RESONANT FORCED OSCILLATIONS IN SYSTEMS WITH PERIODIC NONLINEARITIES

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Dedicated to Jean Mawhin on the occasion of his 70th birthday

Abstract. We present an approach to study degenerate ODE with periodic nonlinearities; for resonant higher order nonlinear equations $L(p)x = f(x) + b(t)$, $p = d/dt$ with 2π-periodic forcing b and periodic f we give multiplicity results, in particular, conditions of existence of infinite and unbounded sets of 2π-periodic solutions.

1. **Introduction.** Consider the equation

$$L(p)x = f(x) + b(t), \qquad p = d/dt \qquad (1)$$

where L is a polynomial with constant coefficients, $\deg L = \ell \geq 2$, f is continuous and periodic with a period T, and b is continuous and periodic with the period 2π. We study 2π-periodic solutions of this equation: their existence, the finiteness and boundedness of the set Π of all 2π-periodic solutions, the asymptotic behavior of 2π-periodic solutions with increasing to infinity amplitudes. If $L(ki) \neq 0$ for integer k, then this problem is non-resonant, the set Π is non-empty and bounded in any reasonable sense (e.g., in C^ℓ). We study the resonant case $L(\pm i) = 0$ and $L(ki) \neq 0$ for integer $k \neq \pm 1$.

In [3] the authors consider related problems with the use of abstract results based on variational arguments, the periodic problem for the case $L(p) = p^2 + 1$ is studied in details. To apply the technique from [3] to 2π-periodic problems for (1), the polynomial L must be even, in this case the linear operator $L(d/dt)$ with periodic boundary conditions is self-adjoint and the equation is Hamiltonian. We study (1) for generic resonant L.

Resonant problems with periodic nonlinearities were also considered in some other papers. For instance, in [5], Section 3, a combination of topological arguments and the stationary phase method [12] is used for a similar problem (see also [3,4,14] and some references therein) with one dimensional kernel of the linear part.

Let $b(t) = \beta \sin(t + \psi) + \tilde{b}(t)$ where the function $\tilde{b}$ does not contain the first harmonics:

$$\int\limits_0^{2\pi} \tilde{b}(t) \cos t \, dt = \int\limits_0^{2\pi} \tilde{b}(t) \sin t \, dt = 0. \qquad (2)$$

If $\beta \neq 0$, then the set Π is *a priori* bounded in C^{ℓ}, moreover it is empty if $|\beta| > 4 \sup |f|/\pi$. We give generic conditions, under which Π is infinite and unbounded if $\beta = 0$, and it is bounded and contains arbitrary finite number of distinct elements if $|\beta| \neq 0$ is small enough. The main results can be generalized in various directions, some of them are discussed in Section 3.

2. **The main results.** Throughout the paper we use the notation $\omega = 2\pi/T$.

According to the Fredholm Alternative, the linear equation $L(p)x = \tilde{b}(t)$ has a unique 2π-periodic solution $b_1(t) \in C^{\ell}$ satisfying

$$\int_0^{2\pi} b_1(t) \cos t \, dt = \int_0^{2\pi} b_1(t) \sin t \, dt = 0. \tag{3}$$

Consider the Fourier series of f:

$$f(x) = \sum_{s=1}^{\infty} \mu_s \sin(s\omega\, x + \psi_s), \qquad \mu_0 = \frac{1}{T} \int_0^T f(x) \, dx = 0, \tag{4}$$

without loss of generality we assume $\mu_0 = 0$: the constant is included in the forcing term b. Such representation is unique if $\mu_s \geq 0$. Fourier series (4) and the function b_1 define the functions

$$B(\varphi) = b_1'(\pi/2 - \varphi) + b_1'(3\pi/2 - \varphi),$$

$$q_1(\varphi, \xi) = \sum_{s=1}^{\infty} \frac{\mu_s}{\sqrt{s}} \sin\left(\omega s\xi - \frac{\pi}{4} + \psi_s + \omega s b_1(\frac{\pi}{2} - \varphi)\right),$$

$$q_2(\varphi, \xi) = \sum_{s=1}^{\infty} \frac{\mu_s}{\sqrt{s}} \sin\left(\omega s\xi - \frac{\pi}{4} - \psi_s - \omega s b_1(\frac{3\pi}{2} - \varphi)\right). \tag{5}$$

All these functions are continuous and periodic, the function $B \in C^{\ell-1}$ is π-periodic, it contains even harmonics only starting from the second, the functions q_j are T-periodic in ξ and 2π-periodic in φ.

Theorem 2.1. *Let there exist a robust zero[1] φ_* of the function B such that $q_1(\varphi_*, \xi_*) \neq 0$ for some robust zero ξ_* of the function $q_1(\varphi_*, \xi) + q_2(\varphi_*, \xi)$. Let*

$$\sum_{s=1}^{\infty} s^{3/2+\nu} \mu_s < \infty \tag{6}$$

for some $\nu > 1/4$. Then, in the case $\beta = 0$ there exists an infinite sequence $x_n \in \Pi$ satisfying $\|x_n\|_{L^2} \to \infty$.

Condition (6) is valid for some ν if f is smooth enough, e.g., (6) holds for $\nu \in (0, 1/2)$ if $f \in C^2$. If (6) holds, then $f \in W_2^2$ and $q_j(\varphi, \cdot) \in C^2$ for any φ.

From the proofs below it follows that 2π-periodic solutions x_n of (1) have the form $x_n(t) = \xi_n \sin(t + \varphi_n) + b_1(t) + h_n(t)$, where n is large enough, $\xi_n - nT - \xi_* \to 0$, $\varphi_n \to \varphi_*$, and $\|h_n\|_C \to 0$.

Let us proceed to the case $\beta \neq 0$.

Theorem 2.2. *Under the conditions of Theorem 2.1 for any integer N there exists a $\sigma > 0$ such that for any $|\beta| \in (0, \sigma)$ equation (1) has at least N distinct 2π-periodic solutions.*

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[1] An isolated zero of a scalar function is call *robust* if the function changes sign in a vicinity of this zero.

If $\beta \to 0$, then under the conditions of Theorem 2.1 the diameter in L^2 of the set Π tends to infinity.

With a linear change of variable $(y = \omega x)$ the equation with a periodic nonlinearity with the period T may be reduced to the case $T = 2\pi$ (we get $L(p)y = \omega b(t) + \omega f(\omega^{-1}y)$, the function $f(\omega^{-1}y)$ is 2π-periodic). We preserve a generic value for the period T in the formulations and proofs to stress the difference between the period 2π of oscillations and the period T of the nonlinearity.

3. Generalizations and comments.

3.1. **Example.** Generically all zeros of the function B are robust, there is a finite number of such zeros. The Sturm–Hurwitz theorem[2] implies the existence of at least four such zeros on $[0, 2\pi)$: constant disappears due to the differentiation, the first harmonics (as well as other odd ones) disappear due to the specific form of the function B. According the same Sturm–Hurwitz theorem, the function $q_1(\varphi_*, \xi) + q_2(\varphi_*, \xi)$ generically has at least two zeros.

If $\beta = 0$, then the change of variables $x = x + b_1$ leads us to the equivalent equation

$$L(p)x = f\big(x(t) + b_1(t)\big) \tag{7}$$

where b_1 satisfies (3). Theorem 2.1 is valid for equation (7) regardless of condition (3): the equation $L(p)y = f(y(t) + \zeta \sin(t + \psi) + b_1(t))$ is equivalent to the equation $L(p)x = f(x(t) + b_1(t))$ if $x(t) = y(t) + \zeta \sin(t + \psi)$.

Consider the simplest example of equation (7): $L(p)x = f(x + b_1(t))$ with $f(x) = \sin x$, $b_1(t) = \lambda(\sin 2t + \sin 3t)$, $\lambda > 0$. In this case the function $B(\varphi) = 4\lambda \cos 2\varphi$ has four robust zeros $\varphi_*^k = \pi(2k + 1)/4$, $k = 0, 1, 2, 3$, on the period $[0, 2\pi)$. The function[3]

$$q_1(\varphi_*^k, \xi) + q_2(\varphi_*^k, \xi) = 2 \sin \big(\xi - \frac{\pi}{4} + \frac{(-1)^{[(k+1)/2]}\lambda\sqrt{2}}{2}\big) \cos \lambda$$

is not identically zero if and only if $\cos \lambda \neq 0$, in this case it has exactly two robust zeros ξ_*^d, $d = 1, 2$ on a period. If $\sin \lambda \neq 0$, then $q_1(\varphi_*^k, \xi_*^d) \neq 0$ for $k = 0, 1, 2, 3$ and $d = 1, 2$. Theorems 2.1 and 2.2 are applicable to each pair (φ_*^k, ξ_*^d) from the set of eight ones, for any $\lambda \in (0, \pi/2)$ there are eight sequences of forced oscillations. If $\lambda \to 0$, then the least amplitude of oscillations tends to infinity.

3.2. **Equations from control theory.** Consider the equation

$$L(p)x = M(p)\big(f(x(t)) + b(t)\big). \tag{8}$$

Such equations[4] with scalar nonlinearities are traditional for control theory, here L and M are real coprime polynomials of the degrees $\ell = \deg L > m = \deg M$. Let the function $f(x) : \mathbb{R} \to \mathbb{R}$ be continuous and periodic with a period T, let $\omega = 2\pi/T$. Let $L(\pm i) = 0$ and $L(ki) \neq 0$ for integer $k \neq \pm 1$. Let $b(t) = \beta \sin(t + \psi) + \tilde{b}(t)$ be a periodic continuous function with the period 2π. Again, the linear equation $L(p)x = M(p)\tilde{b}$ has a unique 2π-periodic solution $b_1 \in C^{\ell-m}$ satisfying (3).

Let $\ell > m + 1$. Define B, q_1, and q_2 by (5).

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[2]The number of sign changes on a period for any periodic continuous function is not less that the lowest order of its harmonics [2, 13].

[3]We denote here the integer part as $[\cdot]$.

[4]To define solutions of (8) we consider an equivalent system of the type $\boldsymbol{z}' = A\boldsymbol{z} + \boldsymbol{b}\big(f(\langle \boldsymbol{z}, \boldsymbol{c} \rangle) + b(t)\big)$ in $\mathbb{R}^\ell$, the $\ell \times \ell$-matrix A and the vectors $\boldsymbol{b}, \boldsymbol{c} \in \mathbb{R}^\ell$ are independent from t (see, e.g., [6, 7]).

Theorem 3.1. *Let all assumptions of Theorem 2.1 be valid. Then for $\beta = 0$ equation (8) has an infinite sequence x_n of 2π-periodic solutions satisfying $\|x_n\|_{L^2} \to \infty$. For any integer N there exists a $\sigma > 0$ such that for $|\beta| \in (0, \sigma)$ equation (8) has at least N distinct 2π-periodic solutions.*

The proof of this theorem almost coincides with the presented proofs of Theorems 2.1–2.2. The assumption $\ell > m + 1$ is particularly used in estimates (10) and (11).

3.3. **Other resonant cases.** It is possible to obtain analogs of Theorems 2.1–2.2 for the case $L(\pm si) = 0$ for some integer $s > 1$ and $L(\pm ki) \neq 0$ for any $k \neq \pm s$, but the final formulations are much more cumbersome. For the case $s = 1$ the principal part of the bifurcation system (it appears in the proofs) has the form

$$q_1(\varphi, \xi) + q_2(\varphi, \xi) = 0, \quad b'(\frac{\pi}{2} - \varphi)q_1(\varphi, \xi) - b'(\frac{3\pi}{2} - \varphi)q_2(\varphi, \xi) = 0. \tag{9}$$

According to the first equation, the second one may be rewritten as $B(\varphi)q_1(\varphi, \xi) = 0$. Since $q_1(\varphi, \xi) \neq 0$ it is equivalent to $B(\varphi) = 0$, and the answer has more or less explicit form as in Theorem 2.1. For $s > 1$ the analogous principal part contains equations with $2s$ terms, it is impossible to separate the variables and to formulate simple enough conditions of solvability for the bifurcation system. The possible condition for the case $s > 1$ may have the form: 'Let the bifurcation system of two variables (φ, ξ) have an isolated zero (φ_*, ξ_*) of a nonzero index'.

3.4. **Almost periodic nonlinearities.** Instead of periodic f it is possible to consider the functions $f = f_1 + f_2$ with a periodic f_1 and sufficiently rapidly decreasing f_2: if the function $f_2(x)|x|^{1+\delta}$ is uniformly bounded for some $\delta > 0$. It is also possible to consider equations with almost periodic $f = \sum f_k$ and T_k-periodic f_k.

Let $f(x) = \sin x + \sin(\sqrt{2}x)$. Put now (compare with (5))

$$q_1(\varphi, \xi) = \sin\left(\xi - \frac{\pi}{4} + b_1(\frac{\pi}{2} - \varphi)\right) + \sin\left(\sqrt{2}\xi - \frac{\pi}{4} + \sqrt{2}b_1(\frac{\pi}{2} - \varphi)\right)$$

$$q_2(\varphi, \xi) = \sin\left(\xi - \frac{\pi}{4} - b_1(\frac{3\pi}{2} - \varphi)\right) + \sin\left(\sqrt{2}\xi - \frac{\pi}{4} - \sqrt{2}b_1(\frac{3\pi}{2} - \varphi)\right).$$

Let φ_* be a robust zero of the function B. Let there exist a sequence $\xi_*^n \to \infty$ of robust zeros of the almost periodic function $q_1(\varphi_*, \xi) + q_2(\varphi_*, \xi)$ such that $|q_1(\varphi_*^n, \xi_*)| \geq \varepsilon_0$ for some $\varepsilon_0 > 0$. Then the conclusions of both Theorems 2.1 and 2.2 remain valid.

It would be interesting to study more general equations $L(p)x = f(t, x)$ with periodic in both variables nonlinearities: $f(t, x) \equiv f(t + 2\pi k_1, x + k_2 T)$, $k_1, k_2 \in \mathbb{Z}$. For the simplest case $f(t, x) = b(t) + a(t)f(x)$ the principal part of the bifurcation system has the form

$$a(\frac{\pi}{2} - \varphi)q_1 + a(\frac{3\pi}{2} - \varphi)q_2 = 0,$$

$$a(\frac{\pi}{2} - \varphi)q_1 B(\varphi) + a'(\frac{\pi}{2} - \varphi)Q_1 - a'(\frac{3\pi}{2} - \varphi)Q_2 = 0$$

where $q_i = q_i(\varphi, \xi)$ and

$$Q_1 = Q_1(\varphi, \xi) = \sum_{s=1}^{\infty} \frac{\mu_s}{s\sqrt{s}} \sin\left(\omega s\xi - \frac{3\pi}{4} + \psi_s + \omega s b_1\left(\frac{\pi}{2} - \varphi\right)\right),$$

$$Q_2 = Q_2(\varphi, \xi) = \sum_{s=1}^{\infty} \frac{\mu_s}{s\sqrt{s}} \sin\left(\omega s\xi - \frac{3\pi}{4} - \psi_s - \omega s b_1\left(\frac{3\pi}{2} - \varphi\right)\right).$$

For various partial cases this system can be studied in an explicit form.

3.5. **Nonlinearities with saturation.** Here we present a result concerning the equation $L(p)x = b(t) + F(x)$ where the function $F = f + g$ is the sum of a T-periodic function f and a function g with saturation:

$$\lim_{|x|\to\infty} |g(x) - \text{sign}(x)| = 0,$$

and $b(t) = \beta \sin(t + \psi) + \tilde{b}(t)$ where $\beta > 0$ and $\tilde{b}$ again satisfies (2).

If $\beta < 4$, then Lazer–Leach condition[5] holds, the set of 2π-periodic solutions is bounded and non-empty, if $\beta > 4$, then the set is bounded and may be empty; it is empty if β is large enough. The case $\beta = 4$ is twice-degenerate: the linear part is degenerate together with the principal nonlinear terms. Twice-degenerate systems without periodic term f were studied in [9] under special one-side conditions on the term $g(x) - \text{sign}(x)$.

Consider the function

$$s(t) = \frac{4}{\pi} \sum_{k=3,5,\ldots} \frac{\sin kt}{k} \equiv \text{sign}(\sin t) - \frac{4}{\pi} \sin t.$$

The equation $L(p)x = \tilde{b}(t) + s(t + \psi)$ has a unique 2π-periodic solution $x = b_2$ satisfying the condition

$$\int_0^{2\pi} b_2(t) \cos t \, dt = \int_0^{2\pi} b_2(t) \sin t \, dt = 0.$$

Put

$$q_*(\xi) = \sum_{s=1}^{\infty} \frac{\mu_s}{\sqrt{s}} \sin\left(\omega s\xi - \frac{\pi}{4} + \psi_s + \omega s b_2\left(\frac{3\pi}{2} - \psi\right)\right)$$

$$+ \sum_{s=1}^{\infty} \frac{\mu_s}{\sqrt{s}} \sin\left(\omega s\xi - \frac{\pi}{4} - \psi_s - \omega s b_2\left(\frac{\pi}{2} - \psi\right)\right).$$

Theorem 3.2. *Let condition* (6) *be valid for some* $\nu > 1/4$. *Let*

$$\limsup_{|x|\to\infty} x^{2/3+\delta} |g(x) - \text{sign}(x)| < \infty$$

for some $\delta > 0$. *Let there exist a robust zero* ξ_* *of the function* q_*. *Then for* $\beta = 4$ *the equation* $L(p)x = b(t) + F(x)$ *has an infinite sequence* x_n *of* 2π-*periodic solutions satisfying* $\|x_n\|_{L^2} \to \infty$. *For any integer* N *there exists a* $\sigma > 0$ *such that for any* $|\beta - 4| \in (0, \sigma)$ *the equation* $L(p)x = b(t) + F(x)$ *has at least* N *different* 2π-*periodic solutions.*

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[5] The Lazer-Leach condition is more known for $f = 0$, but it also works for some other cases, mainly if primitives of f are sublinear (see [1, 10]). For our case (4) the function f has periodic and bounded primitives.

For $\beta = 4$ solutions x_n from Theorem 3.2 have the form

$$x_n(t) = \xi_n \sin(t + \varphi_n) + b_2(t) + h_n(t),$$

where n is large enough, $\xi_n - nT - \xi_* \to 0$, $\varphi_n \to \psi + \pi$, and $\|h_n\|_C \to 0$. To prove Theorem 3.2 it is possible to use the same approaches as in the proof of Theorems 2.1 and 2.2 and partially the same auxiliary statements.

Under the assumptions of Theorems 2.1 and 2.2 the index at infinity for reasonable equivalent vector fields is undefined for $\beta = 0$ and is equal to 0 for $\beta \neq 0$. Under the assumptions of Theorem 3.2 the index is undefined if $\beta = 4$, it is equal to $+1$ or -1 if $\beta \in [0, 4)$, it is equal to 0 if $\beta > 4$.

Unlike Theorems 2.1–2.2, Theorem 3.2 may be easily reformulated for the case $L(\pm si) = 0$ with integer $s > 1$ (see Subsection 3.3).

3.6. **Inverse theorem.** As it follows from the computational part of the proof, Theorem 2.1 is 'almost invertible' in the following sense.

Theorem 3.3. *Let $\beta = 0$ and let the set Π be unbounded in L^2. Let (6) be valid for some $\nu > 1/4$. Then there exist solutions $\varphi_* \in [0, 2\pi), \xi_* \in [0, T)$ of (9) and the sequence $x_n \in \Pi$ of the form $x_n = \xi_n \sin(t + \varphi_n) + b_1 + h_n$, where $\xi_n - nT - \xi_* \to 0$, $\varphi_n \to \varphi_*$, and $\|h_n\|_C \to 0$.*

4. **Proof of Theorem 2.1.**

4.1. **New variables and linear operators.** We use the spaces C, C^1, L^2 and W_2^1 of functions $x = x(t) : [0, 2\pi] \to \mathbb{R}$, $\|x\|_{W_2^1} = \|x\|_C + \|x'\|_{L^2}$, denote the scalar product in L^2 as

$$\langle u, v \rangle_{L^2} = \int_0^{2\pi} u(t)v(t)\,dt, \qquad u, v \in L^2.$$

Denote by $E \subset L^2$ the linear span of the functions $\sin t$ and $\cos t$, denote by $E^\perp \subset L^2$ the orthogonal complement of the plane E. Then

$$\mathcal{P}x(t) = \frac{1}{\pi} \int_0^{2\pi} \cos(t - s)\,x(s)\,ds$$

and $\mathcal{Q} = I - \mathcal{P}$ are orthogonal projectors onto the subspaces E and $E^\perp$ of L^2. Both projectors act in C and in C^1.

We search for 2π-periodic solutions in the form $x(t) = \xi \sin(t + \varphi) + h(t)$ where $\xi > 0$ and $h = \mathcal{Q}x$ does not contain the first harmonics. Below the real variables ξ, φ and the function h are considered as unknowns.

Denote by A the linear operator that maps any function $u \in E^\perp$ to a unique solution $x = Au \in E^\perp$ of the linear equation $L(p)x = u$. The existence of the solution $x = Au$ follows from $u \in E^\perp$, the uniqueness follows from $x \in E^\perp$. The projectors $\mathcal{P}$ and $\mathcal{Q}$ commute with differentiation and with the operator A in W_2^1.

The operator $A : E^\perp \to E^\perp$ is completely continuous. The operator $A\mathcal{Q}$ is well-defined in L^2, it is completely continuous in L^2, in C, and as an operator from L^2 to $C^{\ell-1}$.

The operator $A'\mathcal{Q} : u(t) \mapsto \frac{d}{dt} A\mathcal{Q}u(t)$ is completely continuous in L^2 and in C, it is continuous as an operator from C to $C^{\ell-1}$.

Consider a function $u \in L^2$. If its Fourier coefficients ν_k satisfy the estimate $|\nu_k| \leq \zeta_k$, then the Fourier coefficients $\tilde{\nu}_k$ and $\tilde{\nu}'_k$ of the functions $A\mathcal{Q}u$ and $A'\mathcal{Q}u$ satisfy

$$|\tilde{\nu}_k| \leq r_1 k^{-2}\zeta_k, \quad |\tilde{\nu}'_k| \leq r_1 k^{-1}\zeta_k, \qquad k \geq 2. \tag{10}$$

These estimates follow from the equalities

$$\|A\mathcal{Q}(\sin(kt+\theta))\|_{L^2} = \frac{\sqrt{\pi}}{|L(ik)|}, \quad \|A'\mathcal{Q}(\sin(kt+\theta))\|_{L^2} = \frac{k\sqrt{\pi}}{|L(ik)|}.$$

Constant r_1 may be defined as

$$r_1 = \max_{k=2,3,\ldots} \frac{k^2}{|L(ik)|}.$$

Moreover, from the equalities

$$|\langle A'Q(\sin(kt+\theta)), (\sin(kt+\theta))\rangle_{L^2}| = \frac{\pi k \,|\Im L(ik)|}{|L(ik)|^2}, \qquad k = 2, 3, \ldots$$

it follows that

$$|\langle A'\mathcal{Q}u, \mathcal{Q}u\rangle_{L^2}| \le r_2 \sum_{k=2,3\ldots} k^{-2}\nu_k^2. \tag{11}$$

If the polynomial L is even, the constant r_2 in the last estimate equals zero. Otherwise,

$$r_2 = \sup_{k=2,3,\ldots} \frac{k^3|\Im L(ik)|}{|L(ik)|^2},$$

this value is finite: if $\ell = 2$, then L is even and $r_2 = 0$, if $\ell \ge 3$, then the degree of the polynomial $p^3 L(p)$ is not greater than ℓ^2 (may be $\deg(p^3 L(p)) = \ell^2$ for $\ell = 3$).

4.2. **Topological lemma.** For the sequel, we need the following auxiliary statement on the solvability of a system of two scalar equations and an equation in a Banach space H. This lemma contains the sufficient for our goals part of more general statements from [8].

Consider the system

$$B_1(\varphi, \xi, h) = 0, \quad B_2(\varphi, \xi, h) = 0, \quad h = B_3(\varphi, \xi, h) \tag{12}$$

where the unknowns φ and ξ are scalar, $\varphi \in \bar{\Theta} = [\varphi_1, \varphi_2]$, $\xi \in \bar{\Xi} = [\xi_1, \xi_2]$, and $h \in H$. Suppose the operators $B_1, B_2 : \bar{\Theta} \times \bar{\Xi} \times H \to \mathbb{R}$ are continuous and the operator $B_3 : \bar{\Theta} \times \bar{\Xi} \times H \to H$ is completely continuous (with respect to the set of their arguments). If B_3 is uniformly bounded:

$$\|B_3(\varphi, \xi, h)\|_H \le \rho, \qquad \varphi \in \bar{\Theta}, \ \xi \in \bar{\Xi}, \ h \in H,$$

then from the Schauder fixed point theorem it follows that $\mathcal{H}(\varphi, \xi) = \{h : h = B_3(\varphi, \xi, h)\} \ne \emptyset$ for any $\varphi \in \bar{\Theta}$, $\xi \in \bar{\Xi}$. Put

$$\mathcal{H} = \bigcup_{\varphi \in \bar{\Theta}, \xi \in \bar{\Xi}} \mathcal{H}(\varphi, \xi).$$

Lemma 4.1. *Suppose*

$$B_1(\varphi_1, \xi', h) \cdot B_1(\varphi_2, \xi'', h) < 0, \qquad \xi', \xi'' \in \bar{\Xi}, \ h \in \mathcal{H}, \tag{13}$$

$$B_2(\varphi', \xi_1, h) \cdot B_2(\varphi'', \xi_2, h) < 0, \qquad \varphi', \varphi'' \in \bar{\Theta}, \ h \in \mathcal{H}. \tag{14}$$

Then system (12) *has at least one solution* $\varphi \in \bar{\Theta}, \xi \in \bar{\Xi}, h \in H$.

Lemma 4.1 follows from [8], Theorem 2 that is a generalization of the Rotation Product Formula [11], §7, §23. Under the assumptions of Lemma 4.1 the rotation γ_1 of the infinite-dimensional vector field $h - B_3(\varphi, \xi, h) \in H$ with fixed φ, ξ on the sphere $\{\|h\|_H = \rho + 1\}$ equals 1. The rotation γ_2 of the two-dimensional

vector field $\{B_1(\varphi, \xi, h), \ B_2(\varphi, \xi, h)\}$ with fixed h on the boundary of the rectangular $\mathcal{R} = \{\varphi \in (\varphi_1, \varphi_2), \xi \in (\xi_1, \xi_2)\}$ is either 1 or -1. The rotation γ_0 of the field

$$\{B_1(\varphi, \xi, h), \ B_2(\varphi, \xi, h), \ h - B_3(\varphi, \xi, h)\}$$

on the boundary $\partial \mathcal{R}$ of the domain $\mathcal{R} \times \{\|h\|_H < \rho + 1\}$ in the space $\mathbb{R} \times \mathbb{R} \times H$ equals $\gamma_1 \gamma_2$ ([8]), i.e., $|\gamma_0| = 1$. Hence there exists a solution of system (12) in $\mathcal{R} \times H$. □

4.3. The choice of the rectangle $\mathcal{R}$.

Lemma 4.2. *Under the assumptions of Theorem 2.1 there exists a rectangle $\mathcal{R} = (\varphi_1, \varphi_2) \times (\xi_1, \xi_2)$, $\varphi_* \in (\varphi_1, \varphi_2)$ and $\xi_* \in (\xi_1, \xi_2)$ such that the functions $\sigma_1(\varphi) = q_1(\varphi, \xi_1) + q_2(\varphi, \xi_1)$ and $\sigma_2(\varphi) = q_1(\varphi, \xi_2) + q_2(\varphi, \xi_2)$ preserve the same opposite signs on the interval $\varphi \in [\varphi_1, \varphi_2]$. The rectangle $\mathcal{R} \ni (\varphi_*, \xi_*)$ with this property my be chosen arbitrarily small in both directions.*

The function $\zeta(\xi) = q_1(\varphi_*, \xi) + q_2(\varphi_*, \xi)$ changes the sign at the point ξ_*, without loss of generality suppose that $\zeta(\xi)(\xi - \xi_*) > 0$ for $\xi \in (\xi_1, \xi_2)$, and $\xi_1 < \xi_* < \xi_2$. The continuous function $\sigma_1(\varphi)$ takes the value $\zeta(\xi_1) < 0$ in the point φ_*, therefore the function σ_1 is negative at a vicinity O_1 of φ_*. Analogously, there exists a vicinity O_2 of φ_* such that the function $\sigma_2(\varphi)$ is positive for $\varphi \in O_2$. Put $(\varphi_1, \varphi_2) = O_1 \bigcap O_2$, the interval (ξ_1, ξ_2) may be chosen arbitrarily small, the obtained rectangle $\mathcal{R}$ satisfies all the requirements of Lemma 4.2. □

Below we suppose that the rectangle $\mathcal{R}$ is so small that $q_1(\varphi, \xi) \neq 0$ for $(\varphi, \xi) \in \mathcal{R}$ (this is possible due to the assumption $q_1(\varphi_*, \xi_*) \neq 0$ of the theorem) and so small that $B(\varphi_1)B(\varphi_2) < 0$ (φ_* is a robust zero of the function B).

To prove the theorem we apply Lemma 4.1 to some system of the type (12) on the rectangles $\mathcal{R}_n = (\varphi_1, \varphi_2) \times (\xi_1 + nT, \xi_2 + nT)$. By construction,

$$\big(q_1(\varphi', \xi_1 + nT) + q_2(\varphi', \xi_1 + nT)\big)\big(q_1(\varphi'', \xi_2 + nT) + q_2(\varphi'', \xi_2 + nT)\big) < 0$$

for any integer n and $\varphi', \varphi'' \in [\varphi_1, \varphi_2]$.

4.4. Equivalent systems.

The functions $x(t) = \xi \sin(t + \varphi) + h(t)$ $(h \in E^\perp)$ and $u(t) \in C$ satisfy $L(p)x = b(t) + u$ if and only if

$$\langle \cos(t + \varphi), u(t) \rangle_{L^2} = \langle \sin(t + \varphi), u(t) \rangle_{L^2} = 0, \ h = A\mathcal{Q}(b + u).$$

Therefore, $x(t) = \xi \sin(t + \varphi) + h(t)$ satisfies (1) if and only if

$$\langle \cos(t + \varphi), f(x) \rangle_{L^2} = \langle \sin(t + \varphi), f(x) \rangle_{L^2} = 0, \ h = A\mathcal{Q}(b + f(x)). \qquad (15)$$

From $\cos(t + \varphi) = \xi^{-1}(x'(t) - h'(t))$ it follows that $\langle \cos(t + \varphi), f(x) \rangle_{L^2} = \xi^{-1}\big(\langle x', f(x) \rangle_{L^2} - \langle h', f(x) \rangle_{L^2}\big)$, but $\langle x', f(x) \rangle_{L^2} \equiv 0$ for any x, therefore $\langle \cos(t + \varphi), f(x) \rangle_{L^2} = -\xi^{-1}\langle h', f(x) \rangle_{L^2}$ and $\langle \cos(t + \varphi), f(x) \rangle_{L^2} = -\xi^{-1}\langle h', \mathcal{Q}f(x) \rangle_{L^2}$. The pair of equations $\langle h', \mathcal{Q}f(x) \rangle_{L^2} = 0$ and $h = A\mathcal{Q}(b + f(x))$ is equivalent to the pair $\langle A'\mathcal{Q}(b + f(x)), \mathcal{Q}f(x) \rangle_{L^2} = 0$ and $h = A\mathcal{Q}(b + f(x))$. Since $A'\mathcal{Q}b = b_1'$ system (15) is equivalent to

$$\langle A'\mathcal{Q}f(x), f(x) \rangle_{L^2} + \langle b_1', f(x) \rangle_{L^2} = 0,$$
$$\langle \sin(t + \varphi), f(x) \rangle_{L^2} = 0, \qquad h = A\mathcal{Q}(b + f(x)).$$

Now rewrite this system in the final equivalent form

$$\langle A'\mathcal{Q}f(x), f(x) \rangle_{L^2} + \langle b_1', f(x) \rangle_{L^2} - b_1'\left(\frac{3\pi}{2} - \varphi\right)\langle \sin(t + \varphi), f(x) \rangle_{L^2} = 0,$$
$$\langle \sin(t + \varphi), f(x) \rangle_{L^2} = 0, \qquad h = A\mathcal{Q}(b + f(x)). \qquad (16)$$

4.5. **Auxiliary statements.** The following 3 auxiliary statements (Lemmas 4.3 - 4.5) are proved in Section 6. In their formulations we use a number $\varepsilon \in (1/4, 1/3) \bigcap (1/4, \nu)$, we choose and fix it now and up to the end of the proofs.

Lemma 4.3. *Let* $h = b_1 + h_1$ *satisfies the equation* $h = A\mathcal{Q}(b + f(x))$. *Then for some* $\rho > 0$

$$\|h_1\|_{W_2^1} \leq \rho \xi^{-\varepsilon}.$$

Let $\theta : [0, 2\pi] \to \mathbb{C}$, $\theta \in C^1$ and $\|\theta\|_{C^1} \leq \Theta$.

Lemma 4.4. *There exists* K_1 *such that for any* $\rho > 0$ *and* $\xi \geq 1$

$$\sup_{\|h_1\|_{W_2^1} \leq \rho \xi^{-\varepsilon}} \left| \sqrt{\xi} \int_0^{2\pi} \theta(t) f(\xi \sin(t + \varphi) + b_1(t) + h_1(t)) \, dt - \Delta(\varphi, \xi) \right|$$

$$\leq K_1 (1 + \rho) \Theta \xi^{1/2 - 2\varepsilon} \quad (17)$$

where $\Delta(\varphi, \xi) = \sqrt{\dfrac{2\pi}{\omega}} \left(\theta(\dfrac{\pi}{2} - \varphi) q_1(\varphi, \xi) - \theta(\dfrac{3\pi}{2} - \varphi) q_2(\varphi, \xi) \right)$.

The constant K_1 is independent from θ and ρ.

The most cumbersome parts (Lemma 6.1 and Lemma 6.3) of the proofs of Lemmas 4.3 and 4.4 are related to the Kelvin method of stationary phase ([12], §§11 - 14). We repeat some constructions of the method to obtain necessary uniform estimates.

Lemma 4.5. *There exists* $K_2 = K_2(\gamma)$ *such that for any* φ *and* $\xi \geq 1$

$$\sup_{\|h\|_{C^1} \leq \gamma} |\langle A' \mathcal{Q} f(\xi \sin(t + \varphi) + h(t)), f(\xi \sin(t + \varphi) + h(t)) \rangle_{L^2}|$$

$$\leq K_2 \xi^{-2\varepsilon}. \quad (18)$$

Lemma 4.5 is proved in the end of the paper.

If the degree ℓ of the polynomial L satisfies $\ell \geq 4$ (or $\ell \geq m + 4$ for Theorem 3.1), then computations in Section 6 may be essentially simplified. The estimates (10)–(11) may be rewritten as $|\tilde{\nu}_k| \leq \tilde{r} k^{-4} \zeta_k$, $|\tilde{\nu}_k'| \leq \tilde{r} k^{-3} \zeta_k$, and $|\langle A' \mathcal{Q} u, \mathcal{Q} u \rangle_{L^2}| \leq \tilde{r} \sum_{k=2,3...} k^{-4} \nu_k^2$. This allows to obtain *a priori* estimates for $\|h_1\|_{C^1}$ instead of $\|h_1\|_{W_2^1}$ in Lemma 4.3 and to use $\sup_{\|h_1\|_{C^1} \leq \rho \xi^{-\varepsilon}}$ in Lemma 4.4.

4.6. **Finalization of the proof.** Lemma 4.3 states that any solution of the equation $h = A\mathcal{Q}(b + f(x))$ (for any ξ and φ) has the form $h = b_1 + h_1$ where $\|h_1\|_{W_2^1} \leq \rho \xi^{-\varepsilon}$ for some ρ.

Now apply Lemma 4.4 twice: for $\theta(t) = \sin(t + \varphi))$ and for $\theta(t) = b_1'(t)$. For any solution h_1 of the equation $h_1 = A\mathcal{Q} f(x)$ the relations

$$\sqrt{\frac{\omega \xi}{2\pi}} \langle \sin(t + \varphi), f(x) \rangle_{L^2} = q_1(\varphi, \xi) + q_2(\varphi, \xi) + o_1(1)$$

and

$$\sqrt{\frac{\omega \xi}{2\pi}} \left(\langle b_1', f \rangle_{L^2} - b_1'(\frac{3\pi}{2} - \varphi) \langle \sin(t + \varphi), f \rangle_{L^2} \right) = q_1(\varphi, \xi) B(\varphi) + o_2(1)$$

are valid. In the both formulas the symbol $o_j(1)$ means some infinitesimals (as $\xi \to \infty$) of the order $\xi^{1/2 - 2\varepsilon}$ uniform with respect to φ and h_1 satisfying $h_1 = A\mathcal{Q} f(x)$.

The operator $A\mathcal{Q}$ acts continuously from C to C^1, the function $b(t) + f(x)$ is bounded, therefore any solution h of the equation $h = A\mathcal{Q}(b + f(x))$ satisfies the estimate

$$\|h\|_{C^1} \leq \gamma_0 = \|A\mathcal{Q}\|_{C \to C^1} (\sup |b| + \sup |f|). \tag{19}$$

From Lemma 4.5 it follows that $|\sqrt{\xi}\langle A'\mathcal{Q}f(x), f(x)\rangle_{L^2}| \leq K_2 \xi^{1/2 - 2\varepsilon}$.

Equivalent system (16) by construction satisfies all the assumptions of Lemma 4.1 on the rectangles $\mathcal{R}_n$ for any sufficiently large n and $H = L^2$. Therefore there exists 2π-periodic solution

$$x_n(t) = \xi_n \sin(t + \varphi_n) + b_1(t) + h_n(t), \ (\varphi_n, \xi_n) \in \mathcal{R}_n,$$

by construction $\|x_n\|_{L^2} \geq \sqrt{\pi}\xi_n \to \infty$, Theorem 2.1 is proved.   □

5. Proof of Theorem 2.2. Now $b(t) = \beta \sin(t + \psi) + \tilde{b}(t)$.

Let us follow the proof of Theorem 2.1 for the equation $L(p)x = \tilde{b} + f(x)$. Construct the orthogonal projectors $\mathcal{P}$ and $\mathcal{Q}$, the linear operators A, $A\mathcal{Q}$, and $A'\mathcal{Q}$. Create the equivalent (for the equation $L(p)x = \tilde{b} + f(x)$) system (16) on the set $\mathcal{R}_n \times L^2$.

By construction, there exists a number n_0 such that for any $n \geq n_0$ system (16) satisfies all the assumptions of Lemma 4.1, namely conditions (13) and (14).

Consider the perturbed system (16):

$$\pi\beta \sin(\psi - \varphi) + \langle A'\mathcal{Q}f, f\rangle_{L^2} + \langle b_1', f\rangle_{L^2}$$
$$- b_1'\left(\frac{3\pi}{2} - \varphi\right)\left(\pi\beta \cos(\psi - \varphi) + \langle \sin(t + \varphi), f\rangle_{L^2}\right) = 0, \tag{20}$$
$$\pi\beta \cos(\psi - \varphi) + \langle \sin(t + \varphi), f(x)\rangle_{L^2} = 0, \quad h = A\mathcal{Q}(b + f(x))$$

on the sets $\mathcal{R}_n \times L^2$ for $n = n_0, n_0 + 1, \ldots, n_0 + N - 1$. There exist a $\sigma > 0$ such that for $|\beta| \leq \sigma$ system (20) also satisfies conditions (13) and (14).

System (20) is equivalent to the equation $L(p)x = \beta \sin(t + \psi) + \tilde{b} + f(x)$. From Lemma 4.1 applied to (20) it follows the statement of Theorem 2.2.   □

6. Proof of auxiliary statements.

6.1. Proof of Lemma 4.3. Let h be a solution of $h = A\mathcal{Q}(b + f(x))$. Therefore h satisfies (19).

Lemma 6.1. *For any $\gamma > 0$, $k = 0, 1, 2, \ldots$, and $\varphi \in \mathbb{R}$ the estimate*

$$\sup_{\|h\|_{C^1} \leq \gamma} \left| \int_0^{2\pi} e^{i(\xi \sin t + h(t))} \sin(kt + \varphi) \, dt \right| \leq \frac{20}{\sqrt{\xi}} + \frac{4(k + \gamma)\ln\xi}{\xi} \tag{21}$$

for $\xi \geq 1$ holds.

This lemma is proved in the next Subsection.

From (21) and the trivial relationship

$$\left| \int_0^{2\pi} e^{i(\xi \sin t + h(t))} \sin(kt + \varphi) \, dt \right| \leq 2\pi \tag{22}$$

(it is valid for all ξ, h, k, φ) it follows that

$$\sup_{\|h\|_{C^1} \leq \gamma} \Big| \int_0^{2\pi} e^{i(\xi \sin t + h(t))} \sin(kt + \varphi)\, dt \Big|$$

$$\leq \min\Big\{2\pi, \frac{20}{\sqrt{\xi}} + \frac{4(k+\gamma)\ln\xi}{\xi}\Big\}, \qquad \xi \geq 1. \quad (23)$$

Relation (23) is valid for all ξ, k, γ, φ, put there $\omega s\xi, \omega s h, \omega s\gamma$ instead of ξ, h, γ where $\omega = \frac{2\pi}{T} > 0$ is a real number defined in the beginning of Section 2, $s = 1, 2, \ldots$. The relation

$$\sup_{\|h\|_{C^1} \leq \gamma} \Big| \int_0^{2\pi} e^{i\omega s(\xi \sin t + h(t))} \sin(kt + \varphi)\, dt \Big| \leq Y(k, s, \xi, \gamma) \quad (24)$$

holds for any integer $s > 0$ and real $\xi > \omega^{-1}$ where

$$Y(k, s, \xi, \gamma) \stackrel{\text{def}}{=} \min\Big\{2\pi, \frac{20}{\sqrt{\omega s\xi}} + \frac{4(k+\gamma\omega s)\ln(\omega s\xi)}{\omega s\xi}\Big\}.$$

Put $\alpha_s(t) = \sin\big(\omega s(\xi \sin t + h(t)) + \psi_s\big)$. The function $f(\xi \sin t + h(t))$ by (4) can be represented as

$$f(\xi \sin t + h(t)) = \sum_{s=1}^{\infty} \mu_s \alpha_s(t).$$

Let $a_k(s), c_k(s), c'_k(s)$ be the Fourier coefficients of the functions α_s, $H_s = A\mathcal{Q}\alpha_s$, and $H'_s = \frac{d}{dt}H_s = A'\mathcal{Q}\alpha_s$. Then $|a_k(s)| \leq Y(k, s, \xi, \gamma)$, $k = 0, 1, 2, \ldots$, therefore from (24) it follows the inequality $|c_0(s)| \leq const \cdot \ln(s+1) \cdot (\omega\,\xi)^{-1/2}$ and from (10) it follows that for $k = 2, 3, \ldots$

$$|c_k(s)| \leq r_1 Y(k, s, \xi, \gamma)\, k^{-2}, \quad |c'_k(s)| \leq r_2 Y(k, s, \xi, \gamma)\, k^{-1}. \quad (25)$$

Lemma 6.2. *For some γ_1 the estimate $\|H_s\|_{W_2^1} \leq \gamma_1 \xi^{-\varepsilon} \ln(s+1)$ holds.*

Lemma 6.2 is proved in Subsection 6.3.

We have $h_1 = A\mathcal{Q}f(x)$, the relation $h_1 = \sum_{s=1}^{\infty} \mu_s H_s$ holds and

$$\|h_1\|_{W_2^1} \leq \sum_{s=1}^{\infty} \mu_s \|H_s\|_{W_2^1} \leq \gamma_1 \xi^{-\varepsilon} \sum_{s=1}^{\infty} \mu_s |\ln(s+1)| \leq const \cdot \xi^{-\varepsilon},$$

Lemma 4.3 follows from condition (6) of Theorem 1. $\qquad\qquad\qquad \square$

6.2. **Proof of Lemma 6.1.** Put $q(t) = \sin(kt + \varphi)e^{ih(t)}$, then

$$q'(t) = k\cos(kt + \varphi)e^{ih(t)} + i\sin(kt + \varphi)e^{ih(t)}h'(t);$$

obviously, $\|q\|_C \leq 1$, $\|q'_t\|_C \leq k + \gamma$. Let us estimate the value

$$I(\xi) = I_{(0,\pi/2)}^\xi = \int_0^{\pi/2} e^{i(\xi \sin t + h(t))} \sin(kt + \varphi)\, dt = \int_0^{\pi/2} e^{i\xi \sin t} q(t)dt,$$

analogous integrals $I_{(\pi/2,\pi)}^\xi$, $I_{(\pi,3\pi/2)}^\xi$, and $I_{(3\pi/2,2\pi)}^\xi$ along the corresponding intervals (the function $\sin t$ is monotone on each such interval) can be considered with the use of the same scheme. After the change of variables $v = \sin t$ in the integral $I(\xi)$ we have

$$I(\xi) = \int_0^1 e^{iv\xi} W(v)dv, \qquad W(v) = \frac{q(\arcsin v)}{\sqrt{1 - v^2}}.$$

The function W is continuous on $[0, 1)$, $|W(v)| \leq 1/\sqrt{1-v}$ and

$$|W'(v)| \leq \left|\frac{q'(\arcsin v)}{1-v^2}\right| + \left|\frac{v\, q(\arcsin v)}{(\sqrt{1-v^2}\,)^3}\right| \leq \frac{k+\gamma}{1-v} + \frac{1}{\sqrt{(1-v)^3}}\,.$$

Furthermore, $I(\xi) = I_1(\xi) + I_2(\xi)$;

$$I_1(\xi) = \int_0^{1-\xi^{-1}} e^{i\xi v} W(v)\, dv, \quad I_2(\xi) = \int_{1-\xi^{-1}}^1 e^{i\xi v} W(v)\, dv.$$

Now let us estimate the integrals I_1 and I_2 separately. First of all,

$$|I_2(\xi)| = \left|\int_{1-\xi^{-1}}^1 e^{i\xi v} W(v)\, dv\right| \leq \int_{1-\xi^{-1}}^1 \frac{dv}{\sqrt{1-v}} = \frac{2}{\sqrt{\xi}},$$

then

$$|I_1(\xi)| = \left|\int_0^{1-\xi^{-1}} e^{i\xi v} W(v)\, dv\right| = \frac{1}{\xi}\left|\int_0^{1-\xi^{-1}} W(v)\, d(e^{i\xi v})\right|$$

$$\leq \left|\left(\frac{e^{i\xi v} W(v)}{\xi}\Big|_0^{1-\xi^{-1}}\right)\right| + \frac{1}{\xi}\left|\int_0^{1-\xi^{-1}} e^{i\xi v}\, W'_v(v)\, dv\right|$$

$$\leq \frac{1}{\xi} + \frac{1}{\sqrt{\xi}} + \frac{(k+\gamma)\ln\xi}{\xi} + \frac{2\sqrt{\xi}}{\xi} - \frac{2}{\xi} \leq \frac{3}{\sqrt{\xi}} + \frac{(k+\gamma)\ln\xi}{\xi}\,.$$

Combining the obtained estimate for $I_{(0,\pi/2)}^{\xi}$ with the same estimates for the integrals $I_{(\pi/2,\pi)}^{\xi}$, $I_{(\pi,3\pi/2)}^{\xi}$, and $I_{(3\pi/2,2\pi)}^{\xi}$ we have (21). $\qquad\square$

6.3. **Proof of Lemma 6.2.** From Parseval's Formula

$$\|H'_s\|^2_{L^2} = \sum_{k=1}^\infty |c'_k(s)|^2$$

and (25) it follows the estimate

$$\|H'_s\|^2_{L^2} \leq r_2^2 \sum_{k=1}^\infty |Y(k,s,\xi,\gamma)k^{-1}|^2.$$

Split for any ξ the last series into two parts: a finite part for $k \leq [\xi]$ and an infinite rest part for $k > [\xi]$. For the infinite rest part we have

$$\sum_{k=[\xi]+1}^\infty \left(\frac{Y(k,s,\xi,\gamma)}{k}\right)^2 \leq \sum_{k=[\xi]+1}^\infty \frac{4\pi^2}{k^2}$$

$$\leq \sum_{k=[\xi]+1}^\infty \frac{4\pi^2}{([\xi]+1)^{2\varepsilon}k^{2-2\varepsilon}} \leq \frac{1}{\xi^{2\varepsilon}} \sum_{k=1}^\infty \frac{4\pi^2}{k^{2-2\varepsilon}},$$

the last series converges since $\varepsilon \in (\frac{1}{4}, \frac{1}{2})$ implies $2 - 2\varepsilon > 1$.

The estimates for the finite part follow from the relations

$$
\begin{aligned}
\sum_{k=1}^{[\xi]} \Big(\frac{Y(k,s,\xi,\gamma)}{k}\Big)^2 &\le \sum_{k=1}^{[\xi]} \Big(\frac{20}{k\sqrt{\omega s\xi}} + \frac{4(k+\gamma\omega s)\ln(\omega s\xi)}{k\omega s\xi}\Big)^2 \\
&\le \sum_{k=1}^{\infty} \Big(\frac{20}{k\sqrt{\omega s\xi}} + \frac{4(k+\gamma\omega s)}{k\omega s}\frac{\ln(\omega s\xi)}{\xi^{(1/2-\varepsilon)/2}}\frac{1}{k^{(3-2\varepsilon)/4}\xi^{\varepsilon}}\Big)^2 \\
&\le \frac{\ln^2(s+1)}{\xi^{2\varepsilon}}\sum_{k=1}^{\infty}\Big(\frac{20}{k} + \frac{const}{k^{(3-2\varepsilon)/4}}\Big)^2 = \frac{c\ln^2(s+1)}{\xi^{2\varepsilon}}.
\end{aligned}
$$

We used the evident relations $(3-2\varepsilon)/4 + \varepsilon + (1/2-\varepsilon)/2 = 1$ and $(3-2\varepsilon)/4 > \frac{1}{2}$.

We proved the estimate $\|H'_s\|_{L^2} \le c\xi^{-\varepsilon}\ln(s+1)$, it implies $\|H'_s\|_{L^1} \le \tilde{c}_1\xi^{-\varepsilon}\ln(s+1)$. Any continuous periodic function H_s always takes its mean value c_0 that is its zero harmonics, it satisfies $|c_0| \le \tilde{c}\xi^{-\varepsilon}\ln(s+1)$, let $H_s(t_0) = c_0$. The estimates for the values $\|H_s\|_C$ follow from

$$
|H_s(t)| \le |c_0| + \int_{t_0}^{t} |H'_s(t)|\,dt,
$$

therefore, $\|H_s\|_C \le (\tilde{c}+\tilde{c}_1)\,\xi^{-\varepsilon}\ln(s+1)$. $\qquad\square$

6.4. **Proof of Lemma 4.4.**

Lemma 6.3. *There exists some $S > 0$ such that the relation*

$$
\sup_{\|h_1\|_{W_2^1}\le\tilde{\rho}\xi^{-\varepsilon}} \Big|\sqrt{\xi}\int_0^{2\pi}\tilde{\theta}(t)\,e^{i(\xi\sin t+h_1(t))}\,dt - \Delta_1(\xi)\Big|
$$

$$
\le S(1+\tilde{\rho})(1+\|\tilde{\theta}\|_{C^1})\xi^{1/2-2\varepsilon} \quad (26)
$$

for $\xi \ge 1$ holds for any $\tilde{\rho} > 0$ and for any $\tilde{\theta} \in C^1$ where

$$
\Delta_1(\xi) = \tilde{\theta}(\frac{\pi}{2})e^{(\xi-\pi/4)i}\sqrt{2\pi} + \tilde{\theta}(\frac{3\pi}{2})e^{-(\xi-\pi/4)i}\sqrt{2\pi}.
$$

Lemma 6.3 is proved in Subsection 6.5.

Consider the integral

$$
\begin{aligned}
I &= \int_0^{2\pi}\theta(t-\varphi)\,f(\xi\sin t + b_1(t-\varphi) + h_1(t-\varphi))\,dt \\
&= \sum_{s=1}^{\infty}\mu_s\Im\int_0^{2\pi}\theta(t-\varphi)\,e^{i(\omega s b_1(t-\varphi)+\psi_s)}\,e^{i(\omega s(\xi\sin t+h_1(t-\varphi)))}\,dt
\end{aligned}
$$

Put $\tilde{\theta}(t) = \theta(t-\varphi)\,e^{i(\omega s b_1(t-\varphi)+\psi_s)}$ and apply Lemma 6.3 to the integrals

$$
\int_0^{2\pi}\tilde{\theta}(t)\,e^{i(\omega s(\xi\sin t+h_1(t-\varphi)))}\,dt
$$

for various integer positive s. We obtain

$$
\sup_{\|h_1\|_{W_2^1}\le\rho_1\xi^{-\varepsilon}}\Big|\sqrt{\xi}\int_0^{2\pi}\tilde{\theta}(t)\,e^{i(\omega s(\xi\sin t+h_1(t-\varphi)))}\,dt - \frac{\Delta_1(\omega s\xi)}{\sqrt{\omega s}}\Big|
$$

$$
\le (\omega s)^{-1/2}S(1+\rho_1(\omega s)^{1+\varepsilon})(1+\|\tilde{\theta}\|_{C^1})(\omega s\xi)^{1/2-2\varepsilon}.
$$

Since $\|\tilde{\theta}\|_{C^1} \leq const\, s(1 + \Theta)$ we have

$$\sup_{\|h_1\|_{W_2^1} \leq \rho_1 \xi^{-\varepsilon}} \big| \sqrt{\xi} I - \Im\big(\sum_{s=1}^{\infty} \mu_s \frac{\Delta_1(\omega s \xi)}{\sqrt{\omega s}} \big) \big|$$

$$\leq const\, (\rho + 1)\big(\sum_{s=1}^{\infty} \mu_s s^{3/2+\varepsilon} \big) \xi^{1/2-2\varepsilon},$$

the series in the right-hand side of this formula converges due to (6) and $\varepsilon < \nu$. The statement of the lemma follows from the equalities

$$\Im\big(\sum_{s=1}^{\infty} \mu_s \frac{\Delta_1(\omega s \xi)}{\sqrt{\omega s}} \big)$$

$$= \sqrt{\frac{2\pi}{\omega}} \sum_{s=1}^{\infty} \frac{\mu_s}{\sqrt{s}} \Im\big(\tilde{\theta}(\frac{\pi}{2}) e^{i(\omega s \xi - \frac{\pi}{4})} + \tilde{\theta}(\frac{3\pi}{2}) e^{-i(\omega s \xi - \frac{\pi}{4})} \big) = \sqrt{\frac{2\pi}{\omega}} \Delta(\xi).$$

$\square$

6.5. **Proof of Lemma 6.3.** Consider in detail the integral

$$I(\xi) = \int_0^{\pi/2} e^{i(\xi \sin t + h_1(t))} \tilde{\theta}(t)\, dt,$$

the integrals of the same function $e^{i(\xi \sin t + h_1(t))} \tilde{\theta}(t)$ over the intervals $(\pi/2, \pi)$, $(\pi, 3\pi/2)$, and $(3\pi/2, 2\pi)$ can be considered in a similar way. Lemma 6.3 follows from the relations

$$I(\xi) = \int_{\pi/2}^{\pi} \ldots = \tilde{\theta}(\frac{\pi}{2}) \sqrt{\frac{\pi}{2\xi}}\, e^{(\xi - \frac{\pi}{4})i} + O(\xi^{-2\varepsilon}),$$

$$\int_{\pi}^{3\pi/2} \ldots = \int_{3\pi/2}^{2\pi} \ldots = \tilde{\theta}(\frac{3\pi}{2}) \sqrt{\frac{\pi}{2\xi}}\, e^{-(\xi - \frac{\pi}{4})i} + O(\xi^{-2\varepsilon}),$$

we prove the relation concerning $I(\xi)$ only.

First of all consider the integral

$$\Xi = \int_0^{\pi/2} e^{i\xi \sin t} (e^{ih_1(t)} \tilde{\theta}(t) - \tilde{\theta}(\frac{\pi}{2}))\, dt$$

and prove that $|\Xi| \leq const(1 + \tilde{\rho}) \|\tilde{\theta}\|_{C^1} \xi^{-\varepsilon/2}$. Put $\delta = \xi^{-\varepsilon}$, obviously $\Xi = \Xi_1 + \Xi_2$ where

$$\Xi_1 = \int_{\pi/2-\delta}^{\pi/2} \ldots, \qquad \Xi_2 = \int_0^{\pi/2-\delta} \ldots$$

We estimate the terms Ξ_j, $j = 1, 2$ independently. The term Ξ_1 consists from two parts:

$$\Xi_1 = \tilde{\theta}(\frac{\pi}{2}) \int_{\pi/2-\delta}^{\pi/2} e^{i\xi \sin t} (e^{ih_1(t)} - 1)\, dt + \int_{\pi/2-\delta}^{\pi/2} e^{i\xi \sin t} e^{ih_1(t)} \big(\tilde{\theta}(t) - \tilde{\theta}(\frac{\pi}{2}) \big)\, dt.$$

Since[6] $|1 - e^{ir}| \leq |r|$ and $|\tilde{\theta}(t) - \tilde{\theta}(\pi/2)| \leq \sup |\tilde{\theta}'| (\pi/2 - t)$ we have

$$|\Xi_1| \leq \sup |\tilde{\theta}|\, \delta \|h_1\|_C + \sup |\tilde{\theta}'| \delta^2/2$$

$$\leq \sup |\tilde{\theta}| \xi^{-\varepsilon} \tilde{\rho} \xi^{-\varepsilon} + \sup |\tilde{\theta}'| \xi^{-2\varepsilon} \leq 2\|\tilde{\theta}\|_{C^1} \tilde{\rho} \xi^{-2\varepsilon}.$$

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[6] An arc is always longer than its chord.

Put $\Psi(t) = e^{ih_1(t)}\tilde{\theta}(t) - \tilde{\theta}(\frac{\pi}{2})$ and rewrite the term Ξ_2 in the form

$$\Xi_2 = \int_0^{\pi/2-\delta} e^{i\xi \sin t}\Psi(t)\, dt = \frac{1}{i\xi}\int_0^{\pi/2-\delta} \frac{\Psi(t)}{\cos t}\, d(e^{i\xi \sin t}).$$

After the integration by parts we obtain

$$i\xi\Xi_2 = \frac{\Psi(t)}{\cos t}\, e^{i\xi \sin t}\Big|_0^{\pi/2-\delta} - \int_0^{\pi/2-\delta} e^{i\xi \sin t}\frac{\cos t\Psi'(t) + \sin t\Psi(t)}{\cos^2 t}\, dt.$$

Now

$$\xi|\Xi_2| \le \sup|\Psi|(1 + \frac{1}{\sin\delta}) + \|\Psi\|_C \int_0^{\pi/2-\delta}\frac{dt}{\cos^2 t} + \int_0^{\pi/2-\delta}\frac{\Psi'(t)dt}{\cos t}$$

$$\le \|\Psi\|_C\big(1 + \frac{\pi}{2\delta} + \tan(\frac{\pi}{2} - \delta)\big) + \frac{\|\Psi\|_{L^1}}{\cos(\frac{\pi}{2} - \delta)}$$

and finally $|\Xi_2| \le 9\tilde{\rho}\|\tilde{\theta}\|_{C^1}\xi^{-1+\varepsilon} \le 9\tilde{\rho}\|\tilde{\theta}\|_{C^1}\xi^{-2\varepsilon}$ since $-1+\varepsilon > -2\varepsilon$ for $\varepsilon < 1/3$.

Consider the integral

$$J(\xi) = \int_0^{\pi/2} e^{i\xi \sin t}\, dt = \int_0^1 \frac{e^{iv\xi}dv}{\sqrt{1-v^2}}$$

(we put $v = \sin t$). We have

$$J(\xi) = \int_0^1 \frac{e^{iv\xi}dv}{\sqrt{2(1-v)}} + \int_0^1 e^{iv\xi}E(v)\, dv, \ \ E(v) = \frac{1}{\sqrt{1-v^2}} - \frac{1}{\sqrt{2(1-v)}}$$

Lemma 12.1 from [12] (page 100, formula (12.01)) implies

$$\int_0^\infty \frac{e^{iu\xi}du}{\sqrt{u}} = \frac{1}{\sqrt{\xi}}\int_0^\infty \frac{e^{iu}du}{\sqrt{u}} = \frac{e^{\frac{\pi i}{4}}\Gamma(\frac{1}{2})}{\sqrt{\xi}} = \frac{e^{\frac{\pi i}{4}}\sqrt{\pi}}{\sqrt{\xi}}\ .$$

Combine this with the equalities

$$\int_0^1 \frac{e^{iv\xi}dv}{\sqrt{2(1-v)}} = \frac{e^{i\xi}}{\sqrt{2}}\int_0^1 \frac{e^{-iu\xi}du}{\sqrt{u}} = \frac{e^{i\xi}}{\sqrt{2}}\overline{\int_0^\infty \frac{e^{iu\xi}du}{\sqrt{u}}} - \frac{e^{i\xi}}{\sqrt{2}}\overline{\int_1^\infty \frac{e^{iu\xi}du}{\sqrt{u}}}$$

and with the relations

$$\Big|\int_1^\infty \frac{e^{iu\xi}du}{\sqrt{u}}\Big| = \frac{1}{\xi}\Big|\int_1^\infty \frac{de^{iu\xi}}{\sqrt{u}}\Big| \le \frac{1}{\xi} + \frac{1}{2\xi}\int_1^\infty \frac{du}{\sqrt{u^3}} \le \frac{1}{\xi} + \frac{1}{\xi} = \frac{2}{\xi}$$

to obtain the estimate

$$\Big|\int_0^1 \frac{e^{iv\xi}dv}{\sqrt{2(1-v)}} - \sqrt{\frac{\pi}{2\xi}}\, e^{(\xi-\frac{\pi}{4})i}\Big| \le \frac{\sqrt{2}}{\xi}.$$

The function $E(v)$ is continuous on $[0, 1]$, its derivative E' is continuous on $[0, 1)$, $|E'(v)| \approx const/\sqrt{1-v}$ at $v = 1$, therefore $\int_0^1 |E'(v)|dv = E_0 < \infty$. Now we have

$$\Big|\int_0^1 e^{iv\xi}E(v)\, dv\Big| = \Big|\frac{1}{i\xi}\int_0^1 E(v)\, d(e^{iv\xi})\Big|$$

$$\le \frac{|E(0)|}{\xi} + \frac{1}{\xi}\Big|\int_0^1 E'(v)\, e^{iv\xi}dv\Big| \le \frac{|E(0)| + E_0}{\xi},$$

therefore

$$\Big|J(\xi) - \sqrt{\frac{\pi}{2\xi}}\, e^{(\xi-\frac{\pi}{4})i}\Big| \le \frac{|E(0)| + E_0 + \sqrt{2}}{\xi}.$$

Since $I(\xi) = \Xi + \tilde{\theta}(\pi/2)J(\xi)$ this estimate proves Lemma 6.3.   □

6.6. **Proof of Lemma 4.5.** The proof of this lemma almost repeats the proof of Lemma 4.3. Consider again the functions $\alpha_s(t)$, its Fourier coefficients $a_k(s)$ satisfy $|a_k(s)| \le Y(k, s, \xi, \gamma)$, $k = 2, 3, \ldots$, therefore the Fourier coefficients f_k of the function $f(\xi \sin(t + \varphi) + h(t))$ satisfy the estimates

$$|f_k| \le \sum_{s=1}^\infty \mu_s Y(k, s, \xi, \gamma).$$

Since $\sum \min \le \min \sum$ we have

$$|f_k| \le \min \left\{ 2\pi \sum_{s=1}^\infty \mu_s, \sum_{s=1}^\infty \mu_s \left(\frac{20}{\sqrt{\omega s \xi}} + \frac{4(k + \gamma \omega s) \ln(\omega s \xi)}{\omega s \xi} \right) \right\}$$

$$\le r_4 \min \left\{ 1, \frac{1}{\sqrt{\xi}} + \frac{(k + \xi) \ln(w\xi)}{\xi} \right\}.$$

From (11) it follows the estimate

$$|\langle A'\mathcal{Q}f, f \rangle_{L^2}| \le r_2 r_4^2 \sum_{k=2}^\infty \left(\min \left\{ 1, \frac{1}{\sqrt{\xi}} + \frac{(k + \xi) \ln(w\xi)}{\xi} \right\} \right)^2 k^{-2}. \qquad (27)$$

The last series may be estimated exactly as it was done in Lemma 6.2. The final estimate has the form (18): $|\langle A'\mathcal{Q}f, f \rangle_{L^2}| \le r_2 \xi^{-2\varepsilon}$. $\qquad \square$

REFERENCES

[1] J. M. Alonso and R. Ortega, *Unbounded solutions of semilinear equations at resonance*, Nonlinearity, **9** (1996), 1099–1111.

[2] V. I. Arnold, "Arnold's Problems," Problem 1996-5, comments by S.B.Kuksin, Springer-Verlag, New York, (2005), 580–582.

[3] D. Bonheure, C. Fabry and D. Ruiz, *Problems at resonance for equations with periodic nonlinearities*, Nonlinear Analysis, **55** (2003), 557–581.

[4] A. Cañada and F. Roca, *Existence and multiplicity of solutions of someconservative pendulum-type equations with homogeneous Dirichlet conditions*, Diff. and Int. Eq., **10** (1997), 1113–1122.

[5] E. N. Dancer, *On the use of asymptotics in nonlinear boundary value problems*, Ann. Mat. Pura Appl., **131** (1982), 167–185.

[6] A. Isidori, "Nonlinear Control Systems," Springer-Verlag, London, 1995.

[7] H. K. Khalil, "Nonlinear Systems," Prentice Hall, Upper Saddle River, New Jersey, 2002.

[8] A. M. Krasnosel'skii and M. A. Krasnosel'skii, *Vector fields in the direct product of spaces, and applications todifferential equations*, Differential Equations, **33** (1997), 59–66.

[9] A. M. Krasnosel'skii and J. Mawhin, *The index at infinity of some twice degenerate compact vector fields*, Discrete Contin. Dyn. Syst., **1** (1995), 207–216.

[10] A. M. Krasnosel'skii and J. Mawhin, *Periodic solutions of equations with oscillating nonlinearities*, Mathematical and Computer Modelling, **32** (2000), 1445–1455

[11] M. A. Krasnosel'skii and P. P. Zabreiko, "Geometrical Methods of Nonlinear Analysis," Springer-Verlag, Berlin, Heidelberg, 1984.

[12] F. W. S. Olver, "Asymptotics and Special Functions," Academic Press, New York, 1974.

[13] G. Pólya and G. Szegö, "Aufgaben and Lehrsätze aus der Analysis," Springer, Berlin, **B I-II** (1925).

[14] R. Schaaf and K. Schmitt, *A class of nonlinear Sturm-Liouville problems with infinitely many solutions*, Trans. Amer. Math. Soc., **306** (1988), 853–859.

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