ABSOLUTE STABILITY AND DYNAMIC COMPLEMENTARITY

V.S.KOZYAKIN A.MANDELBAUM
A.A.VLADIMIROV

Abstract. The close connection is established between the DCP theory and the absolute stability theory of collections of matrices. The stability methods are implemented to obtain some new results in DCP theory.

Introduction

The Dynamic Complementarity Problem (DCP) is a dynamic version of the Linear Complementarity Problem (LCP), see [6]. The formulation of DCP is due to Harrison and Reiman [8]. DCP arose as a unifying framework for fluid and diffusion approximations of stochastic flow networks. Such networks encompass open queueing networks, closed queueing networks, finite-buffer networks, assembly networks, dynamic stochastic Leontief systems, and networks with heterogenous population (see [5, 14] for the bibliography).

One of the main mathematical problems in DCP theory is to derive necessary and sufficient conditions for the uniqueness of a solution (such conditions for the existense of solutions are quite well developed). To this end we should implement here a new technique developed in recent years, namely, the so called desynchronization theory, or more specifically, the theory of absolute stability of the matrix collections (see [1, 2, 9]).

We derive some new results in DCP theory by means of this techniques and also give some reasonable conjectures (still unproved) that there can be no "well-formulated" necessary and sufficient conditions for the uniqueness of the DCP solution.

In this paper we consider a DCP on the positive orthant in $\mathbb{R}^N$. This is a convenient model case demonstrating the main features of the problem. More general situations (polyhedrons, general convex sets, etc.) can be often reduced to this one. When this is not the case, we can, nevertheless, sometimes prove the counterparts of the results introduced here.

1 Dynamic complementarity problem

Denote by $\mathbb{R}^N$ a N-dimensional euclidean space with a scalar product $\langle \cdot, \cdot \rangle$ and let $\mathbb{R}^N_+$ be the non-negative orthant of $\mathbb{R}^N$.

Denote by D^N the space of N-dimensional functions on $[0, \infty)$ that are right continuous on $[0, \infty)$ with finite left limits on $(0, \infty)$. We distinguish in D^N the following subsets:

$$D^N_0 = \{x \in D^N : x(0) \geq 0\};$$
$$D^N_+ = \{x \in D^N : x(t) \geq 0 \ \textit{for all} \ t \geq 0\};$$
$$D^N_\wedge = \{x \in D^N : x(0) = 0,\ x(t) \ \textit{is nondecreasing in} \ t \geq 0\}.$$

For any $x \in D^N$, we shall denote by $x^-(t)$ the left limit of the function x at the point t.

Introduce a notion of *continuous dynamic complementarity problem (DCP)*.

Definition 1.1. *Let $x = \{x(t), t \geq 0\}$ be a function in D^N. Assume that $x(0) \geq 0$. The DCP with matrix R is the following:*

given $x(t)$, find functions $y(t)$ and $z(t)$ such that

$$\begin{aligned} z &= x + Ry, \\ z &\in D^N_+ \quad (\textit{nonnegativity}) \\ y &\in D^N_\wedge \quad (\textit{monotonicity}) \\ \langle z', dy \rangle &= 0 \quad (\textit{complementarity}). \end{aligned} \tag{1}$$

Here z' is the derivative of z in t and (1) *is an abbreviation of*

$$\sum_{j=1}^{N} \int_0^\infty z_j dy_j = 0,$$

where the summands are Lebesgue-Stieltjes integrals of components of z with respect to components of y (Rieman-Stieltjes integral are not appropriate here because y and z are allowed to have common points $t > 0$ of discontinuity).

A square matrix R is called a $\mathcal{S}$*-matrix* if there exists a vector $\eta \geq 0$ such that $R\eta > 0$; R is *completely-*$\mathcal{S}$ if all its principle submatrices are $\mathcal{S}$-matrices. Finally, R is a $\mathcal{P}$*-matrix* if all its principal minors are strictly positive.

Lemma 1.2 (see [7]). *Matrix R is not a $\mathcal{P}$-matrix if and only if there exists a non-zero $x \in \mathbb{R}^N$ such that $(Rx)_i x_i \leq 0$, $1 \leq i \leq N$.*

Introduce next two fundumental results in DCP theory (see [14] for the bibliography).

Theorem 1.3. *The continuous DCP has a solution for all $x \in D_0^N$ if and only if the matrix R is completely-$\mathcal{S}$.*

Theorem 1.4. *The continuous DCP has a unique solution for all piecewise-constant $x \in D_0^N$ if and only if R is a $\mathcal{P}$-matrix.*

Uniqueness of a solution of DCP fails to hold for some general $x \in D_0^N$, even with R being a $\mathcal{P}$-matrix. For example, the DCP with the $\mathcal{P}$-matrix

$$R = \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}$$

has multiple solutions [14].

An implicit characterization of the matrices R for which the DCP has a unique solution, for all $x \in D_0^N$, is given by the following

Theorem 1.5 (see [14]). *The continuous DCP has a unique solution for all $x \in D_0^N$ if and only if the system of differential inequalities*

$$(Ru)_j du_j \leq 0, \quad j = 1, \ldots, N, \tag{2}$$

subject to

$$u \in D^N \text{ is locally of bounded variation with } u(0) = 0, \tag{3}$$

has the unique solution $u \equiv 0$.

In full detail, (2) stands for

$$\int_B (Ru)_j du_j \leq 0, \quad \forall B \in \mathcal{B}, \quad j = 1, \ldots, N, \tag{4}$$

where the integrals are Lebesgue-Stieltjes integrals, and $\mathcal{B}$ is the Borel σ-field in $\mathbb{R}^1_+$.

2 Absolute stability theory

We shall consider here the finite collections of $N \times N$-matrices with real entries. Our aim in the present section is to give an outline of a small part of the so called desynchronization theory developed in recent years (see, for example, [9]).

Definition 2.1. *Let us call the collection $\{A_1, \ldots, A_k\}$ of real $N \times N$-matrices absolutely stable if there exists a $M > 0$ such that*

$$\|A_{i_1} \ldots A_{i_m}\| \leq M$$

for any $m > 0$ and $1 \leq i_j \leq k$, $j = 1, \ldots, m$. (here $\|\cdot\|$ is some matrix norm).

We should also say in this case that the difference equation

$$x(n+1) = A(n)x(n) \tag{5}$$

is absolutely stable for the class of matrices

$$\mathcal{A} = \{A_1, A_2, \ldots, A_k\}$$

The meaning of the above definition is the following. Suppose we deal with a discrete-time dynamical system in $\mathbb{R}^N$, governed by the collection of matrices $\mathcal{A} = \{A_1, A_2, \ldots, A_k\}$ as in (5) and the order of appearance of elements from $\mathcal{A}$ in (5) is undetermined. The absolute stability of this collection means that the state of such a system is guaranteed to stay in some bounded set whenever $x(0)$ is contained in $S(1)$ (the zero-centered ball of radius 1).

In the theory of desynchronized systems some more restrictive cases are considered. For instance, the sequence of matrices $A(n)$ in (5) may be allowed to be only quasi-periodic, etc.

Lemma 2.2. *The absolute stability of the collection $\mathcal{A} = \{A_1, \ldots, A_k\}$ is equivalent to the existence of a norm $\|\cdot\|_a$ in $\mathbb{R}^N$ such that $\|A_i x\|_a \leq \|x\|_a$ for any $x \in \mathbb{R}^N$ and $1 \leq i \leq k$.*

Proof. Given an arbitrary matrix norm $\|\cdot\|$ and an absolutely stable collection $\mathcal{A}$ we may construct a "canonical" norm $\|\cdot\|_a$ as follows:

$$\|x\|_a = \sup_{1 \leq m} \sup_{0 \leq i_1, \ldots, i_m \leq k} \|A_{i_1} \ldots A_{i_m} x\|, \tag{6}$$

where $A_0 = I$. We shall assume further that the norm $\|\cdot\|_a$ is always of the form (6) and call it an a-norm produced by the collection $\mathcal{A}$. □

Closely related results can be found in [3, 4].

Next we shall need some technical definitions.

Definition 2.3. *Let $R = \{r_{ij}\}$ be a $N\times N$-matrix with nonzero diagonal elements. Consider the $N\times N$-matrix $\tilde{R}$ with entries $\tilde{r}_{ij} = r_{ij}/r_{ii}$. We shall call this matrix a row normalization of the matrix R. Introduce a K-normalization of R as $A = I - \tilde{R}$, where I is the identity $N\times N$-matrix.*

Clearly, $a_{ii} = 0$ for each $i = 1, \dots, N$.

Definition 2.4. *Let $A = \{a_{ij}\}$ be a $N\times N$-matrix. Consider the collection of N matrices A_i of the following form:*

$$A_i \;=\; \begin{pmatrix} 1 & 0 & \cdot & 0 & \cdot & 0 \\ 0 & 1 & \cdot & 0 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{i1} & a_{i2} & \cdot & a_{ii} & \cdot & a_{iN} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & 0 & \cdot & 1 \end{pmatrix}.$$

We call this collection a row-decomposition of matrix A.

Definition 2.5. *We shall call the collection $\mathcal{A} = \{A_1, \dots, A_N\}$ from above definition a projection decomposition of matrix R if $\mathcal{A}$ is the K-normalization of R.*

Definition 2.6. *A $N\times N$-matrix $R = \{r_{ij}\}$ is called K-stable if its diagonal elements are strictly positive and its projection decomposition is absolutely stable (see Definition 2.1).*

The projection decomposition of matrix R is the collection of matrices A_i of the oblique projection mappings onto the planes $L_i = \{x : \ (Rx)_i = 0\}$ along i-th coordinate vector e_i. The interpretation of the K-stability of the matrix R with strictly positive diagonal elements is the following. Consider the system of N oblique projection operators A_i determined by R as was specified above. We call a sequence of N-vectors x^k, $k = 0, 1, \dots,$ a *path* of this system if $x^{k+1} = A_{i(k)}x^k$, $0 \le i(k) \le N$ for any $k \ge 0$. Matrix R is K-stable if and only if all the paths starting from the unit zero-centered ball ($\|x_0\| \le 1$) are uniformly bounded.

Define finally an auxiliary norm in $\mathbb{R}^N$ (we shall need it in the next section).

Definition 2.7. *Let R be a K-stable matrix. Denote by $\|\cdot\|_R$ the a-norm produced by the collection $\mathcal{A} = \{A_1, \dots, A_N\}$, where $\mathcal{A}$ is the projection decomposition of R.*

3 Sufficient conditions for the uniqueness

The aim of this and the next sections is to establish the equivalence of the uniqueness criterion (Theorem 1.5) to the new criterion formulated in terms of the absolute stability of some collection of matrices.

Lemma 3.1. *Consider a K-stable $N \times N$-matrix R, a vector $x \in \mathbb{R}^N$ and an index i, $1 \le i \le N$, such that $(Rx)_i \ne 0$. Then for any $y = x + \beta e_i$ such that $(Ry)_i (Rx)_i \ge 0$, the following inequality holds:*

$$(\|y\|_R - \|x\|_R)(Rx)_i \beta \ge 0.$$

In other words, if the points x and y lie to one side of the plane $L_i = \{x \in \mathbb{R}^N : (Rx)_i = 0\}$ and differ only in i-th coordinate, then that one of these points has lesser norm $\|\cdot\|_R$ which lies closer to L_i.

Proof. Let $\mathcal{A} = \{A_1, \dots, A_n\}$ be the projection decomposition of R. Consider first the point $z = A_i x$, the oblique projection of x on L_i along e_i. By definition, $(Rz)_i = 0$ and, clearly, $\|z\|_R \le \|x\|_R$ (this follows immediately from the definition of $\|\cdot\|_R$). Furthermore, $z = A_i y$, hence $\|z\|_R \le \|y\|_R$. The convex function $\|\cdot\|_R$ considered on the line $x + \alpha e_i$ attains its minimum at the point z. Since x and y lie to one side from z on this line, then $\|x - z\| \le \|y - z\|$ implies $\|x\|_R \le \|y\|_R$ and $|x - z| \ge |y - z|$ implies $\|x\|_R \ge \|y\|_R$. If the values $(Rx)_i$ and β are of the same sign, then $\|x - z\| \le \|y - z\|$ which finishes the proof. □

Theorem 3.2. *If R is a K-stable matrix without singular principal submatrices, then R is a $\mathcal{P}$-matrix.*

Proof. Suppose the contrary. Without loss of generality we can assume that $\det(R^1) < 0$ for some submatrix

$$R^1 = \{r_{ij}|_{1 \le i,j \le l}\}, \quad l \le N.$$

Then matrix R has a real negative eigenvalue and a real eigenvector belonging to it, i.e. there exist a $y \in \mathbb{R}^N$, $y \ne 0$, $y_i = 0$ for $i = l+1, \dots, N$, and a

$\alpha > 0$ such that $R^1 y = -\alpha y$. Denote by J the set of all indices i such that $|y_i| \neq 0$. Due to Lemma 3.1 the norm $\|\cdot\|_R$ does not increase along any direction $y_i e_i$, $i \in J$ in some neighbourhood of y. Hence there exists a $\beta > 0$ such that for any z from this neighbourhood and any i, $1 \leq i \leq N$, the norm $\|\cdot\|_R$ of the vector $\tilde{z}^i = (z_1, \ldots, (1+\beta)z_i, \ldots, z_N)$ is less or equal to $\|z\|_R$. Thus

$$\|y + \beta y\|_R \leq \|y\|_R,$$

and that implies $\|y\|_R = 0$. □

Lemma 3.3. *For any $\mathcal{P}$-matrix R, there exists a $\delta > 0$ such that, for any $x \in \mathbb{R}^N$, $\|x\| = 1$, the inequalities*

$$|x_i| > \delta, \quad |(Rx)_i| > \delta, \quad x_i (Rx)_i > 0$$

hold for some i, $1 \leq i \leq N$.

Proof. Suppose the contrary. We can choose a sequence of vectors x^j, $1 \leq j < \infty$, $\|x^j\| = 1$, such that

$$\lim_{j \to \infty} \|x^j - x^*\| = 0 \quad \text{for some } x^*, \quad \|x^*\| = 1.$$

and for any i the following condition is fulfilled: if $x_i^j (Rx^j)_i > 0$ then

$$|x_i^j| \leq \alpha_j \text{ or } |Rx^j|_i \leq \alpha_j \quad \text{for} \quad \alpha_j > 0, \quad \lim_{j \to \infty} \alpha_j = 0.$$

Thus for x^* the situation $x_i^* (Rx^*)_i > 0$ is impossible, hence

$$x_i^* (Rx^*)_i > 0, \quad 1 \leq i \leq N \quad \text{and} \quad x^* \neq 0.$$

Applying Lemma 1.2 we show that R is not a $\mathcal{P}$-matrix. The contradiction proves the lemma. □

Remark 3.4. *We can deduce from the above lemma the existense of a $\epsilon > 0$ with the following property: for any $x \in \mathbb{R}^N$, $\|x\| = 1$, an index i can be chosen such that $|x_i| > \epsilon$, $x_i (Rx)_i > 0$ and for any y under restriction $\|x - y\| \leq \epsilon$, the inequality*

$$y_i (Ry)_i \geq 0$$

holds.

Definition 3.5. *Call an ordered pair (x, y) of vectors M-admissible if*

$$(Rx)_i(y_i - x_i) \geq 0, \quad 1 \leq i \leq N,$$

and K-admissible if y differs from x at most in one coordinate and

$$(Ry)_i(y_i - x_i) \geq 0 \quad \text{and} \quad r_{ii}(Rx)_i(y_i - x_i) \geq 0$$

for $1 \leq i \leq N$.

Remark 3.6. *The interpretation of M-admissibility is the following. Consider a solution $u(t)$ of (2), (3) in the statement of Theorem 1.5. If, say, the function u jumps at moment t, then the pair $(u(t), u^-(t))$ is M-admissible. Also if u has a derivative $u'(t)$ at the point t then the pair $(u(t), u(t) - \alpha u'(t))$ is also M-admissible for any $\alpha > 0$. We shall use this interpretation in the proof of Theorem 3.10.*

Lemma 3.7. *For any K-admissible pair (x, y) the inequality*

$$\|x\|_R \leq \|y\|_R$$

holds.

This statement follows directly from Lemma 3.1.

Lemma 3.8. *Let R be a K-stable $\mathcal{P}$-matrix. Then we can choose a $\alpha > 0$ such that for any M-admissible pair (x, y) there exists a $z \in \mathbb{R}^N$ such that the pair (x, z) is M-admissible, the pair (z, y) is K-admissible and*

$$\|x - z\| \leq (1 - \alpha)\|x - y\|. \tag{7}$$

Proof. Due to Remark 3.4 there exist a $\epsilon = \epsilon(R) > 0$ and an index i such that

$$|y_i - x_i| > \epsilon / \|y - x\|$$

and for any p satisfying $\|p - (x - y)\| \leq \epsilon/\|y - x\|$ the inequality $z_i(Az)_i \geq 0$ also holds. Due to the M-admissibility of (x, y) we get $(Rx)_i \geq 0$. Suppose without loss of generality that both $y_i - x_i$ and $R(y - x)_i$ are positive. We can now put $z = y - \epsilon e_i/\|y - x\|$ and the lemma is proved. □

Lemma 3.9. *If matrix R is an absolutely stable $\mathcal{P}$-matrix and the pair (x, y) is M-admissible, then $\|x\|_R \leq \|y\|_R$.*

Proof. We can easily prove this lemma by infinitely repeating the process of vector z choice from Lemma 3.8 and taking it as y on the next step. At each step of this process the distance $\|y-x\|$ is multiplied by the coefficient which does not exceed $(1-\alpha) < 1$ and the value $\|y\|_R$ does not increase. Passing to the limit we get the required inequality. □

Theorem 3.10. *Let R be a K-stable $\mathcal{P}$-matrix. Then the norm $\|\cdot\|_R$ does not increase along any path $u(t)$ which satisfy conditions* (2) *and* (3).

Proof. For any discontinuity point t^* of $u(t)$ we can build a linear "bridge" between left and right limits $u^-(t^*)$ and $u(t^*)$ of $u(t)$ at this point, along which the norm $\|\cdot\|_R$ does not increase due to Lemma 3.9 and Remark 3.6; thus we may suppose $u(t)$ to be continuous. Moreover, it may be supposed absolutely continuous because we can use an isometric parametrization

$$\tau(t) = \mathrm{var}(u(\cdot); 0, t)$$

and consider τ as a new time (u as a function of τ is even Lipschitz continuous with constant 1). Hence $\|u(t)\|_R$ is also absolutely continuous function. Its derivative exists a.e. and equals to minus directional derivative of $\|\cdot\|_R$ along the vector $-u'(t)$ and this one is nonpositive also due to Lemma 3.9 and the same remark. □

Theorem 3.11. *The K-stability of the matrix R without singular principal submatrices is sufficient for the uniqueness of a continuous DCP solution.*

This theorem easily follows from Theorems 3.2 and 3.10. In the next section the reverse statement is proved.

4 Necessary conditions for the uniqueness

Definition 4.1. *Matrix R is called irreducible if it cannot be presented as a block-triangular matrix by some replacement of its rows and corresponding columns.*

The irreducibility of matrix R is equivalent to the nonexistence of non-trivial invariant subspace for this matrix which is simultaneously a linear hull of some subset of basis vectors e_i, $i = 1, 2, ..., N$.

Theorem 4.2 (reduction principle)**.** *Let R be a reducible matrix, i.e. it can be presented as*

$$R = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix} \tag{8}$$

by some replacement of rows and corresponding columns. Then DCP with matrix R has a unique solution if and only if DCP's with matrices B and D are uniquely solvable.

Proof. Due to Theorem 1.5 the DCP has multiple solutions if and only if a nonzero function $x(t) \in \mathbb{R}^N (t \geq 0,\ x(0) = 0)$ of bounded variation can be found such that

$$(Rx(t))_i\,(dx(t))_i \leq 0 \quad (i = 1, 2, \ldots, N); \tag{9}$$

Let us show first that the uniqueness of DCP solution with matrices B and D implies the uniqueness of DCP solution with matrix R. We must demonstrate that if the DCP with R has multiple solutions, then at least one of the DCP's with matrices B or D also has multiple solutions. Let the DCP for matrix R have multiple solutions. Then a nonzero function $x(t)$ of bounded variation can be found such that (9) holds. Denote by K the order of matrix B (then the order of D is $L = N - K$); by definition, $0 < K,\ L < N$. Next denote by $y(t)$ a vector-function with values in $\mathbb{R}^K$ such that its coordinates coincide with the first K coordinates of the function $x(t)$. Let also $z(t)$ be the $\mathbb{R}^L$-valued function with the same coordinates as the last L coordinates of the function $x(t)$. The functions $y(t)$ and $z(t)$ are uniquely determined for $t \geq 0$ and are of bounded variation. Furthermore, the relations

$$(z(t))_i = (x(t))_{i+K}, \quad (dz(t))_i = (dx(t))_{i+K}, \quad (Dz(t))_i = (Rx(t))_{i+K}$$

hold for $1 \leq i \leq L$ and hence relations (9) with matrix D hold for $z(t)$. Due to (9) the DCP for matrix D has multiple solutions provided $z(t)$ is nonzero. Hence we must only consider the case $z(t) = 0$ for all $t \geq 0$. In this case the function $y(t)$ is not identically zero and satisfies the following conditions:

$$(y(t))_i = (x(t))_{i+K}, \quad (dy(t))_i = (dx(t))_{i+K}, \quad (By(t))_i = (Rx(t))_{i+K},$$

for $1 \leq i \leq L$. Thus for the function $y(t)$ the relations (9) hold with the matrix B instead of R and hence the DCP with matrix B has multiple solutions.

We have proved the first part of Theorem 4.2. Let us show next that the uniqueness of the DCP solution with the matrix R implies the uniqueness of the DCP solutions with the matrices B and D. Suppose the contrary. Then there exist functions $y(t) \in \mathbb{R}^K$ and $z(t) \in \mathbb{R}^L$ of bounded variation such that (9) holds with the matrices B and D correspondingly and at least one of these functions is not identically zero for $t \geq 0$. Denote

$$y_*(t) = \{(y(t))_1, (y(t))_2, \dots, (y(t))_K, 0, \dots, 0\} \in \mathbb{R}^N,$$

$$z_*(t) = \{0, \dots, 0, (z(t))_1, (z(t))_2, \dots, (z(t))_L\} \in \mathbb{R}^N.$$

For the functions $y_*(t)$ and $z_*(t)$ of bounded variation the conditions (9) hold with the matrix R and at least one of these functions is not identically zero. Hence the DCP with matrix R has multiple solutions. □

Definition 4.3. *Let $\mathcal{A} = \{A_1, A_2, \dots, A_k\}$ be a finite collection of real-valued $N \times N$-matrices. We say that this collection is quasi-controllable if there is no nontrivial proper subspace of $\mathbb{R}^N$ invariant for each matrix from $\mathcal{A}$. We denote by $\mathcal{A}_m$ $(m = 1, 2, \dots)$ the set of all finite products of matrices of the set $\mathcal{A} \cup \{I\}$ comprizing at most m multipliers. The set $\mathcal{A}_m(x)$ is the set of all vectors Lx, $L \in \mathcal{A}_m$.*

Denote by $\mathrm{co}(W)$, $\mathrm{absco}(W)$ and $\mathrm{span}(W)$, correspondingly, the convex hull, the absolute convex hull, and the linear hull of the vector set $W \subseteq \mathbb{R}^N$. Recall that the set W is *absolutely convex* if it is convex and together with any point x it contains also the point $-x$. Absolute convex hull of W is the intersection of all absolutely convex sets containing W. Let $\|\cdot\|$ be a norm in $\mathbb{R}^N$ and let $S(t)$ be the zero-centered ball of the radius t in this norm.

Lemma 4.4. *Let $p \geq N-1$. The collection of matrices $\mathcal{A}$ is quasi-controllable if and only if for any nonzero $x \in \mathbb{R}^N$ the relation*

$$\mathrm{span}\{\mathcal{A}_p(x)\} = \mathbb{R}^N$$

holds.

Proof. Let the collection $\mathcal{A}$ be quasi-controllable. Take an arbitrary nonzero $x \in \mathbb{R}^N$ and let $L_0 = \mathrm{span}\{x\}$, $L_k = \mathrm{span}\{\mathcal{A}_k(x)\}$ $(k \geq 1)$. Then the relations

$$L_0 \subseteq L_1 \subseteq \cdots \subseteq L_p \subseteq \mathbb{R}^N \tag{10}$$

hold and hence

$$1 \le \dim L_0 \le \dim L_1 \le \cdots \le \dim L_p \le N, \tag{11}$$

is also fullfilled. Furtheremore,

$$A_i L_j \subseteq L_{j+1} \quad (A_i \in \mathcal{A}, \quad 0 \le j \le p-1). \tag{12}$$

If $\dim L_p = N$, then $L_p = \operatorname{span}\{\mathcal{A}_p(x)\} = \mathbb{R}^N$. If, in contrary, $\dim L_p < N$, then owing to (11) and the inequality $p > N-1$ one has $\dim L_j = \dim L_{j+1}$ for some $j \in [0, p-1]$. Then by virtue of (10) we get $L_j = L_{j+1}$ and (12) implies that L_j is a nontrivial subspace, invariant for any matrix from $\mathcal{A}$. The quasi-controllability of $\mathcal{A}$ implies now that this subspace is $\mathbb{R}^N$ itself. Hence

$$L_j = L_{j+1} = \cdots = L_p = \operatorname{span}\{\mathcal{A}(x)\} = \mathbb{R}^N.$$

Suppose next $\operatorname{span}\{\mathcal{A}_p(x)\} = \mathbb{R}^N$ but the collection $\mathcal{A}$ is not quasi-controllable. Then there exists a $\mathcal{A}$-invariant nontrivial subspace L of $\mathbb{R}^N$. In this case, for any vector $x \in L$, the inclusion $\operatorname{span}\{\mathcal{A}(x)\} \subseteq L$ holds and hence $\operatorname{span}\{\mathcal{A}(x)\} \neq \mathbb{R}^N$. The contradiction proves the quasi-controllability of the collection $\mathcal{A}$. □

Definition 4.5. *The value*

$$\operatorname{qcm}(\mathcal{A}) = \inf_{x \in \mathbb{R}^N, \|x\|=1} \sup \{t : S(t) \subseteq \operatorname{absco}[\mathcal{A}_{N-1}(x)]\}$$

is called a quasi-controllability measure of the collection of matrices $\mathcal{A}$.

Lemma 4.6. *The collection of matrices $\mathcal{A}$ is quasi-controllable if and only if* $\operatorname{qcm}(\mathcal{A}) \neq 0$.

Proof. Let $\operatorname{qcm}(\mathcal{A}) \neq 0$. Then for any nonzero vector $x \in \mathbb{R}^N$ the inclusion

$$S(\|x\| \operatorname{qcm}(\mathcal{A})) \subseteq \operatorname{absco}[\mathcal{A}_{N-1}(x)]$$

holds, and hence also the equality $\mathbb{R}^N = \operatorname{span}\{\mathcal{A}_{N-1}(x)\}$ is valid. Thus, due to Lemma 4.4 the collection $\mathcal{A}$ is quasi-controllable.

Let now the collection $\mathcal{A}$ be quasi-controllable and $\operatorname{qcm}(\mathcal{A}) = 0$. Then we can choose

$$x_n \in \mathbb{R}^N \quad (\|x_n\| = 1) \quad \text{and} \quad y_n \in \operatorname{absco}[\mathcal{A}_{N-1}(x_n)]$$

such that

$$y_n \to 0, \quad ty_n \notin \text{absco}[\mathcal{A}_{N-1}(x_n)] \quad \text{for } t > 1.$$

Without loss of generality we may consider the sequences $\{x_n\}$ and $\{y_n/\|y_n\|\}$ converging: $x_n \to x$, $y_n/\|y_n\| \to z$.

Due to Lemma 4.4 the linear hull of vector set $\{\mathcal{A}_{N-1}(x)\}$ coincides with $\mathbb{R}^N$. Therefore matrices $L_1, L_2, \ldots, L_N \in \mathcal{A}_{N-1}$ exist such that the vectors $L_1x, \ldots, L_Nx$ are linearly independent. Then for any n large enough the vectors $L_1x_n, \ldots, L_Nx_n$ are also linearly independent. Hence for any n there exist numbers

$$\alpha_1^{(n)}, \alpha_2^{(n)}, \ldots, \alpha_N^{(n)}, \quad \sum_{i=1}^{N} |\alpha_i^{(n)}| = 1,$$

such that the vector

$$z_n = \sum_{i=1}^{N} \alpha_i^{(n)} L_i x_n \tag{13}$$

is collinear to the vector y_n, i.e. $z_n = \beta_n y_n$ $(\beta_n > 0)$. Taking into account that

$$z_n \in \text{absco}\,\{L_1x_n, L_2x_n, \ldots, L_Nx_n\} \subseteq \text{absco}\{\mathcal{A}_{N-1}(x_n)\},$$

and ty_n by definition is not contained in the set $\mathcal{A}_{N-1}(x_n)$ for $t > 1$, we find that $\beta_n \leq 1$. Thus the condition $y_n \to 0$ implies $z_n \to 0$ as well. Passing to limit in (13) (we can consider the sequences $\{\alpha_1^{(n)}\}, \{\alpha_2^{(n)}\}, \ldots, \{\alpha_N^{(n)}\}$ convergent to some limits $\alpha_1, \alpha_2, \ldots, \alpha_N$, correspondingly) we get:

$$\sum_{i=1}^{N} \alpha_i L_i x = 0, \quad \sum_{i=1}^{N} |\alpha_i| = 1,$$

which is impossible due to the linear independence of the vectors $L_1x, L_2, \ldots, L_Nx$. The contradiction completes the proof of Lemma 4.6. □

Theorem 4.7. *Let a collection of $N\times N$-matrices $\mathcal{A} = \{A_1, A_2, \ldots, A_k\}$ be quasi-controllable. If the difference equation*

$$x(n+1) = A(n)x(n) \tag{14}$$

is not absolutely stable for the class of matrices

$$\mathcal{A}(\lambda) = \{\lambda A_1, \lambda A_2, \ldots, \lambda A_k\} \quad \text{for } \lambda = 1,$$

then it is not absolutely stable for this class also for any value of λ close enough to 1.

Proof. Suppose equation (14) is not absolutely stable for the class of matrices $\mathcal{A}(1)$. Then there exist matrices $A(n) \in \mathcal{A}(1)$ $(n = 0, 1, \dots)$ and a solution $x(\cdot)$ of (14) such that for some $n_0 > 0$ the inequality

$$\|x(n_0)\| > \frac{1}{\mathrm{qcm}[\mathcal{A}(1)]} \|x(0)\|$$

holds.

Denote by $\mathcal{R}(\lambda)$ the set of all finite products of matrices from the class $\mathcal{A}(\lambda)$. Then the vector $x(n_0)$ can be presented as $x(n_0) = Rx(0)$, where R is some matrix from $\mathcal{R}(1)$. Hence a vector $x \in \mathbb{R}^N$, $\|x\| = 1$, can be found such that the strict inequality

$$\|Rx\| > \frac{1}{\mathrm{qcm}[\mathcal{A}(1)]} \tag{15}$$

holds. Let matrix R be of the form

$$R = A_{i_1} A_{i_2} \dots A_{i_q}.$$

As follows from the definition of quasi-controllability measure, the vector $\mathrm{qcm}[\mathcal{A}(1)] \cdot \|Rx\| x$ lies in the absolute convex hull of the vectors $\mathcal{A}_{N-1}(1, Rx)$. Thus numbers $\alpha_1, \alpha_2, \dots, \alpha_Q$ and matrices $L_i \in \mathcal{A}_{N-1}(1)$, $(i = 1, \dots, Q)$, can be chosen such that

$$\sum_{i=1}^{Q} |\alpha_i| \leq 1, \tag{16}$$

$$\sum_{i=1}^{Q} \alpha_i L_i R x = \mathrm{qcm}(\mathcal{A}) \|Rx\| x. \tag{17}$$

Hence the matrix

$$H = \sum_{i=1}^{Q} \alpha_i L_i R \tag{18}$$

has an eigenvalue $\mu = \mathrm{qcm}(\mathcal{A}) \|Rx\|$ which corresponds to the eigenvector x. Due to (15) we get also

$$\mu > 1. \tag{19}$$

Suppose matrix R is a product of q matrices from $\mathcal{A}(1)$ and each matrix L_i $(i = 1, 2, ..., Q)$ is a product of q_i matrices from $\mathcal{A}(1)$. Denote

$$R(\lambda) = \lambda^q R,\ L_i(\lambda) = \lambda^{q_i} L_i \quad (i = 1, 2, \ldots, Q).$$

Then

$$R(\lambda) \in \mathcal{A}(\lambda),\ L_i(\lambda) \in \mathcal{A}(\lambda) \quad (i = 1, 2, \ldots, Q).$$

Due to (18), (19) the matrix

$$H(\lambda) \ = \ \sum_{i=1}^{Q} \alpha_i L_i(\lambda) R(\lambda)$$

has an eigenvalue

(20) $$\mu(\lambda) > 1$$

for all λ close enough to 1. In this case for any $r \geq 1$ the matrix $H^r(\lambda)$ has $\mu^r(\lambda)$ as its eigenvalue, hence

$$\|H^r(\lambda)\| \geq \mu^r(\lambda).$$

At the other hand, matrix $H^r(\lambda)$ is a sum of Q^r terms of the form

$$\alpha_{i_1}\alpha_{i_2}\ldots\alpha_{i_r}[L_{i_1}(\lambda)R(\lambda)]\ldots[L_{i_r}(\lambda)R(\lambda)].$$

Due to (16) we get

$$\sum |\alpha_{i_1}\alpha_{i_2}\ldots\alpha_{i_r}| \ \leq \ 1,$$

therefore there exist indices $i_1, i_2, \ldots, i_r$ such that

(21) $$\|W_r(\lambda)\| \ \geq \ \mu^r(\lambda),$$

where

$$W_r(\lambda) = [L_{i_1}(\lambda)R(\lambda)]\ldots[L_r(\lambda)R(\lambda)] \in \mathcal{R}(\lambda).$$

Hence due to (20) and (21) we get

$$\sup_{R \in \mathcal{R}(\lambda)} \|R\| = \infty,$$

which contradicts the assumption of absolute stability of (14). Theorem 4.7 is thus proved. □

Consider now some $N \times N$-matrix $A = (a_{ij})$ and denote by $\mathcal{P}_1(A)$ its row decomposition (see Definition 2.4).

Theorem 4.8. *The collection of matrices $\mathcal{P}_1(A)$ is quasi-controllable if and only if the value 1 is not an eigenvalue of matrix A and A is irreducible.*

Proof. Let 1 be an eigenvalue of matrix A and x_* be a corresponding eigenvector. Then x_* is also an eigenvector belonging to eigenvalue 1 for any matrices $A_1, A_2, \ldots, A_N$. Hence in this case the collection $\mathcal{P}_1(A)$ is not quasi-controllable.

Suppose matrix A is not irreducible. In this case we can assume without loss of generality that some subspace

$$E_p = \mathrm{span}\{e_1, e_2, \ldots, e_p\}, \quad \text{where } p < N,$$

is invariant for matrix A. Then the subspace E_p is invariant also for each matix $A_1, A_2, \ldots, A_N$. Hence in this case the collection $\mathcal{P}_1(A)$ is also not quasi-controllable.

Prove next that the collection of matrices $\mathcal{A} = \mathcal{P}_1(A)$ is quasi-controllable provided 1 is not an eigenvalue of matrix A and matrix A is irreducible. In order to do that due to Lemma 4.4 we must only prove that for any nonzero vector $x \in \mathbb{R}^N$ the equality

$$\mathrm{span}\{\mathcal{A}_N(x)\} \;=\; \mathbb{R}^N \tag{22}$$

holds.

Choose an arbitrary vector $x \in \mathbb{R}^N$, $\|x\| = 1$, and consider the vectors

$$(A_1 - I)x,\ (A_2 - I)x, \ldots, (A_N - I)x \in \mathrm{span}\{\mathcal{A}_1(x)\}.$$

Due to the equality

$$(A - I)x = (A_1 - I)x + (A_2 - I)x + \cdots + (A_N - I)x$$

and the fact that 1 is not an eigenvalue of matrix A, at least one of the vectors $(A_1 - I)x$, $(A_2 - I)x, \ldots,$ $(A_N - I)x$ is nonzero. Without loss we can suppose that $(A_1 - I)x \neq 0$. Nevertheless, due to the equality

$$(A_i - I)x = \langle \tilde{a}_i, x\rangle e_i, \quad (i = 1, 2, \ldots, N), \tag{23}$$

where vectors $\tilde{a}_i$ are of the form

$$\tilde{a}_i = \{a_{i1}, \ldots, a_{ii} - 1, \ldots, a_{iN}\} \quad (i = 1, 2, \ldots, N),$$

we get $\langle \tilde{a}_1, x\rangle e_1 \neq 0$, $\langle \tilde{a}_1, x\rangle e_1 \in \operatorname{span}\{\mathcal{A}_1(x)\}$. Hence

$$e_1 \in \operatorname{span}\{\mathcal{A}_1(x)\}. \tag{24}$$

Due to the irreducibility of matrix A the subspace $\operatorname{span}\{e_1\}$ is not invariant for A. Hence at least one coordinate of the vector Ae_1 with index not equal to 1 is nonzero; without loss of generality we can assume its second coordinate to be nonzero. The second coordinate of the vector Ae_1 is the same as the second coordinate of the vector A_2e_1, and thus also of the vector $(A_2 - I)e_1$. Hence $(A_2 - I)e_1 \neq 0$ and due to (24) the inclusion $(A_2 - I)e_1 \in \operatorname{span}\{\mathcal{A}_2(x)\}$ holds. Therefore from (23) it follows that

$$e_2 \in \operatorname{span}\{\mathcal{A}_2(x)\}.$$

By a similar argument we can deduce from the irreducibility of matrix A the validity of inclusions

$$e_i \in \operatorname{span}\{\mathcal{A}_i(x)\} \quad (i = 1, 2, \ldots, N)$$

after proper renumeration of the basis vectors $e_1, e_2, \ldots, e_N$. Hence the equality (22) and also the statement of Theorem 4.8 are completely proved. □

Theorem 4.9. *If equation* (14) *is not absolutely stable, then it has a nonzero solution $x(n)$ defined for all integer n and bounded for $n < 0$.*

Proof. To prove Theorem 4.9 we need the following fact (Theorem 1 in [10]). If (14) is not absolutely stable then we can choose a sequence of matrices $A(n) \in \mathcal{A}$, a solution $x(n)$ of (14) and a sequence of integer $n_i \to \infty$ such that

$$\|x(n_i)\| \to \infty,$$

$$\|x(n)\| \leq \|x(n_i)\| \quad \text{for} \quad (0 \leq n \leq n_i). \tag{25}$$

Denote

$$A(i, n) = A(n + n_i), \quad x(i, n) = x(n + n_i)/\|x(n_i)\|.$$

Then for any $i = 1, 2, \ldots$ the sequences of matrices $A(i, n)$ and vectors $x(i, n)$ are defined for $n \geq -n_i$ and due to (14), for these values of n, the equality

$$x(i, n + 1) = A(i, n)x(i, n) \tag{26}$$

holds and the following relations are valid:

$$A(i,n) \in \mathcal{A}, \tag{27}$$

$$\|x(i,n)\| \leq \|x(i,0)\| = 1 \quad (i = 1,2,\ldots; \quad n \geq -n_i) \tag{28}$$

(see (25)). The finiteness of the set $\mathcal{A}$ and relations (27), (28) together imply that for any $n \leq 0$ the sequences of matrices $A(i,n)$ and of vectors $x(i,n)$ without loss of generality may be considered convergent in i:

$$A(i,n) \to B(n) \in \mathcal{A}, \quad x(i,n) \to y(n) \quad \text{when} \quad i \to \infty.$$

Passing to limit in (26) we find that the function $y(n)$ is defined for any $n \leq 0$, that for this function the relation

$$y(n+1) = B(n)y(n), \quad B(n) \in \mathcal{A}$$

holds and that due to (28) the relations

$$\|y(n)\| \leq \|y(0)\| = 1 \quad (n \leq 0)$$

are valid. The claim of Theorem 4.9 is proved. □

Theorem 4.10. *Suppose a matrix $A = (a_{ij})$ is irreducible, 1 is not an eigenvalue of A and equation* (14) *is not absolutely stable for the class of matrices $\mathcal{P}_1(A)$. Then there exist a real number $\lambda < 1$, matrices $A(n) \in \mathcal{P}_1(A)$ and a nonzero solution of* (14) *such that the estimates*

$$\|x(n)\| \leq \lambda^{-n} \quad (n \leq 0) \tag{29}$$

hold.

Proof. The set of matrices $\mathcal{A}(\lambda) = \{\lambda A_1, \ldots, \lambda A_N\}$ depends continuosly on parameter λ. Hence due to Theorem 4.7 and 4.8 a number $\lambda < 1$ can be chosen such that the equation (14) is not absolutely stable for the class $\mathcal{A}(\lambda)$. Now Theorem 4.9 implies the existence of matrices $A(n) \in \mathcal{P}_1(A)$ and of a nonzero solution $y(n)$ of the difference equation

$$y(n+1) = \lambda A(n)y(n) \quad (n \leq 0),$$

such that the following estimates hold:

$$\|y(n)\| \le \|y(0)\| = 1 \quad (n \le 0). \tag{30}$$

Let $x(n) = \lambda^{-n} y(n)$ $(n \le 0)$. Then the function $x(n)$ satisfies the equation

$$x(n+1) = A(n)x(n) \quad (n \le 0)$$

and due to (30) estimate (29) holds true for it. Theorem 4.10 is proved. □

Theorem 4.11. *Let $R = (r_{ij})$ be a matrix with nonzero diagonal elements and suppose matrix A is the K-normalization of R (see Definition 2.3). Suppose the matrix A is irreducible and has not an eigenvalue 1. Let $\mathcal{A}$ be the row decomposition of A. Suppose equation* (14) *is not absolutely stable for the class of matrices $\mathcal{A}$. Then the DCP with matrix R has multiple solutions.*

Proof. Let $x(n)$ be a function defined in the statement of Theorem 4.10. Consider the function $z(t)$ determined for $t \ge 0$ by the relations

$$z(t) = \begin{cases} x(0) & \text{for } \; t \ge 1; \\ x(n) & \text{for } \; \frac{1}{n+1} \le t < \frac{1}{n} \qquad (n = 1, 2, \dots); \\ 0 & \text{for } \; t \le 0; \end{cases} \tag{31}$$

Due to Theorem 4.10 the estimate (29) holds for $x(n)$. Thus the function $x(n)$ is of bounded variation on the interval $(-\infty, 0]$ and hence also the function $z(t)$ is of bounded variation on any interval $[0, \alpha)$.

Let us calculate, for every $t \ge 0$, the values

$$\rho_i(t) = (Rz(t))_i (dz(t))_i \quad (i = 1, 2, \dots, N),$$

assuming that

$$dz(t) = z(t) - z^-(t).$$

If $t \ne 1/n + 1$ $(n = 0, 1, \dots)$, then $dz(t) = 0$. Therefore at these values of t all $\rho_i(t)$ are equal to zero, and hence the condition (9) holds.

Let $t = 1/n + 1$ for some $n = 0, 1, \dots$. Then due to (31) we get

$$dz(t) = x(n) - x(n-1), \quad Rz(t) = Rx(n)$$

We can also choose j $(1 \le j \le N)$ such that $x(n) = A_j x(n-1)$. This means that

$$(dz(t))_i = 0 \quad \text{for } i \ne j. \tag{32}$$

If, in contrast, $i = j$, then, clearly,

(33) $$(Rz(t))_i = (Rx(n))_i = (RA_j x(n-1))_j = 0.$$

Hence due to (32), (33) we get $\rho_i(t) = 0$ for $i = 1, 2, \ldots, N$.

Thus for any $t \geq 0$ the function $z(t)$ of bounded variation satisfies the condition (9), which implies due to Theorem 1.5 the existence of multiple solutions of DCP with matrix R. Theorem 4.11 is proved. □

5 Main results

In this section we formulate necessary and sufficient conditions for the unique solvability of DCP in terms of absolute stability theory. We deduce also a few simple results obtained by means of this criterion.

Theorem 5.1 (Main theorem)**.** *The DCP with matrix R has a unique solution if and only if the following conditions are fulfilled:*

C1. All principal minors of matrix R are nonzero.

C2. Matrix R is K-stable (see Definition 2.6).

Proof. The sufficience of these conditions for the uniqueness of the DCP solution is exactly Theorem 3.11.

In order to prove the reverse implication for irreducible matrices due to Theorem 4.11 we need only demonstrate that 1 is not an eigenvalue of the K-normalization of matrix R. This is equivalent (due to Definition 2.3) to the nondegenerateness of matrix R and the latter, clearly, follows from C1. If, in contrary, matrix R is reducible (say, (8) holds and matrix D is irreducible) then due to Theorem 4.2 we must only prove the existence of multiple solutions to the DCP with matrix D. As easily follows from definitions, the K-stability of R implies the K-stability of D. □

Let us employ Theorem 5.1 to prove a few simple results.

Theorem 5.2. *The continuous DCP with matrix R has a unique solution if and only if the continuous DCP with matrix R' (transposed to R) has a unique solution.*

Proof. Suppose the DCP with matrix R has a unique solution. We must prove that from the validity of conditions C1 and C2 with matrix R follows the validity of these conditions with matrix R' (let us refer to the latter as conditions C1$'$ and C2$'$).

Clearly, C1$'$ follows from C1. Let the matrix R be K-stable and nondegenerate. It means that the collection of matrices $\mathcal{A} = \{\alpha_1, \ldots, A_N\}$ is absolutely stable, where

$$A_i = I - B_i/r_{ii}, \quad B_i = C_i R$$

and C_i is a $N \times N$-matrix with all zero elements except $c_{ii} = 1$. By definition, the absolute stability of $\mathcal{A}$ is invariant with respect to any linear nondegenerate change of coordinates. In other words, if $\mathcal{A}$ is absolutely stable and P is a nondegenerate matrix, then the collection $\mathcal{A}_1 = \{PA_1P^{-1}, \ldots, PA_NP^{-1}\}$ is also absolutely stable. Let us take for P the matrix R itself. Simple argument infers that $RA_iR^{-1} = \tilde{A}_i$, where $\tilde{A}_i = I - RC_{ii}/r_{ii}$. Note now that absolute stability of the collection of matrices is equivalent to absolute stability of the collection consisting of the transposed matrices (that follows from $\|A\| = \|A'\|$). To finish the proof, note that the collection of matrices transposed to the matrices $\tilde{A}_i$ is exactly the projection decomposition of matrix R'. □

Introduce finally three results easily deducible from the analogous results of the desynchronization theory (see [12, 13]).

Theorem 5.3. *The set of all $\mathcal{P}$-matrices R such that the DCP has multiple solutions, is open.*

Theorem 5.4. *Let matrix A be the K-normalization of matrix R and $A = B + C$, B being symmetric and C antisymmetric matrices. Suppose all eigenvalues of B lie in $[-r, r], r < 1$ and, for the spectral radius r_1 of C, the following inequality holds:*

$$r_1 < r[(1-r)/(1+r)]^{1/2}\{\frac{1}{[1-(1-r^2)^N]^{1/2}} - 1\}.$$

Then the DCP with matrix R has a unique solution.

Theorem 5.5. *If the DCP with matrix R has a unique solution then all eigenvalues λ of the matrix A (the K-normalization of R) lie in the circle $|\lambda + (N-1)| \leq N$.*

6 Open problem of algebraic solvability

We have formulated necessary and sufficient conditions for the uniqueness of the DCP in terms of two kinds of stability of a finite collection of matrices. That may help to prove some new theorems but that does not help much to answer the question of the unique solvability of the DCP with a given matrix R. It would be nice to have some criterion which can be algebraically verified, say, the finite system of polynomial equalities and inequalities. There are strong reasons to believe that such criterions do not exist. To clarify it we shall present here a recent result of the desynchronization theory [11].

Let $u = \{u_1, \ldots, u_N\}$ be a vector from $\mathbb{R}^N$. The finite sum

$$p(u) \;=\; \sum p_{i_1 \ldots i_N} u_1^{i_1} \ldots u_N^{i_N}$$

with real coefficients $p_{i_1 \ldots i_N}$ is called a *polynomial* in u. A set $U \subseteq \mathbb{R}^N$ is said to be SA*-set* if there exists a finite collection of polynomials $p_1(u)$, $\ldots, p_k(u), p_{k+1}(u), \ldots, p_{k+l}(u)$ such that U coincides with the set of all elements u satisfying the following conditions:

$$p_1(u) > 0,\ \ldots,\ p_k(u) > 0,\ p_{k+1}(u) = \cdots = p_{k+l}(u) = 0.$$

The set U is called *semialgebraic* if it is a union of a finite number of SA-sets.

Consider now the finite sequences $(A_1, \ldots, A_m)$ of $N \times N$-matrices as elements of $\mathbb{R}^{m \times N \times N}$. Denote by $S(m, N)$ the class of all elements from $\mathbb{R}^{m \times N \times N}$ such that the corresponding collection of matrices $\mathcal{A} = \{A_1, \ldots, A_m\}$ is absolutely stable. We define also the class $A(m, N)$ as a class of all absolutely r-asymptotically stable collections.

Theorem 6.1 (see [1, 11])**.** *Boths classes $S(m, N)$ and $A(m, N)$ are not semialgebraic sets if $m, N \geq 2$.*

This theorem can not be applied directly to the case considered in the present paper because we deal here only with the collections of matrices which are projection decompositions of some matrices, namely, with the classes $S(N, N) \cap P(N, N)$ and $A(N, N) \cap P(N, N)$, where $P(N, N)$ is the class of all projection decompositions of various $n \times N$-matrices. The class $P(N, N)$ is surely semialgebraic but the intersection of a non-semialgebraic set with a semialgebraic one may, certainly, be semialgebraic (consider, for example, the second set to be consisting of one point). So the nonexistence of semialgebraic criterions for the DCP uniqueness problem is only a hypothesis.

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