On global attraction to solitary waves.
Klein-Gordon equation with mean field interaction at several points

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Abstract

We consider the $\mathbf{U}(1)$-invariant Klein-Gordon equation in dimension $n \geq 3$, self-interacting via the mean field mechanism in finitely many regions. We prove that, under certain generic assumptions, each solution converges as $t \to \pm\infty$ to the two-dimensional set of all "nonlinear eigenfunctions" of the form $\phi(x)e^{-i\omega t}$. The proof is based on the analysis of omega-limit trajectories. The Titchmarsh Convolution Theorem allows us to prove that the time spectrum of any omega-limit trajectory of each finite energy solution consists of a single point. This proves the convergence to the attractor in local sub-energy norms.

1 Introduction

The present paper continues the series of papers on the global attraction to solitary waves in $\mathbf{U}(1)$-invariant dispersive systems. In [1], we proved such an attraction for the Klein-Gordon field coupled to one nonlinear oscillator. In [3], we generalized this result for the Klein-Gordon field coupled to several oscillators. In [2], we considered the higher-dimensional model: the Klein-Gordon field with the nonlinear mean field interaction. The ultimate goal is to prove the "soliton resolution conjecture", which could be stated as follows: *for large times, any finite energy solution can be approximated by solitary waves and small dispersive waves*. One expects this effect to take place in a generic nonlinear dispersive system.

In this paper, we establish the global attraction to the set of all solitary waves for the $\mathbf{U}(1)$-invariant Klein-Gordon field $\psi(x,t)$ with the mean field self-interaction at $N \in \mathbb{N}$ different locations:

$$\ddot{\psi}(x,t) = \Delta\psi(x,t) - m^2\psi(x,t) + \sum_{I=1}^{N} \rho_I(x) F_I(\langle \rho_I, \psi(\cdot,t)\rangle), \quad x \in \mathbb{R}^n, \ \ n \geq 3, \ \ t \in \mathbb{R}. \tag{1.1}$$

Above, $\rho_I(x) = \rho(x - X_I)$, with $X_I \in \mathbb{R}^n$, $1 \leq I \leq N$, and ρ a smooth coupling function from the Schwartz class. We will show that under rather mild assumptions any finite energy solution converges to the set of solitary wave solutions of the form $\phi_\pm(x)e^{-i\omega_\pm t}$.

We follow the cairns of the approach we developed in [1, 2, 3]. The main ideas are the absolute continuity of the spectral density for large frequencies, compactness argument to extract the omega-limit trajectories, and then the usage of the Titchmarsh Convolution Theorem to pinpoint the spectrum to just one frequency. The results are presented in Section 2. The proof follows [2], where we proved convergence to solitary waves for the Klein-Gordon equation with mean field self-interaction at just one location. Considering mean field interaction at several regions required substantial modification in the proof of the absolute continuity of the time-spectrum for large frequencies (Section 3) and in the application of the Titchmarsh Convolution Theorem (Section 4). For the completeness, we also give a proof of the local energy decay for the free Klein-Gordon equation; see Appendix A.

The proof contains two new ideas, which allow to tackle (1.1): the local integrability of the Fourier transform of finite energy solutions restricted onto the mass hypersurface $\xi^2 + m^2 = \omega^2$ (see Proposition 3.3 and Remark 3.5) and the analytic continuation of the momentum into the complex domain (see the proof of Proposition 3.8). These ideas are developed in Section 3.

2 Main results

We consider the $\mathbf{U}(1)$-invariant Klein-Gordon equation with the mean field self-interaction at N points:

$$\ddot\psi(x,t)=\Delta\psi(x,t)-m^2\psi(x,t)+\sum_{I=1}^N\rho_I(x)F_I(\langle\rho_I,\psi(\cdot,t)\rangle),\qquad x\in\mathbb{R}^n,\quad n\ge 3,\quad t\in\mathbb{R},\tag{2.1}$$

where

$$\langle\rho_I,\psi(\cdot,t)\rangle=\int_{\mathbb{R}^n}\bar\rho_I(x)\psi(x,t)\,d^n x.$$

We assume that $\rho_I(x)=\rho(x-X_I)$, where $X_I\in\mathbb{R}^n$, $1\le I\le N$, and $\rho(x)$ is a nonzero smooth real-valued function from the Schwartz class: $\rho\in\mathscr{S}(\mathbb{R}^n,\mathbb{R})$, $\rho\not\equiv 0$.

Remark 2.1. Assumptions on ρ could be relaxed, but we will not do this for the sake of simplicity of proofs.

Remark 2.2. In the higher-dimensional case, we need to couple the Klein-Gordon field to nonlinear oscillators using the mean field mechanism. Contrary to the one-dimensional case considered in [1, 3], we can no longer use the δ-function coupling, since the finite energy solutions to the Klein-Gordon equation in higher dimensions are not necessarily continuous and cannot be considered at a particular point.

Assumption 2.3. We assume that for all $1\le I\le N$ one has

$$F_I(z)=-\nabla_{\mathrm{Re}\,z,\mathrm{Im}\,z}u_I(|z|^2)=-2u_I'(|z|^2)z,\qquad z\in\mathbb{C},\quad 1\le I\le N,$$

where

$$u_I(s)=\sum_{q=1}^{p_I}u_{I,q}s^q,\quad u_{I,q}\in\mathbb{R},\qquad u_{I,p_I}>0,\quad\text{and}\quad p_I\ge 2.\tag{2.2}$$

Under Assumption 2.3, equation (2.1) is $\mathbf{U}(1)$-invariant since

$$F_I(e^{i\theta}z)=e^{i\theta}F_I(z),\qquad z\in\mathbb{C},\quad\theta\in\mathbb{R},\quad 1\le I\le N,$$

and can formally can be written as a Hamiltonian system, $\dot\Psi(t)=\boldsymbol{J}\,D\mathcal{H}(\Psi)$, where $\boldsymbol{J}$ is a skew-symmetric matrix, $\Psi=(\mathrm{Re}\,\psi(x),\mathrm{Im}\,\psi(x),\mathrm{Re}\,\pi(x),\mathrm{Im}\,\pi(x))$, with $\pi=\partial_t\psi$. $D\mathcal{H}=D_{\mathrm{Re}\,\psi,\mathrm{Im}\,\psi,\mathrm{Re}\,\pi,\mathrm{Im}\,\pi}\mathcal{H}$ is the Fréchet derivative of the Hamilton functional

$$\mathcal{H}(\Psi)=\frac12\int_{\mathbb{R}^n}\left(|\pi|^2+|\nabla\psi|^2+m^2|\psi|^2\right)d^n x+\sum_{I=1}^N u_I(|\langle\rho_I,\psi\rangle|^2),\qquad\Psi=\begin{bmatrix}\psi(x)\\ \pi(x)\end{bmatrix}.\tag{2.3}$$

Let us introduce the phase space of finite energy states for equation (2.1).

We will use the weighted Sobolev spaces. Denote by $\|\cdot\|_{L^2}$ the norm in $L^2(\mathbb{R}^n)$. Let $\langle x\rangle=(1+x^2)^{1/2}$. For $s\in\mathbb{R}$, $\sigma\in\mathbb{R}$, denote

$$H^s_\sigma(\mathbb{R}^n,\mathbb{C})=\{u\in\mathscr{S}'(\mathbb{R}^n,\mathbb{C})\colon\|\langle x\rangle^\sigma(m^2-\Delta)^{\frac{s}{2}}u\|_{L^2}<\infty\};\quad\|u\|_{H^s_\sigma}=\|\langle x\rangle^\sigma(m^2-\Delta)^{\frac{s}{2}}u\|_{L^2}.\tag{2.4}$$

We will write $H^s=H^s_0$, $L^2_\sigma=H^0_\sigma$.

Definition 2.4. For $\varepsilon\ge 0$, $\sigma\ge 0$ denote by $\mathscr{X}^{-\varepsilon}_{-\sigma}$ the Banach space of states $\Psi=(\psi,\pi)$ with the norm

$$\|\Psi\|^2_{\mathscr{X}^{-\varepsilon}_{-\sigma}}=\|\psi\|^2_{H^{1-\varepsilon}_{-\sigma}}+\|\pi\|^2_{H^{-\varepsilon}_{-\sigma}}.\tag{2.5}$$

We will denote $\mathscr{X}^{-\epsilon}=\mathscr{X}^{-\epsilon}_0$, $\mathscr{X}_{-\sigma}=\mathscr{X}^0_{-\sigma}$, $\mathscr{X}=\mathscr{X}^0_0$. Then $\mathscr{X}=H^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$ is the Hilbert space of states $\Psi=(\psi,\pi)$, with the norm

$$\|\Psi\|^2_{\mathscr{X}}=\|\nabla\psi\|^2_{L^2}+m^2\|\psi\|^2_{L^2}+\|\pi\|^2_{L^2}=\|\psi\|^2_{H^1}+\|\pi\|^2_{L^2}.$$

We fix $\varepsilon>0$ and $\sigma>0$ and denote

$$\mathscr{Y}=\mathscr{X}^{-\varepsilon}_{-\sigma}.\tag{2.6}$$

Equation (2.1) can formally be written as a Hamiltonian system with the phase space $\mathscr{X}$ introduced in Definition 2.4 and the Hamiltonian $\mathcal{H}(\psi,\pi)$ from (2.3), which is a continuous functional on $\mathscr{X}$.

Lemma 2.5. *The embedding $\mathscr{X}\subset\mathscr{Y}=\mathscr{X}^{-\varepsilon}_{-\sigma}$ is compact (for any $\varepsilon>0$ and $\sigma>0$).*

Proof. Let $\Psi_j\in\mathscr{X}$, $j\in\mathbb{N}$ be a sequence such that

$$\|\Psi_j\|_{\mathscr{X}}\le C<\infty,\qquad j\in\mathbb{N}. \tag{2.7}$$

Since $\mathscr{X}$ is a Hilbert space, we can choose a subsequence of Ψ_j which is weakly convergent in $\mathscr{X}$ to some $\Psi_0\in\mathscr{X}$. Let $\mathbb{B}^n_R$ be an open ball of radius R in $\mathbb{R}^n$, centered in the origin. Pick a spherically symmetric function $\varrho\in C_0^\infty(\mathbb{B}^n_2)$ such that $\varrho(x)\equiv 1$ for $|x|\le 1$ and monotonically decreasing to zero as $|x|$ changes from 1 to 2. Fix $R\ge 1$; then the sequence $\{\varrho(x/R)\Psi_j(x)\colon j\in\mathbb{N}\}$ is bounded in $\mathscr{X}$. By the Sobolev embedding theorem, the embedding $\{\Psi\in\mathscr{X}\colon \operatorname{supp}\Psi\subset\mathbb{B}^n_R\}\subset\mathscr{X}^{-\varepsilon}$ is compact; hence we can choose a smaller subsequence of $\Psi_j(x)$, denoted j_k, $k\in\mathbb{N}$, such that $\rho(x/R)\Psi_{j_k}(x)$ converges in the metric $\|\cdot\|_{\mathscr{X}^{-\varepsilon}}$ to $\rho(x/R)\Psi_0(x)$. Note that, for any $1\le r\le R$, $\rho(x/r)\Psi_{j_k}(x)$ converges to $\rho(x/r)\Psi_0(x)$, also in the metric $\|\cdot\|_{\mathscr{X}^{-\varepsilon}}$. By the diagonalization process, we can choose a yet smaller subsequence of Ψ_j, which we also denote Ψ_{j_k}, $k\in\mathbb{N}$, such that for any $R\ge 1$ the sequence $\rho(x/R)\Psi_{j_k}(x)$ converges to $\rho(x/R)\Psi_0$ in $\|\cdot\|_{\mathscr{X}^{-\varepsilon}}$.

Let us show that Ψ_{j_k}, $k\in\mathbb{N}$, converges to Ψ_0 in $\mathscr{X}^{-\varepsilon}_{-\sigma}$. Pick $\delta>0$. Due to the support properties of ϱ, one has

$$\|\big(1-\varrho(x/R)\big)(\Psi_{j_k}-\Psi_0)\|_{\mathscr{X}_{-\sigma}}\le\|\langle x\rangle^{-\sigma}\big(1-\varrho(x/R)\big)(\Psi_{j_k}-\Psi_0)\|_{\mathscr{X}}\le CR^{-\sigma}\|\Psi_{j_k}-\Psi_0\|_{\mathscr{X}}, \tag{2.8}$$

where C only depends on σ and $\|\varrho\|_{H^1}$. Since Ψ_j are uniformly bounded in $\mathscr{X}$, one can choose $R_\delta\ge 1$ large enough so that the right-hand side of (2.8) is bounded by $\delta/2$. At the same time, since $\varrho(x/R)\Psi_{j_k}\to\varrho(x/R)\Psi_0$ in the norm of $\mathscr{X}^{-\varepsilon}$, there is $k_\delta\in\mathbb{N}$ such that $\|\varrho(x/R_\delta)(\Psi_{j_k}-\Psi_0)\|_{\mathscr{X}^{-\varepsilon}}<\delta/2$ for $k\ge k_\delta$. Thus,

$$\|\Psi_{j_k}-\Psi_0\|_{\mathscr{X}^{-\varepsilon}_{-\sigma}}\le\|(1-\varrho(x/R_\delta))(\Psi_{j_k}-\Psi_0)\|_{\mathscr{X}_{-\sigma}}+\|\varrho(x/R_\delta)(\Psi_{j_k}-\Psi_0)\|_{\mathscr{X}^{-\varepsilon}}<\delta,\qquad\forall k\ge k_\delta.$$

□

Theorem 2.6 (Global well-posedness)**.** *Assume that the nonlinearities $F_I(z)$, $1\le I\le N$, satisfy Assumption 2.3.*

1. *For every $\Psi_0=(\psi_0,\pi_0)\in\mathscr{X}$, the Cauchy problem*

$$\begin{cases}\ddot\psi(x,t)=\Delta\psi(x,t)-m^2\psi(x,t)+\sum_{I=1}^N\rho_I(x)F_I(\langle\rho_I,\psi(\cdot,t)\rangle),\\ (\psi,\dot\psi)|_{t=0}=(\psi_0,\pi_0),\end{cases} \tag{2.9}$$

   has a unique solution such that $\Psi=(\psi,\dot\psi)\in C(\mathbb{R},\mathscr{X})$.

2. *The map $W(t):\ \Psi_0\mapsto\Psi(t)=(\psi,\dot\psi)|_t$ is continuous as a map $\mathscr{X}\to\mathscr{X}$ for each $t\in\mathbb{R}$.*
3. *The values of the energy functional are conserved: $\mathcal{H}(\Psi(t))=\mathcal{H}(\Psi_0)$, $t\in\mathbb{R}$.*
4. *The following* a priori *bound holds:*

$$\|\Psi(t)\|_{\mathscr{X}}\le C(\Psi_0),\qquad t\in\mathbb{R}. \tag{2.10}$$

5. *For $E\in\mathbb{R}$, denote $\mathscr{X}_E=\{\Psi\in\mathscr{X}\colon\mathcal{H}(\Psi)\le E\}$. For any $E\in\mathbb{R}$ and $T>0$, the map*

$$W(t):\ \mathscr{X}_E\to\mathscr{X}_E,\quad(\psi_0,\pi_0)\mapsto(\psi(t),\dot\psi(t)),$$

   is continuous in the topology of $\mathscr{X}^{-\varepsilon}_{-\sigma}$, for any $\varepsilon\ge 0$, $\sigma\ge 0$, uniformly in $t\in[-T,T]$.

Proof. The local existence is obtained by standard arguments from the contraction mapping principle. To achieve this, we use the integral representation for the solutions to the Cauchy problem (2.9) for $t\ge 0$:

$$\Psi(t)=W_0(t)\Psi_0+\mathcal{N}[\Psi](t),$$

where $\Psi=(\psi,\dot\psi)$ and

$$\mathcal{N}[\Psi](t):=\sum_{J=1}^N\int_0^t W_0(t-s)\Big[{0\atop \rho_J\,F_J(\langle\rho_J,\psi(\cdot,s)\rangle)}\Big]\,ds.$$

Above, $W_0(t)$ is the dynamical group for the linear Klein-Gordon equation which is a unitary operator in the space $\mathscr{X}^{-\varepsilon}$ for any $\varepsilon \ge 0$. For any $\varepsilon \ge 0$, there exists $C_\varepsilon < \infty$ such that there is a bound

$$\|\mathcal{N}[\Psi_1](t) - \mathcal{N}[\Psi_2](t)\|_{\mathscr{X}^{-\varepsilon}} \le C_\varepsilon |t| \sup_{s\in[0,t]} \|\Psi_1(s) - \Psi_2(s)\|_{\mathscr{X}^{-\varepsilon}}, \quad |t| \le 1, \tag{2.11}$$

which holds for any two functions Ψ_1, $\Psi_2 \in C(\mathbb{R}, \mathscr{X})$. This bound shows that $\mathcal{N}[\psi]$ is a contraction operator in $C_b([0,t], \mathscr{X}^{-\varepsilon})$, $\varepsilon \ge 0$, if $t > 0$ is sufficiently small. The contraction mapping theorem based on the bound (2.11) on the nonlinear term allows us to prove the existence and uniqueness of a local solution in $\mathscr{X}$, as well as the continuity of the map $W(t)$ in $\mathscr{X}$ (continuity with respect to the initial data).

Now let us discuss the a priori bound (2.10). Due to Assumption 2.3, adding to $u_J(|z|^2)$, $1 \le J \le N$, constants if necessary (this does not change equation (2.1)), we may assume that

$$\inf_{z\in\mathbb{C}} u_J(|z|^2) \ge 0, \qquad 1 \le J \le N. \tag{2.12}$$

The conservation of the values of the energy and functional $\mathcal{H}$ is obtained by approximating the initial data in $\mathscr{X}$ with smooth compactly supported initial data and using the continuity of $W(t)$ in $\mathscr{X}$ (which we already know for small times). The a priori bound (2.10) follows from bounding the norm $\|\Psi\|_{\mathscr{X}}$ in terms of the value of the Hamiltonian (2.3), with the aid of (2.12):

$$\|\Psi\|^2_{\mathscr{X}} \le 2\mathcal{H}(\Psi), \qquad \Psi \in \mathscr{X}. \tag{2.13}$$

This bound allows us to extend the existence results for all times, proving the global well-posedness of (2.9) in the energy space. The continuity of $W(t)$ in the topology of $\mathscr{X}^{-\varepsilon}$ for $\varepsilon \ge 0$, follows from the contraction mapping theorem (based on the bound (2.11)). The continuity in the topology of $\mathscr{X}^{-\varepsilon}_{-\sigma}$ for $\sigma \ge 0$ follows from the finite speed of propagation.

More details are in [2]. □

Definition 2.7 (Solitary waves). 1. The solitary waves of equation (2.1) are solutions of the form

$$\phi_\omega(x)e^{-i\omega t}, \qquad \text{where } \omega \in \mathbb{R},\ \ \phi_\omega(x) \in H^1(\mathbb{R}^n). \tag{2.14}$$

2. The solitary manifold is the set $\mathcal{S} = \{(\phi_\omega, -i\omega\phi_\omega) \colon \omega \in \mathbb{R}\}$, where ϕ_ω are the amplitudes of solitary waves.

Remark 2.8. Due to the $\mathbf{U}(1)$-invariance of equation (2.1), the set $\mathcal{S}$ is invariant under multiplication by $e^{i\theta}$, $\theta \in \mathbb{R}$. Let us note that for any $\omega \in \mathbb{R}$ there is a zero solitary wave, $\phi_\omega(x) \equiv 0$.

Define

$$\begin{aligned} \varSigma_I(x,\omega) &= \mathcal{F}_{\xi\to x}\Big[\hat{\varSigma}_I(\xi,\omega)\Big], \\ \hat{\varSigma}_I(\xi,\omega) &= \frac{e^{-i\xi\cdot X_I}\hat{\rho}(\xi)}{\xi^2 + m^2 - \omega^2}, \qquad x,\,\xi \in \mathbb{R}^n, \quad \omega \in \mathbb{C}^+ \cup (-m,m), \end{aligned} \tag{2.15}$$

where $\mathbb{C}^+ = \{\omega \in \mathbb{C} \colon \operatorname{Im}\omega > 0\}$. Note that $\varSigma_I(\cdot,\omega)$ is an analytic function of $\omega \in \mathbb{C}^+$ with the values in $\mathscr{S}(\mathbb{R}^n)$.

Lemma 2.9. *There is $c > 0$ such that $|\varSigma_I(x,\omega)| \le c|\operatorname{Im}\omega|^{-1}$ for $\omega \in \mathbb{C}^+$, $x \in \mathbb{R}^n$.*

Proof. Let us show that

$$|\xi^2 + m^2 - \omega^2| \ge m|\operatorname{Im}\omega| \qquad \text{for all} \quad \xi \in \mathbb{R}^n, \quad \omega \in \mathbb{C}. \tag{2.16}$$

Denoting $a = \operatorname{Re}\omega$, $b = \operatorname{Im}\omega$, we have:

$$|\xi^2 + m^2 - \omega^2|^2 = |\xi^2 + m^2 - a^2 + b^2 - 2iab|^2 = (\xi^2 + m^2 - a^2 + b^2)^2 + 4a^2b^2. \tag{2.17}$$

If $a^2 \le m^2/2$, (2.17) yields $|\xi^2 + m^2 - \omega^2|^2 \ge (\xi^2 + m^2 - a^2 + b^2)^2 \ge (m^2/2 + b^2)^2 \ge m^2b^2$; if instead $a^2 \ge m^2/2$, (2.17) yields $|\xi^2 + m^2 - \omega^2|^2 \ge 4a^2b^2 \ge 2m^2b^2$. This proves the inequality (2.16). This inequality allows us to bound (2.15) by

$$|\varSigma_I(x,\omega)| \le \int_{\mathbb{R}^n} \Big|\frac{\hat{\rho}(\xi)}{\xi^2 + m^2 - \omega^2}\Big| \frac{d^n\xi}{(2\pi)^n} \le \int_{\mathbb{R}^n} \frac{|\hat{\rho}(\xi)|}{m|\operatorname{Im}\omega|} \frac{d^n\xi}{(2\pi)^n} \le \frac{c}{|\operatorname{Im}\omega|}.$$

□

By Lemma 2.9, for any $x \in \mathbb{R}^n$, we can extend the function $\Sigma_I(x,\omega)$ to the entire real line $\omega \in \mathbb{R}$ as the boundary trace:

$$\Sigma_I(x,\omega) = \lim_{\epsilon\to 0+} \Sigma_I(x,\omega+i\epsilon), \qquad \omega \in \mathbb{R}, \tag{2.18}$$

where the limit holds in the sense of tempered distributions.

Define

$$\sigma_{IJ}(\omega) = \langle \rho_I, \Sigma_J(\cdot,\omega)\rangle = \int_{\mathbb{R}^n} \frac{e^{i(X_I-X_J)\cdot\xi}|\hat\rho(\xi)|^2}{\xi^2+m^2-(\omega+i0)^2}\frac{d^n\xi}{(2\pi)^n}, \qquad 1 \le I,J \le N. \tag{2.19}$$

Denote

$$Z_\rho = \{\omega \in \mathbb{R}\backslash[-m,m]\colon \hat\rho(\xi) = 0 \text{ for some } \xi \in \mathbb{R}^n \text{ such that } m^2+\xi^2=\omega^2\}. \tag{2.20}$$

Proposition 2.10 (Existence of solitary waves)**.** *Assume that* $\det_{I,J}\sigma_{IJ}(\omega) \ne 0$ *for* $\omega \in [-m,m]\cup Z_\rho$*, where* $\sigma_{IJ}(\omega)$ *is from* (2.19) *and* Z_ρ *is defined in* (2.20)*.*

Let $\omega \in [-m,m]\cup Z_\rho$*. Assume that there are constants* $C_I \in \mathbb{C}$*,* $1 \le I \le N$*, which satisfy*

$$F_I\Big(\sum_{J=1}^N \sigma_{IJ}(\omega)C_J\Big) = C_I. \tag{2.21}$$

To have solitary waves with $|\omega| = m$ *in dimension* $n \le 4$*, additionally assume that*

$$\int_{\mathbb{R}^n} \frac{|\hat\rho(\xi)|^2}{\xi^4}\,d^n\xi < \infty. \tag{2.22}$$

Then there is a solitary wave solution $\phi_\omega(x)e^{-i\omega t}$ *to* (2.1)*, with* $\phi_\omega(x)$ *such that*

$$\hat\phi_\omega(\xi) = \frac{\sum_{I=1}^N C_I\hat\rho_I(\xi)}{\xi^2+m^2-\omega^2}. \tag{2.23}$$

This describes all nonzero solitary wave solutions to (2.1)*.*

Proof. Substituting the ansatz $\phi_\omega(x)e^{-i\omega t}$ into (2.1), we get the following equation on ϕ_ω:

$$-\omega^2\phi_\omega(x) = \Delta\phi_\omega(x) - m^2\phi_\omega(x) + \sum_{I=1}^N \rho_I(x)F_I(\langle\rho_I,\phi_\omega\rangle), \qquad x \in \mathbb{R}^n.$$

Therefore, all solitary waves satisfy the relation

$$(\xi^2+m^2-\omega^2)\hat\phi_\omega(\xi) = \sum_{I=1}^N \hat\rho_I(\xi)F_I(\langle\rho_I,\phi_\omega\rangle). \tag{2.24}$$

For $\omega \in \mathbb{R}\backslash([-m,m]\cup Z_\rho)$ the relation (2.24) leads to $\phi_\omega \notin L^2(\mathbb{R}^n)$ (unless $\phi_\omega \equiv 0$). We conclude that there are no nonzero solitary waves for $\omega \in \mathbb{R}\backslash([-m,m]\cup Z_\rho)$.

Let us consider the case $\omega \in [-m,m]\cup Z_\rho$. From (2.24), we see that

$$\hat\phi_\omega(\xi) = \sum_{I=1}^N \frac{\hat\rho_I(\xi)}{\xi^2+m^2-\omega^2}F_I(\langle\rho_I,\phi_\omega\rangle). \tag{2.25}$$

Using the functions $\Sigma_I(x,\omega)$ defined in (2.15), we can write $\phi_\omega(x) = \sum_{I=1}^N C_I\Sigma_I(x,\omega)$, with $C_I \in \mathbb{C}$. Substituting this ansatz into (2.25), we can write the condition on C_I in the following form:

$$\sum_{J=1}^N \sigma_{IJ}(\omega)F_J\Big(\sum_{K=1}^N \sigma_{JK}(\omega)C_K\Big) = \sum_{J=1}^N \sigma_{IJ}(\omega)C_J, \tag{2.26}$$

where $\sigma_{IJ}(\omega)$ is defined in (2.19). Since we assumed that $\sigma_{IJ}(\omega)$ is nondegenerate for $\omega \in [-m,m]\cup Z_\rho$, we can rewrite (2.26) in the form (2.21).

For $n \le 4$, the finiteness of the energy of solitons corresponding to $\omega = \pm m$ follows from the condition (2.22). $\square$

Definition 2.11. For $\sigma_{IJ}(\omega)$ from (2.19) and for $1 \le N' \le N$, define

$$Z_\sigma^{N'} = \Big\{\omega \in \mathbb{R}\colon \exists\, \mathcal{I}, \mathcal{J} \subset \{1, \dots, N\},\ |\mathcal{I}| = |\mathcal{J}| = N',\ \det_{I\in\mathcal{I},\, J\in\mathcal{J}} \sigma_{IJ}(\omega) = 0\Big\}.$$

Denote

$$Z_\sigma^* = \Big(\bigcup_{1\le N'\le N/2} Z_\sigma^{N'}\Big) \cup Z_\sigma^N. \tag{2.27}$$

Assumption 2.12. Z_σ^* is a discrete set of points, and $Z_\sigma^* \cap ([-m, m] \cup Z_\rho) = \emptyset$.

Above, Z_ρ is defined in (2.20) and Z_σ^* is defined in (2.27).

Theorem 2.13 (Global attraction for Klein-Gordon with mean field interaction)**.** *Assume that the nonlinearities $F_I(z)$, $1 \le I \le N$, satisfy Assumption 2.3. Assume that the coupling function $\rho(x)$ and the points X_I, $1 \le I \le N$, are such that Assumption 2.12 is satisfied. Then for any $(\psi_0, \pi_0) \in \mathscr{X}$ the solution $\psi(t)$ to equation* (2.1) *with the initial data $(\psi, \dot\psi)|_{t=0} = (\psi_0, \pi_0)$ converges to the solitary manifold $\mathcal{S}$ in the space $\mathscr{Y} = \mathscr{X}_{-\sigma}^{-\varepsilon}$, for any $\varepsilon > 0$, $\sigma > 0$:*

$$\lim_{t\to\pm\infty} \operatorname{dist}_{\mathscr{Y}}((\psi, \dot\psi)|_t, \mathcal{S}) = 0, \tag{2.28}$$

where $\operatorname{dist}_{\mathscr{Y}}(\Psi, \mathcal{S}) := \inf_{\boldsymbol{s}\in\mathcal{S}} \|\Psi - \boldsymbol{s}\|_{\mathscr{Y}}$, *with* $\|\cdot\|_{\mathscr{Y}} = \|\cdot\|_{\mathscr{X}_{-\sigma}^{-\varepsilon}}$ *introduced in* (2.5).

Due to the time reversibility of the equation, it suffices to prove Theorem 2.13 for $t \to +\infty$.
Let us construct examples for which Assumption 2.12 is satisfied.

Example 2.1. Pick $\hat\rho(\xi) = \Lambda^{-n/2} e^{-\xi^2/\Lambda^2}$ with $0 < \Lambda \ll 1$, so that $Z_\rho = \emptyset$; to comply with Assumption 2.12, it suffices to check that $Z_\sigma^* \cap [-m, m] = \emptyset$. For $|\omega| \le m$, (2.19) yields:

$$\sigma_{IJ}(\omega) \approx \int_{\mathbb{R}^n} \frac{|\hat\rho(\xi)|^2}{\xi^2 + m^2 - \omega^2} \frac{d^n\xi}{(2\pi)^n} - \frac{|X_I - X_J|^2}{2} \int_{\mathbb{R}^n} \frac{\frac{1}{n}|\xi|^2 |\hat\rho(\xi)|^2}{\xi^2 + m^2 - \omega^2} \frac{d^n\xi}{(2\pi)^n} + O\left(\int_{\mathbb{R}^n} \frac{\xi^4 |\hat\rho(\xi)|^2\, d^n\xi}{\xi^2 + m^2 - \omega^2}\right),$$

where the factor of $1/n$ is the mean value of $\cos^2\varphi$ integrated over the sphere, with φ the angle between ξ and $X_I - X_J$. Then

$$\sigma_{IJ}(\omega) \approx a_\Lambda(\omega) + b_\Lambda(\omega)|X_I - X_J|^2 + c_\Lambda(\omega, |X_I - X_J|),$$

where

$$a_\Lambda(\omega) = \operatorname{vol}(\mathbb{S}^{n-1}) \int_0^\infty \frac{e^{-\eta^2/\Lambda^2}\eta^{n-1} d\eta}{\eta^2 + m^2 - \omega^2}, \qquad b_\Lambda(\omega) = \operatorname{vol}(\mathbb{S}^{n-1}) \int_0^\infty \frac{e^{-\eta^2\Lambda^2}\eta^{n+1} d\eta}{\eta^2 + m^2 - \omega^2}.$$

Let us show that generically $\det \sigma_{IJ}(\omega) \ne 0$ for $|\omega| \le m$, as long as $\Lambda > 0$ is sufficiently small. For all $|\omega| \le m$ one has $\frac{b_\Lambda(\omega)}{a_\Lambda(\omega)} = O(\Lambda^2)$, $\frac{c_\Lambda(\omega, |X_I - X_J|)}{a_\Lambda(\omega)} = O(\Lambda^4)$, uniformly in $\omega \in [-m, m]$. One has:

$$\det \sigma_{IJ}(\omega) = a_\Lambda(\omega)^N \det\Big(U_{IJ} + \frac{b_\Lambda(\omega)}{a_\Lambda(\omega)} \varDelta_{IJ}^2 + \frac{c_\Lambda(\omega, \varDelta_{IJ})}{a_\Lambda(\omega)}\Big),$$

where $U_{IJ} = 1$ for all I, J between 1 and N, and $\varDelta_{IJ} = |X_I - X_J|$. If the number of oscillators N does not exceed the dimension n, then the quantities $\varDelta_{IJ}$ for different pairs (I, J), with $I \ne J$, could be considered as independent variables (once they satisfy the triangle inequalities $\varDelta_{IJ} + \varDelta_{JK} > \varDelta_{JK}$), and then it follows that $|\det \sigma_{IJ}(\omega)| \sim p(\varDelta_{IJ}) a_\Lambda(\omega) b_\Lambda(\omega)^{N-1}(1 + o(\Lambda))$, where $p(\varDelta_{IJ})$ is a polynomial of $\varDelta_{IJ}$ of degree $N-1$, is nonzero for a generic choice of $\varDelta_{IJ}$ (satisfying the triangle inequalities). The same is true for all minors. We conclude that generically $Z_\sigma^* \cap [-m, m] = \emptyset$, in agreement with Assumption 2.12.

Example 2.2. Let us assume that $k > 0$ and X_I, $1 \le I \le N$, are such that the matrix

$$S_{IJ} = e^{ik\boldsymbol{e}_1\cdot(X_I - X_J)}, \qquad \boldsymbol{e}_1 = (1, 0, \dots, 0) \in \mathbb{S}^{n-1},$$

is non-degenerate and all its $N' \times N'$ minors, $1 \le N' \le N$, are also non-zero. Let

$$\hat\rho_\epsilon(\xi) = \frac{1}{\epsilon^{n/2}} \varrho\Big(\frac{\xi_1 - k}{\epsilon}\Big) \prod_{j=2}^n \varrho\Big(\frac{\xi_j}{\epsilon}\Big) + \epsilon e^{-\xi^2},$$

where $\varrho \in C^\infty_{comp}([-1,1])$, $\varrho|_{(-1,1)} > 0$, $\int_{\mathbb{R}} \varrho(s)\,ds = 1$. Since $\hat\rho_\epsilon$ is strictly positive, one has $Z_\rho = \emptyset$; to comply with Assumption 2.12, it suffices to ensure that $Z^*_\sigma \cap [-m,m] = \emptyset$. If $\epsilon > 0$ is sufficiently small, then $\hat\rho_\epsilon(\xi)$ is concentrated in a sufficiently small neighborhood of $\xi = k\boldsymbol{e}_1$, hence for $|\omega| \le m$ one has

$$\sigma_{IJ}(\omega) = \int_{\mathbb{R}^n} \frac{e^{i(X_I - X_J)\cdot\xi}|\hat\rho_\epsilon(\xi)|^2}{\xi^2 + m^2 - \omega^2}\,\frac{d^n\xi}{(2\pi)^n} = S_{IJ}\int_{\mathbb{R}^n} \frac{|\hat\rho_\epsilon(\xi)|^2}{\xi^2 + m^2 - \omega^2}\,\frac{d^n\xi}{(2\pi)^n} + O(\epsilon),$$

and therefore $Z^*_\sigma \cap [-m,m] = \emptyset$ if $\epsilon > 0$ is sufficiently small.

Remark 2.14. If Assumption 2.12 is not satisfied, then the attractor could be more complicated than the set of all solitary waves of the form (2.14). In particular, there could be "multifrequency solitary waves" (solitary waves with several discrete frequencies), which would also be points of the attractor. Such examples have been constructed in [2].

3 Absolute continuity for large frequencies

According to Theorem 2.6 *(1)*, for any pair $(\psi_0, \pi_0) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ there exists a global solution $\psi(x,t)$, to the Cauchy problem (2.1) with the initial data (ψ_0, π_0),

$$(\psi, \dot\psi)|_{t=0} = (\psi_0, \pi_0) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n). \tag{3.1}$$

By Theorem 2.6 *(4)*,

$$\sup_{t\in\mathbb{R}} \|(\psi,\dot\psi)|_t\|_{\mathscr{X}} < \infty. \tag{3.2}$$

Define $\chi(x,t)$ as the solution to the following Cauchy problem:

$$\ddot\chi(x,t) = \Delta\chi(x,t) - m^2\chi(x,t), \qquad (\chi,\dot\chi)|_{t=0} = (\psi_0,\pi_0), \tag{3.3}$$

where (ψ_0,π_0) is the initial data from (3.1). Since $\chi(t)$ is a finite energy solution to the free Klein-Gordon equation,

$$\sup_{t\in\mathbb{R}} \|(\chi,\dot\chi)|_t\|_{\mathscr{X}} < \infty. \tag{3.4}$$

Define $\varphi(x,t)$ by

$$\varphi(x,t) = \begin{cases} 0, & t < 0, \\ \psi(x,t) - \chi(x,t), & t \ge 0. \end{cases} \tag{3.5}$$

Then $\varphi(x,t)$ satisfies

$$\ddot\varphi(x,t) = \Delta\varphi(x,t) - m^2\varphi(x,t) + \sum_{I=1}^N \rho_I(x) f_I(t), \quad (\varphi,\dot\varphi)|_{t\le 0} = (0,0), \tag{3.6}$$

where $f_I(t) := \Theta(t) F_I(\langle\rho_I, \psi(\cdot,t)\rangle)$. Note that $\langle\rho_I, \psi(\cdot,t)\rangle$ belongs to $C_b(\mathbb{R})$ by (3.2). Hence,

$$\sup_{t\in\mathbb{R}} |f_I(t)| < \infty, \qquad 1 \le I \le N. \tag{3.7}$$

By (3.2) and (3.4), $\varphi(t) = \Theta(t)(\psi(t) - \chi(t))$ satisfies

$$\sup_{t\in\mathbb{R}} \|(\varphi,\dot\varphi)|_t\|_{\mathscr{X}} < \infty. \tag{3.8}$$

Let us consider the complex Fourier transform of $\varphi(x,t)$:

$$\tilde\varphi(x,\omega) = \mathcal{F}_{t\to\omega}[\varphi(x,t)] := \int_0^\infty e^{i\omega t}\varphi(x,t)\,dt, \quad \omega \in \mathbb{C}^+, \quad x \in \mathbb{R}^n, \tag{3.9}$$

where $\mathbb{C}^+ := \{z \in \mathbb{C} :\ \operatorname{Im} z > 0\}$. Due to (3.8), $\tilde\varphi(\cdot,\omega)$ is an H^1-valued analytic function of $\omega \in \mathbb{C}^+$. Equation (3.6) for φ implies that

$$-\omega^2\tilde\varphi(x,\omega) = \Delta\tilde\varphi(x,\omega) - m^2\tilde\varphi(x,\omega) + \sum_{I=1}^N \rho_I(x)\tilde f_I(\omega), \quad \omega \in \mathbb{C}^+, \quad x \in \mathbb{R}^n,$$

where

$$\tilde{f}_I(\omega) = \int_0^\infty e^{i\omega t} f_I(t)\,dt, \qquad \omega \in \mathbb{C}^+. \tag{3.10}$$

The solution $\tilde{\varphi}(x,\omega)$ is analytic for $\omega \in \mathbb{C}^+$ and can be represented by

$$\tilde{\varphi}(x,\omega) = \sum_{I=1}^N \varSigma_I(x,\omega)\tilde{f}_I(\omega), \qquad \omega \in \mathbb{C}^+, \tag{3.11}$$

with $\varSigma_I(x,\omega)$ from (2.15).

Lemma 3.1. *1. For any $\omega \in \mathbb{R}$ and $1 \le I \le N$, there is the convergence $\tilde{f}_I(\omega + i\epsilon) \xrightarrow[\epsilon\to 0+]{\mathscr{S}'(\mathbb{R})} \tilde{f}_I(\omega)$;*

2. For any $\omega \in \mathbb{R}$, $\tilde{\varphi}(\cdot,\omega + i\epsilon) \xrightarrow[\epsilon\to 0+]{\mathscr{S}'(\mathbb{R},H^1(\mathbb{R}^n))} \tilde{\varphi}(\cdot,\omega)$.

Proof. Using $f_I|_{t<0} = 0$, $\varphi|_{t<0} = 0$, and the bounds (3.7) and (3.8), one concludes that $e^{-\epsilon t} f_I(t) \xrightarrow[\epsilon\to 0+]{\mathscr{S}'(\mathbb{R})} f_I(t)$ and $e^{-\epsilon t}\varphi(\cdot,t) \xrightarrow[\epsilon\to 0+]{\mathscr{S}'(\mathbb{R},H^1(\mathbb{R}^n))} \varphi(\cdot,t)$. This finishes the proof. □

Now we can justify the representation (3.11) for $\omega \in \mathbb{R}$, if the multiplication in (3.11) is understood in the sense of distributions.

Lemma 3.2. *There is the following identity, understood in the sense of distributions:*

$$\tilde{\varphi}(x,\omega) = \sum_{I=1}^N \varSigma_I(x,\omega)\tilde{f}_I(\omega), \qquad \omega \in \mathbb{R}. \tag{3.12}$$

Proof. Since we assume that $n \ge 3$, for each $x \in \mathbb{R}^n$, the function $\varSigma_I(x,\omega) = \int_{\mathbb{R}^n} \frac{e^{i\xi\cdot x} e^{-i\xi\cdot X_I}\hat{\rho}(\xi)}{\xi^2+m^2-(\omega+i0)^2}\frac{d^n\xi}{(2\pi)^n}$ is a smooth function of $\omega \in \mathbb{R}$, and hence is a multiplier in the space of tempered distributions in the variable ω. The rest of the proof is based on the relation (3.11) and the convergence stated in Lemma 3.1. □

For $\omega \in \mathbb{C}^+$, let $k(\omega)$ denote the branch of $\sqrt{\omega^2 - m^2}$ such that $\operatorname{Im}\sqrt{\omega^2 - m^2} \ge 0$. The function $k(\omega)$ is analytic for $\omega \in \mathbb{C}^+$. We extend $k(\omega)$ onto $\overline{\mathbb{C}^+}$ by continuity.

Proposition 3.3. *For any finite open interval W such that $\overline{W} \cap ([-m,m] \cup Z_\rho) = \emptyset$ there is a constant $C_W > 0$ such that*

$$\int_{\mathbb{S}^{n-1}\times W} \Big|\sum_{I=1}^N e^{-ik(\omega)\boldsymbol{\theta}\cdot X_I}\tilde{f}_I(\omega)\Big|^2 d\Omega_{\boldsymbol{\theta}}\,d\omega \le C_W. \tag{3.13}$$

Remark 3.4. By Lemma 3.1, $\tilde{f}_I(\omega) = \tilde{f}_I(\omega + i0)$.

Remark 3.5. Proposition 3.3 essentially states that the restriction onto the mass hypersurface $\xi^2 + m^2 = \omega^2$ of the space-time Fourier transform of the source term $\sum_{I=1}^N \rho_I(x) f_I(t)$ from (3.6) is locally integrable. Note that this hypersurface is locally parametrized by $\omega \in \mathbb{R}\backslash[-m,m]$ and $\boldsymbol{\theta} = \frac{\xi}{|\xi|} \in \mathbb{S}^{n-1}$.

Proof. The Parseval identity applied to $\tilde{\varphi}(x,\omega+i\epsilon) = \int_0^\infty \varphi(x,t)e^{i\omega t - \epsilon t}\,dt$, $\epsilon > 0$, yields

$$\int_{-\infty}^\infty \|\tilde{\varphi}(\cdot,\omega+i\epsilon)\|_{L^2}^2 \frac{d\omega}{2\pi} = \int_0^\infty \|\varphi(\cdot,t)\|_{L^2}^2 e^{-2\epsilon t}\,dt, \qquad \epsilon > 0.$$

Since $\sup_{t\ge 0}\|\varphi(\cdot,t)\|_{H^1} < \infty$ by (3.8), we may bound the right-hand side by C_1/ϵ, with some $C_1 > 0$. Taking into account (3.11), we arrive at the key inequality

$$\int_{-\infty}^\infty \|\sum_{I=1}^N \varSigma_I(\cdot,\omega+i\epsilon)\tilde{f}_I(\omega+i\epsilon)\|_{L^2}^2\,d\omega \le \frac{C_1}{\epsilon}. \tag{3.14}$$

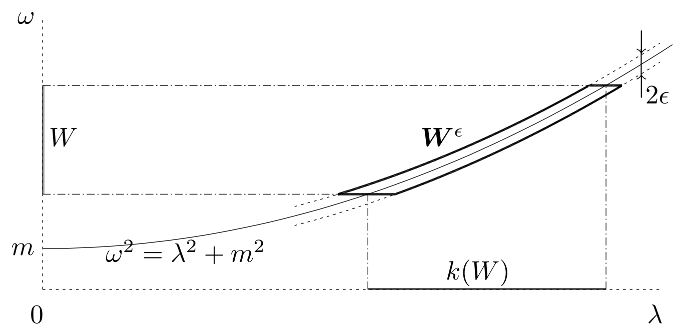

Figure 1: Domain $\boldsymbol{W}^\epsilon$ and intervals W and $k(W)$.

Noting that $\hat\Sigma_I(\xi,\omega+i\epsilon)=e^{-i\xi\cdot X_I}\hat\Sigma(\xi,\omega+i\epsilon)$, with $\hat\Sigma(\xi,\omega+i\epsilon)$ defined in (2.15), we rewrite (3.14) as

$$\int_{-\infty}^{\infty}\epsilon\|\sum_{I=1}^{N}\Sigma_I(\cdot,\omega+i\epsilon)\tilde f_I(\omega+i\epsilon)\|_{L^2}^2\,d\omega=\int\limits_{\mathbb{R}^n\times\mathbb{R}}\epsilon|\hat\Sigma(\xi,\omega+i\epsilon)|^2\Big|\sum_{I=1}^{N}e^{-i\xi\cdot X_I}\tilde f_I(\omega+i\epsilon)\Big|^2\frac{d^n\xi}{(2\pi)^n}\,d\omega\le C_1. \qquad (3.15)$$

Fix a finite open interval W such that $\overline{W}\cap([-m,m]\cup Z_\rho)=\emptyset$. Denote

$$\boldsymbol{W}^\epsilon:=\{(\lambda,\omega)\colon\lambda>0,\ \omega\in W,\ |\omega-\sqrt{\lambda^2+m^2}|<\epsilon\}\subset\mathbb{R}_+\times\mathbb{R}, \qquad (3.16)$$

as on Fig. 1, and also denote

$$\widehat{\boldsymbol{W}}^\epsilon:=\{(\xi,\omega)\in\mathbb{R}^n\times\mathbb{R}\colon(|\xi|,\omega)\in\boldsymbol{W}^\epsilon\}\subset\mathbb{R}^n\times\mathbb{R}. \qquad (3.17)$$

Due to the inequality (3.15), the following weaker inequality also takes place:

$$\int_{\widehat{\boldsymbol{W}}^\epsilon}\epsilon|\hat\Sigma(\xi,\omega+i\epsilon)|^2\Big|\sum_{I=1}^{N}e^{-i\xi\cdot X_I}\tilde f_I(\omega+i\epsilon)\Big|^2\frac{d^n\xi\,d\omega}{(2\pi)^n}\le C_1. \qquad (3.18)$$

Lemma 3.6. *There exists a constant $C_2>0$ such that for any $\epsilon\in(0,1)$ there is the inequality*

$$\int_{\widehat{\boldsymbol{W}}^\epsilon}\epsilon|\hat\Sigma(\xi,\omega+i\epsilon)|^2\Big|\sum_{I=1}^{N}e^{-ik(\omega)\boldsymbol{\theta}_\xi\cdot X_I}\tilde f_I(\omega+i\epsilon)\Big|^2\frac{d^n\xi\,d\omega}{(2\pi)^n}\le C_2, \qquad (3.19)$$

where $\boldsymbol{\theta}_\xi=\frac{\xi}{|\xi|}$.

Proof. We are done if we can prove that the difference between (the square roots of) the left-hand sides of (3.18) and (3.19) is bounded by a constant which depends on W but not on $\epsilon\in(0,1)$. For brevity, denote $\mathrm{f}_I=\tilde f_I(\omega+i\epsilon)$. Using the triangle inequality in the form $|\|a\|-\|b\||\le\|a-b\|$, we get:

$$\left|\left[\int\limits_{\widehat{\boldsymbol{W}}^\epsilon}\epsilon|\hat\Sigma(\xi,\omega+i\epsilon)|^2\Big|\sum_{I=1}^{N}e^{-i\xi\cdot X_I}\mathrm{f}_I\Big|^2\frac{d^n\xi\,d\omega}{(2\pi)^n}\right]^{\frac12}-\left[\int\limits_{\widehat{\boldsymbol{W}}^\epsilon}\epsilon|\hat\Sigma(\xi,\omega+i\epsilon)|^2\Big|\sum_{I=1}^{N}e^{-ik(\omega)\boldsymbol{\theta}_\xi\cdot X_I}\mathrm{f}_I\Big|^2\frac{d^n\xi\,d\omega}{(2\pi)^n}\right]^{\frac12}\right|$$

$$\le\left[\int\limits_{\widehat{\boldsymbol{W}}^\epsilon}\epsilon|\hat\Sigma(\xi,\omega+i\epsilon)|^2\Big|\sum_{I=1}^{N}e^{-ik(\omega)\boldsymbol{\theta}_\xi\cdot X_I}(e^{i(k(\omega)\boldsymbol{\theta}_\xi-\xi)\cdot X_I}-1)\mathrm{f}_I\Big|^2\frac{d^n\xi\,d\omega}{(2\pi)^n}\right]^{\frac12}. \qquad (3.20)$$

According to (3.2), $|f_I(t)|=|F_I(\langle\rho_I,\psi(\cdot,t)\rangle)|$ is bounded uniformly in time. By (3.10), we know that $|\mathrm{f}_I|=|\tilde f_I(\omega+i\epsilon)|\le C\epsilon^{-1}$. We also have $|e^{i(k(\omega)\boldsymbol{\theta}_\xi-\xi)\cdot X_I}-1|\le C\epsilon$ for $(\xi,\omega)\in\widehat{\boldsymbol{W}}^\epsilon$, with some $C\in\mathbb{R}$ independent on $\epsilon\in(0,1)$. Therefore, (3.20) is bounded by

$$\int_{\widehat{\boldsymbol{W}}^\epsilon}\epsilon|\hat\Sigma(\xi,\omega+i\epsilon)|^2\frac{d^n\xi}{(2\pi)^n}\,d\omega\le\int_{\widehat{\boldsymbol{W}}^\epsilon}\epsilon\frac{|\hat\rho(\xi)|^2}{4m^2\epsilon^2}\frac{d^n\xi}{(2\pi)^n}\,d\omega\le\int_{\mathbb{R}^n}\epsilon\frac{|\hat\rho(\xi)|^2}{4m^2\epsilon^2}\frac{d^n\xi}{(2\pi)^n}\,2\epsilon\le\mathrm{const}\,, \qquad (3.21)$$

where const depends on W but not on ϵ. Above, we used the expression $\hat{\Sigma}(\xi,\omega+i\epsilon)=\frac{\hat\rho(\xi)}{\xi^2+m^2-(\omega+i\epsilon)^2}$ (see (2.15)) and the bound

$$|\xi^2+m^2-(\omega+i\epsilon)^2|^2\ge|\mathrm{Im}\,(\xi^2+m^2-(\omega+i\epsilon)^2)|^2\ge 4m^2\epsilon^2,\qquad(\xi,\omega)\in\widehat{\boldsymbol{W}}^\epsilon.$$

The integration in ω contributed 2ϵ, which is the thickness of $\boldsymbol{W}^\epsilon$ in the ω-direction (see Fig. 1). It follows that the right-hand side in (3.20) is bounded by a constant independent on $\epsilon\in(0,1)$. This finishes the proof. □

Lemma 3.7. *There exist $\epsilon_W\in(0,1)$ and $C_3>0$ such that*

$$\int_{\boldsymbol{W}^\epsilon\cap(\mathbb{R}_+\times\{\omega\})}\epsilon|\hat{\Sigma}(\lambda\boldsymbol{\theta},\omega+i\epsilon)|^2\,\lambda^{n-1}\,d\lambda\ge C_3\qquad\textit{for all}\quad\omega\in W,\quad\boldsymbol{\theta}\in\mathbb{S}^{n-1},\quad\epsilon\in(0,\epsilon_W).$$

Proof. First, we note that $k(W)$ is a finite open interval bounded away from 0; see Fig. 1. Since the function $|\hat\rho(\xi)|$ is continuous and strictly positive for $|\xi|\in\overline{k(W)}$, there exist $\epsilon_W>0$ and $c_W>0$ such that $|\hat\rho(\xi)|^2\ge c_W$ for all ξ such that $(|\xi|,\omega)\in\boldsymbol{W}^\epsilon$, $\epsilon\in(0,\epsilon_W)$ (cf. (3.16)). Hence, using (2.15),

$$\int\limits_{\boldsymbol{W}^\epsilon\cap(\mathbb{R}_+\times\{\omega\})}\epsilon|\hat{\Sigma}(\lambda\boldsymbol{\theta},\omega+i\epsilon)|^2\,\lambda^{n-1}\,d\lambda\ge c_W\int\limits_{\boldsymbol{W}^\epsilon\cap(\mathbb{R}_+\times\{\omega\})}\frac{\epsilon\,\lambda^{n-1}\,d\lambda}{|\lambda^2+m^2-(\omega+i\epsilon)^2|^2}.\tag{3.22}$$

Pick $\delta_W<|k(W)|/2$; then, for $\lambda_0\in k(W)$, either $[\lambda_0-\delta_W,\lambda_0]\subset k(W)$, or $[\lambda_0,\lambda_0+\delta_W]\subset k(W)$, or both. Therefore, the integration in λ is over an interval of length at least $\frac{\epsilon}{2}\min_{\omega\in\overline{W}}|k'(\omega)|$. Moreover, for $|\omega-\sqrt{\lambda^2+m^2}|<\epsilon$, the magnitude of the denominator is bounded from above:

$$\begin{aligned}|\lambda^2+m^2-(\omega+i\epsilon)^2|^2&=(\lambda^2+m^2-\omega^2+\epsilon^2)^2+4\omega^2\epsilon^2\\&\le(\sqrt{\lambda^2+m^2}-\omega)^2(\sqrt{\lambda^2+m^2}+\omega)^2+\mathrm{const}\,\epsilon^2\le\mathrm{const}\,\epsilon^2,\end{aligned}$$

where the constant in the right-hand side depends on W but not on ϵ. This shows that the right-hand side of (3.22) is bounded from below by some constant $C_3>0$ which depends on W (and ϵ_W) but not on ω, ϵ, or $\boldsymbol{\theta}$. □

Combining Lemmas 3.6 and 3.7, we get:

$$\int_{\mathbb{S}^{n-1}\times W}\Big|\sum_{I=1}^N e^{-ik(\omega)\boldsymbol{\theta}\cdot X_I}\tilde f_I(\omega+i\epsilon)\Big|^2\frac{d\Omega_{\boldsymbol{\theta}}\,d\omega}{(2\pi)^n}\le\frac{C_2}{C_3},\qquad 0<\epsilon\le\epsilon_W.$$

We conclude that the set of functions

$$g_{W,\epsilon}(\boldsymbol{\theta},\omega)=\sum_{I=1}^N e^{-ik(\omega)\boldsymbol{\theta}\cdot X_I}\tilde f_I(\omega+i\epsilon),\qquad 0<\epsilon\le\epsilon_W,$$

defined for $\boldsymbol{\theta}\in\mathbb{S}^{n-1}$, $\omega\in W$, is bounded in the Hilbert space $L^2(\mathbb{S}^{n-1}\times W)$, and hence is weakly precompact. The convergence of the distributions stated in Lemma 3.1 implies the weak convergence $g_{W,\epsilon}\underset{\epsilon\to 0+}{\rightharpoonup}g_W$ in the Hilbert space $L^2(\mathbb{S}^{n-1}\times W)$. The limit function $g_W\in L^2(\mathbb{S}^{n-1}\times W)$ coincides with the distribution $\sum_{I=1}^N e^{-ik(\omega)\boldsymbol{\theta}\cdot X_I}\tilde f_I(\omega)$ on $\mathbb{S}^{n-1}\times W$. This proves the bound (3.13). □

Proposition 3.8. *The distributions $\tilde f_I(\omega+i0)$, $1\le I\le N$, are locally L^2 for $\omega\in\mathbb{R}\backslash([-m,m]\cup Z_\rho)$.*

Proof. We split the proof into four lemmas.

Lemma 3.9. *Let $k\ne 0$. Assume that the vectors $X_J\in\mathbb{R}^n$, $1\le J\le N$, are pairwise different. Then there exist vectors $\boldsymbol{\theta}_I\in\mathbb{S}^{n-1}$, $1\le I\le N$, such that*

$$\det_{1\le I,J\le N}e^{-ik\boldsymbol{\theta}_I\cdot X_J}\ne 0.$$

Proof. Let us choose a (two-dimensional) plane $\boldsymbol{A}$ through the origin in $\mathbb{R}^n$ such that the orthogonal projections of X_J onto $\boldsymbol{A}$, which we denote by $Y_J=P_{\boldsymbol{A}}(X_J)$, are pairwise different. It suffices to show that we can choose $\boldsymbol{\theta}_J\in\mathbb{S}^{n-1}\cap\boldsymbol{A}$ such that

$$\det_{1\le I,J\le N}e^{-ik\boldsymbol{\theta}_I\cdot Y_J}\ne 0.\tag{3.23}$$

It is enough to consider the case when all Y_J are pairwise linearly independent and have different lengths. Indeed, since Y_J are pairwise different, there exists $Y_0 \in \boldsymbol{A}$ such that $Y_0 + Y_J$ are pairwise linearly independent and have different lengths; at the same time,

$$\det_{1\le I,J\le N} e^{-ik\boldsymbol{\theta}_I\cdot(Y_0+Y_J)} = \Big(\prod_{I=1}^{N} e^{-ik\boldsymbol{\theta}_I\cdot Y_0}\Big) \det_{1\le I,J\le N} e^{-ik\boldsymbol{\theta}_I\cdot Y_J},$$

with the factor $\prod_I e^{-ik\boldsymbol{\theta}_I\cdot Y_0}$ different from zero.

We will prove (3.23) by induction in N, assuming that the vertices X_J are numbered so that

$$|Y_1| < |Y_2| < \dots < |Y_N|. \tag{3.24}$$

The claim is true for $N = 1$ since $e^{-ik\boldsymbol{\theta}_1\cdot Y_1} \ne 0$ for any $\boldsymbol{\theta}_1 \in \mathbb{S}^{n-1}\cap\boldsymbol{A}$. Assume that the statement is true for some $M \ge 1$, $M < N$: there exist vectors $\boldsymbol{\theta}_I \in \mathbb{S}^{n-1}\cap\boldsymbol{A}$, $1\le I\le M$, such that

$$\det_{1\le I,J\le M} e^{-ik\boldsymbol{\theta}_I\cdot Y_J} \ne 0. \tag{3.25}$$

Then we need to check that the statement is also true for $M+1$. That is, we need to show that there exists $\boldsymbol{\theta}_{M+1} \in \mathbb{S}^{n-1}\cap\boldsymbol{A}$ such that

$$\det_{1\le I,J\le M+1} e^{-ik\boldsymbol{\theta}_I\cdot Y_J} \ne 0. \tag{3.26}$$

According to (3.25), there is a unique set of numbers $a_J\in\mathbb{C}$, $1\le J\le M$, such that

$$\sum_{J=1}^{M} a_J \begin{bmatrix} e^{-ik\boldsymbol{\theta}_1\cdot Y_J} \\ \vdots \\ e^{-ik\boldsymbol{\theta}_M\cdot Y_J}\end{bmatrix} + \begin{bmatrix} e^{-ik\boldsymbol{\theta}_1\cdot Y_{M+1}} \\ \vdots \\ e^{-ik\boldsymbol{\theta}_M\cdot Y_{M+1}}\end{bmatrix} = 0. \tag{3.27}$$

To prove (3.26), we need to show that the relation

$$\sum_{J=1}^{M} a_J e^{-ik\boldsymbol{\theta}\cdot Y_J} + e^{-ik\boldsymbol{\theta}\cdot Y_{M+1}} = 0 \tag{3.28}$$

cannot be valid for all $\boldsymbol{\theta}\in\mathbb{S}^{n-1}\cap\boldsymbol{A}$; this, in turn, will imply that there exists $\boldsymbol{\theta}_{M+1}\in\mathbb{S}^{n-1}\cap\boldsymbol{A}$ such that the columns $\begin{bmatrix} e^{-ik\boldsymbol{\theta}_1\cdot Y_J} \\ \vdots \\ e^{-ik\boldsymbol{\theta}_{M+1}\cdot Y_J}\end{bmatrix}$, $1\le J\le M+1$, are linearly independent, leading to (3.26).

We parametrize $\boldsymbol{\theta}\in\mathbb{S}^{n-1}\cap\boldsymbol{A}=\mathbb{S}^1$ by the angle $\vartheta\in[0,2\pi)$. Let $\gamma_J\in[0,2\pi)$, $1\le J\le M+1$, be the angles corresponding to the directions $Y_J/|Y_J|\in\mathbb{S}^1$. Note that since Y_J are pairwise linearly independent, all the angles γ_J are different. The relation (3.28) takes the form

$$f(\vartheta) = 0, \tag{3.29}$$

where

$$f(\vartheta) = \sum_{J=1}^{M} a_J e^{-ik|Y_J|\cos(\vartheta-\gamma_J)} + e^{-ik|Y_{M+1}|\cos(\vartheta-\gamma_{M+1})}. \tag{3.30}$$

For $\vartheta\in\mathbb{C}$, the formula (3.30) defines an entire function; let us show that f is not identically zero. Let $\vartheta = u+iv$, where $u,\ v\in\mathbb{R}$. Since

$$\cos(\vartheta-\gamma) = \cos(u+iv-\gamma) = \cos(u-\gamma)\cosh v - i\sin(u-\gamma)\sinh v, \qquad \gamma\in\mathbb{R},$$

the definition (3.30) takes the form

$$f(\vartheta) = \sum_{J=1}^{M} a_J e^{-k|Y_J|(i\cos(u-\gamma_J)\cosh v+\sin(u-\gamma_J)\sinh v)} + e^{-k|Y_{M+1}|(i\cos(u-\gamma_{M+1})\cosh v+\sin(u-\gamma_{M+1})\sinh v)}. \tag{3.31}$$

Taking into account (3.24), we derive the following asymptotics along the line $\operatorname{Re}\vartheta = \gamma_{M+1} - \frac{\pi}{2}$ (that is, we take $u = \gamma_{M+1} - \frac{\pi}{2}$ and $v \in \mathbb{R}$):

$$f\Big(\gamma_{M+1} - \frac{\pi}{2} + iv\Big) \sim e^{k|Y_{M+1}|\sinh v}, \qquad v \to +\infty. \tag{3.32}$$

It follows that $f(\vartheta)$ is an entire function which is not identically equal to zero. Therefore, (3.29) can hold at no more than finitely many values $\vartheta \in [0, 2\pi)$. We pick $\boldsymbol{\theta}_{M+1}$ so that the corresponding angle ϑ is not a root of (3.29). With this particular value of $\boldsymbol{\theta}_{M+1}$, (3.26) is satisfied. This finishes the induction argument. □

Lemma 3.10. *For any $\omega_0 \in \mathbb{R}\backslash[-m, m]$, there is an open neighborhood $W \subset \mathbb{R}\backslash[-m, m]$ of ω_0 and a family of diffeomorphisms $\boldsymbol{\Theta}_I : \mathbb{B}^{n-1} \to \Omega_I \subset \mathbb{S}^{n-1}$, where $\mathbb{B}^{n-1}$ is a unit open ball in $\mathbb{R}^{n-1}$ and Ω_I are open neighborhoods of $\mathbb{S}^{n-1}$, such that*

$$\det_{1\le I,J\le N} e^{-ik(\omega)\boldsymbol{\Theta}_I(\boldsymbol{\tau})\cdot X_J} \ne 0 \qquad \textit{for all } \ \omega \in \overline{W}, \ \ \boldsymbol{\tau} \in \mathbb{B}^{n-1}.$$

Proof. According to Lemma 3.9, there are $\boldsymbol{\theta}_I \in \mathbb{S}^{n-1}$, $1 \le I \le N$, such that

$$\det_{1\le I,J\le N} e^{-ik(\omega_0)\boldsymbol{\theta}_I\cdot X_J} \ne 0.$$

Therefore, there is an open neighborhood $W \subset \mathbb{R}$ of ω_0 and open neighborhoods $\Omega_I \subset \mathbb{S}^{n-1}$ of $\boldsymbol{\theta}_I$, $1 \le I \le N$, with each of Ω_I being diffeomorphic to $\mathbb{B}^{n-1}$, such that

$$\det_{1\le I,J\le N} e^{-ik(\omega)\boldsymbol{\theta}_I\cdot X_J} \ne 0 \qquad \text{for all } \ \omega \in \overline{W}, \ \ \boldsymbol{\theta}_I \in \overline{\Omega_I}.$$

After we denote the aforementioned diffeomorphisms $\mathbb{B}^{n-1} \xrightarrow{\cong} \Omega_I$ by $\boldsymbol{\Theta}_I$, $1 \le I \le N$, the lemma is proved. □

Fix $\omega_0 \in \mathbb{R}\backslash([-m, m] \cup Z_\rho)$, and let W be an open neighborhood of ω_0 from Lemma 3.10. Taking W smaller if necessary, we may assume that

$$\overline{W} \cap ([-m, m] \cup Z_\rho) = \emptyset. \tag{3.33}$$

Pick a function $\varsigma \in C_0^\infty(\mathbb{B}^{n-1})$ such that $\int_{\mathbb{B}^{n-1}} \varsigma(\boldsymbol{\tau})\,d\boldsymbol{\tau} = 1$. Let

$$R_{IJ}(\omega, \boldsymbol{\tau}), \qquad \omega \in \overline{W}, \quad \boldsymbol{\tau} \in \mathbb{B}^{n-1}, \tag{3.34}$$

be the matrix inverse to $A_{IJ}(\omega, \boldsymbol{\tau}) = e^{-ik(\omega)\boldsymbol{\Theta}_I(\boldsymbol{\tau})\cdot X_J}$. Denote

$$R_I(\omega, \boldsymbol{\theta}) = \int_{\mathbb{B}^{n-1}} \sum_{J=1}^{N} R_{IJ}(\omega, \boldsymbol{\tau})\delta_{\boldsymbol{\Theta}_J(\boldsymbol{\tau})}(\boldsymbol{\theta})\,\varsigma(\boldsymbol{\tau})\,d\boldsymbol{\tau}, \qquad \omega \in \overline{W}, \quad \boldsymbol{\theta} \in \mathbb{S}^{n-1}, \tag{3.35}$$

where $\delta_{\boldsymbol{\theta}_0}(\boldsymbol{\theta})$ is a delta-function on $\mathbb{S}^{n-1}$ supported at $\boldsymbol{\theta}_0 \in \mathbb{S}^{n-1}$.

Lemma 3.11. *For each $1 \le I \le N$, the operator*

$$\mathscr{R}_I : \ u(\omega, \boldsymbol{\theta}) \mapsto \mathscr{R}_I u(\omega) := \int_{\Omega_I} R_I(\omega, \boldsymbol{\theta}) u(\omega, \boldsymbol{\theta})\,d\Omega_{\boldsymbol{\theta}} \tag{3.36}$$

acts continuously from $L^2(W \times \mathbb{S}^{n-1})$ to $L^2(W)$.

Proof. For a given value $\omega \in W$, let $\boldsymbol{T}_I(\boldsymbol{\theta})$ be the inverse function to $\boldsymbol{\Theta}_I(\boldsymbol{\tau})$ which is defined on the neighborhood $\{\boldsymbol{\Theta}_I(\boldsymbol{\tau}) \colon \boldsymbol{\tau} \in \mathbb{B}^{n-1}\} \subset \mathbb{S}^{n-1}$. It suffices to notice that the function $R_I(\omega, \boldsymbol{\theta})$ defined in (3.35) is smooth, since

$$\delta_{\boldsymbol{\Theta}_I(\boldsymbol{\tau})}(\boldsymbol{\theta})\,\varsigma(\boldsymbol{\tau}) = \frac{\delta(\boldsymbol{\tau} - \boldsymbol{T}_I(\boldsymbol{\theta}))}{\Big|\det \frac{\partial \boldsymbol{\Theta}_I(\boldsymbol{\tau})}{\partial \boldsymbol{\tau}}\Big|}\,\varsigma(\boldsymbol{\tau}).$$

□

Lemma 3.12. *For any functions $\tilde f_I \in L^2_{loc}(\mathbb{R})$, $1 \le I \le N$, there is the identity*

$$\mathscr{R}_I\Big(\sum_{K=1}^{N} e^{-ik(\omega)\boldsymbol{\theta}\cdot X_K}\,\tilde f_K(\omega)\Big)\Big|_W = \tilde f_I|_W.$$

Proof. Using (3.34), (3.35) and (3.36), we derive:

$$
\begin{aligned}
&\mathscr{R}_I\Big(\sum_{K=1}^N e^{-ik(\omega)\boldsymbol{\theta}\cdot X_K}\tilde f_K(\omega)\Big)\\
&=\sum_{K=1}^N\int_{\mathbb{S}^{n-1}} R_I(\omega,\boldsymbol{\theta})e^{-ik(\omega)\boldsymbol{\theta}\cdot X_K}\tilde f_K(\omega)\,d\Omega_{\boldsymbol{\theta}}\\
&=\sum_{K=1}^N\int_{\mathbb{S}^{n-1}}\int_{\mathbb{B}^{n-1}}\sum_{J=1}^N R_{IJ}(\omega,\boldsymbol{\tau})\delta(\boldsymbol{\theta}-\boldsymbol{\Theta}_J(\boldsymbol{\tau}))\,\varsigma(\boldsymbol{\tau})e^{-ik(\omega)\boldsymbol{\theta}\cdot X_K}\tilde f_K(\omega)\,d\boldsymbol{\tau}\,d\Omega_{\boldsymbol{\theta}}\\
&=\sum_{K=1}^N\int_{\mathbb{B}^{n-1}}\sum_{J=1}^N R_{IJ}(\omega,\boldsymbol{\tau})\,\varsigma(\boldsymbol{\tau})e^{-ik(\omega)\boldsymbol{\Theta}_J(\boldsymbol{\tau})\cdot X_K}\tilde f_K(\omega)\,d\boldsymbol{\tau}\\
&=\sum_{K=1}^N\int_{\mathbb{B}^{n-1}}\delta_{IK}\,\varsigma(\boldsymbol{\tau})\tilde f_K(\omega)\,d\boldsymbol{\tau}=\tilde f_I(\omega).
\end{aligned}
$$

In the last relation, we used the identity $\int_{\mathbb{B}^{n-1}}\varsigma(\boldsymbol{\tau})\,d\boldsymbol{\tau}=1$. □

By Proposition 3.3 and (3.33),

$$
\sum_{K=1}^N e^{-ik(\omega)\boldsymbol{\theta}\cdot X_K}\tilde f_K(\omega)\in L^2(W\times\mathbb{S}^{n-1}).
$$

Since $\mathscr{R}_I$ is continuous from $L^2(W\times\mathbb{S}^{n-1})$ to $L^2(W)$ by Lemma 3.11, Lemma 3.12 proves that $\tilde f_I\in L^2(W)$. This finishes the proof of Proposition 3.8. □

4 Spectral analysis of omega-limit trajectories

Let S_τ denote the time shift, $S_\tau f(t)=f(t+\tau)$. By (3.2) and Lemma 2.5, for any sequence $t_j\to+\infty$, there exists a subsequence t_{j_k}, $k\in\mathbb{N}$, such that, for any $T>0$,

$$
S_{t_{j_k}}(\psi,\dot\psi)\xrightarrow[k\to\infty]{C_b([-T,T],\mathscr{Y})}(\zeta,\dot\zeta), \tag{4.1}
$$

for some $\zeta\in C(\mathbb{R},H^1(\mathbb{R}^n))$ such that $\dot\zeta\in C(\mathbb{R},L^2(\mathbb{R}^n))$. The function $\zeta(x,t)$, which we call *omega-limit trajectory*, satisfies equation (2.1),

$$
\ddot\zeta(x,t)=\Delta\zeta(x,t)-m^2\zeta(x,t)+\sum_{I=1}^N\rho_I(x)F_I(\langle\rho_I,\zeta(\cdot,t)\rangle),\quad x\in\mathbb{R}^n,\quad t\in\mathbb{R}, \tag{4.2}
$$

which is understood in the sense of distributions, and obeys the following bound:

$$
\sup_{t\in\mathbb{R}}\|(\zeta,\dot\zeta)|_t\|_{\mathscr{X}}<\infty. \tag{4.3}
$$

By the arguments from [2], the proof of Theorem 2.13 will follow if we check that every omega-limit trajectory $\zeta(x,t)$ is one of the solitary waves, so that $\zeta(x,t)=\phi_{\omega_\star}(x)e^{-i\omega_\star t}$, with some $\omega_\star\in\mathbb{R}$.

For a particular omega-limit trajectory $\zeta(x,t)$ which appears in (4.1), we denote

$$
\zeta_I(t)=\langle\rho_I,\zeta(\cdot,t)\rangle;\qquad g_I(t)=F_I(\langle\rho_I,\zeta(\cdot,t)\rangle)=F_I(\zeta_I(t)). \tag{4.4}
$$

Due to Assumption 2.3, we have

$$
g_I(t)=-2u_I'(|\zeta_I(t)|^2)\zeta_I(t) \tag{4.5}
$$

(cf. Assumption 2.3). According to the convergence (4.1), for any $T>0$ and any $1\le I\le N$,

$$
f_I(t_{j_k}+t)=F_I(\langle\rho_I,\psi(\cdot,t_{j_k}+t)\rangle)\xrightarrow[k\to\infty]{C_b([-T,T])}F_I(\langle\rho_I,\zeta(\cdot,t)\rangle)=g_I(t). \tag{4.6}
$$

Due to the well-known local energy decay for the free Klein-Gordon equation (see Proposition A.1 in Appendix A), $\lim\limits_{t\to\infty}\|(\chi,\dot\chi)|_t\|_{\mathscr{Y}}=0$. Together with (4.1), this yields

$$S_{t_{j_k}}(\varphi,\dot\varphi)\ \xrightarrow[k\to\infty]{C_b([-T,T],\mathscr{Y})}\ (\zeta,\dot\zeta). \tag{4.7}$$

Lemma 4.1. *There is the following identity, understood in the sense of distributions:*

$$\tilde\zeta(x,\omega)=\sum_{I=1}^{N}\varSigma_I(x,\omega)\tilde g_I(\omega),\qquad x\in\mathbb{R}^n,\qquad \omega\in\mathbb{R}. \tag{4.8}$$

Proof. This follows from Lemma 3.2, (4.6), (4.7), and from $\varSigma_I(x,\omega)$ being a multiplier in $\mathscr{S}'(\mathbb{R})$ (see the proof of Lemma 3.2). □

Coupling (4.8) with ρ, we deduce that

$$\tilde\zeta_I(\omega)=\sum_{J=1}^{N}\sigma_{IJ}(\omega)\tilde g_J(\omega),\qquad \omega\in\mathbb{R}. \tag{4.9}$$

Lemma 4.2. *Let* $u\in\mathscr{S}'(\mathbb{R})$ *and let* $\{t_j\colon j\in\mathbb{N}\}$ *be such that* $\lim_{j\to\infty}t_j=\infty$. *If* $e^{i\omega t_j}u\xrightarrow{\mathscr{S}'}v\in\mathscr{S}'(\mathbb{R})$ *and* $u|_{\mathcal{I}}\in L^1_{loc}(\mathcal{I})$ *for some open set* $\mathcal{I}\subset\mathbb{R}$, *then* $v|_{\mathcal{I}}=0$.

Proof. This is an immediate consequence of the Riemann-Lebesgue lemma. □

Lemma 4.3. *For each* $1\le I\le N$, $\operatorname{supp}\tilde g_I\subset[-m,m]\cup Z_\rho$, *where* Z_ρ *is defined in* (2.20).

Proof. By (4.6) and the continuity of the Fourier transform in the space of tempered distributions,

$$\tilde f_I(\omega)e^{-i\omega t_{j_k}}\ \xrightarrow[k\to\infty]{\mathscr{S}'}\ \tilde g_I(\omega),\qquad 1\le I\le N.$$

By Proposition 3.8, $\tilde f_I(\omega)$ is locally L^2 for $\omega\in\mathbb{R}\backslash([-m,m]\cup Z_\rho)$. Therefore, Lemma 4.2 completes the proof. □

Proposition 4.4. *There exists* $\omega_\star\in Z_\rho\cup[-m,m]$ *such that* $\operatorname{supp}\tilde\zeta_I\subset\{\omega_\star\}$, $1\le I\le N$.

Proof. Denote

$$\omega^-=\min_{1\le I\le N}\inf\operatorname{supp}\tilde\zeta_I,\quad \omega^+=\max_{1\le I\le N}\sup\operatorname{supp}\tilde\zeta_I. \tag{4.10}$$

We claim that the assumption $\omega^-<\omega^+$ leads to a contradiction.

Lemma 4.5. *For each* $1\le I\le N$, $\quad\operatorname{supp}\tilde\zeta_I\subset[-m,m]\cup Z_\rho$, $\quad\omega^\pm\in[-m,m]\cup Z_\rho$.

Proof. This follows from Lemma 4.3, the relation (4.9), and the definition (4.10). □

Lemma 4.6. *For each* $1\le I\le N$,

$$\sup\operatorname{supp}\tilde\zeta_I+(p_I-1)(\sup\operatorname{supp}\tilde\zeta_I-\inf\operatorname{supp}\tilde\zeta_I)\le\omega^+,$$

$$\inf\operatorname{supp}\tilde\zeta_I-(p_I-1)(\sup\operatorname{supp}\tilde\zeta_I-\inf\operatorname{supp}\tilde\zeta_I)\ge\omega^-.$$

Remark 4.7. In particular, Lemma 4.6 states that if $\sup\operatorname{supp}\tilde\zeta_I=\omega^+$, then $\operatorname{supp}\tilde\zeta_I=\{\omega^+\}$; if $\inf\operatorname{supp}\tilde\zeta_I=\omega^-$, then $\operatorname{supp}\tilde\zeta_I=\{\omega^-\}$.

Proof. Let us assume that, on the contrary, there is I, $1\le I\le N$, such that

$$\sup\operatorname{supp}\tilde\zeta_I+(p_I-1)(\sup\operatorname{supp}\tilde\zeta_I-\inf\operatorname{supp}\tilde\zeta_I)>\omega^+. \tag{4.11}$$

After the Fourier transform in time, (4.5) takes the form

$$\tilde g_I=-2\widetilde{u_I'(|\zeta_I|^2)\zeta_I}=-2\sum_{q=1}^{p_I}q\,u_{I,q}\underbrace{(\tilde\zeta_I*\tilde{\bar\zeta}_I)*\dots*(\tilde\zeta_I*\tilde{\bar\zeta}_I)}_{q}*\tilde\zeta_I, \tag{4.12}$$

where the coefficients $u_{I,q} \in \mathbb{R}$ are from (2.2).

The Titchmarsh Convolution Theorem [4] states that for any compactly supported distributions $f,\ g \in \mathscr{E}'(\mathbb{R})$, one has

$$\inf \operatorname{supp} f + \inf \operatorname{supp} g = \inf \operatorname{supp} f * g, \qquad \sup \operatorname{supp} f + \sup \operatorname{supp} g = \sup \operatorname{supp} f * g. \tag{4.13}$$

In the case at hand, the Titchmarsh Convolution Theorem shows that the nonlinearity applied to a function with compact spectrum always expands this spectrum unless the spectrum consists of a single point. Applying the Titchmarsh Convolution Theorem (namely, we use the second equality from (4.13)) to (4.12), we get:

$$\begin{aligned} \sup \operatorname{supp} \tilde{g}_I &= \sup \operatorname{supp} \tilde{\zeta}_I + (p_I - 1)(\sup \operatorname{supp} \tilde{\zeta}_I + \sup \operatorname{supp} \tilde{\bar{\zeta}}_I) \\ &= \sup \operatorname{supp} \tilde{\zeta}_I + (p_I - 1)(\sup \operatorname{supp} \tilde{\zeta}_I - \inf \operatorname{supp} \tilde{\zeta}_I), \end{aligned} \tag{4.14}$$

where we used the mutual symmetry of the supports $\operatorname{supp} \tilde{\zeta}_I$ and $\operatorname{supp} \tilde{\bar{\zeta}}_I$ with respect to $\omega = 0$. By (4.11) and (4.14), $\sup \operatorname{supp} \tilde{g}_I > \omega^+$. Then the right-hand side of (4.14) is strictly greater than $\sup \operatorname{supp} \tilde{\zeta}_I$, hence there exists

$$\omega_* > \omega^+ := \max_{1 \le J \le N} \sup \operatorname{supp} \tilde{\zeta}_J \tag{4.15}$$

such that $\omega_* \in \operatorname{supp} \tilde{g}_I$. By Lemma 4.3 and (4.9),

$$\operatorname{supp} \tilde{\zeta}_I \subset Z_\rho \cup [-m, m], \qquad \operatorname{supp} \tilde{g}_I \subset Z_\rho \cup [-m, m].$$

By Assumption 2.12, $\det\limits_{1 \le I,J \le N} \sigma_{IJ}(\omega) \ne 0$ for $\omega \in \cup_I \operatorname{supp} \tilde{g}_I$, hence (4.9) implies that $\omega_* \in \cup_I \operatorname{supp} \tilde{\zeta}_I$, contradicting (4.15). Thus, our assumption (4.11) cannot be true. ◻

Lemma 4.8. *If I is such that $\omega^+ \in \operatorname{supp} \tilde{\zeta}_I$, then $\operatorname{supp} \tilde{\zeta}_I = \{\omega^+\}$. Similarly, if $\omega^- \in \operatorname{supp} \tilde{\zeta}_I$, then $\operatorname{supp} \tilde{\zeta}_I = \{\omega^-\}$.*

Proof. The statement of the lemma immediately follows from Lemma 4.6. ◻

Lemma 4.9. *If the point ω^+ belongs to the support of $\tilde{g}_I(\omega)$, then it is an isolated point of the support. The same conclusion takes place for ω^-.*

Proof. One can make the desired conclusion on the support of distributions $\tilde{g}_I$ using the relation (4.9), Lemma 4.8, and recalling that, by Assumption 2.12, $\det\limits_{1 \le I,J \le N} \sigma_{IJ}(\omega)$ vanishes only at a discrete set of points, which we denoted Z_σ^N. Let us give a detailed proof. By Lemma 4.8, there is an open neighborhood $\mathcal{O}$ of ω^+ such that $\mathcal{O} \cap \operatorname{supp} \tilde{\zeta}_I \subset \{\omega^+\}$, $1 \le I \le N$. We may assume that $\mathcal{O}$ is so small that it does not contain a single point from the discrete set Z_σ^N, except perhaps ω^+ (if it itself belongs to Z_σ^N). Let $\mathcal{O}'$ be an open neighborhood such that $\overline{\mathcal{O}'} \subset \mathcal{O} \backslash \{\omega^+\}$. Then, by the choice of $\mathcal{O}$ and $\mathcal{O}'$, $\tilde{\zeta}_I(\omega) = \sum_J \sigma_{IJ}(\omega) \tilde{g}_J(\omega) = 0$ in $\mathcal{O}'$ and $\det\limits_{1 \le I,J \le N} \sigma_{IJ}(\omega) \ne 0$ for $\omega \in \overline{\mathcal{O}'}$. Let $r_{IJ}(\omega)$ be the matrix inverse to $\sigma_{IJ}(\omega)$, $\omega \in \overline{\mathcal{O}'}$. For any test function $\varrho \in C_0^\infty(\mathcal{O}')$ and any $1 \le K \le N$, using the properties of multipliers in $\mathscr{S}'(\mathbb{R})$, one has:

$$0 = \sum_{I=1}^N \left\langle \varrho\, r_{KI}, \sum_J \sigma_{IJ} \tilde{g}_J \right\rangle = \sum_{I,J} \langle \varrho\, r_{KI} \sigma_{IJ}, \tilde{g}_J \rangle = \sum_J \langle \varrho\, \delta_{KJ}, \tilde{g}_J \rangle = \langle \varrho, \tilde{g}_K \rangle.$$

Due to the arbitrariness of the choice of ϱ, one concludes that $\tilde{g}_K|_{\mathcal{O}'} = 0$, $1 \le K \le N$. We are done. ◻

By Lemma 4.8 and Lemma 4.9, there exist disjoint open neighborhoods $\mathcal{O}^-$ and $\mathcal{O}^+$ of $\omega = \omega^-$ and $\omega = \omega^+$, respectively, so that

$$\mathcal{O}^\pm \cap \operatorname{supp} \tilde{\zeta}_I \subset \{\omega^\pm\}, \qquad \mathcal{O}^\pm \cap \operatorname{supp} \tilde{g}_I \subset \{\omega^\pm\}; \qquad 1 \le I \le N.$$

Let $\eta^\pm \in C_0^\infty(\mathbb{R})$ be such that $\operatorname{supp} \eta^\pm \subset \mathcal{O}^\pm$.

Lemma 4.10. *There exist $B_I^\pm, G_I^\pm \in \mathbb{C}$, $1 \le I \le N$, such that*

$$\eta^\pm(\omega) \tilde{\zeta}_I(\omega) = 2\pi B_I^\pm \delta(\omega - \omega^\pm), \qquad \eta^\pm(\omega) \tilde{g}_I(\omega) = 2\pi G_I^\pm \delta(\omega - \omega^\pm), \qquad 1 \le I \le N.$$

Proof. One uses the inclusions $\operatorname{supp} \eta^\pm \tilde{\zeta}_I \subset \{\omega^\pm\}$, $\operatorname{supp} \eta^\pm \tilde{g}_I \subset \{\omega^\pm\}$, and argues that the expressions for $\eta^\pm(\omega)\tilde{\zeta}_I(\omega)$ and $\eta^\pm(\omega)\tilde{g}_I(\omega)$ in terms of $\delta(\omega - \omega^\pm)$ and its derivatives cannot contain terms with $\delta^{(k)}(\omega - \omega^\pm)$, $k \ge 1$, due to the boundedness of $(\check{\eta}^\pm * \zeta_I)(t)$ and $(\check{\eta}^\pm * g_I)(t)$, where $\check{\eta}^\pm(t)$ is the inverse Fourier transform of $\eta^\pm$. This boundedness takes place in view of the definition (4.4) and the bound (4.3). ◻

Now let us finish the proof of Proposition 4.4. Introduce the sets

$$\mathcal{I}^- = \{I\colon \operatorname{supp}\tilde\zeta_I = \{\omega^-\}\} \subset \mathbb{N}, \quad \mathcal{I}^+ = \{I\colon \operatorname{supp}\tilde\zeta_I = \{\omega^+\}\} \subset \mathbb{N}. \tag{4.16}$$

Due to the assumption that $\omega^- < \omega^+$ and Lemma 4.8, one has $\mathcal{I}^- \cap \mathcal{I}^+ = \emptyset$. This implies that at least one of the sets $\mathcal{I}^-, \mathcal{I}^+$ contains no more than $N/2$ points:

$$\min(|\mathcal{I}^-|, |\mathcal{I}^+|) \le N/2. \tag{4.17}$$

Let us assume that $|\mathcal{I}^-| \le |\mathcal{I}^+|$ (the other case is treated similarly). Multiplying (4.9) by $\eta^-(\omega)$ (and factoring out $\delta(\omega-\omega^-)$), we obtain the following relations:

$$B_I^- = \sum_{J=1}^N \sigma_{IJ}(\omega^-) G_J^-. \tag{4.18}$$

For $I \in \mathcal{I}^+$, by Lemma 4.8, one has $\operatorname{supp}\tilde\zeta_I = \{\omega^+\}$; by (4.5), this leads to the inclusion $\operatorname{supp}\tilde g_I \subset \{\omega^+\}$, hence $B_I^- = G_I^- = 0$. Therefore, (4.18) yields

$$B_I^- = \sum_{J\in\{1,\,\dots,\,N\}\backslash\mathcal{I}^+} \sigma_{IJ}(\omega^-) G_J^-. \tag{4.19}$$

Since $|\mathcal{I}^-| \le |\mathcal{I}^+|$, and, by (4.17), $|\mathcal{I}^-| \le N/2$, there exists a set $\mathcal{I}_1 \subset \{1,\,\dots,\,N\}\backslash\mathcal{I}^-$ such that $|\mathcal{I}_1| = N - |\mathcal{I}^+|$. Therefore, considering the relations (4.19) with $I \in \mathcal{I}_1$ (when $B_I^- = 0$ due to $\mathcal{I}_1 \cap \mathcal{I}^- = \emptyset$), we conclude that

$$0 = B_I^- = \sum_{J\in\{1,\,\dots,\,N\}\backslash\mathcal{I}^+} \sigma_{IJ}(\omega^-) G_J^-, \qquad I \in \mathcal{I}_1. \tag{4.20}$$

We know from (4.18) that not all G_I^-, $I \in \{1,\,\dots,\,N\}\backslash\mathcal{I}^+$, are equal to zero; hence, (4.20) implies that

$$\det_{I\in\mathcal{I}_1, J\in\{1,\,\dots,\,N\}\backslash\mathcal{I}^+} \sigma_{IJ}(\omega^-) = 0. \tag{4.21}$$

Thus, according to Definition 2.11, one has $\omega^- \in \mathbb{Z}_\sigma^*$. On the other hand, by Lemma 4.5, $\omega^- \in [-m,m] \cap Z_\rho$, contradicting Assumption 2.12.

The case $|\mathcal{I}^-| > |\mathcal{I}^+|$ is treated similarly; in that case, the conclusion is that $\omega^+ \in Z_\sigma^*$, again leading to a contradiction with Assumption 2.12.

Thus, the assumption that $\omega^- < \omega^+$ cannot be true. It follows that there is $\omega_\star = \omega^- = \omega^+$ such that $\operatorname{Spec}\zeta_I \subset \{\omega_\star\}$ for $1 \le I \le N$, finishing the proof of Proposition 4.4. □

Due to Proposition 4.4, $\tilde\zeta_I(\omega)$ are finite linear combinations of $\delta(\omega-\omega_\star)$ and its derivatives. As the matter of fact, the derivatives could not be present because of the boundedness of $\zeta_I(t) := \langle\rho_I, \zeta(\cdot,t)\rangle$ which follows from (4.3). Therefore, $\tilde\zeta_I = 2\pi C_I \delta(\omega-\omega_\star)$, with some $C_I \in \mathbb{C}$. This implies that

$$\zeta_I(t) = C_I e^{-i\omega_\star t}, \qquad C_I \in \mathbb{C}, \qquad t \in \mathbb{R}. \tag{4.22}$$

It follows that $g_I(t) = F(C_I)e^{-i\omega_\star t}$, $\tilde g_I(\omega) = 2\pi F(C_I)\delta(\omega-\omega_\star)$. By Lemma 4.1,

$$\tilde\zeta(x,\omega) = 2\pi\delta(\omega-\omega_\star)\sum_{I=1}^N \varSigma_I(x,\omega_\star)F(C_I),$$

hence $\zeta(x,t) = \phi(x)e^{-i\omega_\star t}$ with $\phi(x) = \sum_{I=1}^N \varSigma_I(x,\omega_\star)F(C_I)$. Therefore, equation (4.2) and the bound (4.3) imply that $\zeta(x,t)$ is a solitary wave. By the arguments from [2], this completes the proof of Theorem 2.13.

A Local energy decay

Proposition A.1 (Local energy decay). *Let $n\in\mathbb{N}$, $m>0$. If χ solves*

$$\ddot\chi=\Delta\chi-m^2\chi,\qquad x\in\mathbb{R}^n,\quad (\chi,\dot\chi)|_{t=0}=(\psi_0,\pi_0)\in H^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n),$$

then, for any $\sigma>0$, $\lim\limits_{t\to\infty}\big(\|\chi(t)\|_{H^1_{-\sigma}}+\|\dot\chi(t))\|_{L^2_{-\sigma}}\big)=0$.

Proof. For the Fourier transform of $\chi(x,t)$ in x, we have:

$$\hat\chi(\xi,t)=\hat\psi_0(\xi)\cos(\omega(\xi)t)+\hat\pi_0(\xi)\frac{\sin(\omega(\xi)t)}{\omega(\xi)},\qquad \omega(\xi)=\sqrt{m^2+\xi^2}.$$

We will only prove that $\lim\limits_{t\to\infty}\|\chi(t)\|_{H^1_{-\sigma}}=0$; the limit $\lim\limits_{t\to\infty}\|\dot\chi(t)\|_{L^2_{-\sigma}}=0$ is computed similarly.

Pick $\epsilon>0$. We split the initial data ψ_0 and π_0 into $\psi_0=u_1+u_2$, $\pi_0=v_1+v_2$, so that

$$\|u_1\|_{H^1}+\|v_1\|_{L^2}<\epsilon/3 \tag{A.1}$$

and

$$\hat u_2,\ \hat v_2\in\mathscr{S}(\mathbb{R}^n),\quad \operatorname{supp}\hat u_2\cup\operatorname{supp}\hat v_2\subset\{\xi\in\mathbb{R}^n\colon|\xi|\ge\lambda\}, \tag{A.2}$$

for some $\lambda>0$. Let χ_1 and χ_2 be the solutions to the linear Klein-Gordon equation with the initial data

$$(\chi_1,\dot\chi_1)|_{t=0}=(u_1,v_1),\qquad (\chi_2,\dot\chi_2)|_{t=0}=(u_2,v_2).$$

Due to (A.1) and the energy conservation, $\|\chi_1(t)\|_{H^1}\le\epsilon/3$ for $t\in\mathbb{R}$.

Let $\rho_1\in C_0^\infty(\mathbb{B}^n_2)$, $\rho_1|_{\mathbb{B}^n_1}\equiv 1$. For $R\ge1$, denote $\rho(x)=\rho_1(x/R)$. Since $\|\chi_2(t)\|_{H^1}$ remains uniformly bounded, while $\|\langle x\rangle(1-\rho(x))\|_{C^1_b(\mathbb{R}^n)}\to0$ as $R\to\infty$, one can choose $R\ge1$ large enough so that

$$\|(1-\rho(\cdot))\chi_2(\cdot,t)\|_{H^1_{-\sigma}}\le\epsilon/3,\qquad t\ge0. \tag{A.3}$$

It suffices to show that

$$\lim_{t\to\infty}\|\rho(\cdot)\chi_2(\cdot,t)\|_{H^1_{-\sigma}}=0. \tag{A.4}$$

We have:

$$\|\rho\chi_2(\cdot,t)\|^2_{L^2}\le\|\rho\|_{L^2}\|\chi_2(\cdot,t)\|_{L^2}\|\rho\chi_2(\cdot,t)\|_{L^\infty}. \tag{A.5}$$

The first two factors in the right-hand side of (A.5) are bounded uniformly in time. For the last factor in the right-hand side of (A.5), we have:

$$\|\rho(\cdot)\chi_2(\cdot,t)\|_{L^\infty}\le\Big\|\hat\rho*\Big(\hat u_2(\cdot)\cos(\omega(\cdot)t)+\hat v_2(\cdot)\frac{\sin(\omega(\cdot)t)}{\omega(\cdot)}\Big)\Big\|_{L^1}. \tag{A.6}$$

Lemma A.2. *Let $f,\,g\in\mathscr{S}(\mathbb{R}^n)$, and $0\notin\operatorname{supp}g$. Then, for any $N\in\mathbb{N}$, there is $C_N>0$ so that*

$$\|f*\big(g(\cdot)e^{i\omega(\cdot)t}\big)\|_{L^1}\le C_N(1+|t|)^{-N},\qquad t\in\mathbb{R}.$$

Proof. Since $0\notin\operatorname{supp}g$, $|\nabla_\eta\omega(\eta)|$ is bounded away from zero on the support of g. Therefore, the expression

$$\|f*\big(g(\cdot)e^{i\omega(\cdot)t}\big)\|_{L^1}=\int\Big|\int f(\xi-\eta)g(\eta)e^{i\omega(\eta)t}\,d\eta\Big|\,d\xi \tag{A.7}$$

decays faster than any negative power of t due to the stationary phase method. Namely, one can place the operator $L=\frac{1}{i|\nabla\omega(\eta)|^2t}\nabla_\eta\omega\cdot\nabla_\eta$ in front of the exponential factor $e^{i\omega(\eta)t}$ under the inner integral in the right-hand side of (A.7), and then integrate by parts in η. This gives a factor of t^{-1}. The procedure could be repeated arbitrarily many times. □

By (A.2), $\hat u_2$ and $\hat v_2$ vanish in the vicinity of $\xi=0$, thus we can apply Lemma A.2 to the right-hand side of (A.6), getting $\lim\limits_{t\to\infty}\|\rho\chi_2(\cdot,t)\|_{L^\infty}=0$. Now (A.5) yields

$$\lim_{t\to\infty}\|\rho\chi_2\|^2_{L^2}=0. \tag{A.8}$$

Similarly, one proves that

$$\lim_{t\to\infty} \|\nabla_x(\rho\chi_2(\cdot,t))\|^2_{L^2} = 0. \tag{A.9}$$

Indeed, we notice that each of the terms in the right-hand side of (A.5) could accommodate a derivative in x: $\|\nabla\rho\|_{L^2}$ is bounded, $\|\nabla\chi_2(\cdot,t)\|_{L^2}$ is bounded uniformly in time, while $\|\nabla(\rho\chi_2(\cdot,t))\|_{L^\infty}$ is sent to zero by the stationary phase method of Lemma A.2. By (A.8) and (A.9), $\lim\limits_{t\to\infty}\|\rho(\cdot)\chi_2(\cdot,t)\|_{H^1} = 0$; (A.4) follows, finishing the proof. □

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References

[1] Alexander I. Komech and Andrew A. Komech, *Global attractor for a nonlinear oscillator coupled to the Klein-Gordon field*, Arch. Ration. Mech. Anal. **185** (2007), 105–142.

[2] ______, *Global attraction to solitary waves for Klein-Gordon equation with mean field interaction*, Ann. Inst. H. Poincaré Anal. Non Linéaire **26** (2008), no. 2, 855–868.

[3] ______, *Global attractor for the Klein-Gordon field coupled to several nonlinear oscillators*, J. Math. Pures Appl. **93** (2010), 91–111.

[4] E.C. Titchmarsh, *The zeros of certain integral functions*, Proc. of the London Math. Soc. **25** (1926), 283–302.