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On global attraction to quantum stationary states. Dirac equation with mean field interaction

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Abstract. We consider a $\mathbf{U}(1)$-invariant nonlinear Dirac equation, interacting with itself via the mean field mechanism. We analyze the long-time asymptotics of solutions and prove that, under certain generic assumptions, each finite charge solution converges as $t \to \pm\infty$ to the two-dimensional set of all "nonlinear eigenfunctions" of the form $\phi(x)e^{-i\omega t}$. This global attraction is caused by the nonlinear energy transfer from lower harmonics to the continuous spectrum and subsequent dispersive radiation. The research is inspired by Bohr's postulate on quantum transitions and Schrödinger's identification of the quantum stationary states to the nonlinear eigenfunctions of the coupled $\mathbf{U}(1)$-invariant Maxwell-Schrödinger and Maxwell-Dirac equations.

Keywords. Dirac equation, solitary waves, unitary invariance, global attractor, nonlinear spectral analysis, Titchmarsh convolution theorem.
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1 Introduction. Bohr's transitions as global attraction

In the present paper we continue our research on the global attraction to solitary wave solutions in $\mathbf{U}(1)$-invariant dispersive nonlinear systems. Our aim is a dynamical interpretation of "quantum jumps" postulated by N. Bohr in 1913 to explain the stability of the Rutherford model of the atom (1911).

The long time asymptotics for nonlinear field theory equations, and in particular the nonlinear Klein-Gordon equation, have been the subject of intensive research, starting with the pioneering papers by Segal [Seg63], Strauss [Str68], and Morawetz and Strauss [MS72], where the nonlinear scattering and the local attraction to zero solution were considered. The orbital stability of solitary wave solutions of nonlinear Schrödinger and Klein-Gordon equations is well-understood (see [GSS87]), and there is presently an active research on local attraction to solitary waves, or *asymptotic stability* [BS03, Cuc08].

Traditionally, the Dirac equation has presented difficulties. The existence of solitary waves in the nonlinear Dirac equation was proved in [CV86]. The existence of solitary waves in the Maxwell-Dirac system is proved in [EGS96].

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The stability of solitary waves with respect to a particular class of perturbations was analyzed in [SV86]. The spectral stability of small amplitude solitary waves of a particular nonlinear Dirac equation in one dimension, known as the massive Gross-Neveu model [GN74, LG75], has been confirmed numerically in [CP06]. Neither the orbital stability nor asymptotic stability of solitary waves is presently understood in the context of the Dirac equation.

The next question is related to the structure of a global attractor of all finite energy solutions. Global attraction to *static*, stationary solutions in dispersive systems *without* $\mathbf{U}(1)$ *symmetry* was first established in [Kom95, KSK97, Kom99, KS00]. The attraction of any finite energy solution to the set of all solitary waves in the context of the Klein-Gordon equation coupled to one and to several nonlinear oscillators was proved in [KK07, KK10]. In [KK10], we generalized this result for the Klein-Gordon field coupled to several oscillators. In [KK08], a similar result was generalized to a higher dimensional setting for the Klein-Gordon equation with the mean field interaction. In the present paper, we consider the structure of the attractor for the Dirac equation, in the model with the mean field interaction. We will show that under rather general assumptions the attractor coincides with the set of all solitary waves.

According to Bohr's postulates [Boh13], an unperturbed electron runs forever along certain *stationary orbit*, which we denote $|E\rangle$ and call *quantum stationary state*. Once in such a state, the electron has a fixed value of energy E, with the energy not being lost via emitted radiation. Under a perturbation, the electron can jump from one quantum stationary state to another, $|E_-\rangle \longmapsto |E_+\rangle$, radiating (or absorbing) a quantum of light. Bohr's postulate suggests the dynamical interpretation of Bohr's transitions as a long-time asymptotics

$$\psi(t) \longrightarrow |E_\pm\rangle, \qquad t \to \pm\infty \tag{1.1}$$

for any trajectory $\psi(t)$ of the corresponding dynamical system, where the limiting states $|E_\pm\rangle$ depend on the trajectory. Then the *quantum stationary states* should be viewed as points of the *global attractor*.

At first glance, the global attraction (1.1) seems incompatible with the energy conservation and time reversibility of Hamiltonian systems. We intend to verify that such asymptotics in principle are possible for nonlinear Hamiltonian field equations. In this paper, we verify this asymptotics for equations of Dirac type.

Developing de Broglie's ideas, Schrödinger identified quantum stationary states of energy E with the solutions of type

$$\psi(x,t) = e^{-iEt/\hbar}\psi(x), \qquad x \in \mathbb{R}^3. \tag{1.2}$$

Then the Schrödinger equation

$$i\hbar\partial_t\psi = H\psi := -\frac{\hbar^2}{2m}\Delta\psi + V(x)\psi, \qquad \psi = \psi(x,t) \in \mathbb{C}, \quad x \in \mathbb{R}^3, \tag{1.3}$$

becomes the *eigenvalue problem* $E\psi(x) = H\psi(x)$. One of the original Schrödinger's ideas [Sch26a] was to identify the integers in the Debye-Sommerfeld-Wilson quantum rules with the integers arising in the eigenvalue problems for PDEs.

For the case of the *free particles*, this identification agrees with the de Broglies wave function $\psi(x,t) = e^{ip\cdot x/\hbar}e^{-iEt/\hbar}$, where $p \in \mathbb{R}^3$ is the momentum of the particle. For the *bound particles* in an external potential, the identification (1.2) reflects the fact that the space is "twisted" by the external field, while the time remains "free".

Thus, the reason for Schrödinger's choice of quantum stationary states in the form (1.2) seems to be rather algebraic. At the same time, the attraction (1.1) suggests the dynamical interpretation of the quantum stationary states as asymptotic states in the long time limit, that is,

$$\psi(x,t) \sim \phi_{\omega_\pm}(x)e^{-i\omega_\pm t}, \qquad t \to \pm\infty. \tag{1.4}$$

Such asymptotics should hold for each finite charge solution. The asymptotics of type (1.4) are generally impossible for the linear autonomous equation (1.3) because of the superposition principle. The asymptotics would mean that the solitary waves (1.2) form the global attractor for the coupled nonlinear Maxwell-Schrödinger and Maxwell-Dirac systems.

An adequate description of "quantum jumps" in an atom requires that we consider the equation for the electron wave function (Schrödinger or Dirac equation) coupled to the Maxwell system which governs the time evolution of the four-potential $A^\mu(x,t)=(\varphi(x,t),\mathbf{A}(x,t))$. The corresponding Maxwell-Schrödinger system was initially introduced by Schrödinger in [Sch26b]. Its global well-posedness was considered in [GNS95]. The results on local existence of solutions to the Dirac-Maxwell system were obtained in [Bou96]. The coupled systems are $\mathbf{U}(1)$-invariant with respect the (global) gauge group $\big(\psi(x),A^\mu(x)\big)\mapsto\big(\phi(x)e^{-i\theta},A^\mu(x)\big)$, $\theta\in\mathbb{R}$. Respectively, one might expect the following natural generalization of asymptotics (1.4) for solutions to the coupled Maxwell-Schrödinger or Maxwell-Dirac equations:

$$\big(\psi(x,t),A^\mu(x,t)\big)\sim\big(\phi_\pm(x)e^{-i\omega_\pm t},A^\mu_\pm(x)\big),\qquad t\to\pm\infty. \tag{1.5}$$

The asymptotics (1.5) would mean that the set of all solitary waves forms a global attractor for the coupled system. The asymptotics of this type are not available yet in the context of coupled systems.

2 Model and the result

In the present paper we consider the Dirac equation with the mean field interaction:

$$i\dot\psi(x,t)=\big(-i\boldsymbol{\alpha}\cdot\nabla+\beta m\big)\psi+\rho(x)F(\langle\rho,\psi(\cdot,t)\rangle),\quad \psi\in\mathbb{C}^4,\quad x\in\mathbb{R}^3, \tag{2.1}$$

where $\boldsymbol{\alpha}\cdot\nabla=\sum_{j=1}^3\alpha_j\partial_j$, the Hermitian 4×4 matrices α_j and β satisfy $\{\alpha_j,\alpha_k\}=2\delta_{jk}\boldsymbol{I}$, $\{\alpha_j,\beta\}=\mathbf{0}$, $\beta^2=\boldsymbol{I}$. We write $\langle\rho,\psi(\cdot,t)\rangle=\int_{\mathbb{R}^3}\rho^\dagger(x)\psi(x,t)\,d^3x$, where ρ is a spinor-valued coupling function from the Schwartz class:

$$\rho=\left[\begin{smallmatrix}\rho_1\\ \dots\\ \rho_4\end{smallmatrix}\right]\in\mathscr{S}(\mathbb{R}^3,\mathbb{C}^4),\qquad \rho\not\equiv\mathbf{0},\qquad \rho^\dagger=[\bar\rho_1,\dots,\bar\rho_4].$$

Assumption 2.1. $F(z)=a(|z|^2)z$, with $a(s)=\sum\limits_{k=0}^p a_k s^k$, $s\ge0$, where $a_k\in\mathbb{R}$, $p\ge2$, and $a_p\ne0$.

Since one has $F(e^{i\theta}z)=e^{i\theta}F(z)$ for $\theta\in\mathbb{R}$, $z\in\mathbb{C}$, equation (2.1) is $\mathbf{U}(1)$-invariant, where $\mathbf{U}(1)$ stands for the unitary group $e^{i\theta}$, $\theta\in\mathbb{R}\mod 2\pi$. By the Nöther theorem, the charge functional $Q(\psi)=\int_{\mathbb{R}^3}|\psi|^2\,d^3x$ is conserved.

Equation (2.1) is the simplest model sharing the following features with the Maxwell-Dirac system, which we consider the key for global attraction to solitary waves: The model has $\mathbf{U}(1)$-symmetry; It is dispersive; It is nonlinear. The first feature allows for solitary waves, while the last two features are responsible for the "friction by dispersion" mechanism: The nonlinearity moves the perturbations into the continuous spectrum of the linearized equation, and thereafter the perturbations are dispersed to infinity. Let us also mention that the modelling of the interaction of the matter with gauge fields in terms of local self-interaction takes its origin back to at least as early as [Sch51], where the Lorentz-invariant nonlinear Klein-Gordon equation originally appeared.

Definition 2.2. The space of states of finite charge is $\mathscr{X}=L^2(\mathbb{R}^3,\mathbb{C}^4)$, with the standard norm denoted by $\|\cdot\|_{L^2}$.

Equation (2.1) formally can be written as a Hamiltonian system with the phase space $\mathscr{X}$, $\dot\psi(t)=JD\mathcal{H}(\psi)$, where $D\mathcal{H}$ is the variational derivative with respect to $\mathrm{Re}\,\psi_\mu$, $\mathrm{Im}\,\psi_\mu$, $\mu=1,\dots,4$, of the Hamiltonian

$$\mathcal{H}(\psi)=\frac12\int_{\mathbb{R}^3}\psi^\dagger\big(-i\boldsymbol{\alpha}\cdot\nabla+\beta m\big)\psi\,d^3x+U(\langle\rho,\psi\rangle),\qquad \psi^\dagger=[\bar\psi_1,\dots,\bar\psi_4].$$

Let $\hat{\mathscr{D}}(\xi)=\boldsymbol{\alpha}\cdot\xi+\beta m$, $\xi\in\mathbb{R}^3$. Since $\hat{\mathscr{D}}(\xi)$ is self-adjoint and $\hat{\mathscr{D}}(\xi)^2=\xi^2+m^2$, its eigenvalues are $\pm\sqrt{\xi^2+m^2}$. Define the projectors $\Pi_\pm(\xi)=\frac12\Big(1\pm\frac{\hat{\mathscr{D}}(\xi)}{\sqrt{\xi^2+m^2}}\Big)$ onto the corresponding eigenspaces, and denote $\hat\rho_\pm(\xi):=\Pi_\pm(\xi)\hat\rho(\xi)$.

Assumption 2.3. (i) For any $\lambda>0$, $\hat\rho_+(\xi)$ and $\hat\rho_-(\xi)$ do not vanish identically on the sphere $|\xi|=\lambda$.

(*ii*) The function

$$\sigma(\omega)=\int_{\mathbb{R}^3}\frac{\hat\rho^\dagger(\xi)(\omega+\boldsymbol{\alpha}\cdot\xi+\beta m)\hat\rho(\xi)}{\omega^2-\xi^2-m^2}\,\frac{d^3\xi}{(2\pi)^3}\tag{2.2}$$

is nonzero for all $\omega\in[-m,m]$, except perhaps at one point which will be denoted $\omega_\sigma\in[-m,m]$.

Remark 2.4. Note that limits $\sigma(m-0)$, $\sigma(-m+0)$ exist.

Let us give examples of such functions $\rho\in\mathscr{S}(\mathbb{R}^3,\mathbb{C}^4)$. We take a spherically symmetric function ρ such that $\hat\rho(\xi)\ne0$ for $\xi\ne0$, with $\hat\rho(\xi)$ being an eigenvector of β (since β is Hermitian and $\beta^2=1$, its eigenvalues are ±1). Then $|\hat\rho_\pm(\xi)|^2=\frac12\hat\rho^\dagger(\xi)\big(1\pm\frac{\boldsymbol{\alpha}\cdot\xi+\beta m}{\sqrt{\xi^2+m^2}}\big)\hat\rho(\xi)$, hence

$$\int_{|\xi|=\lambda}|\hat\rho_\pm(\xi)|^2\,d^2\Omega\ge\int_{|\xi|=\lambda}\hat\rho(\xi)^\dagger\big(1-\frac{m}{\sqrt{\xi^2+m^2}}\big)\hat\rho(\xi)\,d^2\Omega>0,\qquad\lambda>0.$$

The integral of the term with $\boldsymbol{\alpha}\cdot\xi$ dropped out since $\hat\rho(\xi)$ is spherically symmetric.

Let us now consider $\sigma(\omega)$. Due to $\hat\rho$ being spherically symmetric, $\boldsymbol{\alpha}\cdot\xi$-term cancels out from the integration in (2.2). Since we require that $\hat\rho(\xi)$ be an eigenvector of β, one has $\hat\rho(\xi)^*(\omega+\beta m)\hat\rho(\xi)=(\omega\pm m)\hat\rho(\xi)^*\hat\rho(\xi)$, hence the expression under the integral in (2.2) is sign-definite for all $\omega\in[-m,m]$ except at $\omega=m$ or at $\omega=-m$. Therefore, $\sigma(\omega)$ does not vanish on $[-m,m]$ except at one of the endpoints.

Definition 2.5. The solitary waves of equation (2.1) are solutions of the form $\psi(x,t)=\phi_\omega(x)e^{-i\omega t}$, where $\omega\in\mathbb{R}$, $\phi_\omega(x)\in\mathscr{X}$. The solitary manifold is the set $\mathbf{S}=\big\{\phi_\omega\colon\phi_\omega(x)e^{-i\omega t}\text{ is a solitary wave}\big\}\subset\mathscr{X}$.

For $\omega\in\mathbb{C}^+=\{\omega\in\mathbb{C}\colon\operatorname{Im}\omega>0\}$, introduce

$$\hat\Sigma(\xi,\omega)=\frac{(\omega+\boldsymbol{\alpha}\cdot\xi+\beta m)\hat\rho(\xi)}{\omega^2-\xi^2-m^2},\qquad\Sigma(x,\omega)=\mathcal{F}^{-1}_{\xi\to x}\big[\hat\Sigma(\cdot,\omega)\big].\tag{2.3}$$

$\Sigma(x,\omega)$ is an analytic function of $\omega\in\mathbb{C}^+$ with the values in $\mathscr{S}(\mathbb{R}^3,\mathbb{C}^4)$. One can show that for any $x\in\mathbb{R}^3$, $\Sigma(x,\omega)$ can be extended to the real line $\omega\in\mathbb{R}$ as the boundary trace (in the sense of tempered distributions):

$$\Sigma(x,\omega)=\lim_{\varepsilon\to0+}\Sigma(x,\omega+i\varepsilon),\qquad\omega\in\mathbb{R}.\tag{2.4}$$

One has $\Sigma(\cdot,\omega)\in\mathscr{X}$ for $\omega\in(-m,m)$. For $\omega=\pm m$, $\Sigma(\cdot,\omega)\in\mathscr{X}$ only when $\hat\rho_\pm(0)=0$.

Proposition 2.6. *Let Assumptions 2.1 and 2.3 be satisfied. Then there are no nonzero solitary wave solutions to* (2.1) *for $\omega\notin[-m,m]$. For $\omega\in(-m,m)$, there is a nonzero solitary wave $\phi_\omega(x)e^{-i\omega t}$ if and only if there is $C\in\mathbb{C}\backslash0$ such that*

$$\sigma(\omega)a(|C\sigma(\omega)|^2)=1,\tag{2.5}$$

with $a(\cdot)$ from Assumption 2.1. One has $\phi_\omega(x)=C\Sigma(x,\omega)$. For $\omega=\pm m$, $\phi_\omega(x)\in\mathscr{X}$ only when $\hat\rho_\pm(0)=0$.

Remark 2.7. Due to Assumption 2.1, the set $\mathbf{S}$ from Definition 2.5 is invariant under multiplication by $e^{i\theta}$, $\theta\in\mathbb{R}$. Generically, the solitary manifold $\mathbf{S}$ is two-dimensional. It may contain disconnected components.

Remark 2.8. Since $F(0)=0$, the zero solitary wave is an element of $\mathbf{S}$, formally corresponding to any $\omega\in\mathbb{R}$.

Remark 2.9. It is possible that for some ω equation (2.5) has nonzero roots $C_j\in\mathbb{C}$ of different magnitude; then there are solitary wave solutions corresponding to each of these roots.

Definition 2.10. Fix $\varepsilon>0$ and $\chi\in C^\infty_{comp}(\mathbb{R}^3)$, $0\le\chi\le1$, $\chi(x)\equiv1$ for $|x|<1$, with $\operatorname{supp}\chi$ in the ball of radius 2. Denote by $\mathscr{Y}$ the space $\mathscr{X}$ endowed with the metric $\|\psi\|_{\mathscr{Y}}:=\sum_{R\in\mathbb{N}}2^{-R}\|\chi(x/R)\psi(x)\|_{H^{-\varepsilon}}$, where the norm in the Sobolev space H^s is given by $\|\psi\|_{H^s}=\|(m^2-\Delta)^{s/2}\psi\|_{L^2}$.

The Sobolev Embedding Theorem implies that the embedding $\mathscr{X} \subset \mathscr{Y}$ is compact.

Theorem 2.11 (Main Theorem)**.** *Assume that the nonlinearity $F(z)$ is as in Assumption 2.1 and that the coupling function $\rho(x)$ satisfies Assumption 2.3. Then for any $\psi_0 \in \mathscr{X}$ there is a global solution $\psi(t) \in C(\mathbb{R}, \mathscr{X})$ to equation* (2.1)*, such that $\psi|_{t=0} = \psi_0$, which converges to $\mathbf{S}$ in the space $\mathscr{Y}$:*

$$\lim_{t\to\pm\infty} \operatorname{dist}_{\mathscr{Y}}(\psi(t),\mathbf{S}) = 0, \qquad \textit{where} \quad \operatorname{dist}_{\mathscr{Y}}(\psi,\mathbf{S}) := \inf_{\boldsymbol{s}\in\mathbf{S}} \|\psi - \boldsymbol{s}\|_{\mathscr{Y}}.$$

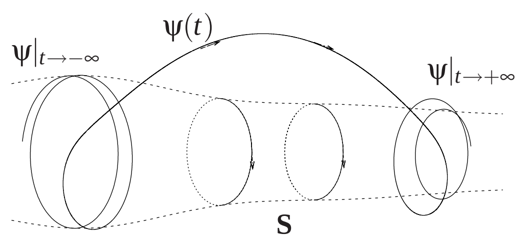

Figure 1. For $t \to \pm\infty$, a finite charge solution $\psi(t)$ approaches the global attractor which coincides with the set of all solitary waves $\mathbf{S}$.

We work in the L^2 setting because of the absence of existence results for finite energy solutions. As a result, the convergence to the attractor holds just below L^2, in local $H^{-\varepsilon}$-norm. The convergence to $\mathbf{S}$ is due to the "friction by dispersion": the nonlinearity carries the excess energy into the continuous spectrum, where it is subsequently dispersed to infinity, and the remaining part of the solution settles to one of the nonlinear eigenstates (or zero). By the Titchmarsh Convolution Theorem (see [Tit26] and [Yos80]), this process can only stop when the time-spectrum of the solution consists of the single frequency, that is, when the limiting solution coincides with a nonlinear solitary wave (a nonlinear Schrödinger eigenstate). It is only for such states that the energy transfer from lower harmonics into the continuous spectrum (the *spectral inflation*) is absent.

Let us comment on our methods. We follow the path developed in [KK07, KK08, KK10]: we prove the absolute continuity of the spectral density for large frequencies, use the compactness argument to extract the omega-limit trajectories, and show that these trajectories have the finite time spectrum. Then we use the Titchmarsh Convolution Theorem to pinpoint the spectrum to a single frequency. Let us note that the Titchmarsh theorem is only applicable if we assume that F admits a real-valued polynomial potential (Assumption 2.1).

3 Conclusion

We demonstrated that in a particular nonlinear dispersive equation, whose linearization is the Dirac equation, the convergence to Schrödinger states for long positive and negative times takes place as a consequence of "friction by dispersion", so that the solution settles to one of the nonlinear eigenstates (or to zero). We conjecture that a similar consideration will allow to prove convergence to nonlinear eigenstates in systems such as coupled Maxwell-Dirac equations, providing a dynamical description of Bohr's "quantum jumps".

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References

[Boh13] N. Bohr, *On the constitution of atoms and molecules*, Phil. Mag. **26** (1913), pp 1–25.

[Bou96] N. Bournaveas, *Local existence for the Maxwell-Dirac equations in three space dimensions*, Comm. Partial Differential Equations **21** (1996), pp 693–720.

[BS03] V. S. Buslaev and C. Sulem, *On asymptotic stability of solitary waves for nonlinear Schrödinger equations*, Ann. Inst. H. Poincaré Anal. Non Linéaire **20** (2003), pp 419–475.

[CP06] M. Chugunova and D. Pelinovsky, *Block-diagonalization of the symmetric first-order coupled-mode system*, SIAM J. Appl. Dyn. Syst. **5** (2006), pp 66–83.

[Cuc08] S. Cuccagna, *On asymptotic stability in energy space of ground states of NLS in 1D*, J. Differential Equations **245** (2008), pp 653–691.

[CV86] T. Cazenave and L. Vázquez, *Existence of localized solutions for a classical nonlinear Dirac field*, Comm. Math. Phys. **105** (1986), pp 35–47.

[EGS96] M. J. Esteban, V. Georgiev, and E. Séré, *Stationary solutions of the Maxwell-Dirac and the Klein-Gordon-Dirac equations*, Calc. Var. Partial Differential Equations **4** (1996), pp 265–281.

[GN74] D. J. Gross and A. Neveu, *Dynamical symmetry breaking in asymptotically free field theories*, Phys. Rev. D **10** (1974), pp 3235–3253.

[GNS95] Y. Guo, K. Nakamitsu, and W. Strauss, *Global finite-energy solutions of the Maxwell-Schrödinger system*, Comm. Math. Phys. **170** (1995), pp 181–196.

[GSS87] M. Grillakis, J. Shatah, and W. Strauss, *Stability theory of solitary waves in the presence of symmetry. I*, J. Funct. Anal. **74** (1987), pp 160–197.

[KK07] A. I. Komech and A. A. Komech, *Global attractor for a nonlinear oscillator coupled to the Klein-Gordon field*, Arch. Ration. Mech. Anal. **185** (2007), pp 105–142.

[KK08] A. I. Komech and A. A. Komech, *Global attraction to solitary waves for Klein-Gordon equation with mean field interaction*, Ann. Inst. H. Poincaré Anal. Non Linéaire **26** (2008), pp 855–868.

[KK10] A. I. Komech and A. A. Komech, *Global attractor for the Klein-Gordon field coupled to several nonlinear oscillators*, J. Math. Pures Appl. **93** (2010), pp 91–111.

[Kom95] A. I. Komech, *On stabilization of string-nonlinear oscillator interaction*, J. Math. Anal. Appl. **196** (1995), pp 384–409.

[Kom99] A. I. Komech, *On transitions to stationary states in one-dimensional nonlinear wave equations*, Arch. Ration. Mech. Anal. **149** (1999), pp. 213–228.

[KS00] A. I. Komech and H. Spohn, *Long-time asymptotics for the coupled Maxwell-Lorentz equations*, Comm. Partial Differential Equations **25** (2000), pp 559–584.

[KSK97] A. I. Komech, H. Spohn, and M. Kunze, *Long-time asymptotics for a classical particle interacting with a scalar wave field*, Comm. Partial Differential Equations **22** (1997), pp 307–335.

[LG75] S. Y. Lee and A. Gavrielides, *Quantization of the localized solutions in two-dimensional field theories of massive fermions*, Phys. Rev. D **12** (1975), pp 3880–3886.

[MS72] C. S. Morawetz and W. A. Strauss, *Decay and scattering of solutions of a nonlinear relativistic wave equation*, Comm. Pure Appl. Math. **25** (1972), pp 1–31.

[Sch26a] E. Schrödinger, *Quantisierung als eigenwertproblem*, Ann. d. Phys. **79** (1926), pp 361–376.

[Sch26b] E. Schrödinger, *Quantisierung als eigenwertproblem*, Ann. d. Phys. **81** (1926), p. 109.

[Sch51] L. I. Schiff, *Nonlinear meson theory of nuclear forces. I. Neutral scalar mesons with point-contact repulsion*, Phys. Rev. **84** (1951), pp 1–9.

[Seg63] I. E. Segal, *The global Cauchy problem for a relativistic scalar field with power interaction*, Bull. Soc. Math. France **91** (1963), pp. 129–135.

[Str68] W. A. Strauss, *Decay and asymptotics for* $\Box u = f(u)$, J. Functional Analysis **2** (1968), pp 409–457.

[SV86] W. A. Strauss and L. Vázquez, *Stability under dilations of nonlinear spinor fields*, Phys. Rev. D (3) **34** (1986), pp 641–643.

[Tit26] E. Titchmarsh, *The zeros of certain integral functions*, Proc. of the London Math. Soc. **25** (1926), pp 283–302.

[Yos80] K. Yosida, *Functional analysis*, vol. 123 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*, Springer-Verlag, Berlin, 1980, sixth edn.