Let

$$\begin{aligned} &P_1 = (X_1^{(1)}, X_2^{(1)}, \ldots, X_{2s+2}^{(1)}), \quad P_2 = (X_1^{(2)}, X_2^{(2)}, \ldots, X_{2s+2}^{(2)}),\\ &P_3 = (X_1^{(3)}, X_2^{(3)}, \ldots, X_{2s+2}^{(3)}). \end{aligned}$$

Let $v_j = \min\{v \mid X_v^{(j)} \neq 0\}$. Assume without loss of generality that $v_1 \le v_2 \le v_3$. Note that it is impossible that $v_1 < v_2$, as in this case the line through P_2 and P_3 is contained in the subspace $T : X_1 = X_2 = \cdots = X_{v_2-1} = 0$, whereas $P_1 \notin T$. Then $v_1 = v_2 \le v_3$. Note also that $v_1 = v_2 = v_3$ cannot occur, as in this case $\{P_1, P_2, P_3\} \subset A_j$ for some $j = 1, 2, 3$. Moreover, as it is not possible that $\{P_1, P_2, P_3\} \subset A_3$, we have that $v_1 \le 2$ holds. Then we are left with the following two cases:

Case (1): $v_1 = v_2 = 1, v_3 \ge 2$. Write $P_3 = \lambda P_1 + \mu P_2$, $P_1 = (1, m_k, a_1, a_1^2, \ldots, a_s, a_s^2)$, $P_2 = (1, m_l, b_1, b_1^2, \ldots, b_s, b_s^2)$, with $k, l \in \{1, 2\}$. Then clearly $\lambda = \mu$. Note that

$$P_3 = \lambda(P_1 + P_2) = (0, \lambda(m_k + m_l), \lambda c_1, \lambda c_1^2, \ldots, \lambda c_s, \lambda c_s^2),$$

where $c_u = a_u + b_u$, $u = 1, \ldots, s$.

Assume that $m_k = m_l$. Note that $c_u \neq 0$ for some u, otherwise $P_1 = P_2$. Let v be the minimum u for which $c_u \neq 0$. Then $v < s$, otherwise

$$P_3 = \lambda(P_1 + P_2) = (0, \ldots, 0, \lambda c_s, \lambda c_s^2) = (0, \ldots, 0, 1, c_s) \notin K\,.$$

Therefore $P_3=(0,\ldots,0,1,m_k,d_{v+1},d_{v+1}^2,\ldots,d_s,d_s^2)$ for some $k\in\{1,2\}$, $d_{v+1},\ldots,d_s\in\mathbb{F}_q$, whence

$$(1,m_k,d_{v+1},d_{v+1}^2,\ldots,d_s,d_s^2)=(\lambda c_v,\lambda c_v^2,\lambda c_{v+1},\lambda c_{v+1}^2,\ldots,\lambda c_s,\lambda c_s^2)$$

holds. This implies $\lambda=1/c_v$, $m_k=c_v$, and therefore $d_s=c_s/m_k$, $d_s^2=c_s^2/m_k$, which is impossible as $m_k\neq 1$.

Assume then that $m_k\neq m_l$. Then $P_3\in A_2$, and hence $P_3=(0,1,d_1,d_1^2,\ldots,d_s,d_s^2)$ for some $d_1,\ldots,d_s\in\mathbb{F}_q$. Then

$$(1,d_1,d_1^2,\ldots,d_s,d_s^2)=(\lambda(m_1+m_2),\lambda c_1,\lambda c_1^2,\ldots,\lambda c_s,\lambda c_s^2)$$

holds. As $P_3\in K$, $d_u\neq 0$ for some u. Then $d_u=c_u/(m_1+m_2)$, $d_u^2=c_u^2/(m_1+m_2)$, which yields $(m_1+m_2)^2=m_1+m_2$. But this is impossible as $m_1+m_2\notin\{0,1\}$.

Case (2): $v_1=v_2=2$, $v_3\geq 3$. Write $P_3=\lambda P_1+\mu P_2$, with $P_1=(0,1,a_1,a_1^2,\ldots,a_s,a_s^2)$, $P_2=(0,1,b_1,b_1^2,\ldots,b_s,b_s^2)$. Then clearly $\lambda=\mu$. Note that

$$P_3=\lambda(P_1+P_2)=(0,0,\lambda c_1,\lambda c_1^2,\ldots,\lambda c_s,\lambda c_s^2),$$

where $c_l=a_l+b_l$. Then a contradiction can be obtained as in case (1), $m_k=m_l$.

To complete the proof we only need to show that the point U is not collinear with two points P_2, P_3 in $K\setminus\{U\}$. Clearly, either P_2 or P_3 belongs to A_1. Assume without loss of generality that $P_2=(1,m_k,a_1,a_1^2,\ldots,a_s,a_s^2)\in A_1$. We first deal with the case $P_3\in A_1$. Let $P_3=(1,m_l,b_1,b_1^2,\ldots,b_s,b_s^2)$. Write $U=\lambda P_2+\mu P_3$. If $a_u=0$ and $b_u\neq 0$ for some u, then $\mu=0$, which is impossible as $U\neq P_2$. Similarly the case $a_u\neq 0$ and $b_u=0$ can be ruled out. By definition $a_u\neq 0$ for some u. Therefore $b_u\neq 0$. Note that

$$D\begin{pmatrix}1 & 0 & 0\\ 1 & a_u & a_u^2\\ 1 & b_u & b_u^2\end{pmatrix}=0,$$

yields $a_u=b_u$. Then, from

$$D\begin{pmatrix}1 & 0 & 0\\ 1 & m_k & a_u\\ 1 & m_l & b_u\end{pmatrix}=0$$

it follows $m_k=m_l$. Then $P_2=P_3$ follows, a contradiction. Assume then that $P_3\in A_2\cup A_3$. In this case

$$P_3=\lambda(U+P_2)=(0,m_k,a_1,a_1^2,\ldots,a_s,a_s^2).$$

Therefore $P_3\in A_2$, and hence

$$P_3=(0,1,b_1,b_1^2,\ldots,b_s,b_s^2)$$

for some $b_u\in\mathbb{F}_q$. Then, for any u with $b_u\neq 0$, we have $a_u/m_k=b_u$, $a_u^2/m_k=b_u^2$, which yields $m_k=1$, a contradiction. □

The completeness properties of the cap $K_{m_1,m_2}^{(2s+1)}$ can be now investigated.

Lemma A4. *Assume that $q>4$. The cap $K=K^{(2s+1)}_{m_1,m_2}$ covers all the points in $PG(2s+1,q)$, with the exception of points*

$$(1,m,0,0\ldots,0), \quad m\in\mathbb{F}_q, \ m\neq 0,$$

when $s>0$.

Proof. We prove the assertion by induction on s. The case $s=0$ is trivial. Assume then that $s>0$. Note that by inductive hypothesis any point $P\in L_2$ is covered by the secants of $A_3^{(2s+1)}$, provided that P is not of type $(0,0,1,m,0,\ldots,0)$, $m\neq 0$. We first prove that any $P=(0,0,1,m,0,\ldots,0)$, $m\neq 0$, is actually covered by the secants of K. If $s=1$, that is $P=(0,0,1,m)$, then clearly

$$P=(0,0,1,0)+m(0,0,0,1);$$

otherwise,

$$P=(1,m_1,a,a^2,0,\ldots,0)+(1,m_1,b,b^2,0,\ldots,0)$$

for any a,b such that $a+b=m$. This proves that all the points in L_2 are covered by the secants of K.

We now consider points $P\in L_1\setminus L_2$. From Lemma A1 together with (A1), it follows that if $P\notin A_2^{(2s+1)}$, $P\neq(0,1,0\ldots,0)$, then P is contained in $(q-2)/2$ distinct lines joining two distinct points in $A_2^{(2s+1)}\cup\{(0,1,0\ldots,0)\}$. The hypothesis $q>4$ implies $(q-2)/2>1$, whence if $P\neq(0,1,0,\ldots,0)$, then P is covered by the secants of $A_2^{(2s+1)}$. Actually, also $P=(0,1,0,\ldots,0)$ is covered by the secants of K, as

$$P=\frac{1}{m_1+m_2}((1,m_1,a_i,a_i^2,\ldots,a_s^2)+(1,m_2,a_i,a_i^2,\ldots,a_s^2)).$$

Now let $P=(1,m,d_1,d_1',\ldots,d_s,d_s')\in PG(2s+1,q)\setminus L_1$, with either $d_l\neq 0$ or $d_l'\neq 0$ for some l. Assume that $m=m_j$, $j\in\{1,2\}$. Let $L^{(m_j)}$ be the hyperplane of equation $X_2=m_jX_1$. Note that $L^{(m_j)}\setminus L_2$ is an affine space $AG(2s,q)$, and that

$$(K\cap(L^{(m_j)}\setminus L_2))\cup\{(1,m_j,0,\ldots,0)\}=\mathcal{P}^s.$$

Then by Lemma A1, through every point $P=(1,m_j,d_1,d_1',\ldots,d_s,d_s')\in L^{(m_j)}\setminus L_2$ with $d_u'\neq d_u^2$ for some $u\in\{1,\ldots,s\}$, there pass $(q-2)/2$ lines joining two distinct points in $(K\cap(L^{(m_j)}\setminus L_2))\cup\{(1,m_j,0,\ldots,0)\}$. The hypothesis $q>4$ ensures $(q-2)/2>1$, whence P is covered by the secants of $K\cap(L^{(m_j)}\setminus L_2)$.

Finally, we consider points $P=(1,m,d_1,d_1',\ldots,d_s,d_s')\in PG(2s+1,q)\setminus L_1$, $m\notin\{m_1,m_2\}$, with either $d_u\neq 0$ or $d_u'\neq 0$ for some u. Let l be the smallest index for which either $d_l\neq 0$ or $d_l'\neq 0$. Let $t_m=(m+m_1)/(m_1+m_2)$, $s_m=m+m_1$, $r_m=m+m_2$. From Lemma A2 together with (A1), it follows that $A_1^{(2s+1)}\cup\{(1,m_1,0,\ldots,0),(1,m_2,0,\ldots,0)\}$ is a cap which covers all the points in $PG(2s+1)\setminus L_1$. Therefore, P is covered by the secants of $A_1^{(2s+1)}$ provided that P does not belong to $C_1\cup C_2$, where C_j is the union of lines joining $(1,m_j,0,\ldots,0)$ to some point in $A_1^{(2s+1)}$.

Straightforward computation yields that points in $(C_1 \cup C_2) \setminus L_1$ such that $X_2 = mX_1$ are

$$\left(1, m, \frac{m+m_j}{m_1+m_2}a_1, \frac{m+m_j}{m_1+m_2}a_1^2, \ldots, \frac{m+m_j}{m_1+m_2}a_s, \frac{m+m_j}{m_1+m_2}a_s^2\right),$$

$a_1 \in \mathbb{F}_q$, $(a_1, \ldots, a_s) \neq (0, \ldots, 0)$, $j = 1, 2$. If P is a point of this type, then either $d'_l = d_l^2/t_m$ or $d'_l = d_l^2/(t_m+1)$, according to whether $j=1$ or $j=2$. Therefore P is covered by the secants of $K \setminus L_1$ provided that $d'_l \neq d_l^2/t_m$ and $d'_l \neq d_l^2/(t_m+1)$.

Now we establish whether P belongs to a line joining a point in $K \setminus L_1$ to a point in $K \cap L_1$. We first look for points

$$P_1 = (1, m_1, a_1, a_1^2, \ldots, a_s, a_s^2), \quad P_3 = (0, 1, b_1, b_1^2, \ldots, b_s, b_s^2)$$

such that $P = P_1 + s_m P_3$, that is,

$$a_u + s_m b_u = d_u, \quad (a_u^2 + s_m b_u^2) = d'_u, \quad u = 1, \ldots, s.$$

The system

$$\begin{cases} a_u^2 + s_m^2 b_u^2 = d_u^2 \\ a_u^2 + s_m b_u^2 = d'_u \end{cases}$$

has precisely one solution when $s_m \neq 1$, that is, $m \neq m_1 + 1$. This solution corresponds to two points in K provided that not all a_u and not all b_u are equal to 0, that is

$$P \notin \{(1, m, a_1, a_1^2, \ldots, a_s, a_s^2), (1, m, s_m b_1, s_m b_1^2, \ldots, s_m b_s, s_m b_s^2)\}.$$

Next we look for points

$$P_2 = (1, m_2, a_1, a_1^2, \ldots, a_s, a_s^2), \quad P_3 = (0, 1, b_1, b_1^2, \ldots, b_s, b_s^2)$$

such that $P = P_2 + r_m P_3$. Arguing as above, we deduce that P_2 and P_3 exist, and both belong to K, provided that $m \neq m_2 + 1$ and

$$P \notin \{(1, m, a_1, a_1^2, \ldots, a_s, a_s^2), (1, m, r_m b_1, r_m b_1^2, \ldots, r_m b_s, r_m b_s^2)\}.$$

To sum up, if P is not covered by the secants of K then all the conditions (A), (B) and (C) below hold:

(A) either $d'_l = d_l^2/t_m$ or $d'_l = d_l^2/(1+t_m)$;
(B) either $m = m_1 + 1$, or $d'_l = d_l^2$ or $d'_l = d_l^2/s_m$;
(C) either $m = m_2 + 1$, or $d'_l = d_l^2$ or $d'_l = d_l^2/r_m$.

As either $d_l \neq 0$ or $d'_l \neq 0$, from (A) it follows that actually both d_l and d'_l are different from 0. Let $\rho = d_l^2/d'_l$. Let $E_1 = \{t_m, 1+t_m\}$, $E_2 = \{1, s_m\}$, $E_3 = \{1, r_m\}$.

Assume first that $m \neq m_1 + 1$ and $m \neq m_2 + 1$. Then ρ belongs to all sets E_1, E_2, E_3. But this is impossible as the intersection $E_1 \cap E_2 \cap E_3$ is empty.

Assume now that $m = m_1 + 1$. Then $\rho \in E_1 \cap E_3$. This yields that either $r_m = t_m$ or $r_m = t_m + 1$; that is, either

$$m_1 + m_2 + 1 = \frac{1}{m_1 + m_2} \quad \text{or} \quad m_1 + m_2 = \frac{1}{m_1 + m_2}.$$

The former case yields that (m_1+m_2) is a root of T^2+T+1; the latter $m_1+m_2=1$. Both are not possible as $(m_1+m_2)^3\neq 1$.

Finally, assume that $m=m_2+1$. Then $\rho\in E_1\cap E_2$, which yields that either $s_m=t_m$ or $s_m=t_m+1$; that is, either

$$m_1+m_2+1=\frac{m_1+m_2+1}{m_1+m_2} \quad \text{or} \quad m_1+m_2=\frac{m_1+m_2+1}{m_1+m_2}.$$

A contradiction is then obtained as for the case $m=m_1+1$. This completes the proof. □

Now we are in a position to conclude the proof of Proposition 3.2. It is enough to note that in the proof of Lemma A4 the point $(1,0,\ldots,0)$ is not used to prove that each point of $PG(2s+1,q)$ different from $(1,m,0,0,\ldots,0)$ is covered by the secant of $K^{(2s+1)}_{m_1,m_2}$.