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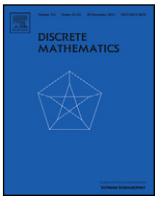

On sizes of complete arcs in $PG(2, q)$

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ABSTRACT

New upper bounds on the smallest size $t_2(2, q)$ of a complete arc in the projective plane $PG(2, q)$ are obtained for $853 \leq q \leq 5107$ and $q \in T_1 \cup T_2$, where $T_1 = \{173, 181, 193, 229, 243, 257, 271, 277, 293, 343, 373, 409, 443, 449, 457, 461, 463, 467, 479, 487, 491, 499, 529, 563, 569, 571, 577, 587, 593, 599, 601, 607, 613, 617, 619, 631, 641, 661, 673, 677, 683, 691, 709\}$, and $T_2 = \{5119, 5147, 5153, 5209, 5231, 5237, 5261, 5279, 5281, 5303, 5347, 5641, 5843, 6011, 8192\}$. From these new bounds it follows that for $q \leq 2593$ and $q = 2693, 2753$, the relation $t_2(2, q) < 4.5\sqrt{q}$ holds. Also, for $q \leq 5107$ we have $t_2(2, q) < 4.79\sqrt{q}$. It is shown that for $23 \leq q \leq 5107$ and $q \in T_2 \cup \{2^{14}, 2^{15}, 2^{18}\}$, the inequality $t_2(2, q) < \sqrt{q}\ln^{0.75} q$ is true. Moreover, the results obtained allow us to conjecture that this estimate holds for all $q \geq 23$. The new upper bounds are obtained by finding new small complete arcs with the help of a computer search using randomized greedy algorithms. Also new constructions of complete arcs are proposed. These constructions form families of k-arcs in $PG(2, q)$ containing arcs of all sizes k in a region $k_{\min} \leq k \leq k_{\max}$, where $k_{\min}$ is of order $\frac{1}{3}q$ or $\frac{1}{4}q$ while $k_{\max}$ has order $\frac{1}{2}q$. The completeness of the arcs obtained by the new constructions is proved for $q \leq 2063$. There is reason to suppose that the arcs are complete for all $q > 2063$. New sizes of complete arcs in $PG(2, q)$ are presented for $169 \leq q \leq 349$ and $q = 1013, 2003$.

1. Introduction

Let $PG(2, q)$ be the projective plane over the Galois field F_q. A k-arc is a set of k points no three of which are collinear. A k-arc is called complete if it is not contained in a $(k + 1)$-arc of $PG(2, q)$. For an introduction in projective geometries over finite fields, see [26,47,49].

In [27,28] the close relationship between the theory of k-arcs, coding theory and mathematical statistics is presented. In particular, a complete arc in a plane $PG(2, q)$, points of which are treated as 3-dimensional q-ary columns, defines a parity check matrix of a q-ary linear code with codimension 3, Hamming distance 4, and covering radius 2. Arcs can be interpreted as linear maximum distance separable (MDS) codes [53,54] and they are related to optimal coverings arrays [24] and to superregular matrices [29].

One of the main problems in the study of projective planes, which is also of interest in coding theory, is finding the spectrum of possible sizes of complete arcs.

A great part of this work is devoted to upper bounds on $t_2(2, q)$, the smallest size of a complete arc in $PG(2, q)$. Also we propose new constructions of complete arcs in $PG(2, q)$ and consider the spectrum of their possible sizes.

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Surveys of results on the sizes of plane complete arcs, methods of their construction and comprehension of the relating properties can be found in [3–5,16,25–28,36,42–45,47–53]. In particular, as it is noted in [27,28], the following idea of Segre [48] and Lombardo-Radice [36] is fruitful: the points of the arc are chosen, with some exceptions, among the points of a conic or a cubic curve. We use this idea for constructions of complete arcs and for finding the spectrum of arc sizes; see Sections 2, 5 and 6.

The maximum size $m_2(2, q)$ of a complete arc in $PG(2, q)$ is well known. It holds that

$$m_2(2,q) = \begin{cases} q+1 & \text{if } q \text{ odd} \\ q+2 & \text{if } q \text{ even.} \end{cases}$$

On the other hand, finding an estimation of the minimum size $t_2(2, q)$ is a hard open problem.

Problems connected with small complete plane arcs are considered in [3–5,8,9,14,17,19–23,26,30,35,37–39,46–48,50–53]; see also the references therein.

We denote the aggregates of q values:

$$\begin{aligned} T_1 = \{&173, 181, 193, 229, 243, 257, 271, 277, 293, 343, 373, 409, 443, 449, 457, 461, 463,\\ &467, 479, 487, 491, 499, 529, 563, 569, 571, 577, 587, 593, 599, 601, 607, 613, 617,\\ &619, 631, 641, 661, 673, 677, 683, 691, 709\}; \end{aligned}$$

$$T_2 = \{5119, 5147, 5153, 5209, 5231, 5237, 5261, 5279, 5281, 5303, 5347, 5641, 5843, 6011, 8192\};$$

$$T_3 = \{2^{14}, 2^{15}, 2^{18}\};$$

$$Q = \{961, 1024, 1369, 1681, 2401\} = \{31^2, 2^{10}, 37^2, 41^2, 7^4\};$$

$$N = \{601, 5^4, 661, 3^6, 29^2, 31^2, 2^{10}, 37^2, 41^2, 7^4, 2^{18}\}.$$

The exact values of $t_2(2, q)$ are known only for $q \le 32$, see [37] and the recent work [38], where the equalities $t_2(2, 31) = t_2(2, 32) = 14$ are proved. Also, there are the following lower bounds (see [46,47]):

$$t_2(2,q) > \begin{cases} \sqrt{2q}+1 & \text{for any } q \\ \sqrt{3q}+\frac{1}{2} & \text{for } q = p^h, \quad p \text{ prime}, \; h = 1, 2, 3. \end{cases}$$

Let $\bar{t}_2(2, q)$ be the smallest *known* size of a complete arc in $PG(2, q)$. For $q \le 841$ the values of $\bar{t}_2(2, q)$ (up to June 2009) are collected in [5, Tab. 1] whence it follows that $\bar{t}_2(2, q) < 4\sqrt{q}$ for $q \le 841$. In [19], see also [23], complete $(4\sqrt{q}-4)$-arcs are obtained for $q = p^2$ odd, $q \le 1681$ or $q = 2401$. In [9,14] complete $(4\sqrt{q} - 4)$-arcs are obtained for even $q = 64, 256, 1024$. By the results cited above, it holds that

$$t_2(2,q) < 4\sqrt{q} \quad \text{for } 2 \le q \le 841, \; q \in Q. \tag{1.1}$$

For even $q = 2^h$, $11 \le h \le 15$, the smallest known sizes of complete n-arcs in $PG(2, q)$ are obtained in [9]; see also [5, p. 35]. They are as follows: $\bar{t}_2(2, 2^{11}) = 201$, $\bar{t}_2(2, 2^{12}) = 307$, $\bar{t}_2(2, 2^{13}) = 461$, $\bar{t}_2(2, 2^{14}) = 665$, $\bar{t}_2(2, 2^{15}) = 993$. Also, $(6\sqrt{q} - 6)$-arcs in $PG(2, q)$, $q = 4^{2h+1}$, are constructed in [8]; for $h \le 4$ it is proved that they are complete. It gives a complete 3066-arc in $PG(2, 2^{18})$.

Let $t(\mathcal{P}_q)$ be the size of the smallest complete arc in any (not necessarily Galois) projective plane $\mathcal{P}_q$ of order q. In [30], for *sufficiently large* q, the following result is proved (we give it in the form of [28, Tab. 2.6]):

$$t(\mathcal{P}_q) \le d\sqrt{q}\log^c q, \quad c \le 300, \tag{1.2}$$

where c and d are constants independent of q (i.e. universal constants). The logarithm basis is not noted as the estimate is asymptotic.

In this work, by computer search using randomized greedy algorithms (see Section 2), new small complete arcs in $PG(2, q)$ are obtained for all $q \in T_1 \cup T_2$ and for $853 \le q \le 5107$. For $q = 601, 661$, the new complete arcs arose from a theoretical study on orbits of subgroups, helped by computer [22]. From the sizes of the new arcs, with the use of (1.1) and [5, Tab. 1], [8], the following theorems result (see also Theorems 3.2, 3.3 and 4.1 for more details).

Theorem 1.1. *In $PG(2, q)$, the following holds.*

$$t_2(2,q) < 4.5\sqrt{q} \quad \textit{for } q \le 2593, \; q = 2693, 2753.$$

$$t_2(2,q) < 4.79\sqrt{q} \quad \textit{for } q \le 5107.$$

$$t_2(2,q) < 4.98\sqrt{q} \quad \textit{for } q \in T_2.$$

Theorem 1.2. *In $PG(2, q)$,*

$$t_2(2,q) < \sqrt{q}\ln^{0.75} q \quad \textit{for } 23 \le q \le 5107, \; q \in T_2 \cup T_3. \tag{1.3}$$

Moreover, the study of the values of $\bar{t}_2(2, q)$ allows us to conjecture that the estimate (1.3) holds for all $q \ge 23$ and the last estimate of Theorem 1.1 is right for all $q \le 8192$.

Conjecture 1.3. *In* $PG(2,q)$,

$$\begin{aligned} & t_2(2,q) < \sqrt{q}\ln^{0.75} q \quad \textit{for } q \ge 23. \\ & t_2(2,q) < 5\sqrt{q} \quad \textit{for } q \le 8192. \end{aligned} \tag{1.4}$$

Regarding the spectrum of complete arc sizes, we note (going after [28, p. 209]) that in the literature, complete arcs have been constructed with sizes approximately $\frac{1}{2}q$ (see [2,5,13,18,26,33,34,36,42,44,45,49]), $\frac{1}{3}q$ (see [1,31,50,51,55]), $\frac{1}{4}q$ (see [31,52]), $2q^{0.9}$ (see [50] where such arcs are constructed for $q > 7^{10}$). In particular, for even $q \ge 8$ there is a complete $\frac{1}{2}(q+4)$-arc [26]. Important results on the spectrum of complete arc sizes are collected in [28, Th. 2.6] where it is noted, for example, that in $PG(2,q)$ with q square there exists a complete $(q-\sqrt{q}+1)$-arc. In [32], *large complete arcs* in $PG(2,q^n)$ are defined and new infinite families of such arcs are constructed.

Much attention is given to $\frac{1}{2}(q+5)$-arcs and $\frac{1}{2}(q+7)$-arcs sharing $\frac{1}{2}(q+3)$ points with a conic for q odd [2,4,5], [12, Rem. 3], [13,18,33,34,44]. It is proved that for all odd q there is a complete $\frac{1}{2}(q+5)$-arc [34]; see also [2]. Also, a complete $\frac{1}{2}(q+7)$-arc exists at least for the following *odd q*:

$$\begin{aligned} & 25 \le q \le 167 \text{[3, Tab. 2], [5, Sec. 2, Tab. 2], [16, Tab. 2.4], [34, Introduction];} \\ & q \equiv 2 \pmod 3,\ q \le 4523 \text{[5,13];} \\ & q \equiv 1 \pmod 4,\ q \le 337 \text{[18];} \\ & q = 2bt-1,\ t \text{ odd prime},\ b = 1, 2 \text{[34, Introduction];} \\ & q \equiv 3 \pmod 4 \text{and condition of [34, Cor. 4.16(i)] holds, see [34, Introduction];} \\ & q^2 \equiv 1 \pmod{16} \text{and condition of [34, Cor. 4.17] holds, see [34, Introduction].} \end{aligned} \tag{1.5}$$

For $q \le 167$ the known sizes of complete arcs in $PG(2,q)$ are collected in [3, Tab. 2], [5, Tab. 2], [16, Tab. 2.4].

In this work new Constructions A–C of complete arcs are proposed; see Section 5. These constructions form families of complete k-arcs in $PG(2,q)$ containing arcs of all sizes k in a region $k_{\min} \le k \le k_{\max}$ where $k_{\min}$ is of order $\frac{1}{3}q$ or $\frac{1}{4}q$ while $k_{\max}$ has order $\frac{1}{2}q$. The completeness of the arcs obtained by the new constructions is proved for $q \le 2063$. Moreover, there is reason to suppose that the arcs are complete for all $q > 2063$. From Theorems 5.8, 5.18 and 5.25 the following theorem results.

Theorem 1.4. *Constructions A–C of Section 5 form families of complete k-arcs in* $PG(2,q)$ *containing arcs of all sizes k in the following regions:*

$$\text{(i) Construction A}: \left\lfloor \frac{q+8}{3} \right\rfloor \le k \le \frac{q+5}{2},\ q \text{ prime},$$

$$109 \le q \le 2063,\ \ q = 73, 97, 101, 103.$$

$$\text{(ii) Construction B}: \left\lfloor \frac{q+8}{3} \right\rfloor \le k \le \left\lfloor \frac{q+4}{2} \right\rfloor,\ \ q \not\equiv 3 \pmod 4\ \ \textit{is a} \text{ prime power},$$

$$128 \le q \le 2063,\ \ q = 89, 109, 113, 121.$$

$$\text{(iii) Construction C}: \frac{q+13}{4} \le k \le \frac{q+5}{2},\ \ q \equiv 3 \pmod 4\ \textit{is a} \text{ prime power},$$

$$347 \le q \le 2063,$$

$$q = 199, 227, 239, 243, 251, 263, 271, 283, 307, 311, 331.$$

For the given q, in order to show that arcs obtained by a construction are complete, we should calculate by computer some special value, say $\overline{L}_q$ (see Definitions 5.4, 5.14 and 5.21) and check if $\overline{L}_q \le R_q$, where $R_q = \lfloor \frac{1}{3}(q-1) \rfloor$ for Constructions A and B and $R_q = \frac{1}{4}(q-3)$ for Construction C. The calculations are relatively simple. Moreover, for $q \le 2063$ it holds that $\overline{L}_q < \sqrt{q}\ln q$ and the difference $R_q - \overline{L}_q$ has a tendency to increase when q grows; see Theorems 5.7(iii), (iv), 5.17(iii), (iv), 5.24(iii), (iv). It allows us to conjecture the following; cf. Conjecture 5.26.

Conjecture 1.5. *The assertions of Theorem 1.4 hold also for all* $q > 2063$.

A k-arc of Constructions A and B contains $k-2$ points in common with a conic and two points lying on a tangent to the conic (Construction A) or on a bisecant of the conic (Construction B). A k-arc of Construction C contains $k-3$ points in common with a conic, two points lying on a bisecant and one point on a tangent. Note that in [1], k-arcs containing $k-2$ points of a hyperoval (the nucleus among them) and two points on its bisecant are constructed. Also, in the space $PG(3,q)$, k-caps with $k-2$ points in common with a quadric are considered in [15,40,41]. In [15,40], the cap contains two points on a tangent to the quadric while in [41] two points lie on an external line.

The complete arcs of Constructions A–C can be used as starting objects for the inductive constructions of [9,32]; see Remark 5.28. In that way, using results of [9] together with Constructions A–C, one can generate infinite sets of families of complete caps in projective spaces $PG(v, 2^n)$ of growing dimensions v. Also, infinite families of large complete arcs in $PG(2,q^n)$ with growing n can be obtained by constructions of [32] using arcs of Constructions A–C.

In this work, using Constructions A–C and randomized greedy algorithms, new complete arcs in $PG(2, q)$ are obtained for $169 \le q \le 349$ and $q = 1013, 2003$.

In Section 2 we describe the greedy algorithms used for obtaining new arcs. In Section 3 we collect the known and new upper bounds on $t_2(2, q)$ for $q \le 5107$ and $q \in T_2 \cup T_3$. The bounds are represented by tables, where values of $\bar{t}_2(2, q)$ are written, and by the corresponding relations. In Section 4 we give the upper bounds on $t_2(2, q)$ in the form of (1.3) and substantiate Conjecture 1.3. In Section 5 new Constructions A–C of complete arcs are described. Finally, in Section 6 we present new sizes of complete arcs in $PG(2, q)$ with $169 \le q \le 349$ and $q = 1013, 2003$.

Some of the results of this work were briefly presented without proofs in [6]; see also [7].

2. An approach to computer search

In this paper for computer search we use the randomized greedy algorithms [3, Sec. 2], [11, Sec. 2] that are convenient for relatively large q and for obtaining examples of different sizes of complete arcs. At every step an algorithm minimizes or maximizes an objective function f but some steps are executed in a random manner. The number of these steps and their ordinal numbers have been taken intuitively. Also, if the same extremum of f can be obtained in distinct ways, one way is chosen randomly.

We begin to construct a complete arc by using a starting set of points S_0. At the i-th step one point is added to the set and we obtain a point set S_i. As the value of the objective function f we consider the number of points in $PG(2, q)$ that lie on bisecants of the set obtained. For small arcs we look for the maximum of the objective function f. For the spectrum of arc sizes we use both the maximum and the minimum of f.

On every "random" steps we take d_q of randomly chosen uncovered points of $PG(2, q)$ and compute the objective function f adding each of these d_q points to S_i. The point providing the extremum is included into S_i. The value of d_q is given intuitively depending upon q, upon the number of chosen points (i.e. $|S_{i-1}|$), and upon the current task (small arcs or the spectrum of arc sizes). For example, one can put $d_q = 1$ for finding the spectrum and $d_q = 100\beta$ with $\beta = 1, 2, \dots$ for small arcs.

We can use S_0 as a subset of points of an arc obtained in previous stages of the search. Also, for finding the spectrum of arc sizes it is fruitful to take S_0 as a part of points of a conic. A generator of random numbers is used for a random choice. To get arcs with distinct sizes, starting conditions of the generator are changed for the same set S_0. In this way the algorithm works in a convenient limited region of the search space to obtain examples improving the size of the arc from which the fixed points have been taken.

In order to obtain arcs with new sizes one should make sufficiently many attempts with the randomized greedy algorithms. For small arcs, the so called predicted sizes considered in Section 4 are useful for understanding if a good result has been obtained. If the result is not close to the predicted size, the attempts should be continued.

Note also that arcs with sizes close to $\bar{t}_2(2, q)$ usually are obtained as a byproduct when we execute the computer search for the smallest arcs using a few attempts.

3. Small complete k-arcs in $PG(2, q)$, $q \le 5107$, $q \in T_2$

Throughout the paper, in all tables we denote $A_q = \lfloor a_q\sqrt{q} - \bar{t}_2(2, q)\rfloor$, where

$$a_q = \begin{cases} 4 & \text{if } q \le 841,\ q \in Q \\ 4.5 & \text{if } 853 \le q \le 2593,\ \ q = 2693, 2753,\ \ q \notin Q \\ 5 & \text{if } 2609 \le q \le 8192,\ \ q \notin \{2693, 2753\}. \end{cases}$$

Also, in all tables, B_q is a superior approximation of $\bar{t}_2(2, q)/\sqrt{q}$.

For $q \le 841$, the values of $\bar{t}_2(2, q)$ (up to June 2009) are collected in [5, Tab. 1]. In this work we obtained small arcs with new sizes for $q \in T_1$. The new arcs are obtained by computer search, based on the randomized greedy algorithms. Complete 90-arcs for $q = 601, 661$ came from a theoretical study on orbits of subgroups, helped by computer; see [22]. These arcs are announced also in [10, Tab. 1]. A complete 104-arc for $q = 709$ is obtained by the greedy algorithm with the starting point set taking from [22]. The current values of $\bar{t}_2(2, q)$ for $q \le 841$ are given in Table 1. The data for $q \in T_1$ improving results of [5, Tab. 1] are written in Table 1 in bold font. The exact values $\bar{t}_2(2, q) = t_2(2, q)$ are marked by the dot "**.**". In particular, due to the recent result [38] we noted the values $t_2(2, 31) = t_2(2, 32) = 14$.

From Table 1 and the results of [19,23], on complete $(4\sqrt{q} - 4)$-arcs for $q = p^2$ (see Introduction) we obtain Theorem 3.1 improving and extending the results of [5, Th. 1].

Theorem 3.1. *In $PG(2, q)$, the following holds.*

$$\begin{aligned} & t_2(2, q) < 4\sqrt{q} \quad \text{for } 2 \le q \le 841,\ q \in Q. \\ & t_2(2, q) \le 3\sqrt{q} \quad \text{for } 2 \le q \le 89,\ q = 101; \\ & t_2(2, q) < 3.5\sqrt{q} \quad \text{for } 2 \le q \le 277; \\ & t_2(2, q) < 3.8\sqrt{q} \quad \text{for } 2 \le q \le 509,\ q = 521, 523, 529, 601, 661. \end{aligned} \tag{3.1}$$

Table 1
The smallest known sizes $\bar{t}_2 = \bar{t}_2(2,q) < 4\sqrt{q}$ of complete arcs in planes $PG(2,q)$, $q \le 841$. $A_q = \lfloor 4\sqrt{q} - \bar{t}_2(2,q) \rfloor$, $B_q \ge \bar{t}_2(2,q)/\sqrt{q}$.

q	$\bar{t}_2$	A_q	B_q	q	$\bar{t}_2$	A_q	B_q	q	$\bar{t}_2$	A_q	B_q	q	$\bar{t}_2$	A_q	B_q
2	4.	1	2.83	128	34	11	3.01	347	67	7	3.60	**599**	**94**	**3**	**3.85**
3	4.	2	2.31	131	36	9	3.15	349	67	7	3.59	**601**	**90**	**8**	**3.68**
4	6.	2	3.00	137	37	9	3.17	353	68	7	3.62	**607**	**95**	**3**	**3.86**
5	6.	2	2.69	139	37	10	3.14	359	69	6	3.65	**613**	**96**	**3**	**3.88**
7	6.	4	2.27	149	39	9	3.20	361	69	7	3.64	**617**	**96**	**3**	**3.87**
8	6.	5	2.13	151	39	10	3.18	367	70	6	3.66	**619**	**96**	**3**	**3.86**
9	6.	6	2.00	157	40	10	3.20	**373**	**70**	**7**	**3.63**	625	96	4	3.84
11	7.	6	2.12	163	41	10	3.22	379	71	6	3.65	**631**	**97**	**3**	**3.87**
13	8.	6	2.22	167	42	9	3.26	383	71	7	3.63	**641**	**98**	**3**	**3.88**
16	9.	7	2.25	169	42	10	3.24	389	72	6	3.66	643	99	2	3.91
17	10.	6	2.43	**173**	**43**	**9**	**3.27**	397	73	6	3.67	647	99	2	3.90
19	10.	7	2.30	179	44	9	3.29	401	74	6	3.70	653	100	2	3.92
23	10.	9	2.09	**181**	**44**	**9**	**3.28**	**409**	**74**	**6**	**3.66**	659	100	2	3.90
25	12.	8	2.40	191	46	9	3.33	419	76	5	3.72	**661**	**90**	**12**	**3.51**
27	12.	8	2.31	**193**	**46**	**9**	**3.32**	421	76	6	3.71	**673**	**101**	**2**	**3.90**
29	13.	8	2.42	197	47	9	3.35	431	77	6	3.71	**677**	**102**	**2**	**3.93**
31	**14.**	8	2.52	199	47	9	3.34	433	77	6	3.71	**683**	**102**	**2**	**3.91**
32	**14.**	8	2.48	211	49	9	3.38	439	78	5	3.73	**691**	**103**	**2**	**3.92**
37	15	9	2.47	223	51	8	3.42	**443**	**78**	**6**	**3.71**	701	104	1	3.93
41	16	9	2.50	227	51	9	3.39	**449**	**79**	**5**	**3.73**	**709**	**104**	**2**	**3.91**
43	16	10	2.45	**229**	**51**	**9**	**3.38**	**457**	**80**	**5**	**3.75**	719	106	1	3.96
47	18	9	2.63	233	52	9	3.41	**461**	**80**	**5**	**3.73**	727	106	1	3.94
49	18	10	2.58	239	53	8	3.43	**463**	**80**	**6**	**3.72**	729	104	4	3.86
53	18	11	2.48	241	53	9	3.42	**467**	**81**	**5**	**3.75**	733	107	1	3.96
59	20	10	2.61	**243**	**53**	**9**	**3.40**	**479**	**82**	**5**	**3.75**	739	107	1	3.94
61	20	11	2.57	251	55	8	3.48	**487**	**83**	**5**	**3.77**	743	108	1	3.97
64	22	10	2.75	256	55	9	3.44	**491**	**83**	**5**	**3.75**	751	108	1	3.95
67	23	9	2.81	**257**	**55**	**9**	**3.44**	**499**	**84**	**5**	**3.77**	757	109	1	3.97
71	22	11	2.62	263	56	8	3.46	503	85	4	3.79	761	109	1	3.96
73	24	10	2.81	269	57	8	3.48	509	85	5	3.77	769	110	0	3.97
79	26	9	2.93	**271**	**57**	**8**	**3.47**	512	86	4	3.81	773	111	0	4.00
81	26	10	2.89	**277**	**58**	**8**	**3.49**	521	86	5	3.77	787	112	0	4.00
83	27	9	2.97	281	59	8	3.52	523	86	5	3.77	797	112	0	3.97
89	28	9	2.97	283	59	8	3.51	**529**	**87**	**5**	**3.79**	809	113	0	3.98
97	30	9	3.05	289	60	8	3.53	541	89	4	3.83	811	113	0	3.97
101	30	10	2.99	**293**	**60**	**8**	**3.51**	547	89	4	3.81	821	114	0	3.98
103	31	9	3.06	307	62	8	3.54	557	90	4	3.82	823	114	0	3.98
107	32	9	3.10	311	63	7	3.58	**563**	**91**	**3**	**3.84**	827	115	0	4.00
109	32	9	3.07	313	63	7	3.57	**569**	**91**	**4**	**3.82**	829	115	0	4.00
113	33	9	3.11	317	63	8	3.54	**571**	**92**	**3**	**3.86**	839	115	0	3.98
121	34	10	3.10	331	65	7	3.58	**577**	**92**	**4**	**3.84**	841	112	4	3.87
125	35	9	3.14	337	66	7	3.60	**587**	**93**	**3**	**3.84**				
127	35	10	3.11	**343**	**66**	**8**	**3.57**	**593**	**94**	**3**	**3.87**				

Also,

$$
\begin{aligned}
&t_2(2,q) \le 4\sqrt{q} - 9 \quad \textit{for } 37 \le q \le 211,\ q = 23, 227, 229, 233, 241, 243, 256, 257, 661;\\
&t_2(2,q) \le 4\sqrt{q} - 8 \quad \textit{for } 23 \le q \le 307,\ q = 317, 343, 601, 661;\\
&t_2(2,q) \le 4\sqrt{q} - 7 \quad \textit{for } 19 \le q \le 353,\ q = 16, 361, 373, 383, 601, 661;\\
&t_2(2,q) \le 4\sqrt{q} - 6 \quad \textit{for } 9 \le q \le 409,\ q = 421, 431, 433, 443, 463, 601, 661;\\
&t_2(2,q) \le 4\sqrt{q} - 5 \quad \textit{for } 8 \le q \le 499,\ q = 509, 521, 523, 529, 601, 661;\\
&t_2(2,q) \le 4\sqrt{q} - 4 \quad \textit{for } 7 \le q \le 557,\ q = 569, 577, 601, 625, 661, 729, 841,\ q \in Q;\\
&t_2(2,q) < 4\sqrt{q} - 3 \quad \textit{for } 7 \le q \le 641,\ q = 661, 729, 841,\ q \in Q;\\
&t_2(2,q) \le 4\sqrt{q} - 2 \quad \textit{for } 3 \le q \le 691,\ q = 709, 729, 841,\ q \in Q;\\
&t_2(2,q) < 4\sqrt{q} - 1 \quad \textit{for } 2 \le q \le 761,\ q = 841,\ q \in Q.
\end{aligned}
$$

In Table 2, the current values of $\bar{t}_2(2,q)$ for $853 \le q \le 2593$ are given. The data for $q = p^2$ with $\bar{t}_2(2,q) = 4\sqrt{q} - 4$ [19,23] are written in bold font.

From Table 2, we obtain Theorem 3.2.

Theorem 3.2. *In $PG(2,q)$, the following holds.*

$$t_2(2,q) < 4.5\sqrt{q} \quad \textit{for } q \le 2593,\ q = 2693, 2753. \tag{3.2}$$

Table 2
The smallest known sizes $\bar{t}_2 = \bar{t}_2(2, q) < 4.5\sqrt{q}$ of complete arcs in planes $PG(2, q)$, $853 \le q \le 2593$, $A_q = \lfloor a_q\sqrt{q} - \bar{t}_2(2, q)\rfloor$, $B_q \ge \bar{t}_2(2, q)/\sqrt{q}$.

q	$\bar{t}_2$	A_q	B_q	q	$\bar{t}_2$	A_q	B_q	q	$\bar{t}_2$	A_q	B_q	q	$\bar{t}_2$	A_q	B_q
853	117	14	4.01	1277	150	10	4.20	1693	178	7	4.33	2141	205	3	4.44
857	118	13	4.04	1279	150	10	4.20	1697	178	7	4.33	2143	205	3	4.43
859	118	13	4.03	1283	150	11	4.19	1699	178	7	4.32	2153	205	3	4.42
863	118	14	4.02	1289	151	10	4.21	1709	178	8	4.31	2161	205	4	4.41
877	119	14	4.02	1291	151	10	4.21	1721	179	7	4.32	2179	207	3	4.44
881	120	13	4.05	1297	151	11	4.20	1723	180	6	4.34	2187	207	3	4.43
883	120	13	4.04	1301	152	10	4.22	1733	180	7	4.33	2197	208	2	4.44
887	120	14	4.03	1303	151	11	4.19	1741	181	6	4.34	2203	208	3	4.44
907	122	13	4.06	1307	152	10	4.21	1747	181	7	4.34	2207	208	3	4.43
911	122	13	4.05	1319	153	10	4.22	1753	181	7	4.33	2209	208	3	4.43
919	123	13	4.06	1321	153	10	4.21	1759	182	6	4.34	2213	209	2	4.45
929	124	13	4.07	1327	153	10	4.21	1777	183	6	4.35	2221	209	3	4.44
937	124	13	4.06	1331	154	10	4.23	1783	183	7	4.34	2237	210	2	4.45
941	125	13	4.08	1361	156	10	4.23	1787	183	7	4.33	2239	210	2	4.44
947	125	13	4.07	1367	156	10	4.22	1789	184	6	4.36	2243	210	3	4.44
953	126	12	4.09	**1369**	**144**	**4**	**3.90**	1801	184	6	4.34	2251	211	2	4.45
961	**120**	**4**	**3.88**	1373	157	9	4.24	1811	184	7	4.33	2267	211	3	4.44
967	127	12	4.09	1381	157	10	4.23	1823	186	6	4.36	2269	212	2	4.46
971	127	13	4.08	1399	158	10	4.23	1831	186	6	4.35	2273	212	2	4.45
977	127	13	4.07	1409	159	9	4.24	1847	187	6	4.36	2281	212	2	4.44
983	128	13	4.09	1423	160	9	4.25	1849	187	6	4.35	2287	213	2	4.46
991	127	14	4.04	1427	160	9	4.24	1861	188	6	4.36	2293	213	2	4.45
997	129	13	4.09	1429	160	10	4.24	1867	188	6	4.36	2297	213	2	4.45
1009	130	12	4.10	1433	161	9	4.26	1871	189	5	4.37	2309	214	2	4.46
1013	130	13	4.09	1439	161	9	4.25	1873	189	5	4.37	2311	214	2	4.46
1019	131	12	4.11	1447	162	9	4.26	1877	189	5	4.37	2333	215	2	4.46
1021	131	12	4.10	1451	162	9	4.26	1879	189	6	4.37	2339	216	1	4.47
1024	**124**	**4**	**3.88**	1453	162	9	4.25	1889	190	5	4.38	2341	216	1	4.47
1031	132	12	4.12	1459	162	9	4.25	1901	190	6	4.36	2347	216	2	4.46
1033	132	12	4.11	1471	163	9	4.25	1907	191	5	4.38	2351	216	2	4.46
1039	132	13	4.10	1481	164	9	4.27	1913	191	5	4.37	2357	217	1	4.47
1049	133	12	4.11	1483	164	9	4.26	1931	192	5	4.37	2371	217	2	4.46
1051	133	12	4.11	1487	164	9	4.26	1933	192	5	4.37	2377	216	3	4.44
1061	134	12	4.12	1489	164	9	4.26	1949	193	5	4.38	2381	217	2	4.45
1063	134	12	4.11	1493	165	8	4.28	1951	193	5	4.37	2383	218	1	4.47
1069	135	12	4.13	1499	165	9	4.27	1973	195	4	4.40	2389	218	1	4.47
1087	136	12	4.13	1511	166	8	4.28	1979	195	5	4.39	2393	218	2	4.46
1091	136	12	4.12	1523	167	8	4.28	1987	196	4	4.40	2399	219	1	4.48
1093	136	12	4.12	1531	167	9	4.27	1993	196	4	4.40	**2401**	**192**	**4**	**3.92**
1097	137	12	4.14	1543	167	9	4.26	1997	196	5	4.39	2411	220	0	4.49
1103	137	12	4.13	1549	168	9	4.27	1999	196	5	4.39	2417	220	1	4.48
1109	138	11	4.15	1553	169	8	4.29	2003	197	4	4.41	2423	220	1	4.47
1117	138	12	4.13	1559	169	8	4.29	2011	197	4	4.40	2437	221	1	4.48
1123	139	11	4.15	1567	170	8	4.30	2017	197	5	4.39	2441	221	1	4.48
1129	139	12	4.14	1571	170	8	4.29	2027	198	4	4.40	2447	221	1	4.47
1151	141	11	4.16	1579	170	8	4.28	2029	198	4	4.40	2459	222	1	4.48
1153	141	11	4.16	1583	171	8	4.30	2039	199	4	4.41	2467	223	0	4.49
1163	142	11	4.17	1597	172	7	4.31	2048	199	4	4.40	2473	223	0	4.49
1171	142	11	4.15	1601	172	8	4.30	2053	199	4	4.40	2477	223	0	4.49
1181	143	11	4.17	1607	172	8	4.30	2063	200	4	4.41	2503	224	1	4.48
1187	144	11	4.18	1609	172	8	4.29	2069	200	4	4.40	2521	225	0	4.49
1193	144	11	4.17	1613	173	7	4.31	2081	201	4	4.41	2531	226	0	4.50
1201	145	10	4.19	1619	173	8	4.30	2083	201	4	4.41	2539	226	0	4.49
1213	145	11	4.17	1621	173	8	4.30	2087	201	4	4.40	2543	226	0	4.49
1217	146	10	4.19	1627	174	7	4.32	2089	201	4	4.40	2549	226	1	4.48
1223	146	11	4.18	1637	174	8	4.31	2099	202	4	4.41	2551	227	0	4.50
1229	146	11	4.17	1657	175	8	4.30	2111	203	3	4.42	2557	227	0	4.49
1231	147	10	4.19	1663	176	7	4.32	2113	203	3	4.42	2579	228	0	4.49
1237	147	11	4.18	1667	176	7	4.32	2129	204	3	4.43	2591	229	0	4.50
1249	148	11	4.19	1669	176	7	4.31	2131	204	3	4.42	2593	229	0	4.50
1259	149	10	4.20	**1681**	**160**	**4**	**3.91**	2137	204	4	4.42				

$$t_2(2,q) < 4.1\sqrt{q} \quad \textit{for } q \le 1013,\ q = 1021, 1024, 1039, 1369, 1681, 2401;$$

$$t_2(2,q) < 4.2\sqrt{q} \quad \textit{for } q \le 1283,\ q = 1297, 1303, 1369, 1681, 2401;$$

$$t_2(2,q) < 4.3\sqrt{q} \quad \textit{for } q \le 1583,\ q = 1601, 1607, 1609, 1619, 1621, 1657, 1681, 2401;$$

$$t_2(2,q) < 4.4\sqrt{q} \quad \textit{for } q \le 1999,\ q = 2011, 2017, 2027, 2029, 2048, 2053, 2069, 2087, 2089, 2401.$$

Also,

$$t_2(2,q) < 4.5\sqrt{q} - 12 \quad \text{for } q \le 1103,\ q = 1117, 1129, 1369, 1681, 2401;$$
$$t_2(2,q) < 4.5\sqrt{q} - 11 \quad \text{for } q \le 1193,\ q = 1213, 1223, 1229, 1237, 1249, 1283, 1297, 1303, 1369, 1681, 2401;$$
$$t_2(2,q) < 4.5\sqrt{q} - 10 \quad \text{for } q \le 1369,\ q = 1381, 1399, 1429, 1681, 2401;$$
$$t_2(2,q) < 4.5\sqrt{q} - 9 \quad \text{for } q \le 1489,\ q = 1499, 1531, 1543, 1549, 1681, 2401;$$
$$t_2(2,q) < 4.5\sqrt{q} - 8 \quad \text{for } q \le 1583,\ q = 1601, 1607, 1609, 1619, 1621, 1637, 1657, 1681, 1709, 2401;$$
$$t_2(2,q) < 4.5\sqrt{q} - 7 \quad \text{for } q \le 1721,\ q = 1733, 1747, 1753, 1783, 1787, 1811, 2401;$$
$$t_2(2,q) < 4.5\sqrt{q} - 6 \quad \text{for } q \le 1867,\ q = 1879, 1901, 2401;$$
$$t_2(2,q) < 4.5\sqrt{q} - 5 \quad \text{for } q \le 1951,\ q = 1979, 1997, 1999, 2017, 2401;$$
$$t_2(2,q) < 4.5\sqrt{q} - 4 \quad \text{for } q \le 2099,\ q = 2137, 2161, 2401;$$
$$t_2(2,q) < 4.5\sqrt{q} - 3 \quad \text{for } q \le 2187,\ q = 2203, 2207, 2209, 2221, 2243, 2267, 2377, 2401;$$
$$t_2(2,q) < 4.5\sqrt{q} - 2 \quad \text{for } q \le 2333,\ q = 2347, 2351, 2371, 2377, 2381, 2393, 2401;$$
$$t_2(2,q) < 4.5\sqrt{q} - 1 \quad \text{for } q \le 2401,\ q = 2417, 2423, 2437, 2441, 2447, 2459, 2503, 2549.$$

In Table 3, the current values of $\bar{t}_2(2,q)$ for $2609 \le q \le 5107$ are given. The data with $\bar{t}_2(2,q) < 4.5\sqrt{q}$ are written in bold font.

Values of $\bar{t}_2(2,q)$ for relatively great $q \in T_2 \cup T_3$ are given in Table 4. The notation $\overline{D}_q\left(\frac{3}{4}\right)$ is explained in the next section.

In Table 2 for $q \in Q$, we use the results of [9,19,23]; see also [5, p. 35]. In Table 4, for $q \in T_3$ we use the results of [8,9], see also [5, p. 35] and Introduction. The remaining sizes k for small complete k-arcs in Tables 2–4 are obtained in this work by computer search with the help of the randomized greedy algorithms.

Note that a complete 199-arc in $PG(2,2048)$ of Table 2, a complete 301-arc in $PG(2,4096)$ of Table 3, and a complete 450-arc in $PG(2,8192)$ of Table 4 improve the results of [9] for $q = 2^{11}, 2^{12}, 2^{13}$; see Introduction.

From Tables 3 and 4, we obtain Theorem 3.3.

Theorem 3.3. *In $PG(2,q)$, the following holds.*

$$t_2(2,q) < 4.79\sqrt{q} \quad \text{for } q \le 5107. \tag{3.3}$$
$$t_2(2,q) < 4.6\sqrt{q} \quad \text{for } q \le 3209,\ q = 3221, 3229, 3251, 3253, 3257, 3271, 3299, 3301, 3319, 3323, 3343, 3347;$$
$$t_2(2,q) < 4.7\sqrt{q} \quad \text{for } q \le 4093,\ q = 4099, 4111, 4129, 4133, 4139, 4153, 4157, 4159, 4177, 4217, 4219, 4229, 4241, 4243, 4253, 4271, 4273, 4297, 4363, 4423.$$

Also,

$$t_2(2,q) < 5\sqrt{q} - 22 \quad \text{for } q \le 3391,\ q = 3413, 3433, 3457, 3461, 3463, 3467, 3469, 3481, 3491, 3499, 3511, 3517, 3529, 3533, 3539, 3559, 3583, 3607, 3617, 3643, 3739;$$
$$t_2(2,q) < 5\sqrt{q} - 21 \quad \text{for } q \le 3659,\ q = 3673, 3677, 3697, 3701, 3709, 3721, 3727, 3733, 3739, 3779, 3797, 3803, 3821, 3853, 3919, 4021, 4027, 4153;$$
$$t_2(2,q) < 5\sqrt{q} - 20 \quad \text{for } q \le 3911,\ q = 3919, 3923, 3929, 3931, 3947, 3967, 4001, 4003, 4013, 4019, 4021, 4027, 4049, 4057, 4073, 4099, 4153, 4159, 4177, 4229, 4253, 4273, 4363, 4423;$$
$$t_2(2,q) < 5\sqrt{q} - 19 \quad \text{for } q \le 4297,\ q = 4357, 4363, 4423, 4447, 4463, 4481, 4489, 4517;$$
$$t_2(2,q) < 5\sqrt{q} - 16 \quad \text{for } q \le 4969,\ q = 4987, 4993, 5003, 5009, 5011, 5021, 5041, 5059, 5107, 5119.$$

4. Observations on $\bar{t}_2(2,q)$ values

We look for upper estimates of the collection of $\bar{t}_2(2,q)$ values from Tables 1–4 in the form (1.2), see [30] and [28, Tab. 2.6]. For definiteness, we use the natural logarithms. Let c be a constant independent of q. We introduce $D_q(c)$ and $\overline{D}_q(c)$ as follows:

$$t_2(2,q) = D_q(c)\sqrt{q}\ln^c q,$$
$$\bar{t}_2(2,q) = \overline{D}_q(c)\sqrt{q}\ln^c q. \tag{4.1}$$

Let $\overline{D}_{\text{aver}}(c, q_0)$ be the average value of $\overline{D}_q(c)$ calculated in the region $q_0 \le q \le 5107$ and $q \in T_2$ under condition $q \notin N$. From Tables 1–4, we obtain Observation 1.

Table 3
The smallest known sizes $\bar{t}_2 = \bar{t}_2(2, q) < 4.79\sqrt{q}$ of complete arcs in planes $PG(2, q)$, $2609 \le q \le 5107$, $A_q = \lfloor a_q\sqrt{q} - \bar{t}_2(2, q) \rfloor$, $B_q \ge \bar{t}_2(2, q)/\sqrt{q}$.

q	$\bar{t}_2$	A_q	B_q	q	$\bar{t}_2$	A_q	B_q	q	$\bar{t}_2$	A_q	B_q	q	$\bar{t}_2$	A_q	B_q
2609	230	25	4.51	3221	261	22	4.60	3847	290	20	4.68	4493	317	18	4.73
2617	231	24	4.52	3229	260	24	4.58	3851	290	20	4.68	4507	318	17	4.74
2621	231	24	4.52	3251	262	23	4.60	3853	289	21	4.66	4513	318	17	4.74
2633	231	25	4.51	3253	261	24	4.58	3863	290	20	4.67	4517	317	19	4.72
2647	232	25	4.51	3257	262	23	4.60	3877	291	20	4.68	4519	318	18	4.74
2657	233	24	4.53	3259	263	22	4.61	3881	291	20	4.68	4523	318	18	4.73
2659	233	24	4.52	3271	263	22	4.60	3889	291	20	4.67	4547	319	18	4.74
2663	233	25	4.52	3299	264	23	4.60	3907	292	20	4.68	4549	319	18	4.73
2671	233	25	4.51	3301	264	23	4.60	3911	292	20	4.67	4561	319	18	4.73
2677	234	24	4.53	3307	265	22	4.61	3917	293	19	4.69	4567	320	17	4.74
2683	234	24	4.52	3313	265	22	4.61	3919	292	21	4.67	4583	320	18	4.73
2687	234	25	4.52	3319	265	23	4.60	3923	293	20	4.68	4591	321	17	4.74
2689	234	25	4.52	3323	265	23	4.60	3929	293	20	4.68	4597	321	18	4.74
2693	**233**	**0**	**4.49**	3329	266	22	4.62	3931	293	20	4.68	4603	322	17	4.75
2699	235	24	4.53	3331	266	22	4.61	3943	294	19	4.69	4621	323	16	4.76
2707	235	25	4.52	3343	265	24	4.59	3947	294	20	4.68	4637	322	18	4.73
2711	235	25	4.52	3347	266	23	4.60	3967	294	20	4.67	4639	323	17	4.75
2713	235	25	4.52	3359	267	22	4.61	3989	296	19	4.69	4643	323	17	4.75
2719	236	24	4.53	3361	267	22	4.61	4001	296	20	4.68	4649	323	17	4.74
2729	236	25	4.52	3371	268	22	4.62	4003	296	20	4.68	4651	323	17	4.74
2731	236	25	4.52	3373	268	22	4.62	4007	297	19	4.70	4657	323	18	4.74
2741	237	24	4.53	3389	268	23	4.61	4013	296	20	4.68	4663	323	18	4.74
2749	237	25	4.53	3391	269	22	4.62	4019	296	20	4.67	4673	325	16	4.76
2753	**236**	**0**	**4.50**	3407	270	21	4.63	4021	296	21	4.67	4679	324	18	4.74
2767	238	25	4.53	3413	269	23	4.61	4027	296	21	4.67	4691	324	18	4.74
2777	238	25	4.52	3433	270	22	4.61	4049	298	20	4.69	4703	325	17	4.74
2789	239	25	4.53	3449	272	21	4.64	4051	299	19	4.70	4721	326	17	4.75
2791	239	25	4.53	3457	271	22	4.61	4057	298	20	4.68	4723	326	17	4.75
2797	240	24	4.54	3461	272	22	4.63	4073	299	20	4.69	4729	327	16	4.76
2801	240	24	4.54	3463	272	22	4.63	4079	300	19	4.70	4733	326	17	4.74
2803	240	24	4.54	3467	272	22	4.62	4091	300	19	4.70	4751	328	16	4.76
2809	240	25	4.53	3469	272	22	4.62	4093	300	19	4.69	4759	327	17	4.75
2819	241	24	4.54	3481	272	23	4.62	4096	301	19	4.71	4783	328	17	4.75
2833	242	24	4.55	3491	273	22	4.63	4099	300	20	4.69	4787	329	16	4.76
2837	242	24	4.55	3499	273	22	4.62	4111	301	19	4.70	4789	329	17	4.76
2843	242	24	4.54	3511	274	22	4.63	4127	302	19	4.71	4793	330	16	4.77
2851	243	23	4.56	3517	274	22	4.63	4129	302	19	4.70	4799	330	16	4.77
2857	243	24	4.55	3527	275	21	4.64	4133	302	19	4.70	4801	329	17	4.75
2861	243	24	4.55	3529	275	22	4.63	4139	302	19	4.70	4813	330	16	4.76
2879	244	24	4.55	3533	275	22	4.63	4153	301	21	4.68	4817	329	18	4.75
2887	243	25	4.53	3539	275	22	4.63	4157	303	19	4.70	4831	329	18	4.74
2897	245	24	4.56	3541	276	21	4.64	4159	302	20	4.69	4861	331	17	4.75
2903	245	24	4.55	3547	276	21	4.64	4177	303	20	4.69	4871	332	16	4.76
2909	245	24	4.55	3557	277	21	4.65	4201	305	19	4.71	4877	332	17	4.76
2917	246	24	4.56	3559	276	22	4.63	4211	305	19	4.71	4889	333	16	4.77
2927	245	25	4.53	3571	277	21	4.64	4217	305	19	4.70	4903	334	16	4.77
2939	246	25	4.54	3581	278	21	4.65	4219	305	19	4.70	4909	334	16	4.77
2953	248	23	4.57	3583	277	22	4.63	4229	305	20	4.70	4913	334	16	4.77
2957	248	23	4.57	3593	278	21	4.64	4231	306	19	4.71	4919	334	16	4.77
2963	248	24	4.56	3607	278	22	4.63	4241	306	19	4.70	4931	335	16	4.78
2969	248	24	4.56	3613	279	21	4.65	4243	306	19	4.70	4933	335	16	4.77
2971	249	23	4.57	3617	278	22	4.63	4253	306	20	4.70	4937	335	16	4.77
2999	250	23	4.57	3623	279	21	4.64	4259	307	19	4.71	4943	334	17	4.76
3001	250	23	4.57	3631	280	21	4.65	4261	307	19	4.71	4951	335	16	4.77
3011	251	23	4.58	3637	280	21	4.65	4271	307	19	4.70	4957	335	17	4.76
3019	251	23	4.57	3643	278	23	4.61	4273	306	20	4.69	4967	336	16	4.77
3023	251	23	4.57	3659	281	21	4.65	4283	308	19	4.71	4969	336	16	4.77
3037	252	23	4.58	3671	282	20	4.66	4289	308	19	4.71	4973	337	15	4.78
3041	252	23	4.57	3673	282	21	4.66	4297	308	19	4.70	4987	336	17	4.76
3049	252	24	4.57	3677	282	21	4.66	4327	310	18	4.72	4993	337	16	4.77
3061	253	23	4.58	3691	283	20	4.66	4337	311	18	4.73	4999	338	15	4.79
3067	253	23	4.57	3697	283	21	4.66	4339	311	18	4.73	5003	337	16	4.77
3079	253	24	4.56	3701	283	21	4.66	4349	311	18	4.72	5009	337	16	4.77
3083	253	24	4.56	3709	283	21	4.65	4357	311	19	4.72	5011	337	16	4.77
3089	254	23	4.58	3719	284	20	4.66	4363	310	20	4.70	5021	338	16	4.78
3109	255	23	4.58	3721	284	21	4.66	4373	312	18	4.72	5023	339	15	4.79
3119	255	24	4.57	3727	284	21	4.66	4391	313	18	4.73	5039	339	15	4.78
3121	255	24	4.57	3733	284	21	4.65	4397	313	18	4.73	5041	339	16	4.78
3125	256	23	4.58	3739	283	22	4.63	4409	314	18	4.73	5051	340	15	4.79
3137	257	23	4.59	3761	286	20	4.67	4421	314	18	4.73	5059	339	16	4.77

(*continued on next page*)

Table 3 (*continued*)

q	$\bar{t}_2$	A_q	B_q	q	$\bar{t}_2$	A_q	B_q	q	$\bar{t}_2$	A_q	B_q	q	$\bar{t}_2$	A_q	B_q
3163	257	24	4.57	3767	286	20	4.66	4423	312	20	4.70	5077	341	15	4.79
3167	258	23	4.59	3769	286	20	4.66	4441	315	18	4.73	5081	341	15	4.79
3169	258	23	4.59	3779	286	21	4.66	4447	314	19	4.71	5087	341	15	4.79
3181	259	23	4.60	3793	287	20	4.67	4451	316	17	4.74	5099	342	15	4.79
3187	258	24	4.58	3797	287	21	4.66	4457	315	18	4.72	5101	342	15	4.79
3191	259	23	4.59	3803	287	21	4.66	4463	315	19	4.72	5107	341	16	4.78
3203	259	23	4.58	3821	288	21	4.66	4481	315	19	4.71				
3209	260	23	4.59	3823	289	20	4.68	4483	317	17	4.74				
3217	261	22	4.61	3833	289	20	4.67	4489	316	19	4.72				

Table 4

The smallest known sizes $\bar{t}_2 = \bar{t}_2(2,q)$ of complete arcs in planes $PG(2,q)$ with $q \in T_2 \cup T_3$. $A_q = \lfloor 5\sqrt{q} - \bar{t}_2(2,q) \rfloor$, $B_q > \bar{t}_2(2,q)/\sqrt{q}$.

q	$\bar{t}_2$	A_q	B_q	$\bar{D}_q\left(\frac{3}{4}\right)$	q	$\bar{t}_2$	A_q	B_q	$\bar{D}_q\left(\frac{3}{4}\right)$	q	$\bar{t}_2$	A_q	B_q	$\bar{D}_q\left(\frac{3}{4}\right)$
5119	341	16	4.77	0.9540	5261	348	14	4.80	0.9581	5843	370	12	4.85	0.9578
5147	343	15	4.79	0.9566	5279	349	14	4.81	0.9589	6011	377	10	4.87	0.9598
5153	344	14	4.80	0.9587	5281	349	14	4.81	0.9587	8192	450	2	4.98	0.9560
5209	346	14	4.80	0.9582	5303	349	15	4.80	0.9564	2^{14}	665		5.20	0.9449
5231	347	14	4.80	0.9585	5347	352	13	4.82	0.9599	2^{15}	993		5.49	0.9474
5237	347	14	4.80	0.9579	5641	363	12	4.84	0.9592	2^{18}	3066		5.99	0.9020

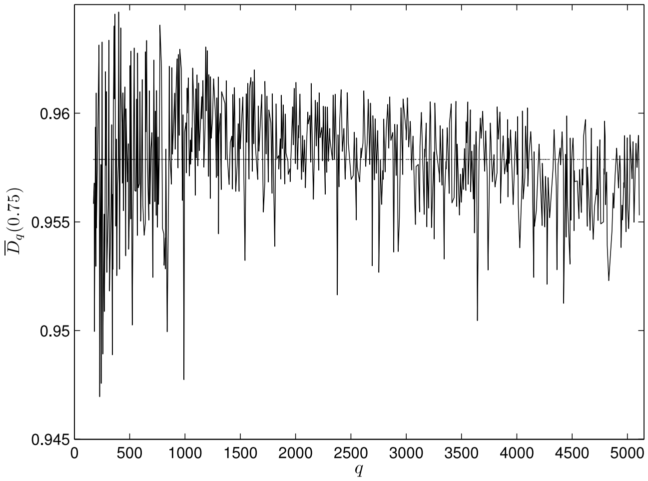

Fig. 1. The values of $\bar{D}_q(0.75)$ for $173 \le q \le 5107$, $q \notin N$. $\bar{D}_{\text{aver}}(0.75, 173) = 0.95787$.

Observation 1. *Let* $173 \le q \le 5107$ *or* $q \in T_2$, *under condition* $q \notin N$. *Then*

(i) *When q grows,* $\bar{D}_q(0.8)$ *has a tendency to decrease.*

(ii) *When q grows,* $\bar{D}_q(0.5)$ *has a tendency to increase.*

(iii) *When q grows, the values of* $\bar{D}_q(0.75)$ *oscillate about the average value* $\bar{D}_{\text{aver}}(0.75, 173) = 0.95787$ *(see Fig. 1). Also,*

$$\begin{array}{ll} 0.946 < \bar{D}_q(0.75) < 0.9647 & \text{if } 173 \le q \le 997, \\ 0.953 < \bar{D}_q(0.75) < 0.9631 & \text{if } 1009 \le q \le 1999, \\ 0.951 < \bar{D}_q(0.75) < 0.9615 & \text{if } 2003 \le q \le 2999, \\ 0.950 < \bar{D}_q(0.75) < 0.9608 & \text{if } 3001 \le q \le 3989, \\ 0.951 < \bar{D}_q(0.75) < 0.9603 & \text{if } 4001 \le q. \end{array} \tag{4.2}$$

Data for relatively big q, collected in Table 4, in large confirm Observation 1.

By Observation 1 it seems that the values of $D_q(0.75)$ and $\bar{D}_q(0.75)$ are sufficiently convenient for estimates of $t_2(2,q)$ and $\bar{t}_2(2,q)$.

From Tables 1–4, we obtain Theorem 4.1.

Theorem 4.1. *In* $PG(2,q)$,

$$t_2(2,q) < 0.9987\sqrt{q}\ln^{0.75} q \quad \text{for } 23 \le q \le 5107,\ q \in T_2 \cup T_3. \tag{4.3}$$

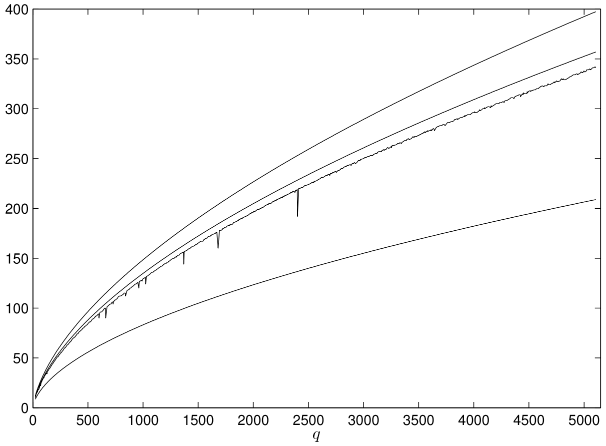

Fig. 2. The values of $\sqrt{q}\ln^{0.8} q$ (the top curve), $\sqrt{q}\ln^{0.75} q$ (the 2-nd curve), $\overline{t}_2(2, q)$ (the 3-rd curve), and $\sqrt{q}\ln^{0.5} q$ (the bottom curve) for $23 \le q \le 5107$.

In Theorem 1.2 we slightly rounded the estimate (4.3).

The graphs of values of $\sqrt{q}\ln^{0.8} q$, $\sqrt{q}\ln^{0.75} q$, $\overline{t}_2(2, q)$, and $\sqrt{q}\ln^{0.5} q$ are shown in Fig. 2, where $\sqrt{q}\ln^{0.8} q$ is the top curve and $\sqrt{q}\ln^{0.5} q$ is the bottom one.

One can see in Fig. 2 that always $\overline{t}_2(2, q) < \sqrt{q}\ln^{0.75} q$ and, moreover, when q grows, the graphs $\sqrt{q}\ln^{0.75} q$ and $\overline{t}_2(2, q)$ diverge so that positive difference $\sqrt{q}\ln^{0.75} q - \overline{t}_2(2, q)$ increases.

We denote

$$\widehat{t}_2(2, q) = \overline{D}_{\text{aver}}(0.75, 173)\sqrt{q}\ln^{0.75} q, \qquad \overline{\Delta}_q = \overline{t}_2(2, q) - \widehat{t}_2(2, q), \qquad \overline{P}_q = \frac{100\overline{\Delta}_q}{\overline{t}_2(2, q)}\%. \tag{4.4}$$

One can treat $\widehat{t}_2(2, q)$ as a *predicted* value of $t_2(2, q)$. Then $\overline{\Delta}_q$ is the difference between the smallest known size $\overline{t}_2(2, q)$ of complete arcs and the predicted value. Finally, $\overline{P}_q$ is this difference in percentage terms of the smallest known size.

Observation 2. *Let* $173 \le q \le 5107$ *or* $q \in T_2$, $q \notin N$. *Then*

$$-2.18 < \overline{\Delta}_q < 0.78. \tag{4.5}$$

$$\begin{aligned} &-1.16\% < \overline{P}_q < 0.71\% && \text{if } 173 \le q \le 997,\\ &-0.49\% < \overline{P}_q < 0.54\% && \text{if } 1009 \le q \le 1999,\\ &-0.66\% < \overline{P}_q < 0.37\% && \text{if } 2003 \le q \le 2999,\\ &-0.79\% < \overline{P}_q < 0.30\% && \text{if } 3001 \le q \le 3989,\\ &-0.70\% < \overline{P}_q < 0.25\% && \text{if } 4001 \le q. \end{aligned} \tag{4.6}$$

By (4.5) and (4.6), see also Figs. 3 and 4, the upper bounds of $\overline{\Delta}_q$ and $\overline{P}_q$ are relatively small. Moreover, the upper bound of $\overline{P}_q$ decreases when q grows. Therefore the values of $\overline{\Delta}_q$ and $\overline{P}_q$ are useful for computer search of small arcs.

The relations (4.2)–(4.6), Theorems 1.2 and 4.1, and Figs. 1–4 are the foundation for Conjecture 1.3.

Remark 4.2. By above, $\sqrt{q}\ln^{0.75} q$ seems to be a reasonable upper bound on the current collection of $\overline{t}_2(2, q)$ values. It gives some reference points for computer search and foundations for Conjecture 1.3 on the upper bound for $t_2(2, q)$. In principle, the constant $c = 0.75$ can be sightly reduced to move the curve $\sqrt{q}\ln^{c} q$ near to the curve of $\overline{t}_2(2, q)$; see Fig. 2. For example, from Tables 1–4, Theorem 4.3 holds.

Theorem 4.3. *In* $PG(2, q)$,

$$t_2(2, q) < \sqrt{q}\ln^{0.735} q \quad \text{for } 32 \le q \le 5107,\ q \in T_2 \cup T_3. \tag{4.7}$$

5. Constructions of families of complete arcs in *PG*(2, *q*)

In the homogeneous coordinates of a point (x_0, x_1, x_2) we put $x_0 \in \{0, 1\}, x_1, x_2 \in F_q$. Let $F_q^* = F_q \setminus \{0\}$. Let ξ be a primitive element of F_q. Recall that *indexes of powers of* ξ *are calculated modulo* $q - 1$.

Throughout this section we use the conic $\mathcal{C}$ of equation $x_1^2 = x_0x_2$. We denote points of $\mathcal{C}$ as follows:

$$A_i = (1, i, i^2), \quad i \in F_q; \qquad \overline{A}_d = (1, \xi^d, \xi^{2d}), \quad d \in \{0, 1, \dots, q-2\}; \qquad A_\infty = (0, 0, 1).$$

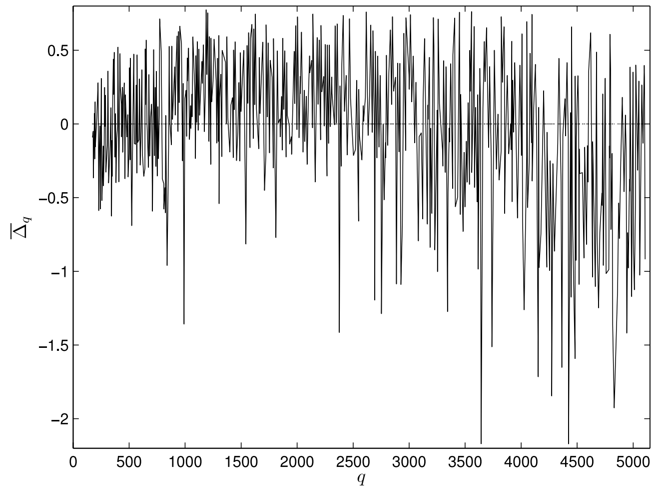

Fig. 3. The values of $\overline{\Delta}_q = \overline{t}_2(2,q) - \widehat{t}_2(2,q)$ for $173 \le q \le 5107$, $q \notin N$.

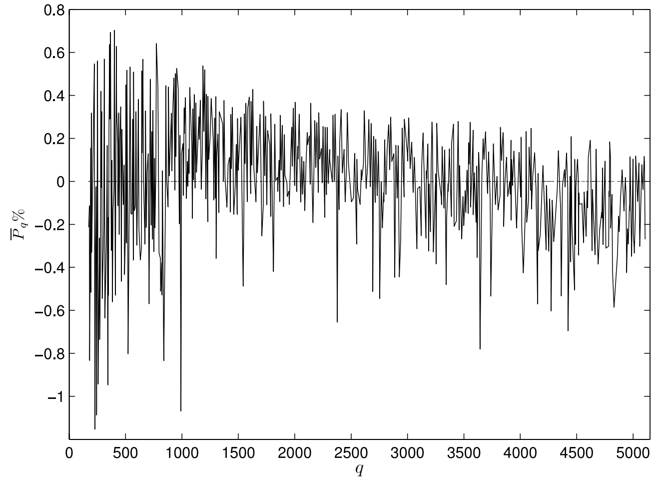

Fig. 4. The values of $\overline{P}_q = 100\overline{\Delta}_q/\overline{t}_2(2,q)\%$ for $173 \le q \le 5107$, $q \notin N$.

5.1. Arcs with two points on a tangent to a conic

Throughout this subsection, $q \ge 19$ is an *odd prime*. Let H be an integer in the region

$$\left\lfloor \frac{q-1}{3} \right\rfloor \le H \le \frac{q-1}{2}. \tag{5.1}$$

We denote by $\mathcal{V}_H$ the following $(H+1)$-subset of the conic $\mathcal{C}$:

$$\mathcal{V}_H = \{A_i : i = 0, 1, 2, \dots, H\} \subset \mathcal{C}. \tag{5.2}$$

We denote the points of $PG(2,q)$:

$$P = (0,1,0), \qquad T_H = (0,1,b_H), \qquad b_H = \begin{cases} 2H+1 & \text{if } H = \left\lfloor \frac{1}{3}(q-1) \right\rfloor \\ 2H & \text{if } \left\lfloor \frac{1}{3}(q-1) \right\rfloor < H \le \frac{1}{2}(q-1). \end{cases} \tag{5.3}$$

Let ℓ_0 be the line of equation $x_0 = 0$. It is the *tangent* to $\mathcal{C}$ at A_∞. It holds that $\{P, T_H\} \subset \ell_0$.

Construction A. *Let q be an* odd prime. *Let H, $\mathcal{V}_H$, P and T_H be given by (5.1)–(5.3). We construct a point $(H+3)$-set $\mathcal{K}_H$ in the plane $PG(2,q)$ as follows:*

$$\mathcal{K}_H = \mathcal{V}_H \cup \{P, T_H\}.$$

The following lemma can be proved by elementary calculations.

Lemma 5.1. (i) *Let $i \neq j$. A point $(0, 1, b)$ is collinear with points A_i, A_j if and only if*

$$b = i + j. \tag{5.4}$$

(ii) *Let $i \neq j$, $a, b \in F_q$, $b \neq a^2$. Then a point $(1, a, b)$ is collinear with A_i, A_j if and only if $b = a(i + j) - ij$.*
(iii) *Let $a \in F_q$, $a \neq i$. Then a point $(1, a, i^2)$ is collinear with P, A_i.*

Theorem 5.2. *The $(H + 3)$-set $\mathcal{K}_H$ of Construction A is an arc in $PG(2, q)$.*

Proof. By (5.1), (5.2), the sum $i + j$ in (5.4) is running on $\{1, 2, \ldots, 2H - 1\}$, where $2H - 1 \le q - 2$ if $\lfloor \frac{1}{3}(q - 1) \rfloor < H$ and $2H - 1 < \frac{2}{3}(q - 1)$ if $H = \lfloor \frac{1}{3}(q - 1) \rfloor$. So, $\{0, b_H\} \cap \{1, 2, \ldots, 2H - 1\} = \emptyset$, see (5.3). Therefore P and T_H do not lie on bisecants of $\mathcal{V}_H$. In other side, any point of $\mathcal{V}_H$ does not lie on the line PT_H as PT_H is a tangent to $\mathcal{C}$ in A_∞. □

Theorem 5.3. *Let H be given by (5.1). Then all points of $\ell_0 \cup \mathcal{C} \setminus \mathcal{V}_H$ lie on bisecants of $\mathcal{K}_H$.*

Proof. All points of ℓ_0 are covered as two points P and T_H of this line belong to $\mathcal{K}_H$.

Let $\mathcal{R}$ and $\mathcal{S}$ be sets of integers modulo q, i.e. $\mathcal{R} \cup \mathcal{S} \subset F_q$.

Let $\mathcal{R} = \{-H, -(H - 1), \ldots, -1\} = \{q - H, q - (H - 1), \ldots, q - 1\}$. By Lemma 5.1(i), points A_j, A_{-j}, P are collinear. Therefore, a point A_j of $\mathcal{C} \setminus \mathcal{V}_H$ with $j \in \mathcal{R}$ lies on the bisecant of $\mathcal{K}_H$ through P and A_{-j}, where $-j \in \{H, H - 1, \ldots, 1\}$, $A_{-j} \in \mathcal{V}_H$.

Let $\mathcal{S} = \{b_H - H, b_H - (H - 1), \ldots, b_H - 1, b_H\}$. By Lemma 5.1(i), a point A_j of $\mathcal{C} \setminus \mathcal{V}_H$ with $j \in \mathcal{S}$ lies on the bisecant $T_H A_{b_H - j}$, where $b_H - j \in \{H, H - 1, \ldots, 1, 0\}$, $A_{b_H - j} \in \mathcal{V}_H$.

Let $b_H = 2H + 1$. Then $\mathcal{S} = \{H + 1, H + 2, \ldots, 2H + 1\}$. Also, by (5.3), $H = \lfloor \frac{1}{3}(q - 1) \rfloor$ whence $H = \frac{1}{3}(q - v)$, where $v \in \{1, 2\}$ and $v \equiv q \pmod 3$. Hence, $3H = q - v$, $2H + 1 = q - H - v + 1 \in \{q - H, q - H - 1\}$.

Let $b_H = 2H$. Then $\mathcal{S} = \{H, H + 1, \ldots, 2H\}$. Also, by (5.3), $H > \lfloor \frac{1}{3}(q - 1) \rfloor$ whence $H > \frac{1}{3}(q - v)$, where $v \in \{1, 2\}$ is as above. Therefore $3H > q - v$, $2H > q - H - v$, $2H \ge q - H - v + 1 \in \{q - H, q - H - 1\}$.

We proved that $\{H + 1, H + 2, \ldots, q - 1\} \subseteq \mathcal{S} \cup \mathcal{R}$. Also we showed that the points A_j of $\mathcal{C} \setminus \mathcal{V}_H$ with $j \in \mathcal{S} \cup \mathcal{R}$ are covered by bisecants of $\mathcal{K}_H$ through P (if $j \in \mathcal{R}$) or through T_H (if $j \in \mathcal{S}$). In the other side, $\mathcal{C} \setminus \mathcal{V}_H = \{A_j : j = H + 1, H + 2, \ldots, q - 1\} \cup \{A_\infty\}$, where $A_\infty \in \ell_0$. So, all points of $\mathcal{C} \setminus \mathcal{V}_H$ are covered. □

Definition 5.4. Let q be an *odd prime*. Let $\overline{H}$ be an integer and let

$$\mathcal{P}_{\overline{H}} = \{P\} \cup \{A_i : i = 0, 1, 2, \ldots, \overline{H}\}.$$

We call *critical value of* $\overline{H}$ and denote by $\overline{H}_q$ the *smallest* value of $\overline{H}$ such that all points of the form $(1, a, b)$, $a, b \in F_q$, $b \neq a^2$, lie on bisecants of $\mathcal{P}_{\overline{H}}$.

Theorem 5.5. *Let $q \ge 19$ be an* odd prime. *Let $\overline{H}_q \le \frac{1}{2}(q - 1)$ and let*

$$\max\left\{\overline{H}_q, \left\lfloor \frac{q - 1}{3} \right\rfloor\right\} \le H \le \frac{q - 1}{2}.$$

Then the arc $\mathcal{K}_H$ of Construction A *is complete.*

Proof. We use Theorem 5.3 and Definition 5.4. □

In this subsection, we put $q \ge 19$ as we checked by computer that $\frac{1}{2}(q - 1) < \overline{H}_q$ if $q \le 17$.

Corollary 5.6. *Let $q \ge 19$ be an* odd prime. *Let $\overline{H}_q \le \frac{1}{2}(q - 1)$. Then* Construction A *forms a family of complete k-arcs in $PG(2, q)$ containing arcs of all sizes k in the region*

$$\max\left\{\overline{H}_q, \left\lfloor \frac{q - 1}{3} \right\rfloor\right\} + 3 \le k \le \frac{q + 5}{2}.$$

If $\overline{H}_q \le \lfloor \frac{1}{3}(q - 1) \rfloor$ then the cardinality of this family is equal to $\lceil \frac{1}{6}(q + 5) \rceil$ and the size of the smallest complete arc of the family is $\lfloor \frac{1}{3}(q + 8) \rfloor$.

By computer search using Lemma 5.1(ii), (iii) we obtained the following theorem.

Theorem 5.7. *Let $q \ge 19$ be an* odd prime. *Let $\overline{H}_q$ be given by* Definition 5.4. *We introduce D_q and Δ_q as follows: $\overline{H}_q = D_q \sqrt{q} \ln q$, $\Delta_q = \lfloor \frac{1}{3}(q - 1) \rfloor - \overline{H}_q$. Then the following holds.*

Table 5
The values $\overline{H}_q, \overline{G}_q, \bar{J}_q$ for cases $\overline{H}_q, \overline{G}_q > \left\lfloor \frac{1}{3}(q-1) \right\rfloor, \bar{J}_q > \frac{1}{4}(q-3)$.

q	$\overline{H}_q$	$\overline{G}_q$	$\bar{J}_q$	q	$\overline{H}_q$	$\overline{G}_q$	$\bar{J}_q$	q	$\overline{H}_q$	$\overline{G}_q$	$\bar{J}_q$	q	$\overline{H}_q$	$\overline{G}_q$	$\bar{J}_q$	q	$\bar{J}_q$
19	9			43	19		16	71	24		25	103			34	167	42
23	11			47	18		18	73		27		107	36		30	179	62
27			12	49		18		79	29		27	125		43		191	49
29	13			53	20	19		81		29		127			39	211	54
31	14		14	59	23		24	83	29		29	131			38	223	60
32		15		61	22	22		89	32			139			38	343	86
37	16	16		64		24		97		33		151			42		
41	16	19		67	24		23	101		35		163			41		

$$\text{(i)} \left\lfloor \frac{q-1}{3} \right\rfloor < \overline{H}_q \le \frac{q-1}{2} \quad \textit{if } 19 \le q \le 71 \textit{ and } q = 79, 83, 89, 107.$$

$$\text{(ii)} \, \overline{H}_q \le \left\lfloor \frac{q-1}{3} \right\rfloor \quad \textit{if } 109 \le q \le 2063,\ q = 73, 97, 101, 103.$$

$$\text{(iii)} \, \overline{H}_q < 0.84\sqrt{q}\ln q \quad \textit{if } 19 \le q \le 2063. \tag{5.5}$$

$$\text{(iv)} \begin{array}{lll} 0 \le \Delta_q \le 99, & 0.59 < D_q < 0.82, & \textit{if } 109 \le q \le 599; \\ 89 \le \Delta_q \le 198, & 0.62 < D_q < 0.78, & \textit{if } 599 < q \le 1049; \\ 187 \le \Delta_q \le 288, & 0.59 < D_q < 0.78, & \textit{if } 1049 < q \le 1367; \\ 241 \le \Delta_q \le 464, & 0.61 < D_q < 0.84, & \textit{if } 1367 < q \le 2063. \end{array}$$

For situations $\left\lfloor \frac{1}{3}(q-1) \right\rfloor < \overline{H}_q$, the values of $\overline{H}_q$ are given in Table 5.

Theorem 5.8. *Let q be an* odd prime *with $109 \le q \le 2063$, or $q = 73, 97, 101, 103$. Then Construction A forms a family of complete k-arcs in $PG(2, q)$ containing arcs of all sizes k in the region*

$$\left\lfloor \frac{q+8}{3} \right\rfloor \le k \le \frac{q+5}{2}.$$

Proof. We use Corollary 5.6 and Theorem 5.7(ii). □

5.2. *Arcs with two points on a bisecant of a conic*

Throughout this subsection, $q \ge 32$ is a *prime power*. Let G be an integer in the region

$$\left\lfloor \frac{q-1}{3} \right\rfloor \le G \le \left\lceil \frac{q-3}{2} \right\rceil. \tag{5.6}$$

We denote by $\mathcal{D}_G$ the following $(G+1)$-subset of the conic $\mathcal{C}$:

$$\mathcal{D}_G = \{\overline{A}_d : d = 0, 1, 2, \ldots, G\} \subset \mathcal{C}. \tag{5.7}$$

Clearly, $A_0 \notin \mathcal{D}_G$. Let

$$\gamma \in F_q, \qquad \gamma = \begin{cases} -1 = \xi^{(q-1)/2} & \text{for } q \text{ odd} \\ 1 & \text{for } q \text{ even.} \end{cases} \tag{5.8}$$

We denote the points of $PG(2, q)$:

$$Z = (1, 0, \gamma\xi^0), \qquad B_G = (1, 0, \gamma\xi^{\beta_G}), \tag{5.9}$$

where

$$\beta_G = \begin{cases} 2G+1 & \text{if } \left\lfloor \dfrac{1}{3}(q-1) \right\rfloor = G,\ q \equiv 0 \pmod 3 \\ 2G & \text{if } \left\lfloor \dfrac{1}{3}(q-1) \right\rfloor = G,\ q \not\equiv 0 \pmod 3 \\ 2G & \text{if } \left\lfloor \dfrac{1}{3}(q-1) \right\rfloor < G \le \left\lceil \dfrac{1}{2}(q-3) \right\rceil, \text{ arbitrary } q. \end{cases} \tag{5.10}$$

Let ℓ_1 be the line of equation $x_1 = 0$. It is the *bisecant* $A_\infty A_0$ of $\mathcal{C}$. We have $\{Z, B_G\} \subset \ell_1$.

Using (5.6), (5.8), by elementary calculations we obtained the following lemma.

Lemma 5.9. (i) *Let $d \neq t$. A point $(0, 1, b)$ is collinear with points $\overline{A}_d, \overline{A}_t$ if and only if*

$$b = \xi^d + \xi^t. \tag{5.11}$$

(ii) *A point $(0, 1, b)$ is collinear with points $\overline{A}_d$ and $(1, 0, \gamma U)$ if and only if*

$$b = \frac{\xi^{2d} + U}{\xi^d}. \tag{5.12}$$

Corollary 5.10. (i) *For all q, the point $P = (0, 1, 0)$ does not lie on any bisecant of $\mathcal{D}_G$.*
(ii) *Let q be even. Then the points $P, Z, \overline{A}_d$ are collinear if and only if $d = 0$.*
(iii) *Let $q \equiv 1 \pmod 4$. Then the points $P, Z, \overline{A}_d$ are collinear if and only if $d \in \left\{\frac{1}{4}(q-1), \frac{3}{4}(q-1)\right\}$.*
(iv) *Let $q \equiv 3 \pmod 4$ and let $\beta_G = 2G$. Then the points $P, Z, \overline{A}_d$ and $P, B_G, \overline{A}_d$ are not collinear for any d.*

Proof. (i) In (5.11), the case $b = 0$ implies $\xi^d + \xi^t = 0$. For even q, it is impossible. For odd q, we obtain $\xi^d = -\xi^t$ whence, by (5.8), $d = t + (q-1)/2$. By (5.6), it is impossible.

(ii)–(iv) In (5.12), the case $b = 0$ implies $\xi^{2d} + U = 0$ whence $\xi^{2d} + 1 = 0$ if $(1, 0, \gamma U) = Z$ and $\xi^{2d} + \xi^{\beta_G} = 0$ if $(1, 0, \gamma U) = B_G$. Recall that $d \le q - 2$.

(ii) Here q is even but $q - 1$ is odd. For $(1, 0, \gamma U) = Z$, we have $\xi^{2d} = 1$ whence $d = 0$.

(iii) Here both $q-1$ and $\frac{1}{2}(q-1)$ are even. If $(1, 0, \gamma U) = Z$, then $\xi^{2d} = -1 = \xi^{(q-1)/2}$ whence $2d \equiv \frac{1}{2}(q-1) \pmod{q-1}$. It is possible if $d = \frac{1}{4}(q-1)$ or $d = \frac{3}{4}(q-1)$.

(iv) Here $q - 1$ is even whereas $\frac{1}{2}(q-1)$ is odd. If $(1, 0, \gamma U) = Z$, then $\xi^{2d} = -1 = \xi^{(q-1)/2}$ whence $2d \equiv \frac{1}{2}(q-1) \pmod{q-1}$. This is impossible. For $(1, 0, \gamma U) = B_G$, it holds that $\xi^{2d} = -\xi^{\beta_G} = \xi^{2G+(q-1)/2}$, which is impossible. □

Construction B. *Let q be a* prime power. *Assume that $q \not\equiv 3 \pmod 4$. Let $G, \mathcal{D}_G, Z$, and B_G be given by* (5.6)–(5.10). *We construct a point $(G+3)$-set $\mathcal{W}_G$ in $PG(2, q)$ as follows:*

$$\mathcal{W}_G = \mathcal{D}_G \cup \{Z, B_G\}.$$

From Lemma 5.1 it follows.

Lemma 5.11. *Let $d \neq t$. A point $(1, 0, \gamma\xi^\beta)$ is collinear with $\overline{A}_d, \overline{A}_t$ if and only if*

$$\beta = d + t. \tag{5.13}$$

Theorem 5.12. *The $(G+3)$-set $\mathcal{W}_G$ of* Construction B *is an arc.*

Proof. By (5.6), (5.7), the sum $d + t$ in (5.13) is running on $\{1, 2, \dots, 2G-1\}$, where $2G - 1 \le q - 3$. So, $\{0, \beta_G\} \cap \{1, 2, \dots, 2G-1\} = \emptyset$, see (5.10). Therefore Z and B_G do not lie on bisecants of $\mathcal{D}_G$. In other side, any point of $\mathcal{D}_G$ does not lie on the line ZB_G as ZB_G is the bisecant of $\mathcal{C}$ through A_∞ and A_0, where $\{A_\infty, A_0\} \cap \mathcal{D}_G = \emptyset$. □

Theorem 5.13. *Let q be a* prime power. *Assume that $q \not\equiv 3 \pmod 4$. Let G be given by* (5.6). *Then all points of $\{P\} \cup \ell_1 \cup \mathcal{C} \setminus \mathcal{D}_G$ lie on bisecants of the arc $\mathcal{W}_G$ of* Construction B.

Proof. By (5.6), (5.7), $\{\overline{A}_0, \overline{A}_{(q-1)/4}\} \subset \mathcal{D}_G$. So, the point P is covered by Corollary 5.10(ii), (iii).

All points of ℓ_1 are covered as two points Z and B_G of this line belong to $\mathcal{W}_G$.

Throughout this proof, $\mathcal{R}$ and $\mathcal{S}$ are sets of integers modulo $q-1$. It can be said that $\mathcal{R}$ and $\mathcal{S}$ are sets of indexes of powers of ξ.

Let $\mathcal{R} = \{-G, -(G-1), \dots, -1\} = \{q-1-G, q-1-(G-1), \dots, q-2\}$. By Lemma 5.11, points $\overline{A}_t, \overline{A}_{-t}, Z$ are collinear. Therefore, a point $\overline{A}_t$ of $\mathcal{C} \setminus \mathcal{D}_G$ with $t \in \mathcal{R}$ lies on the bisecant of $\mathcal{W}_G$ through $\overline{A}_{-t}$ and Z, where $-t \in \{G, G-1, \dots, 1\}$, $\overline{A}_{-t} \in \mathcal{D}_G$.

Let $\mathcal{S} = \{\beta_G - G, \beta_G - (G-1), \dots, \beta_G - 1, \beta_G\}$. By Lemma 5.11, a point $\overline{A}_t$ of $\mathcal{C} \setminus \mathcal{D}_G$ with $t \in \mathcal{S}$ lies on the bisecant $B_G\overline{A}_{\beta_G - t}$, where $\beta_G - t \in \{G, G-1, G-2, \dots, 1, 0\}$, $\overline{A}_{\beta_G - t} \in \mathcal{D}_G$.

Let $\beta_G = 2G + 1$. Then $\mathcal{S} = \{G+1, G+2, \dots, 2G+1\}$. Also, by (5.10), $q \equiv 0 \pmod 3$ and $G = \left\lfloor \frac{1}{3}(q-1) \right\rfloor = \frac{1}{3}(q-3)$ whence $3G = q - 3$ and $2G + 1 = q - G - 2$.

Let $\beta_G = 2G$. Then $\mathcal{S} = \{G, G+1, \dots, 2G\}$. Let $v \in \{0, 1, 2\}$ and $v \equiv q \pmod 3$. For $G = \left\lfloor \frac{1}{3}(q-1) \right\rfloor$, we have $v \neq 0$, see (5.10), whence $G = \frac{1}{3}(q - v)$ and $2G = q - G - v \ge q - G - 2$. For $G > \left\lfloor \frac{1}{3}(q-1) \right\rfloor$, we have $G > \frac{1}{3}(q-3)$ whence $2G > q - G - 3$ and $2G \ge q - G - 2$.

We proved that $\{G+1, G+2, \dots, q-2\} \subseteq \mathcal{S} \cup \mathcal{R}$. Also we showed that the points $\overline{A}_t$ of $\mathcal{C} \setminus \mathcal{D}_G$ with $t \in \mathcal{S} \cup \mathcal{R}$ are covered by bisecants of $\mathcal{W}_G$ either through Z (if $t \in \mathcal{R}$) or through B_G (if $t \in \mathcal{S}$). In the other side, $\mathcal{C} \setminus \mathcal{D}_G = \{\overline{A}_t : t = G+1, G+2, \dots, q-2\} \cup \{A_\infty, A_0\}$, where $\{A_\infty, A_0\} \subset \ell_1$. So, all points of $\mathcal{C} \setminus \mathcal{D}_G$ are covered. □

Definition 5.14. Let q be a *prime power*. Let $q \not\equiv 3 \pmod 4$. For integer $\overline{G}$, let

$$\mathcal{Z}_{\overline{G}} = \{Z\} \cup \{\overline{A}_d : d = 0, 1, 2, \ldots, \overline{G}\}.$$

We call *critical value of* $\overline{G}$ and denote by $\overline{G}_q$ the *smallest* value of $\overline{G}$ such that all points $(1, a, b)$ with $a \in F_q^*$, $b \in F_q$, $b \neq a^2$, and all points $(0, 1, b)$ with $b \in F_q^*$, lie on bisecants of $\mathcal{Z}_{\overline{G}}$.

Theorem 5.15. *Let $q \geq 32$ be a* prime power. *Let $q \not\equiv 3 \pmod 4$. If $\overline{G}_q \leq \lceil \frac{1}{2}(q-3) \rceil$ and*

$$\max\left\{\overline{G}_q, \left\lfloor \frac{q-1}{3} \right\rfloor\right\} \leq G \leq \left\lceil \frac{q-3}{2} \right\rceil,$$

then the arc $\mathcal{W}_G$ of Construction B *is complete.*

Proof. We use Theorem 5.13 and Definition 5.14. □

In this subsection, we put $q \geq 32$ as we checked by computer that $\lceil \frac{1}{2}(q-3) \rceil < \overline{G}_q$ if $q \leq 29$.

Corollary 5.16. *Let $q \not\equiv 3 \pmod 4$ be a* prime power. *If $\overline{G}_q \leq \lceil \frac{1}{2}(q-3) \rceil$, then* Construction B *forms a family of complete k-arcs in $PG(2, q)$ containing arcs of all sizes k in the region*

$$\max\left\{\overline{G}_q, \left\lfloor \frac{q-1}{3} \right\rfloor\right\} + 3 \leq k \leq \left\lceil \frac{q-3}{2} \right\rceil + 3 = \begin{cases} \frac{1}{2}(q+3) & \text{if } q \text{ odd} \\ \frac{1}{2}(q+4) & \text{if } q \text{ even}. \end{cases}$$

If $\overline{G}_q \leq \lfloor \frac{1}{3}(q-1) \rfloor$, then size of the smallest complete arc of the family is $\lfloor \frac{1}{3}(q+8) \rfloor$.

By computer search using Lemmas 5.1, 5.9 and 5.11 we obtained the following theorem.

Theorem 5.17. *Let $q \not\equiv 3 \pmod 4$ be a* prime power. *Let $\overline{G}_q$ be given by* Definition 5.14. *We introduce d_q and δ_q as follows: $\overline{G}_q = d_q\sqrt{q}\ln q$, $\delta_q = \lfloor \frac{1}{3}(q-1) \rfloor - \overline{G}_q$. Then the following holds.*

$$\text{(i)} \left\lfloor \frac{q-1}{3} \right\rfloor < \overline{G}_q \leq \left\lceil \frac{q-3}{2} \right\rceil \quad \text{if } 32 \leq q \leq 81 \text{ and } q = 97, 101, 125.$$

$$\text{(ii)}\ \overline{G}_q \leq \left\lfloor \frac{q-1}{3} \right\rfloor, \quad \text{if } 128 \leq q \leq 2063,\ q = 89, 109, 113, 121.$$

$$\text{(iii)}\ \overline{G}_q < 0.87\sqrt{q}\ln q \quad \text{if } 32 \leq q \leq 2063. \tag{5.14}$$

$$\text{(iv)} \begin{array}{lll} 1 \leq \delta_q \leq 93, & 0.60 < d_q < 0.81, & \text{if } 128 \leq q \leq 593; \\ 89 \leq \delta_q \leq 196, & 0.64 < d_q < 0.84, & \text{if } 593 < q \leq 1049; \\ 183 \leq \delta_q \leq 267, & 0.62 < d_q < 0.80, & \text{if } 1049 < q \leq 1361; \\ 271 \leq \delta_q \leq 464, & 0.62 < d_q < 0.87, & \text{if } 1361 < q \leq 2063. \end{array}$$

For situations $\lfloor \frac{1}{3}(q-1) \rfloor < \overline{G}_q$, the values of $\overline{G}_q$ are given in Table 5.

Theorem 5.18. *Let $q \not\equiv 3 \pmod 4$ be a* prime power. *Let $128 \leq q \leq 2063$, or $q = 89, 109, 113, 121$. Then* Construction B *forms a family of complete k-arcs in $PG(2, q)$ containing arcs of all sizes k in the region*

$$\left\lfloor \frac{q+8}{3} \right\rfloor \leq k \leq \begin{cases} \frac{1}{2}(q+3) & \text{if } q \text{ odd} \\ \frac{1}{2}(q+4) & \text{if } q \text{ even}. \end{cases}$$

Proof. We use Corollary 5.16 and Theorem 5.17(ii). □

5.3. Arcs with three points outside a conic

Throughout this subsection, $q \geq 27$ is a *prime power* and also $q \equiv 3 \pmod 4$.

Let J be an integer in the region

$$\frac{q-3}{4} \leq J \leq \frac{q-3}{2}. \tag{5.15}$$

Let

$$\beta_J = 2J. \tag{5.16}$$

Notations $\mathcal{D}_J$ and B_J are taken from (5.7) and (5.9) with substitution of G by J and by applying (5.16). Using (5.15) and (5.16), it is easy to see that Corollary 5.10(i), (iv), Theorem 5.12 and their proofs hold for $\mathcal{D}_J$, B_J, and β_J as well as for $\mathcal{D}_G$, B_G, and β_G.

Construction C. *Let $q \equiv 3 \pmod 4$ be a* prime power. *Let $P, J, \mathcal{D}_J, Z, B_J$, and β_J be given by* (5.3), (5.15), (5.7), (5.9) *and* (5.16). *We construct a point $(J+4)$-set $\mathcal{E}_J$ in $PG(2,q)$ as follows:*

$$\mathcal{E}_J = \mathcal{D}_J \cup \{P, Z, B_J\}.$$

Theorem 5.19. *The $(J+4)$-set $\mathcal{E}_J$ of* Construction C *is an arc.*

Proof. The set $\mathcal{D}_J \cup \{Z, B_J\}$ is an arc due to Theorem 5.12. By Corollary 5.10(i), (iv), the point P does not lie on bisecants of $\mathcal{D}_J$ and $\mathcal{D}_J \cup \{Z, B_J\}$. Finally, P, Z, B_J are not collinear. □

Theorem 5.20. *Let $q \equiv 3 \pmod 4$ be a* prime power. *Let J and β_J be given by* (5.15) *and* (5.16). *Then all points of $\ell_1 \cup \mathcal{C} \setminus \mathcal{D}_J$ lie on bisecants of the arc $\mathcal{E}_J$ of* Construction C.

Proof. All points of ℓ_1 are covered as two points Z and B_J of this line belong to $\mathcal{E}_J$.

Throughout this proof, $\mathcal{R}$, $\mathcal{S}$, and $\mathcal{T}$ are sets of integers modulo $q-1$. It can be said that $\mathcal{R}$, $\mathcal{S}$, and $\mathcal{T}$ are sets of indexes of powers of ξ. We act similarly to the proof of Theorem 5.13.

Let $\mathcal{R} = \{-J, -(J-1), \ldots, -1\} = \{q-1-J, q-1-(J-1), \ldots, q-2\}$. By Lemma 5.11, a point $\bar{A}_t$ of $\mathcal{C} \setminus \mathcal{D}_J$ with $t \in \mathcal{R}$ lies on the bisecant of $\mathcal{E}_J$ through $\bar{A}_{-t}$ and Z, where $-t \in \{J, J-1, \ldots, 1\}$, $\bar{A}_{-t} \in \mathcal{D}_J$.

Let $\mathcal{S} = \{\beta_J - J, \beta_J - (J-1), \ldots, \beta_J - 1, \beta_J\}$. By Lemma 5.11, a point $\bar{A}_t$ of $\mathcal{C} \setminus \mathcal{D}_J$ with $t \in \mathcal{S}$ lies on the bisecant $B_J\bar{A}_{\beta_J - t}$, where $\beta_J - t \in \{J, J-1, J-2, \ldots, 1, 0\}$, $\bar{A}_{\beta_J - t} \in \mathcal{D}_J$.

Let $\mathcal{T} = \left\{\frac{1}{2}(q-1), \frac{1}{2}(q-1)+1, \ldots, \frac{1}{2}(q-1)+J\right\}$. By (5.8), (5.11), points P, $\bar{A}_t$, and $\bar{A}_{t+(q-1)/2}$ are collinear. Therefore, a point $\bar{A}_t$ of $\mathcal{C} \setminus \mathcal{D}_J$ with $t \in \mathcal{T}$ lies on the bisecant $P\bar{A}_{t+(q-1)/2}$, where $t + \frac{1}{2}(q-1) \in \{q-1, q, \ldots, q-1+J\} = \{0, 1, \ldots, J\}$, $\bar{A}_{t+(q-1)/2} \in \mathcal{D}_J$.

As $\beta_J = 2J$, we have $\mathcal{S} = \{J, J+1, \ldots, 2J\}$, where $2J \geq \frac{1}{2}(q-1) - 1$; see (5.15). Also, by (5.15), $\frac{1}{2}(q-1) + J \geq \frac{1}{4}(3q-5)$ while $q - 1 - J \leq \frac{1}{4}(3q-5) + 1$.

We proved that $\{J+1, J+2, \ldots, q-2\} \subseteq \mathcal{S} \cup \mathcal{R} \cup \mathcal{T}$. Also we showed that the points $\bar{A}_t$ of $\mathcal{C} \setminus \mathcal{D}_J$ with $t \in \mathcal{S} \cup \mathcal{R} \cup \mathcal{T}$ are covered by bisecants of $\mathcal{E}_J$ either through Z (if $t \in \mathcal{R}$) or through B_J (if $t \in \mathcal{S}$) or, finally, through P (if $t \in \mathcal{T}$). In the other side, $\mathcal{C} \setminus \mathcal{D}_J = \{\bar{A}_t : t = J+1, J+2, \ldots, q-2\} \cup \{A_\infty, A_0\}$, where $\{A_\infty, A_0\} \subset \ell_1$. So, all points of $\mathcal{C} \setminus \mathcal{D}_J$ are covered. □

Definition 5.21. Let $q \equiv 3 \pmod 4$ be a *prime power*. For integer $\bar{J}$, let

$$\mathcal{Q}_{\bar{J}} = \{P, Z\} \cup \{\bar{A}_d : d = 0, 1, 2, \ldots, \bar{J}\}.$$

We call *critical value of* $\bar{J}$ and denote by $\bar{J}_q$ the *smallest* value of $\bar{J}$ such that all points $(1, a, b)$ with $a \in F_q^*$, $b \in F_q$, $b \neq a^2$, and all points $(0, 1, b)$ with $b \in F_q^*$, lie on bisecants of $\mathcal{Q}_{\bar{J}}$.

Theorem 5.22. *Let $q \geq 27$ be a* prime power. *Let $q \equiv 3 \pmod 4$. If $\bar{J}_q \leq \frac{1}{2}(q-3)$ and*

$$\max\left\{\bar{J}_q, \frac{q-3}{4}\right\} \leq J \leq \frac{q-3}{2},$$

then the arc $\mathcal{E}_J$ of Construction C *is complete.*

Proof. We use Theorem 5.20 and Definition 5.21. □

In this subsection, we put $q \geq 27$ as we checked by computer that $\frac{1}{2}(q-3) < \bar{J}_q$ if $q \leq 23$.

Corollary 5.23. *Let $q \equiv 3 \pmod 4$ be a* prime power. *If $\bar{J}_q \leq \frac{1}{2}(q-3)$, then* Construction C *forms a family of complete k-arcs in $PG(2,q)$ containing arcs of all sizes k in the region*

$$\max\left\{\bar{J}_q, \frac{q-3}{4}\right\} + 4 \leq k \leq \frac{q+5}{2}.$$

If $\bar{J}_q \leq \frac{1}{4}(q-3)$, then the cardinality of this family is equal to $\frac{1}{4}(q-3)$ and the size of the smallest complete arc of the family is $\frac{1}{4}(q+13)$.

By computer search using Lemmas 5.1, 5.9 and 5.11 we obtained the following theorem.

Theorem 5.24. *Let $q \equiv 3 \pmod 4$ be a* prime power. *Let $\bar{J}_q$ be given by Definition* 5.21. *We introduce r_q and θ_q as follows: $\bar{J}_q = r_q\sqrt{q}\ln q$, $\theta_q = \frac{1}{4}(q-3) - \bar{J}_q$. Then it holds that*

$$\begin{array}{ll}
\text{(i) } \dfrac{q-3}{4} < \bar{J}_q \le \dfrac{q-3}{2} \quad \textit{if } 27 \le q \le 191 \textit{ and } q = 211, 223, 343; & \\
\text{(ii) } \bar{J}_q \le \dfrac{q-3}{4}, \quad \textit{if } 347 \le q \le 2063,\ q = 199, 227, 239, 243, 251, 263, 271, 283, 307, 311, 331; & \\
\text{(iii) } \bar{J}_q < 0.85\sqrt{q}\ln q \quad \textit{if } 27 \le q \le 2063; & \\
\text{(iv) } \begin{array}{lll}
3 \le \theta_q \le 44, & 0.61 < r_q < 0.78, & \textit{if } 347 \le q \le 599; \\
18 \le \theta_q \le 111, & 0.61 < r_q < 0.85, & \textit{if } 599 < q \le 991; \\
89 \le \theta_q \le 167, & 0.61 < r_q < 0.76, & \textit{if } 991 < q \le 1367; \\
163 \le \theta_q \le 296, & 0.59 < r_q < 0.73, & \textit{if } 1367 < q \le 2063.
\end{array} &
\end{array} \tag{5.17}$$

For situations $\frac{1}{4}(q-3) < \bar{J}_q$, the values of $\bar{J}_q$ are given in Table 5.

Theorem 5.25. *Let* $q \equiv 3 \pmod 4$ *be a* prime power. *Let* $347 \le q \le 2063$, *or* $q = 199, 227, 239, 243, 251, 263, 271, 283, 307, 311, 331$. *Then* Construction C *forms a family of complete k-arcs in* $PG(2,q)$ *containing arcs of all sizes k in the region*

$$\frac{q+13}{4} \le k \le \frac{q+5}{2}.$$

Proof. We use Corollary 5.23 and Theorem 5.24(ii). □

Based on Theorems 5.7, 5.17 and 5.24, we conjecture the following; cf. Conjecture 1.5.

Conjecture 5.26. *Let* $\overline{H}_q, \overline{G}_q, \bar{J}_q$ *be given by* Definitions 5.4, 5.14 *and* 5.21. *Let for* $\overline{H}_q$, q *be prime while for* $\overline{G}_q$ *and* $\bar{J}_q$ *it holds that* q *is a prime power. Finally, let* $q \not\equiv 3 \pmod 4$ *for* $\overline{G}_q$ *and* $q \equiv 3 \pmod 4$ *for* $\bar{J}_q$. *Then the following holds.*

$$\begin{array}{lll}
\overline{H}_q \le \left\lfloor \dfrac{q-1}{3} \right\rfloor \quad \textit{if } q \ge 109; & \overline{G}_q \le \left\lfloor \dfrac{q-1}{3} \right\rfloor \quad \textit{if } q \ge 128; & \bar{J}_q \le \dfrac{q-3}{4} \quad \textit{if } q \ge 347. \\
\overline{H}_q < \sqrt{q}\ln q \quad \textit{if } q \ge 19; & \overline{G}_q < \sqrt{q}\ln q \quad \textit{if } q \ge 32; & \bar{J}_q < \sqrt{q}\ln q \quad \textit{if } q \ge 27.
\end{array} \tag{5.18}$$

Remark 5.27. It is interesting to compare the relations (5.5), (5.14), (5.17) and (5.18) with Theorems 1.2, 4.1 and 4.3 and Conjecture 1.3 and to compare also computer results provided by Tables 1 and 2 with Theorems 5.7, 5.17 and 5.24. One can see that the upper estimates of $t_2(2,q)$, $\overline{H}_q$, $\overline{G}_q$, and $\bar{J}_q$ have the same structure and the values of $\bar{t}_2(2,q)$, $\overline{H}_q$, $\overline{G}_q$, and $\bar{J}_q$ have a close order. This seems to be natural as *almost all* points of $PG(2,q)$ lie on bisecants of $\mathcal{P}_{\overline{H}_q}$, $\mathcal{Z}_{\overline{G}_q}$, and $\mathcal{Q}_{\bar{J}_q}$; see Definitions 5.4, 5.14 and 5.21.

Remark 5.28. The complete arcs of Constructions A–C can be used as starting objects in inductive constructions. For example, for even q, arcs of Construction B can be used in constructions of [9, Ths. 1.1, 3.14–3.17, 4.6–4.8]. In that way, one can generate infinite sets of families of complete caps in projective spaces $PG(v, 2^n)$ of growing dimensions v. For every v, constructions of [9] can obtain a complete cap from every complete arc of Construction B. Also, it can be shown that in Constructions A–C all points not on conic are *external*. So, the arcs of Constructions A and C for $q \equiv 3 \pmod 4$ can be used as starting objects in constructions of [32]; see [32, Th. 23]. Thereby, infinite families of large complete arcs in $PG(2, q^n)$ with growing n can be obtained.

6. On the spectrum of possible sizes of complete arcs in $PG(2,q)$

The main known results on the spectrum of possible sizes of complete arcs in $PG(2,q)$ are given in Introduction with the corresponding references. Taking into account the results cited in Introduction, we denote

$$M_q = \begin{cases} \dfrac{1}{2}(q+4) & \text{for even } q \\ \dfrac{1}{2}(q+7) & \text{for odd } q \text{ included to (1.5)} \\ \dfrac{1}{2}(q+5) & \text{for odd } q \text{ not included to (1.5).} \end{cases}$$

We suppose that the smallest known sizes $\bar{t}_2(2,q)$ are given in Tables 1–4 of this paper.

Theorem 6.1. *In* $PG(2,q)$ *with* $25 \le q \le 349$, *and* $q = 1013, 2003$, *there are complete k-arcs of* all *the sizes in the region*

$$\bar{t}_2(2,q) \le k \le M_q.$$

Proof. For $25 \le q \le 167$ the assertion of the theorem follows from [3, Tab. 2] and [5, Tab. 2]. For $169 \le q \le 349$ and $q = 1013, 2003$, we used Constructions A–C of Section 5. The sizes not following from the constructions are obtained in this work by the randomized greedy algorithms. □

An experience obtained in computer search for the proof of Theorem 6.1 allows us to do Conjecture 6.2. Here we took into account sizes that can be got by Constructions A–C and the remark on sizes close to $\bar{t}_2(2, q)$ in the end of Section 2. Note also that the remaining sizes for all $169 \le q \le 349$ and $q = 1013, 2003$ were relatively easy obtained by the greedy algorithms with point subset of a conic taken as the starting set S_0. For this we used consequently subsets of cardinality approximately 15%, 20%, 25%, 30% of the conic cardinality $q + 1$.

Conjecture 6.2. *Let $353 \le q \le 5107$, $q \in T_2 \cup T_3$, be a prime power. Then in $PG(2, q)$ there are complete k-arcs of* all *the sizes in the region $\bar{t}_2(2, q) \le k \le M_q$. Moreover, complete k-arcs with $\bar{t}_2(2, q) \le k \le \frac{1}{2}(q + 5)$ can be obtained either by Constructions A–C or by the randomized greedy algorithms.*

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