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PERIODIC CANARD TRAJECTORIES WITH MULTIPLE SEGMENTS FOLLOWING THE UNSTABLE PART OF CRITICAL MANIFOLD

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Abstract. We consider a scalar fast differential equation which is periodically driven by a slowly varying input. Assuming that the equation depends on n scalar parameters, we present simple sufficient conditions for the existence of a periodic canard solution, which, within a period, makes n fast transitions between the stable branch and the unstable branch of the folded critical curve. The closed trace of the canard solution on the plane of the slow input variable and the fast phase variable has n portions elongated along the unstable branch of the critical curve. We show that the length of these portions and the length of the time intervals of the slow motion separated by the short time intervals of fast transitions between the branches are controlled by the parameters.

1. **Introduction.** Canard solutions appearing in singularly perturbed systems are an interesting mathematical object and an important model of critical phenomena observed, for example, in chemical kinetics [2,5,8,9,15,16,37]. Recall that a canard trajectory is characterized by the property that it follows the unstable branch of the critical manifold for some time after following the stable branch of this manifold. Most of the literature explores periodic canard solutions.

In systems with a scalar slow component, such as a planar slow-fast autonomous system, or a scalar fast nonautonomous differential equation with a slowly varying periodic forcing, canard solutions are not generic unless the system includes a parameter; *i.e.*, one scalar parameter is required to unfold the canard phenomenon. Moreover, a canard trajectory typically exists for a small interval of the parameter values[1] and is sensitive to the variation of the parameter within this interval [2,9,35,36]. If we assume, for simplicity, that the critical manifold (which is a

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[1] The phenomenon known as 'the short life of canards'.

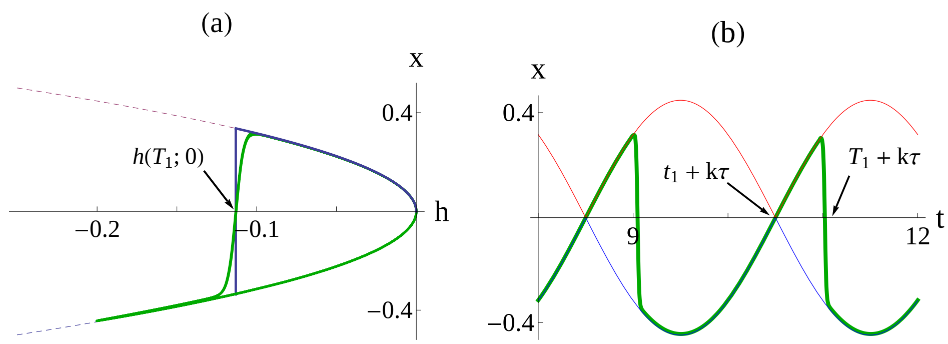

FIGURE 1. Periodic stable canard solution of the equation $\varepsilon\dot{x} = h(t;a) + x^2$ with the input $h(t;a) = 0.1\sin(\pi t) - 0.1 + a$ for $a = 0.0034969174919347302$ and $\varepsilon = 0.005$; the period is $\tau = 2$. (a) The trace $(h(t), x(t))$ of the canard solution (solid line) on the (h,x)-plane. The critical curve S defined by $h + x^2 = 0$ is shown by the dashed line. (b) The graph of the same canard solution on the (t,x)-plane (thick). Thin lines represent the stable (below the time axis) and unstable (above the time axis) branches $x = x^s(h(t;a^*))$ and $x = x^u(h(t;a^*))$ of the critical curve, respectively; they are obtained from the equation $h(t;a) + x^2 = 0$ with the parameter a set to $a = a^* = 0$, that is the value for which the maximum of h is zero. The self-intersecting curve $h(t;a^*) + x^2 = 0$ allows for formation of the canard solutions for $a = a(\varepsilon)$ close to a^* (see, for example, [37]). For $a < a^*$ this curve, $h(t;a) + x^2 = 0$, splits into two infinite branches in the half-planes $x < 0$ and $x > 0$; for $a > a^*$, it splits into a periodic sequence of closed curves going to infinity along the t-axis.

curve, in this case) consists of a stable branch S_s and an unstable branch S_u meeting at one fold point, then, within one period, a periodic canard solution makes one fast transition from the unstable curve S_u to the stable curve S_s and follows slowly the critical curve for the rest of the time. That is, the canard solution follows the stable curve S_s, passes the fold point and continues along the unstable curve S_u for a while, whereupon it makes a fast transition back towards the stable curve S_s and, finally, continues along S_s to the initial point, see Fig. 1. By tuning the parameter, one controls the amount of time which the canard solution spends near the curve S_u from the moment it passes the fold point until the fast transition towards the curve S_s (or, equivalently, the length of the portion of the canard trajectory elongated along the unstable curve S_u).

In this paper, we assume that, in the above simple setting, the system includes several control parameters. We ask a question, whether, by tuning those parameters, one can obtain a periodic canard solution which has multiple portions elongated along the unstable curve S_u and makes multiple transitions between S_u and S_s during the period. That is, within one period, the solution passes the fold of the critical curve several times, from the stable part S_s to the unstable part S_u, and remains near the unstable curve S_u after passing the fold point for varying intervals of time before returning quickly to the stable curve S_s.

Taking the scalar equation of the form $\varepsilon\dot{x} = h(t; a_1, \dots, a_n) - f(x)$, $\varepsilon \ll 1$, with a periodic input h that depends on n scalar parameters a_i as the simplest example, we present generic sufficient conditions for the existence of a periodic canard solution which makes n fast transitions from the unstable curve S_u to the stable curve S_s within a period. The canard solution exists for a small range of the parameter values. The closed trace of this solution on the plane (h, x) has n portions elongated along the unstable branch S_u of the critical curve. We show that the lengths of these portions and the time intervals between the fast transitions are controlled by the parameters.

The paper is organized as follows. In the next section we describe the setting. The main result is presented in Section 3 followed by the proofs in Section 4. The last section contains a discussion.

2. Problem statement.

Consider the equation

$$\varepsilon \frac{dx}{dt} = h(t) - f(x), \qquad x \in \mathbb{R}, \tag{1}$$

with a small parameter $0 < \varepsilon \ll 1$. Here $h(t)$ is a smooth periodic input depending on one or more scalar parameters; $f = f(x)$ has one maximum and no minima. Without loss of generality, assume that f reaches its maximum at the point $x = 0$ and $f(0) = 0$. For example,

$$\varepsilon \frac{dx}{dt} = h(t) + x^2.$$

We also assume that the minimum of $h(t)$ belongs to the range of each of the branches $f(x), x \le 0$, and $f(x), x \ge 0$, of the function f. These assumptions about the nonlinearity f can be summarised as follows.

(i) *The Lipschitz continuous nonpositive function $f = f(x)$ strictly increases for $x < 0$, strictly decreases for $x > 0$ and satisfies $f(0) = 0$ at its maximum point. The input satisfies*

$$\inf h > \inf_{x \le 0} f(x), \qquad \inf h > \inf_{x \ge 0} f(x). \tag{2}$$

For example, condition (2) is satisfied for a bounded input h if $f(x) \to -\infty$ as $x \to \pm\infty$.

As the derivative of x in equation (1) is multiplied by $\varepsilon \ll 1$, the variable x varies with time much faster (roughly speaking $1/\varepsilon$ times faster) than the input h unless the trajectory is close to the curve $h - f(x) = 0$ where the right-hand side of (1) vanishes. Therefore, the trajectories are almost vertical whenever away from this curve, which is called the critical curve S (critical manifold) of equation (1), see Fig. 1. Note that the equation $h - f(x) = 0$ for the critical curve is obtained by setting formally ε to zero in (1).

A useful perspective on the solutions is obtained by considering dynamics on the characteristic time scale of the fast variable x. In terms of the so-called fast time $s = t/\varepsilon$, equation (1) has the form

$$\frac{dx}{ds} = h(\varepsilon s) - f(x)$$

with a slowly varying input h. Neglecting the variation of h on time intervals of order $O(1)$ of the fast time s, one obtains the autonomous equation

$$\frac{dx}{ds} = \tilde{h} - f(x), \tag{3}$$

where the parameter $\tilde{h}$ is the frozen value of the input. This equation approximates dynamics of equation (1) on the fast time scale for small ε. Equation (3) has a pair of equilibria for negative $\tilde{h}$ which collide and disappear in a fold bifurcation for $\tilde{h} = 0$. The negative root $x^s(\tilde{h})$ of the equation $\tilde{h} = f(x)$ with $\tilde{h} < 0$ is a stable equilibrium of equation (3); the positive root $x^u(\tilde{h})$ is an unstable equilibrium; and, for $\tilde{h} > 0$ equation (3) does not have equilibria. The graph of the stable equilibrium, S_s, and the graph of the unstable equilibrium, S_u, together with their meeting point at the origin, form the critical curve S of equation (1) on the plane $(\tilde{h}, x) = (h, x)$.

If $\tilde{h} < 0$ and $x_0 < x^u(\tilde{h})$, then the solution of equation (3), which starts from the point x_0, will be attracted by the stable equilibrium $x^s(\tilde{h})$ of this equation. Tikhonov's theorem applied to the simple equation (1) defines conditions under which solutions of (1) follow the branch of stable equilibria of the associated system (3). Namely, if $h(t_0) < 0$ and $x(t_0) = x_0 < x^u(h(t_0))$, then the solution of equation (1) starting from the point x_0 at the moment t_0 will quickly (on the time scale t) approach the stable equilibrium of system (3) with $\tilde{h} = h(t_0)$; then, the solution will follow the branch S_u of stable equilibria $x^s(h(t))$ of (3) as long as $h(t)$ is negative. In other words, the solution approaches to, and then follows, the branch S_u of the critical curve S for $h(t) < 0$. If $h(t)$ moves to the domain of positive values then the solution passes near the tip of the critical curve S at the origin and explodes (quickly grows to large positive values), as the right-hand side of (1) is positive everywhere in the domain $h > 0$. More precisely, if $h(t)$ gets positive, then the solution explodes for any sufficiently small ε.

Interesting solutions, known as canard solutions, can appear when $h(t)$ has a maximum value close to zero at a turning point t_*. A canard solution follows the branch S_s of stable equilibria of equation (3) towards the meeting point of S_s with S_u at the origin before the instant t_* and then continues along the unstable branch S_u for a certain while after the moment t_*, before it either explodes or returns to the stable branch. If the maximum of the input $h(t)$ is negative, then assumption (i) ensures that the projection of the stable branch S_s of the critical curve S on the h-axis contains the range of the input $h(t)$ in its interior, and, consequently, Tikhonov's theorem implies that any solution of (1) which, at some point, follows the branch S_s will stay near S_s forever if ε is sufficiently small. On the other hand, if the maximum of $h(t)$ is positive, then any solution following S_s explodes when $h(t)$ gets positive and before it returns to the negative values, if ε is sufficiently small, because both branches S_s and S_u are situated in the left half-plane $h \leq 0$ of the (h, x) plane. In both cases, a solution cannot be a canard. Hence, a canard solution can be obtained only if the maximum of $h(t)$ is placed in a small neighborhood of zero, which becomes smaller and vanishes with $\varepsilon \to 0$. Such fine-tuning of the maximum value of the input near zero can be achieved by introducing a control parameter. A typical statement for equation

$$\varepsilon \frac{dx}{dt} = h(t; a) - f(x), \qquad x \in \mathbb{R}, \tag{4}$$

where the function f is smooth and the input h depends on a scalar parameter a is as follows. Suppose that the function $h = h(t; a)$ is smooth, and for every a this function strictly increases on a time interval $[t_0, T(a)]$ and strictly decreases on the time interval $[T(a), t_1]$ with $t_0 < T(a) < t_1$. Suppose that for some $a = a^*$ the value of h at the maximum point $t = T(a^*)$ is zero, *i.e.* $h(T(a^*); a^*) = 0$, and

$$\frac{\partial h}{\partial a}(T(a^*); a^*) \neq 0.$$

This inequality is a generic transversality condition, which ensures that the maximum value $h(T(a); a)$ of the input gets from negative to positive values as a gets across the value a_*, that is we can efficiently manipulate the maximum of the input by controlling a. Under these assumptions, given any initial value $x_0 < x^u(h(t_0; a^*))$ and any instant T_1 satisfying $T(a^*) < T_1 < t_1$, for every sufficiently small $\varepsilon > 0$ there is a parameter value $a = a(\varepsilon)$ close to a^* such that the solution $x(t; a)$ of equation (4) starting from the initial condition $x(t_0; a) = x_0$ satisfies $x(T_1; a) = 0$. The latter condition $x(T_1; a) = 0$ defines the canard shape of the curve $(h(t; a), x(t; a))$ on the (h, x) plane, which we can call the trace Γ of the solution $x(t; a)$. As $\varepsilon \to 0$, this curve approaches the curve Γ_{lim} (see Fig. 2), which consists of the vertical segment connecting the initial point $(h(t_0; a^*), x_0)$ with the graph of the equilibrium curve $h = f(x)$ of equation (3), the arc of the equilibrium curve connecting the point $(f(x_0), x_0)$ on the lower stable branch of this curve with the point $(h(T_1; a^*), x_1)$ on the upper unstable branch of this curve, the vertical segment connecting the latter point with the lower stable branch of the equilibrium curve at the point $(h(T_1; a^*), x_2)$, and the arc of the stable branch of the equilibrium curve between the points $(h(T_1; a^*), x_2)$ and $(h(t_1; a^*), x_3)$. For a given small ε the parameter a controls the length of the time interval $T(a) < t < T_1$ during which the solution stays close to the unstable equilibrium of the associated equation (equivalently, a controls the length of those segment of the solution trace Γ where it follows the unstable branch of equilibria of the associated system).

A similar result holds for periodic canard solutions of equation (4) with a periodic input. Conditions of Theorem 1 below ensure that equation (4) has a periodic canard solution $x(t)$ satisfying $x(T_1 + k\tau) = 0$ for some $a = a(\varepsilon, T_1)$ (see, Fig. 1(b)), where T_1 can be chosen arbitrarily, τ is the period of h and x, and $a(\varepsilon, T_1) \to a^*$ as $\varepsilon \to 0$. In the example on Fig. 1, the canard solution has one segment following the unstable manifold and one segment of the fast motion per period. The goal of this paper is to show that if the input h has n local maxima on a period and depends on n scalar parameters which control the local maximum values and can place them to zero simultaneously, then equation (4) has a periodic canard solution that has n segments following the unstable manifold x^u and the length of these segments can be controlled by the parameters of the input. Double canards in predator-prey systems have been studied in [32].

3. **Main result.** Consider equation (4) with the input $h(t; a) = h(t; a_1, \ldots, a_n)$ depending on a vector parameter $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$. Suppose that the input satisfies the following assumptions.

(ii) *The continuous function $h(t; a)$ is twice continuously differentiable with respect to t and continuously differentiable with respect to $a = (a_1, \ldots, a_n)$. This function is τ-periodic in t for every a with a period $\tau > 0$:*

$$h(t; a) = h(t + \tau; a), \qquad t \in \mathbb{R}, \ a \in \mathbb{R}^n.$$

(iii) *For $a = 0$, there are exactly n points $0 < t_1 < \cdots < t_n < \tau$ in the interval $0 \le t \le \tau$, where the nonpositive function $h(t; 0)$ achieves the value zero:*

$$h(t_i; 0) = 0, \quad i = 1, \ldots, n; \qquad h(t; 0) < 0 \quad \text{for} \quad t \neq t_i + \tau m, \ m \in \mathbb{Z}.$$

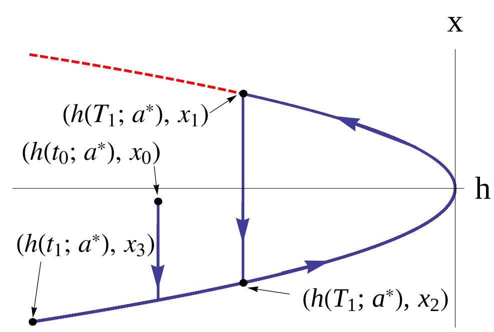

FIGURE 2. The schematic limit Γ_{lim} of the trace Γ of a simple (non-periodic) canard solution on the (h, x)-plane as $\varepsilon \to 0$ and $a(\varepsilon) \to a^*$. In this example, the canard solution starts near the point $(h(t_0; a^*), x_0)$, follows closely the vertical segment down to the parabola, continues along the lower branch of the parabola to the right towards the origin, then goes to the left along the upper branch of the parabola, drops along the vertical segment back to the lower branch, and proceeds to the left along the lower branch. The vertical segment connecting the branches of the parabola can be placed anywhere between the point $(h(t_1; a^*), x_3)$, corresponding to the minimum of h, and the origin by a proper choice of the moment T_1 when the canard solution is zero, $x(T_1) = 0$. Once T_1 is fixed, there is an $a = a(\varepsilon)$ close to a^* for every small $\varepsilon > 0$, such that the trace of the canard solution follows the schematic on this picture, approaching it in the limit $\varepsilon \to 0$, $a(\varepsilon) \to a^*$.

Moreover,

$$\frac{\partial^2 h}{\partial t^2}(t_i; 0) < 0, \quad i = 1, \dots, n. \tag{5}$$

(iv) *The matrix* $A = \{\alpha_{ij}\}$ *defined by*

$$\alpha_{ij} = \frac{\partial h}{\partial a_i}(t_j; 0), \quad i, j = 1, \dots, n, \tag{6}$$

is nondegenerate,

$$\det A \neq 0. \tag{7}$$

For parameter values from some vicinity of the point $a = 0$, the nondegeneracy conditions (5), (7) ensure that there are exactly n isolated points in the interval $0 < t < \tau$ where the input $h(\cdot; a)$ achieves maximum values, which are sufficiently close to zero. Moreover, the parameters have full control of these maximum values. In particular, given any set of small values $b_1, \dots, b_n$ there is a vector of parameters $a = (a_1, \dots, a_n)$ such that the n largest maximum values of the twice continuously differentiable input $h(\cdot; a)$ on the interval $0 < t < \tau$ equal $b_1, \dots, b_n$.

We are interested in τ-periodic canard solutions of equation (4).

Theorem 3.1. *Suppose assumptions* (i) – (iv) *are satisfied with* $\inf h = \inf\{h(t; a) : t \in \mathbb{R}, a \in \mathbb{R}^n\}$ *in relations* (2). *Then, given any* n *points* $T_k \in (t_k, t_{k+1})$ *with* $t_{n+1} = t_1 + \tau$, *for every sufficiently small* $\varepsilon > 0$ *there is an* $a = a(\varepsilon)$, *where*

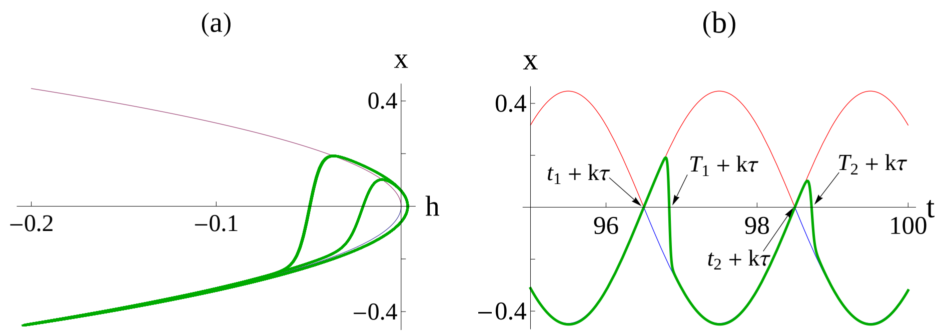

FIGURE 3. Periodic double canard solution of the equation $\varepsilon\dot{x} = h(t;a)+x^2$ with the input $h(t;a) = (0.1+a_1)\sin(\pi t)+a_2\sin(\pi t/2)-0.1$ for $a_1 = 0.00355314$, $a_2 = 8\times10^{-6}$ and $\varepsilon = 0.005$; the period is $\tau = 4$. (a) The trace of the canard solution on the (h,x)-plane. (b) The time series $x_\varepsilon(t)$ of the canard solution (thick). Thin lines above the time axis and below the time axis are the stable branch $x = x^s(h(t;0))$ and the unstable branch $x = x^u(h(t;0))$ of the critical curve, respectively.

$a(\varepsilon) \to 0$ as $\varepsilon \to 0$, such that equation (4) has a τ-periodic solution $x = x_\varepsilon$ satisfying $x_\varepsilon(T_k) = 0$ for $k = 1,\dots,n$.

Fig. 3, 4 show examples of double and triple canards for equation (4) with the quadratic nonlinearity $f(x) = -x^2$.

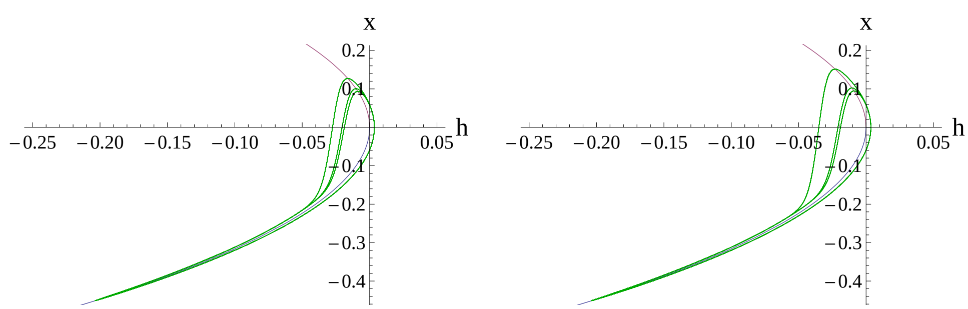

FIGURE 4. Examples of the periodic triple canard for the equation $\varepsilon\dot{x} = h(t;a) + x^2$ with the input $h(t;a) = (0.1 + a_1)\sin(\pi t) + a_2\sin(\pi t/3) + a_3\cos(\pi t/3) - 0.1$. The parameter a_1 is $a_1 = 0.0035480$ for the left panel and $a_1 = 0.0035494$ for the right panel. Other parameters are $a_2 = 10^{-6}$, $a_3 = 10^{-5}$, $\varepsilon = 0.005$.

The τ-periodic canard solution x_ε defined in Theorem 3.1 satisfies the pointwise limit relations

$$x_\varepsilon(t) \to x^u(h(t;0)), \qquad t \in \bigcup_{k=1}^{n}(t_k, T_k),$$

$$x_\varepsilon(t) \to x^s(h(t;0)), \qquad t \in \bigcup(T_k, t_{k+1})$$

as $\varepsilon \to 0$, see Fig. 3(b). The moments $T_1, \dots, T_n$ of switching from the unstable branch of equilibria of equation (3) to the stable branch of equilibria (and thus the length of the segments of the unstable branch followed by the solution) are controlled by the parameters $a_1, \dots, a_n$; these moments can be chosen arbitrarily from the intervals (t_k, t_{k+1}). In particular, if a is a scalar parameter, then x_ε is a usual periodic canard solution which stays near the stable branch of equilibria during the time interval $T_1 - \tau < t < t_1$ and near the unstable branch of equilibria during the time interval $t_1 < t < T_1$, and makes a fast transition from the unstable branch to the stable branch approximately at the moment $t = T_1$, see Fig. 1(b).

The proof of Theorem 3.1 consists in showing the solvability of the simultaneous system of equations $x_\varepsilon(T_k; a_1, \dots, a_n) = 0$, $k = 1, \dots, n$, with respect to the unknown variables $a_1, \dots, a_n$, where x_ε is a solution of equation (4) satisfying the initial condition $x_\varepsilon(T_n - \tau; a_1, \dots, a_n) = 0$. The proof is presented in Section 4.

4. **Proof of Theorem 3.1.**

4.1. **Evolution operator.** Denote by $x = x(\cdot; t_*, x_*, \varepsilon, a)$ the solution of equation (4) satisfying the initial condition $x(t_*) = x_*$. Solutions are not necessarily nonlocally extendable. Under the conditions of the theorem, a solution x is always bounded from below on every finite interval of time where it is defined; however, it can be unbounded from above on a finite interval $t_* \le t < t_{max} < \infty$, in which case $x(t) \to +\infty$ as $t \to t_{max} - 0$. In order to have a well defined time evolution operator (map) for arbitrary intervals of time, we define a *quasisolution* of equation (4) by the formula

$$x_R(\cdot; t_*, x_*, \varepsilon, a) = \begin{cases} x(\cdot; t_*, x_*, \varepsilon, a), & t_* \le t < t_R, \\ R, & t \ge t_R, \end{cases}$$

where $x_* < R$, $R > 0$ is a fixed number satisfying

$$f(R) < \inf h,$$

and

$$t_R = t_R(t_*, x_*, \varepsilon, a) = \inf\{t \ge t_* : x(t; t_*, x_*, \varepsilon, a) = R\}.$$

In particular, $t_R = \infty$ and $x_R(\cdot) = x(\cdot)$ for all $t \ge t_*$ whenever $x(t) < R$ for all $t \ge t_*$. Therefore, every periodic quasisolution x_R with $x_R(t_*) = x_* < R$ is a solution of equation (4). Moreover, condition (i) implies that if $t_R < \infty$ then the solution $x(\cdot)$ strictly increases after the moment t_R, *i.e.,* after entering the domain $x \ge R$. Hence, every periodic solution is a quasisolution.

4.2. **Scheme of the proof.** If $F\colon \mathbb{R}^n \mapsto \mathbb{R}^n$ is a continuous mapping, $\Omega \subset \mathbb{R}^n$ is a bounded open set, and $y \in \mathbb{R}^n$ does not belong to the image $F(\partial\Omega)$ of the boundary $\partial\Omega$ of Ω, then the symbol $\deg(F, \Omega, y)$ denotes the *topological degree* [10] of F at y with respect to Ω. If $0 \notin F(\partial\Omega)$, then the integer number $\gamma(F, \Omega) = \deg(F, \Omega, 0)$, called the *rotation* of the vector field F at $\partial\Omega$, is well defined. It measures the algebraic number of zeros of the mapping F in Ω. A detailed description of properties of the number $\gamma(F, \Omega)$ can be found, for example, in [29].

We use the topological degree argument to prove the existence of a τ-periodic quasisolution x_R satisfying the relations

$$x_R(T_1) = \dots = x_R(T_n) = 0 \tag{8}$$

for a given set of moments $T_k \in (t_k, t_{k+1})$. If relations (8) hold, then the τ-periodicity of x_R is equivalent to the additional relationship $x_R(T_0) = 0$ at the moment $T_0 = T_n - \tau < T_1$. Hence, we define the vector field

$$\Psi(a) = \Psi_\varepsilon(a) = (x_R(T_1; T_0, 0, \varepsilon, a), x_R(T_2; T_1, 0, \varepsilon, a), \ldots, x_R(T_n; T_{n-1}, 0, \varepsilon, a))$$

in the space of vectors $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$. Suppose that this vector field is nonzero on the boundary $\partial\Omega$ of an open bounded domain $\Omega \subset \mathbb{R}^n$ and, hence, the rotation $\gamma(\Psi, \Omega)$ is defined. If this rotation is nonzero, then the equation $\Psi(a) = 0$ has a solution $a_* \in \Omega$ [29]. The definition of Ψ implies that $x_R(\cdot; T_0, 0, \varepsilon, a_*)$ is a τ-periodic solution of equation (4) satisfying (8); here the periodicity follows from $x_R(T_0; T_0, 0, \varepsilon, a_*) = x_R(T_n; T_0, 0, \varepsilon, a_*) = 0$.

The rest of the proof consists in constructing the domain Ω and proving the inequality $\gamma(\Psi, \Omega) \neq 0$. We use a simple version of the Rotation Product Theorem [29], which is applied to the direct sum of scalar vector fields.[2] Similar constructions based on the homotopy of a vector field to the direct sum of scalar vector fields and an infinite dimensional identity field were applied in functional spaces to prove the existence of periodic solutions and cycles in [1, 4, 6, 7, 13, 14, 20–28] (however, they addressed systems without slow-fast structure). More complex topological degree argument was used to prove the existence of canard solutions in three-dimensional systems with a two-dimensional slow component in [31, 33, 34].

A rather general product formula for the topological degree of vector fields in the product of two Banach spaces was obtained in abstract setting, and applied to an existence problem for differential equations, in [19].

4.3. **Definition of the domain Ω.** Consider the linear functions

$$\ell_k(a) = \sum_{i=1}^{n} \alpha_{ik} a_i, \qquad a = (a_1, \ldots, a_n) \in \mathbb{R}^n,$$

where the coefficients α_{ij} are defined in (6). Define the parallelepiped

$$\Omega = \Omega_\delta = \{a \in \mathbb{R}^n : |\ell_k(a)| < \delta,\ k = 1, \ldots, n\}$$

and its closure $\bar{\Omega}$, which depend on a parameter $\delta > 0$. Assumption $\det A \neq 0$ in (7) ensures that Ω is an open domain.

Lemma 4.1. *There exists a $\delta > 0$ and closed intervals $\Delta_k \subset (T_{k-1}, T_k)$, $k = 1, \ldots, n$, of nonzero length with $t_k \in \Delta_k$ such that*

$$h(t; a) > \delta/2 \qquad \text{if} \qquad t \in \Delta_k,\ \ell_k(a) = \delta,\ a \in \bar{\Omega}, \tag{9}$$

$$h(t; a) < -\delta/2 \qquad \text{if} \qquad t \in [T_{k-1}, T_k],\ \ell_k(a) = -\delta,\ a \in \bar{\Omega}. \tag{10}$$

To prove this lemma, we first note that $|a| \leq C\delta$ for $a \in \bar{\Omega} = \bar{\Omega}_\delta$ with $C > 0$ independent of δ. As the function h is continuously differentiable with respect to a, the relations $h(t_k; 0) = 0$ and $\alpha_{ik} = \frac{\partial h}{\partial a_i}(t_k; 0)$ imply that $h(t_k; a) = \ell_k(a) + o(\delta)$ as $\delta \to 0$ for all a from the ball $|a| \leq C\delta$ which contains the parallelepiped $\bar{\Omega}_\delta$. Hence, $h(t_k; a) > \delta/2$ whenever $\ell_k(a) = \delta$, $a \in \bar{\Omega}_\delta$ and δ is sufficiently small. Due to the continuity of h, it follows that for any small δ one can find an interval $\Delta_k \ni t_k$ such that (9) is valid.

To prove (10), we consider separately the neighborhood $\Lambda_\delta^k = (t_k - \delta^{1/3}, t_k + \delta^{1/3})$ of the point t_k and the rest of the interval $[T_{k-1}, T_k]$.

[2] Alternatively, the conclusion of the theorem can be obtained by using the Implicit Function Theorem.

Since the interval Λ_δ^k and the parallelepiped $\bar\Omega_\delta$ shrink to the point t_k and the point $a = 0$, respectively, as δ decreases, the continuity of the derivatives $\frac{\partial h}{\partial a_i}$ implies that $h(t;a) = h(t;0) + \ell_k(a) + o(\delta)$ as $\delta \to 0$ for all $t \in \Lambda_\delta^k$, $a \in \bar\Omega_\delta$. Therefore, the relation $h(t;0) \le 0$ ensures that $h(t;a) < -\delta/2$ whenever $\ell_k(a) = -\delta$ and $t \in \Lambda_\delta^k$, $a \in \bar\Omega_\delta$ if δ is sufficiently small.

The relations $h(t_k;0) = \frac{\partial h}{\partial t}(t_k;0) = 0$ and $\frac{\partial^2 h}{\partial t^2}(t_k;0) < 0$ imply that $h(t;0) \le -c\delta^{2/3}$ for all $t \in [T_{k-1}, T_k] \setminus \Lambda_\delta^k$ if δ is sufficiently small, with some constant $c > 0$ independent of δ. Combining this inequality with the relation $|h(t;a) - h(t;0)| \le C_1|a|$, which is valid for all t, and taking into account that $|a| \le C\delta$ for $a \in \bar\Omega_\delta$, we conclude that $h(t;a) < -\delta/2$ in the domain $t \in [T_{k-1}, T_k] \setminus \Lambda_\delta^k$, $a \in \bar\Omega_\delta$ too, if δ is small, thus completing the proof.

Lemma 4.2. *Relations* (9) *and* (10) *imply that if* $\varepsilon > 0$ *is sufficiently small, then the components* $\Psi_k(a) = x_R(T_k; T_{k-1}, 0, \varepsilon, a)$, $k = 1, \ldots, n$, *of the vector field* $\Psi = \Psi_\varepsilon$ *satisfy the following relationships:*

$$\Psi_k(a) = R > 0 \qquad \text{if} \qquad \ell_k(a) = \delta,\ a \in \bar\Omega, \tag{11}$$

$$\Psi_k(a) < 0 \qquad \text{if} \qquad \ell_k(a) = -\delta,\ a \in \bar\Omega. \tag{12}$$

Indeed, if $\ell_k(a) = \delta$, then relations (9) and $f(x) \le 0$, $x \in \mathbb{R}$, imply that the derivative of the solution $x(\cdot) = x(\cdot; T_{k-1}, 0, \varepsilon, a)$ satisfies $\varepsilon\dot x(t) = h(t;a) - f(x) > \delta/2$ for $t \in \Delta_k$ as long as $x(t) \le R$. Hence, for sufficiently small ε, the solution $x(\cdot)$ reaches the value R within the time interval Δ_k or earlier, which implies (11). If $\ell_k(a) = -\delta$, then relations (10) and $f(0) = 0$ imply that $\dot x(t) < 0$ whenever $x(t) = 0$ within the whole time interval $[T_{k-1}, T_k]$. As $x(T_{k-1}) = 0$ at the initial moment of this time interval, it follows that $x(t) < 0$ for all $t \in (T_{k-1}, T_k]$ and hence (12) is valid.

4.4. **Completion of the proof.** Let us introduce new variables (new parameters) $\sigma_k = \ell_k(a) = \ell_k(a_1, \ldots, a_n)$. In the space $\mathbb{R}^n$ of these variables define the parallelepiped $\Pi = \{\sigma \in \mathbb{R}^n : |\sigma_k| < \delta,\ k = 1, \ldots, n\}$. The nondegenerate linear mapping $(\sigma_1, \ldots, \sigma_n) = (a_1, \ldots, a_n)A$, *i.e.*, $\sigma = aA$, maps the parallelepiped $\bar\Omega$ to the parallelepiped $\bar\Pi$ and the vector field $\Psi(a) = (\Psi_1(a), \ldots, \Psi_n(a))$ to the vector field $\Phi(\sigma) = (\Phi_1(\sigma), \ldots, \Phi_n(\sigma)) = \Psi(\sigma A^{-1})A$. Hence, $\gamma(\Phi, \Pi) = \gamma(\Psi, \Omega)$. According to Lemma 4.2, the components $\Theta_k(\sigma) = \Psi_k(\sigma A^{-1})$ of the vector field $\Theta(\sigma) = \Phi(\sigma)A^{-1}$ satisfy the conditions of the Rotation Product Theorem [29] in the parallelepiped $\bar\Pi$, *i.e.*,

$$\Theta_k(\sigma_1, \ldots, \sigma_n) < 0 \quad \text{if} \quad \sigma_k = -\delta; \quad \Theta_k(\sigma_1, \ldots, \sigma_n) > 0 \quad \text{if} \quad \sigma_k = \delta \tag{13}$$

for all $\sigma \in \bar\Pi$, $k = 1, \ldots, n$. Hence, $\gamma(\Theta, \Pi) = \gamma_1\gamma_2\cdots\gamma_n$, where γ_k is the rotation of the scalar vector field $\Theta_k(0, \ldots, 0, \sigma_k, 0, \ldots, 0)$ on the boundary of the interval $-\delta < \sigma_k < \delta$. Moreover, relations (13) imply $\gamma_k = 1$. Hence, the rotation $\gamma(\Theta, \Pi)$ of the vector field $\Theta(\sigma) = \Phi(\sigma)A^{-1}$ on the boundary of the parallelepiped Π equals 1. The relation $\gamma(\Phi(\sigma)A^{-1}, \Pi) = 1$ implies $\gamma(\Phi, \Pi) = \det A$ and, hence, $\gamma(\Psi, \Omega) = \det A \ne 0$. This completes the proof.

5. Discussion.

5.1. **Fold bifurcation of canards.** Fig. 5 illustrates the bifurcation of canard solutions for the equation $\varepsilon\dot x = x^2 + \cos t - 1 + a$ with a scalar parameter a. For negative a, the equation has a negative stable periodic solution following the stable branch S_s of the critical curve S, and the positive unstable periodic solution following the

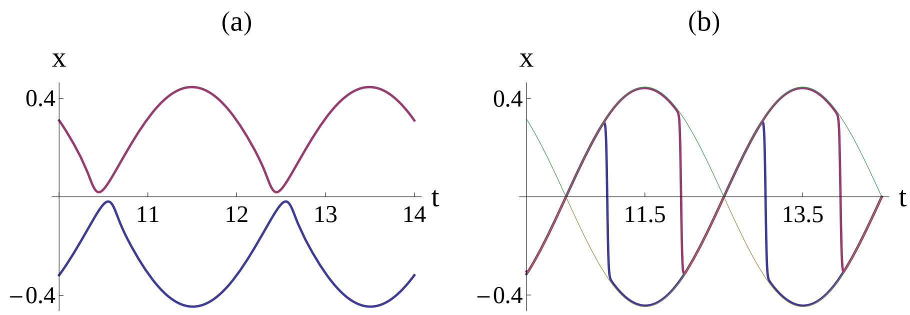

FIGURE 5. Stable (below) and unstable (above) periodic solutions of the equation $\varepsilon\dot{x} = x^2 + 0.1\sin(\pi t) - 0.1 + a$ with $\varepsilon = 0.005$ for (a) $a = 0.001$; and, (b) $a = 0.0034969174919347302$ (the same a as in Fig. 1). The periodic solutions are non-canards for $a < 0$ and canards for $a > 0$. As a increases, the fast segments of the canards move towards one another until the canards collide and disappear in the fold bifurcation. Thin lines in (b) are the stable (below the time axis) and unstable (above the time axis) branches of the critical curve.

unstable branch S_u of S. Fig. 5(a) shows the graphs of these solutions on the (t, x) plane. They are not canards as neither of them passes the meeting points of the branches S_s and S_u (*i.e.*, the self-intersection points of S). However, as a increases and becomes positive, the stable periodic solution acquires a segment elongated along the unstable branch S_u of the critical curve in the positive half-plane $x > 0$, while the unstable periodic solution acquires a negative segment elongated along the stable branch S_s, see Fig. 5(b) (Fig. 1 presents one of these canards and its trace on the (h, x)-plane). The two canard periodic solutions change continuously with a. The segments of fast motion of the periodic solutions on Fig. 5(b) move toward one another as a increases, until the periodic canard solutions coalesce and disappear in the fold bifurcation. Note that the closer $a < a_F$ to the fold bifurcation value a_F of the parameter, the longer are the canards, *i.e.*, the longer the portion of the stable periodic solution following the unstable curve S_u and the portion of the unstable periodic solution following the stable curve S_s. The longest semistable canard is achieved at the bifurcation point $a = a_F$; in this example, the longest canard also has the longest portion of fast motion. For $a > a_F$, the equation does not have periodic (or bounded) solutions, and for larger values of a all the solutions become increasing.

In this example, the graphs of the stable and the unstable periodic canard solutions are symmetric to each other as the equation $\varepsilon\dot{x} = x^2 + \cos t - 1 + a$ is invariant with respect to the transformation $t \mapsto -t$, $x \mapsto -x$ of the variables. However, under the conditions of Theorem 1, the similar bifurcation scenario is generic for equation (4) with a scalar parameter a and an asymmetric function f. In case of the two dimensional parameter a, we expect to observe fold bifurcation lines meeting at the co-dimension two cusp bifurcation point.

Fig. 5 shows that the mechanism of formation of canards near the points of self-intersection of the critical curve S is associated with the delayed exchange of stability effect (see, [37] for some explicit examples).

Theorem 1 describes both stable and unstable canards. However, an analysis of stability of the canard solutions is beyond the scope of this paper. For the example presented in Fig. 5, the canards with $T_1 < t_1 + 1$ are unstable, the canards with $t_1+1 < T_1 < t_1+2$ are stable. The unstable canard solution in Fig. 5 was obtained by solving the equation in backward time, as this solution becomes stable in the time reversed equation. All the Figs. 1-4 present asymptotically stable canards; the plots were obtained by numerical integration of the equation $\varepsilon\dot{x}(t) = h(t;a) + x^2$ in Mathematica 8 (with an arbitrarily chosen initial condition from the basin of attraction of the stable branch of the critical curve) and truncating the transient.

All our figures present relatively short canards. The closer to the bifurcation point and the longer the canard, the narrower its basin of attraction and the shorter the parameter window for which it exists. Moreover, the numerical solution becomes increasingly sensitive to the numerical error as the canard becomes longer, requiring higher precision, smaller time step and longer computation time (which also increases exponentially with decreasing ε). As a result, longer canards cannot be obtained by Initial Value Problem solvers. Instead, one can use algorithms based on Boundary Value Problem solvers to calculate canards of arbitrary length [11,17,18].

5.2. **Extensions of the theorem.** The conditions $x_\varepsilon(T_k) = 0$ of Theorem 3.1 can be replaced by the condition that the trace $(h(t;a), x_\varepsilon(t))$ of the canard solution on the (h,x)-plane passes through the points $(H_k, 0)$, $k = 1, \dots, n$, with any given H_k satisfying

$$\min\{h(t;0) : t_k \le t \le t_{k+1}\} < H_k < 0.$$

The strict inequalities here ensure that the point $H_k = h(T_k, a)$ belongs to the range of the restriction of the input $h(t;a)$" to the interval $t_k < t < t_{k+1}$ for every value of the parameter a which is sufficiently close to the reference value $a = 0$.

The Lipschitz continuity of the function f requested by condition (i) ensures uniqueness of solutions of equation (4). A standard limit argument can be used to extend Theorem 3.1 to equations with continuous functions f. Assumptions (ii) about the smoothness of the input h can also be relaxed as follows from the proof of Theorem 3.1 below. The function f and the period of the input can be made dependent on the parameters a_i.

5.3. **Remark on higher order systems.** Suppose the input h is generated as a scalar component of a stable cycle of an autonomous system. Statements similar to Theorem 3.1 can be formulated for canard cycles of the triangular slow-fast system

$$\begin{aligned} \varepsilon\dot{x} &= \langle c, z\rangle - f(x), \qquad x \in \mathbb{R}, \\ \dot{z} &= g(z;a), \qquad z \in \mathbb{R}^m, \end{aligned}$$

where the slow z-component generates a cycle; $\langle\cdot,\cdot\rangle$ is a scalar product in $\mathbb{R}^m$; and, the vector of parameter $a = (a_1, \dots, a_n)$ controls the shape of the slow cycle and the input $h = \langle c, z\rangle$ to the fast x-equation. It would be interesting to obtain analogs of Theorem 3.1 for systems with a multidimensional fast component.

5.4. **Critical curve with multiple folds.** Natural modifications of Theorem 3.1 extend it to the case when the critical curve of equation (4) has multiple folds. For example, assume that the function f has n points x_k of local extremum, see Fig. 6. If, for some value a_* of the vector parameter $a = (a_1, \dots, a_n)$, the input $h(\cdot;a_*)$ has a point t_k of local maximum with $h(t_k;a_*) = f_k$ for each local maximum value $f_k = f(x_k)$ of the function f and $h(\cdot;a_*)$ has a point t_k of local minimum with $h(t_k;a_*) = f_k$ for each local minimum value $f_k = f(x_k)$ of f, then one can obtain

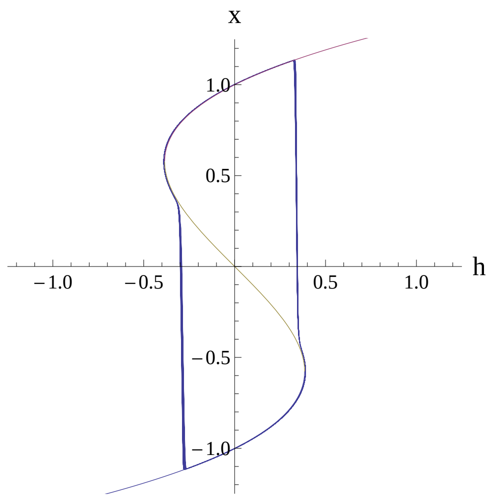

FIGURE 6. The periodic double canard solution of the equation $\varepsilon\dot{x} = h(t; a_1, a_2) + x\,(x-1)\,(x+1)$ with the cubic nonlinearity and the input $h(t; a_1, a_2) = a_1 \sin(\pi t) + a_2 \max\{\sin(\pi t), 0\}$. Parameters are $a_1 = 0.390169329$, $a_2 = 2 \times 10^{-6}$ and $\varepsilon = 0.005$.

sufficient conditions for the existence of a periodic canard solution with n portions elongated along the unstable branches of the critical curve, which are controlled by the parameters a_i.

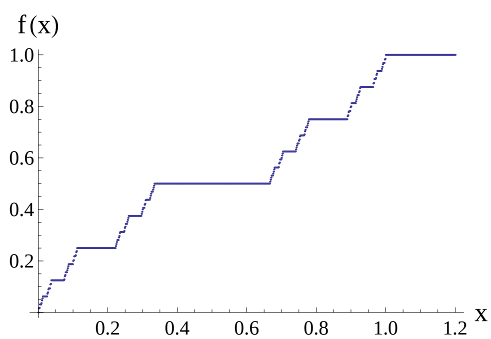

FIGURE 7. The graph of the Cantor function $f(x)$.

5.5. **Remark on nonsmooth systems.** As one illustration to the previous remark and the second remark of this section, we consider the system of equations

$$\begin{aligned} y(t) &= f(h(t; a) + kx(t)), \\ \varepsilon\dot{x} &= y(t) - x(t) \end{aligned}$$

where f is the Cantor function[3] [12], see Fig. 7. These equations were introduced in [30] as a simple model of a system of interacting agents where the Cantor function

[3]We assume that the Cantor function $f = f(x)$ defined on the segment $0 \le x \le 1$ is extended to the whole real axis by $f = 0$ for $x < 0$ and $x = 1$ for $x > 1$.

arises as a lumped model of multiple negative feedback loops which characterise local interactions [3], while the kx term introduces the global mean field type positive feedback loop into the system. The variable h is interpreted as the input (control), x is the state, and y is the output of the system; all the variables are scalar. Equivalently, the system can be written as one equation

$$\varepsilon \frac{dx}{dt} = f(h(t;a) + kx) - x \tag{14}$$

where $k > 0$, $0 < \varepsilon \ll 1$, and a is either a scalar or a vector parameter.

The Cantor function is an increasing continuous singular function, which has zero derivative almost everywhere, as well as infinitely many points with $f' = \infty$; its graph is self-similar. As a consequence of these properties, the critical curve

$$x = f(h + kx) \tag{15}$$

of equation (14) makes infinitely many folds on the (h, x) plane. In particular, equation (14) has multiple stable equilibria for infinitely many stationary inputs h. As the Cantor function is not smooth, the critical curve is not smooth either. We are going to formulate an existence result for a simple periodic canard solution in this non-smooth setting.

Assume that the input is periodic, $h(t; a) \equiv h(t + \tau; a)$, and continuous with respect to the set of its arguments t, a, where a is a scalar parameter. For simplicity, suppose that the input has exactly one local maximum and one local minimum on a period and $h_m(a) = \min\{h(t; a) : t \in \mathbb{R}\} < 0$ for every a; moreover, the function $h_M(a) = \max\{h(t; a) : t \in \mathbb{R}\}$ is unbounded from above.

It is easy to see that, under these assumptions, equation (14) has a periodic solution $x_\varepsilon(t; a) \equiv x_\varepsilon(t+\tau; a)$ for each a. Moreover, for most values of the parameter a, all the slow parts of the trace of the periodic solution on the (h, x) plane follow the stable branches of the curve (15), see [30]. More specifically, this is true for sufficiently small ε whenever the point $(h_M(a), \xi_-(h_M(a)))$, where $x = \xi_-(h_M(a))$ is the smallest solution of equation (15) for $h = h_M(a)$, lies in the interior of one of the 'steps' of the graph of f. However, it turns out that careful tuning of the maximum value $h_M(a)$ of the input close to an end of a 'step' of f, leads to the existence of a periodic canard solution, like in the smooth case. To make this statement precise, we define a nonempty open set

$$S = \{(h, x) : \xi_-(h) < x < \xi_+(h),\ -\infty < h < \infty\}$$

on the (h, x) plane, where ξ_- and ξ_+ are the smallest and the largest solutions of equation (15),

$$\xi_-(h) = \min\{x :\ x = f(h + kx)\}, \qquad \xi_+(h) = \max\{x :\ x = f(h + kx)\}.$$

Solutions of equation (14) which pass through the set S are canard solutions.

Proposition 1. *Let $(h, x) \in S$. Than for every sufficiently small $\varepsilon > 0$ there is an $a = a(h, x, \varepsilon)$ such that equation (14) with the input $h(t) = h(t; a)$ has a T-periodic solution passing through the point (h, x).*

This proposition and similar statements about multiple canards of equation (14) with a vector parameter a can be proved by modification of the method presented in the previous section. The proof of Proposition 1 is omitted.

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