Advances in Mathematics of Communications
Volume 5, No. 1, 2011, 119–147

doi:10.3934/amc.2011.5.119

LINEAR NONBINARY COVERING CODES AND SATURATING SETS IN PROJECTIVE SPACES

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(Communicated by Simon Litsyn)

Abstract. Let $\mathcal{A}_{R,q}$ denote a family of covering codes, in which the covering radius R and the size q of the underlying Galois field are fixed, while the code length tends to infinity. The construction of families with small asymptotic covering densities is a classical problem in the area of Covering Codes.

In this paper, infinite sets of families $\mathcal{A}_{R,q}$, where R is fixed but q ranges over an infinite set of prime powers are considered, and the dependence on q of the asymptotic covering densities of $\mathcal{A}_{R,q}$ is investigated. It turns out that for the upper limit $\mu_q^*(R,\mathcal{A}_{R,q})$ of the covering density of $\mathcal{A}_{R,q}$, the best possibility is $\mu_q^*(R,\mathcal{A}_{R,q}) = O(q)$. The main achievement of the present paper is the construction of *optimal* infinite sets of families $\mathcal{A}_{R,q}$, that is, sets of families such that relation $\mu_q^*(R,\mathcal{A}_{R,q}) = O(q)$ holds, for any covering radius $R \geq 2$.

We first showed that for a given R, to obtain optimal infinite sets of families it is enough to construct R infinite families $\mathcal{A}_{R,q}^{(0)}, \mathcal{A}_{R,q}^{(1)}, \ldots, \mathcal{A}_{R,q}^{(R-1)}$ such that, for all $u \geq u_0$, the family $\mathcal{A}_{R,q}^{(\gamma)}$ contains codes of codimension $r_u = Ru + \gamma$ and length $f_q^{(\gamma)}(r_u)$ where $f_q^{(\gamma)}(r) = O(q^{(r-R)/R})$ and u_0 is a constant. Then, we were able to construct the necessary families $\mathcal{A}_{R,q}^{(\gamma)}$ for any covering radius $R \geq 2$, with q ranging over the (infinite) set of R-th powers. A result of independent interest is that in each of these families $\mathcal{A}_{R,q}^{(\gamma)}$, the lower limit of the covering density is bounded from above by a constant independent of q.

The key tool in our investigation is the design of new small saturating sets in projective spaces over finite fields, which are used as the starting point for the q^m-concatenating constructions of covering codes. A new concept of N-fold strong blocking set is introduced. As a result of our investigation, many new asymptotic and finite upper bounds on the length function of covering codes and on the smallest sizes of saturating sets, are also obtained. Updated tables for these upper bounds are provided. An analysis and a survey of the known results are presented.

2000 *Mathematics Subject Classification:* Primary: 94B05, 51E22; Secondary: 94B25, 94B27, 94B65, 51E21.

Key words and phrases: Linear covering codes, nonbinary codes, saturating sets in projective spaces, covering density.

1. Introduction

1.1. Covering codes and covering densities. Let F_q be the Galois field with q elements. Let F_q^n be the n-dimensional vector space over F_q. Denote by $[n, n-r]_q$ a q-ary linear code of length n and codimension (redundancy) r, that is, a subspace of F_q^n of dimension $n-r$. For an introduction to coding theory, see [51, 54].

The Hamming distance $d(v, c)$ of vectors v and c in F_q^n is the number of positions in which v and c differ. The smallest Hamming distance between distinct code vectors is called the minimum distance of the code. An $[n, n-r]_q$ code with minimum distance d is denoted as an $[n, n-r, d]_q$ code. The sphere of radius R with center c in F_q^n is the set $\{v : v \in F_q^n, d(v, c) \leq R\}$.

Definition 1.1. *i)* The *covering radius* of an $[n, n-r]_q$ code is the least integer R such that the space F_q^n is covered by spheres of radius R centered at codewords.

ii) A linear $[n, n-r]_q$ code has *covering radius* R if every vector of F_q^r is equal to a linear combination of at most R columns of a parity check matrix of the code, and R is the smallest value with such property.

Definition 1.1i makes sense for both linear and nonlinear codes. For linear codes Definitions 1.1i and 1.1ii are equivalent. An $[n, n-r]_q R$ code ($[n, n-r, d]_q R$ code, resp.) is an $[n, n-r]_q$ code ($[n, n-r, d]_q$ code, resp.) with covering radius R. For an introduction to coverings of vector Hamming spaces over finite fields, see [9, 11, 12, 41].

The covering problem for codes is that of finding codes with small covering radius with respect to their lengths and dimensions. Codes investigated from the point of view of the covering problem are usually called *covering codes* (in contrast to error-correcting codes) [41].

Problems connected with covering codes are considered in numerous works, see e.g. [2, 3], [9]–[21], [27, 28, 29, 31, 32, 33, 35, 41, 45, 46, 53, 55] and the references therein, and the online bibliography of [50]. In this work, we mainly give references to researches on nonbinary codes; some papers on binary codes are also mentioned as they contain useful general ideas. It should be noted that the monographs [9, 11] mostly deal with binary covering codes, and that no surveys on nonbinary covering codes have been recently published. In this work we try to make up for this deficiency for linear codes; in particular, for infinite linear code families. We obtain a number of new asymptotic optimal results, essentially improving the known estimates for both finite and infinitely growing code lengths. The description of new results is provided, along with a survey of the known ones and their updates.

Studying covering codes is a classical combinatorial task. Covering codes are connected with many areas of information theory and combinatorics, see e.g. [11, Sec. 1.2] where problems of data compression, decoding errors and erasures, football pools, write-once memories, Berlekamp-Gale game, and Caley graphs are mentioned. Covering codes can also be used in problems of reduction of switching noise and in steganography, see [6, Chap. 14], [7, 36] and the references therein. Codes of covering radius 2 and codimension 3 are relevant, for example, for defining sets of block designs [8] and the degree/diameter problem in graph theory [47]. Covering codes can be used in databases [44]. There are connections between covering codes and a popular game puzzle, called "Hats-on-a-line" [1]. Covering codes can be also used to construct identifying codes [34] .

It should be particularly emphasized that linear covering codes are deeply connected with *saturating sets* in *projective spaces* over finite fields. Let $PG(v, q)$ be the v-dimensional projective space over F_q. For an introduction to such spaces and the related geometrical objects, see [42, 43, 49].

A set of points $S \subseteq PG(v,q)$ is said to be *ϱ-saturating* if for any point $x \in PG(v,q)$ there exist $\varrho+1$ points in S generating a subspace of $PG(v,q)$ containing x, and ϱ is the smallest value with such property, cf. [15, Def. 1.1], [21, 26, 27, 30, 52, 56].

As usual, by an n-set of $PG(v,q)$ we mean a point set of size n. Homogenous coordinates of points of an $(R-1)$-saturating n-set K in the projective space $PG(r-1,q)$ can be treated as columns of a parity-check matrix of an $[n,n-r]_qR$ *related* covering code $\mathcal{C}_K$ [15, 26, 27, 30, 31].

In the literature, saturating sets are also called "saturated sets" [15, 17, 48, 56], "spanning sets" [10], and "dense sets"[8, 37, 40].

Let $V_q(n,R)$ be the cardinality of the sphere of radius R in the vector space F_q^n.

$$V_q(n,R) = \sum_{i=0}^{R}(q-1)^i\binom{n}{i}. \tag{1.1}$$

The covering quality of an $[n,n-r(\mathcal{C})]_qR$ code $\mathcal{C}$ of codimension $r(\mathcal{C})$ can be measured by its *covering density*

$$\mu_q(n,R,\mathcal{C}) = \frac{V_q(n,R)}{q^{r(\mathcal{C})}}. \tag{1.2}$$

We will write $\mu_q(n,R)$ for $\mu_q(n,R,\mathcal{C})$ when the code $\mathcal{C}$ is clear from the context. Note that $\mu_q(n,R,\mathcal{C}) \geq 1$ and equality holds when $\mathcal{C}$ is a perfect code. From the point of view of the covering problem, the best codes are those with small covering density.

For fixed parameters r, R, and q, the smaller is the length n of an $[n,n-r]_qR$ code, the smaller is its covering density. The *length function* $\ell_q(r,R)$ is the smallest length of a q-ary linear code with codimension r and covering radius R [9, 10] . The *smallest known* length of such code is denoted by $\overline{\ell}_q(r,R)$. Clearly, $\ell_q(r,R) \leq \overline{\ell}_q(r,R)$, and the existence of an $[n,n-r]_qR$ code or, equivalently, of an $(R-1)$-saturating n-set in $PG(r-1,q)$ implies the upper bound $\overline{\ell}_q(r,R) \leq n$.

Fact 1.2. *If there is an $[n,n-r]_qR$ code then there is an $[n+1,n+1-r]_qR$ code.*

One can obtain an $[n+1,n+1-r]_qR$ code by attaching an arbitrary column to a parity check matrix of an $[n,n-r]_qR$ code $\mathcal{C}$, or, equivalently, by adding an information symbol. Clearly, by repeating the process it is possible to obtain an $[n+\delta,n+\delta-r]_qR$ code from $\mathcal{C}$ for any integer $\delta \geq 1$. We will call such a code a *δ-extension* of $\mathcal{C}$.

For a given $R \geq 1$ and for a fixed prime power q, let $\mathcal{A}_{R,q}$ denote an infinite sequence of q-ary linear $[n,n-r_n]_qR$ codes $\mathcal{C}_n$, $n \geq R$. Throughout the paper, an infinite sequence $\mathcal{A}_{R,q}$ will be called an *infinite family of covering codes* or an infinite code family, or simply an infinite family.

Given an infinite family $\mathcal{A}_{R,q}$, we consider its *asymptotic covering densities*

$$\overline{\mu}_q(R,\mathcal{A}_{R,q}) = \liminf_{n\to\infty}\mu_q(n,R,\mathcal{C}_n), \tag{1.3}$$

$$\mu_q^*(R,\mathcal{A}_{R,q}) = \limsup_{n\to\infty}\mu_q(n,R,\mathcal{C}_n). \tag{1.4}$$

We will write $\overline{\mu}_q(R)$ ($\mu_q^*(R)$ resp.) for $\overline{\mu}_q(R,\mathcal{A}_{R,q})$ ($\mu_q^*(R,\mathcal{A}_{R,q})$ resp.) if the family $\mathcal{A}_{R,q}$ is clear from the context.

For an infinite family $\mathcal{A}_{R,q}$ the sequence of codimensions r_n will be assumed to be non-decreasing. In fact, if $r_{n+1} < r_n$ for some n, then any 1-extension $\mathcal{C}^*$ of $\mathcal{C}_n$ has a better covering density than $\mathcal{C}_{n+1}$, and therefore it is convenient to replace $\mathcal{C}_{n+1}$ with $\mathcal{C}^*$.

A code $\mathcal{C}_n$ will be called a *supporting code* of $\mathcal{A}_{R,q}$ if $r_n > r_{n-1}$, a *filling code* otherwise. It is immediately seen that a filling code must have the same parameters as a δ-extension of some supporting code, and this motivates our notation. The subsequence of supporting codes will be denoted as $\mathcal{C}_{n_i}$.

Throughout the paper, when we construct an infinite family, we will only describe supporting codes; the filling codes will be assumed to be obtained via δ-extension. The expression "to construct a family" will mean "to construct the supporting codes of a family".

In this work we will mainly deal with infinite families $\mathcal{A}_{R,q}$ for which the lengths and the codimensions of the supporting codes $\mathcal{C}_{n_i}$ are related by some function, namely $n_i = f_q(r_i)$ where f_q is an increasing function for a fixed q. In most cases, an explicit expression for the function f_q will be given.

By (1.2), the covering density of an $[n+1, n+1-r]_qR$ is greater than that of an $[n, n-r]_qR$ one. Therefore,

$$\overline{\mu}_q(R, \mathcal{A}_{R,q}) = \liminf_{i\to\infty} \mu_q(n_i, R, \mathcal{C}_{n_i}), \tag{1.5}$$

$$\mu^*_q(R, \mathcal{A}_{R,q}) = \limsup_{i\to\infty} \mu_q(n_{i+1}-1, R, \mathcal{C}_{n_{i+1}-1}). \tag{1.6}$$

Note that by (1.5), (1.6), the lower limit of the asymptotic covering density depends only on the supporting codes, while the upper limit depends on the filling codes.

1.2. Open Problems 1 and 2. The size q of the base field F_q is fixed for a given family $\mathcal{A}_{R,q}$. It is however natural to consider an *infinite set of families* $\mathcal{A}_{R,q}$ with fixed R and infinitely growing q. In most constructions, $f_q(r)$ is an increasing function of q for a fixed r. Therefore, a central problem for *linear* covering codes is the following:

For a fixed covering radius R, find a set of families $\mathcal{A}_{R,q}$ of q-ary codes with q ranging over an infinite set of prime power, such that the covering densities (1.5) *and* (1.6) *are asymptotically as small as possible with respect to the size of the base field q.*

This problem has distinct perspectives and solutions for lower and upper limits.

As to the lower limit (1.5), it can happen that the asymptotic covering density of a family $\mathcal{A}_{R,q}$ is bounded from above by a constant independent of q. In this case $\overline{\mu}_q(R, \mathcal{A}_{R,q}) = O(1)$ and the family $\mathcal{A}_{R,q}$ is said to be "*good*". Accordingly, an $[n, n-r]_qR$ covering code is called "*short*" if $n = O(q^{(r-R)/R})$. By (1.2) and (1.3), a family $\mathcal{A}_{R,q}$ consisting of short codes is good. In this case, $f_q(r) = O(q^{(r-R)/R})$. A saturating set K will be said to be "*small*" if the related covering code $\mathcal{C}_K$ is short.

A classical example is the direct sum [11] of R copies of the $[\frac{q^i-1}{q-1}, \frac{q^i-1}{q-1} - i]_q1$ perfect Hamming code, which gives an infinite family $\mathcal{A}_{R,q}$ of $[n_i, n_i - r_i]_qR$ codes with parameters

$$\mathcal{A}_{R,q} : n_i = R\frac{q^i-1}{q-1},\ r_i = Ri,\ i = 1, 2, 3, \ldots;\ \overline{\mu}_q(R) = O\left(\frac{R^R}{R!}\right). \tag{1.7}$$

When the upper limit is considered, it is not possible to obtain an upper bound independent on q. This depends on the fact that

$$\mu_q(n_{i+1}-1,R,C_{n_{i+1}-1}) = \frac{V_q(n_{i+1}-1,R)}{q^{r_i}} = \\ = \mu_q(n_{i+1},R,C_{n_{i+1}})\frac{V_q(n_{i+1}-1,R)}{V_q(n_{i+1},R)}\cdot\frac{q^{r_{i+1}}}{q^{r_i}}.$$

Since $r_{i+1} > r_i$, this implies that the optimal case is $\mu_q^*(R,\mathcal{A}_{R,q}) = O(q)$. Then the following natural issue arises.

Open Problem 1. *For any covering radius $R \geq 2$, construct an infinite code family $\mathcal{A}_{R,q}$ with $\mu_q^*(R,\mathcal{A}_{R,q}) = O(q)$.*

To solve Open Problem 1 it is convenient to proceed as follows. For any given integer γ with $0 \leq \gamma \leq R-1$, construct an infinite family $\mathcal{A}_{R,q}^{(\gamma)}$ whose supporting codes are $[n_u, n_u - r_u]_q R$ codes with codimension $r_u = Ru + \gamma$ and lenght $n_u = f_q^{(\gamma)}(r_u)$, for any integer $u \geq u_0$, with u_0 a constant which may depend on the family. Considering families of type $\mathcal{A}_{R,q}^{(\gamma)}$ is a standard method of investigation of *linear* covering codes, see [11, 15, 17, 19, 27, 28, 29, 31] and the references therein; families $\mathcal{A}_{R,q}^{(\gamma)}$ with distinct values of γ often have distinct properties.

Assume that we have R good infinite code families $\mathcal{A}_{R,q}^{(\gamma)}$, $\gamma = 0, 1, \dots, R-1$. Let us consider the infinite family $\widehat{\mathcal{A}}_{R,q}$, whose set of supporting codes is the union of those of all the families $\mathcal{A}_{R,q}^{(\gamma)}$. The family $\widehat{\mathcal{A}}_{R,q}$ contains an infinite sequence of $[n_j, n_j - j]_q R$ codes $\mathcal{C}_j$ with length $n_j = f_q^{(\gamma_j)}(j)$, $\gamma_j \equiv j \pmod R$, for any $j \geq j_0$, with j_0 a constant depending on the constants u_0 of the starting families. Note that it may occur that $n_{v+1} \leq n_v$ for some v. In this case we replace the code $\mathcal{C}_v$ by an $[n_{v+1}-1, n_{v+1}-1-v]_q R$ code that can always be obtained from $\mathcal{C}_{v+1}$ by removing a redundancy symbol and a suitable parity check. Arguing as before,

$$\mu_q^*(R,\widehat{\mathcal{A}}_{R,q}) = \limsup_{j\to\infty}\frac{V_q(n_{j+1},R)}{q^{j+1}}\cdot\frac{V_q(n_{j+1}-1,R)}{V_q(n_{j+1},R)}\cdot\frac{q^{j+1}}{q^j}.$$

Since all families $\mathcal{A}_{R,q}^{(\gamma)}$ are good, we have $V_q(n_{j+1},R)/q^{j+1} = O(1)$. Hence,

$$\mu_q^*(R,\widehat{\mathcal{A}}_{R,q}) = O(q).$$

Therefore, to solve Open Problem 1 it is sufficient to find a solution to Open Problem 2.

Open Problem 2. *For any covering radius $R \geq 2$, construct R infinite code families $\mathcal{A}_{R,q}^{(0)}, \mathcal{A}_{R,q}^{(1)}, \dots, \mathcal{A}_{R,q}^{(R-1)}$ such that for each $\gamma = 0, 1, \dots, R-1$ the supporting codes of $\mathcal{A}_{R,q}^{(\gamma)}$ are $[n_u, n_u - r_u]_q R$ codes with codimension $r_u = Ru + \gamma$ and length $n_u = f_q^{(\gamma)}(r_u)$ with $f_q^{(\gamma)}(r) = O(q^{(r-R)/R})$ for any $u \geq u_0$, with u_0 a constant which may depend on the family.*

On one hand, infinite families $\mathcal{A}_{R,q}^{(0)}$ are provided by example (1.7), and for $R = 2, 3$, families $\mathcal{A}_{R,q}^{(0)}$ with better parameters are obtained in [15, 17, 31]. On the other hand, for $\gamma \geq 1$, code families $\mathcal{A}_{R,q}^{(\gamma)}$ with density $\overline{\mu}_q(R,\mathcal{A}_{R,q}^{(\gamma)}) = O(1)$ are only known for $R = 2$, $\gamma = 1$, $q = (q')^2$ [17] , and $R = 3$, $\gamma = 1$, $q = (q')^3$ [27].

1.3. Main results and tools. In this paper, Open Problem 2 (and hence Open Problem 1) is solved for an arbitrary covering radius $R \geq 2$ and $q = (q')^R$, with q' a prime power.

Our main tools are the q^m-concatenating constructions of covering codes, and the connection between covering codes and saturating sets in projective spaces.

The q^m-concatenating constructions are proposed in [14] and are developed in [15]–[21], [27, 28, 29, 31, 32], [55, Supplement], see also [9], [11, Sec. 5.4] and the references in these works. These constructions are the fundamental tool for obtaining infinite families of covering codes with a fixed covering radius. By using a starting code as a "seed", a q^m-concatenating construction yields an infinite family of new codes with the same covering radius and with almost the same covering density. If the starting code is short then the obtained infinite family is good.

Linear codes arising from small saturating sets are a convenient choice for the starting codes of q^m-concatenating constructions [15, 17, 21, 27, 31].

The achievements of the present paper are mainly a consequence of new constructions of small saturating sets, some of which rely on the concept of a *multifold strong blocking set* which is introduced in this work. By using new small saturating sets as starting codes for the q^m-concatenating constructions we solve Open Problem 2 As a result of our previously mentioned constructions, many new upper bounds on the length function are obtained for radii $R \geq 2$.

It should be noted that by solving Open Problem 2 we have constructed, for all $1 \leq \gamma \leq R$, all $R \geq 2$, and $q = (q')^R$, infinite families $\mathcal{A}_{R,q}^{(\gamma)}$ with density $\overline{\mu}_q(R, \mathcal{A}_{R,q}^{(\gamma)}) = O(1)$, i.e. such that the *lower limit of their covering densities is bounded from above by a constant independent of* q.

Codes with $R = 2, 3$ play an important role in some constructions for $R \geq 4$, see Sections 6 and 7. Therefore we have thoroughly analyzed and collected the known results on the upper bounds on the length function for $R = 2$. We also give for $R = 2$ many new results including updated tables about the upper bounds for infinite code families. For $R = 3$ we provide a few new infinite families and some upper bounds. Regarding to the known codes with $R = 3$, the reader is referred to the recent paper [32].

The paper is organized as follows. In Section 2 the q^m-concatenating constructions, used in this work, are recalled. In Section 3 new constructions of small ϱ-saturating sets, including those relying on the new concept of strong blocking sets, are described. Section 4.1 contains updated tables of the upper bounds on $\ell_q(3, 2)$. In Sections 4, 5, and 6 we consider codes with radii $R = 2$, $R = 3$, and $R \geq 4$. Section 7 provides results for nonprime covering radius.

Some of the results from this work were briefly presented without proofs in [23, 24].

2. q^m-concatenating constructions lengthening covering codes

In this section we describe the common ideas and the popular versions of the q^m-concatenating constructions. Other versions can be found in [9], [11, Sec. 5.4], [14]–[21], [27, 28, 29, 31], [55, Supplement]. Specific constructions for $R = 2$ and $R = 3$ are given in detail in [17] and [32].

Starting with an $[n_0, n_0 - r_0]_q R$ code of length n_0, the q^m-concatenating constructions yield an infinite family of $[n, n - (r_0 + Rm)]_q R$ codes with the same covering radius R and length $n = q^m n_0 + N_m$, where m ranges over an infinite set of integers.

Here $N_m \leq R\theta_{m,q}$, where

$$\theta_{m,q} = \frac{q^m - 1}{q - 1}.$$

It should be noted that for all q^m-concatenating constructions the length n is at least $q^m n_0$; two q^m-concatenating constructions may differ by the value of N_m.

Throughout this paper, all matrices and columns are q-ary. An element of F_{q^m} written in a q-ary matrix denotes an m-dimensional q-ary column, that is a q-ary representation of this element; vice versa, an m-dimensional q-ary column can be viewed as an element of F_{q^m}.

Definition 2.1. Let $\mathbf{H}$ be a parity-check matrix of an $[n, n-r]_q R$ code V and let $0 \leq \ell \leq R$.

i) A partition of the column set of the matrix $\mathbf{H}$ into nonempty subsets is called an (R,ℓ)-*partition* if every column of F_q^r (including the zero column) is equal to a linear combination with nonzero coefficients of at least ℓ and at most R columns of $\mathbf{H}$ belonging to distinct subsets. For an $(R, 0)$-partition we can formally treat the zero column as the linear combination of 0 columns.

Any (R,ℓ)-partition is called also an R-*partition*.

ii) If $\mathbf{H}$ admits an (R,ℓ)-partition, the code V is called an (R,ℓ)-*object* and is denoted as an $[n, n-r]_q R, \ell$ code or an $[n, n-r, d]_q R, \ell$ code, where d is the minimum distance of V.

Clearly, the *trivial* partition of a parity-check matrix of an $[n, n-r]_q R, \ell$ code into n one-element subsets is an (R,ℓ)-partition.

Note that in Definition 2.1, it is not necessary that ℓ is the greatest value with the properties considered. Therefore, any (R,ℓ)-partition with $\ell > 0$ is also an (R,ℓ_1)-partition with $\ell_1 = 0, 1, \dots, \ell - 1$.

We use the term "R-partition" instead of "(R,ℓ)-partition" when the value of ℓ is not relevant or not known.

Lemma 2.2. [14, 15, 19] *An $[n, n-r, d]_q R$ code is an $[n, n-r, d]_q R, \ell$ code with $\ell \geq 1$ if and only if $d \leq R$. If $d > R$ the maximum possible value of ℓ is zero.*

A *spherical* (R,ℓ)-*capsule* with center c in F_q^n is the set $\{v : v \in F_n^q,\ 0 \leq \ell \leq d(v, c) \leq R\}$ [14]. Spherical (R,ℓ)-capsules centered at vectors of an (R,ℓ)-object cover the space F_q^n.

We give a basic q^m-concatenating construction QM based on ideas in [14, 15, 19].

Basic Construction QM. Let $\mathbf{H}_0 = [\mathbf{h}_1\mathbf{h}_2 \dots \mathbf{h}_{n_0}]$, with $\mathbf{h}_j \in F_q^{r_0}$, be a parity check matrix of an $[n_0, n_0 - r_0]_q R, \ell_0$ *starting* code V_0. Assume that $\mathbf{H}_0$ has a starting (R,ℓ_0)-partition $\mathcal{P}_0$ into p_0 subsets. Let $m \geq 1$ be an integer depending on p_0 and n_0. To each column $\mathbf{h}_j$ we associate an element $\beta_j \in F_{q^m} \cup \{*\}$ so that $\beta_i \neq \beta_j$ if columns $\mathbf{h}_i$ and $\mathbf{h}_j$ belong to *distinct* subsets of $\mathcal{P}_0$. If $\mathbf{h}_i$ and $\mathbf{h}_j$ belong to the same subset we are free to assign either $\beta_i = \beta_j$ or $\beta_i \neq \beta_j$. We call β_j an *indicator* of column $\mathbf{h}_j$. Let $\mathcal{B} = \{\beta_1, \beta_2, \dots, \beta_{n_0}\}$ be an *indicator set.* It is necessary that $|\mathcal{B}| \geq p_0$. Also, let $\mathbf{C}$ be an $(r_0 + Rm) \times N_m$ matrix with $N_m \leq (R - \ell_0)\theta_{m,q}$. Finally, define V as the $[n, n - (r_0 + Rm)]_q R_V$ code with $n = q^m n_0 + N_m$ and parity-check matrix of the following form:

$$\mathbf{H}_V = [\mathbf{C}\ \mathbf{B}_1\ \mathbf{B}_2\ \ldots\ \mathbf{B}_{n_0}], \tag{2.1}$$

$$\mathbf{B}_j = \begin{bmatrix} \mathbf{h}_j & \mathbf{h}_j & \cdots & \mathbf{h}_j \\ \xi_1 & \xi_2 & \cdots & \xi_{q^m} \\ \beta_j\xi_1 & \beta_j\xi_2 & \cdots & \beta_j\xi_{q^m} \\ \beta_j^2\xi_1 & \beta_j^2\xi_2 & \cdots & \beta_j^2\xi_{q^m} \\ \vdots & \vdots & \vdots & \vdots \\ \beta_j^{R-1}\xi_1 & \beta_j^{R-1}\xi_2 & \cdots & \beta_j^{R-1}\xi_{q^m} \end{bmatrix} \text{ if } \beta_j \in F_{q^m},$$

$$\mathbf{B}_j = \begin{bmatrix} \mathbf{h}_j & \mathbf{h}_j & \cdots & \mathbf{h}_j \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \\ \xi_1 & \xi_2 & \cdots & \xi_{q^m} \end{bmatrix} \text{ if } \beta_j = *,$$

where $\{\xi_1, \xi_2, \ldots, \xi_{q^m}\} = F_{q^m}$, $\xi_1 = 0$, $\xi_2 = 1$; the submatrix $\mathbf{C}$ is absent if $\ell_0 = R$.

If $m, \mathbf{C}$, and $\mathcal{B}$ are carefully chosen, then the covering radius R_V of the new code V is equal to the covering radius R of the starting code V_0. Examples are shown in Constructions QM_1 – QM_8 below where we use the following notation:

$\mathbf{W}_m$ is a parity-check matrix of the $[\theta_{m,q}, \theta_{m,q} - m]_q 1$ Hamming code;

$\mathbf{A}_{R',m}$ is a parity-check matrix of an $[n', n' - R'm]_q R'$ code $V_{R',m}$ (in most cases we will assume that either $n' = \overline{\ell}_q(R'm, R')$ or $n' = \ell_q(R'm, R')$);

$\mathbf{0}_k$ is the zero matrix with k rows (the number of columns will be clear by the context);

$\mathbf{\Sigma}_{R'',m}$ is the "direct sum" of R'' matrices $\mathbf{W}_m$, i.e. an $R''m \times R''\theta_{m,q}$ matrix of the form

$$\mathbf{\Sigma}_{R'',m} = \begin{bmatrix} \mathbf{W}_m & \mathbf{0}_m & \cdots & \mathbf{0}_m \\ \mathbf{0}_m & \mathbf{W}_m & \cdots & \mathbf{0}_m \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0}_m & \mathbf{0}_m & \cdots & \mathbf{W}_m \end{bmatrix}. \tag{2.2}$$

Note that $\mathbf{\Sigma}_{R'',m}$ is a parity-check matrix of an $[R''\theta_{m,q}, R''\theta_{m,q} - R''m]_q R''$ code, see Section 4.2.

Construction $\mathbf{QM}_1$. Here $R \geq 2$, $\ell_0 = 0$, $q^m + 1 \geq p_0$, $\mathcal{B} \subseteq F_{q^m} \cup \{*\}$,

$$\mathbf{C} = \begin{bmatrix} \mathbf{0}_{r_0} \\ \mathbf{\Sigma}_{R,m} \end{bmatrix}, \ n = q^m n_0 + R\theta_{m,q}.$$

Construction $\mathbf{QM}_2$. Here $R \geq 2$, $1 \leq \ell_0 < R$, $q^m \geq p_0$, $\mathcal{B} \subseteq F_{q^m}$,

$$\mathbf{C} = \begin{bmatrix} \mathbf{0}_{r_0 + m\ell_0} \\ \mathbf{\Sigma}_{R-\ell_0,m} \end{bmatrix}, \ n = q^m n_0 + (R - \ell_0)\theta_{m,q}.$$

Construction $\mathbf{QM}_3$. Here $R \geq 2$, $\ell_0 = R$, $q^m + 1 \geq p_0$, $\mathcal{B} \subseteq F_{q^m} \cup \{*\}$, $\mathbf{C}$ is absent, $n = q^m n_0$.

Construction $\mathbf{QM}_4$. Here $R \geq 2$, $\ell_0 = 0$, $q^m - 1 \geq p_0$, $\mathcal{B} \subseteq F^*_{q^m}$,

$$\mathbf{C} = \begin{bmatrix} \mathbf{0}_{r_0} & \mathbf{0}_{r_0} \\ \mathbf{\Sigma}_{\lfloor R/2 \rfloor,m} & \mathbf{0}_{m\lfloor R/2 \rfloor} \\ \mathbf{0}_{m\lceil R/2 \rceil} & \mathbf{A}_{\lceil R/2 \rceil,m} \end{bmatrix}, \ n \leq q^m n_0 + \lfloor R/2 \rfloor \theta_{m,q} + \overline{\ell}_q(m\lceil R/2 \rceil, \lceil R/2 \rceil).$$

Now we describe q^m-concatenating constructions with a **complete set of indicators** (CSI). In these versions of the basic Construction QM we *must* use *all* elements of F_{q^m} or $F_{q^m} \cup \{*\}$ as indicators β_j. To this end, sometimes, we need to assign distinct indicators to columns from the same subset of an R-partition. As a result the size of the submatrix $\mathbf{C}$ is reduced.

Construction QM$_5$. Here $R \geq 2$, $\ell_0 = 0$, $n_0 \geq q^m \geq p_0$,

$$\mathcal{B} = F_{q^m},\ \mathbf{C} = \begin{bmatrix} \mathbf{0}_{r_0+m} \\ \mathbf{\Sigma}_{R-1,m} \end{bmatrix},\ n = q^m n_0 + (R-1)\theta_{m,q}.$$

Construction QM$_6$. Here $R \geq 3$, $\ell_0 = 0$, $n_0 \geq q^m + 1 \geq p_0$,

$$\mathcal{B} = F_{q^m} \cup \{*\},\ \mathbf{C} = \begin{bmatrix} \mathbf{0}_{r_0+m} \\ \mathbf{\Sigma}_{R-1,m} \end{bmatrix},\ n = q^m n_0 + (R-1)\theta_{m,q}.$$

Construction QM$_7$. Here $R = 3$, $\ell_0 = 0$, $n_0 \geq q^m + 1 \geq p_0$, $q = 2^i$,

$$\mathcal{B} = F_{q^m} \cup \{*\},\ \mathbf{C} = \begin{bmatrix} \mathbf{0}_{r_0+m} \\ \mathbf{W}_m \\ \mathbf{0}_m \end{bmatrix},\ n = q^m n_0 + \theta_{m,q}.$$

Construction QM$_8$. Here $R = 4$, $\ell_0 = 0$, $n_0 \geq q^m \geq p_0$, 3 does not divide $q^m - 1$,

$$\mathcal{B} = F_{q^m},\ \mathbf{C} = \begin{bmatrix} \mathbf{0}_{r_0+m} & \mathbf{0}_{r_0+m} \\ \mathbf{A}_{2,m} & \mathbf{0}_{2m} \\ \mathbf{0}_m & \mathbf{W}_m \end{bmatrix},\ n \leq q^m n_0 + \overline{\ell}_q(2m, 2) + \theta_{m,q}.$$

Other constructions with a CSI including these with $n_0 < q^m$ can be found in [15, 18].

Theorem 2.3. [11, 14, 15, 17, 27, 31] *In all Constructions QM$_i$ the new code V is an $[n, n-(r_0+Rm), 3]_q R, \ell$ code with covering radius R and $\ell \geq \ell_0$.*

Corollary 2.4. *The length function $\ell_q(r_0 + Rm, R)$ satisfies*

$$\ell_q(r_0 + Rm, R) \leq q^m \ell_q(r_0, R) + \tag{2.3}$$
$$\begin{cases} R\theta_{m,q} & \text{if } q^m \geq \ell_q(r_0, R) - 1 \\ \lfloor R/2 \rfloor \theta_{m,q} + \ell_q(m\lceil R/2 \rceil, \lceil R/2 \rceil) & \text{if } q^m \geq \ell_q(r_0, R) + 1 \end{cases}.$$

Proof. In Constructions QM$_1$ and QM$_4$ we put $n_0 = \ell_q(r_0, R)$ and then use the trivial R-partition. In the code $V_{R',m}$ with $R' = \lceil R/2 \rceil$ we put $n' = \ell_q(m\lceil R/2 \rceil, \lceil R/2 \rceil)$. □

Constructions QM$_1$ – QM$_4$ provide *infinite families* of *new* $[n, n-(r_0+Rm)]_q R$ codes V with growing codimension $r = r_0 + Rm$. In Constructions QM$_5$ – QM$_8$ instead, the value of m cannot assume arbitrarily large values. However, these construction can be used in an iterative process where the new codes are the starting ones for the subsequent steps [15, 17]. As a result we obtain an *infinite code family,* see e.g. [17, Rem. 5]. For this iterative process, it is important that in the new codes obtained by the q^m-concatenating constructions the value of ℓ is increasing and eventually reaches R, see [15, Sec. IV] and the examples in Section 5.

Remark 2.5. *i)* By (1.2), (1.3), (1.5), in Construction QM$_1$ the covering density of the starting code V_0 and the lower limit of the asymptotic covering density of the infinite family of the new codes V are $\mu_q(n_0, R) \approx (q-1)^R n_0^R q^{-r_0}/R!$ and $\overline{\mu}_q(R) \approx (q-1)^R(n_0 + R/q)^R q^{-r_0}/R!$, respectively. We have $\overline{\mu}_q(R)/\mu_q(n_0, R) \approx (1 + R/qn_0)^R$. This shows that for the q^m-concatenating constructions the lower limit of the asymptotic covering density of the new family is somewhat greater than covering density of the starting code. However, it should be noted that the difference is not significant when the value of R/qn_0 is small.

ii) In every Construction QM_1 – QM_8, if the starting code V_0 is "short", i.e. $n_0 = O(q^{(r_0-R)/R})$, then the all new $[n, n-(r_0+Rm)]_q R$ codes V, obtained by the q^m-concatenating constructions, are "short" too, i.e. $n = O(q^{(r_0+Rm-R)/R})$. This means that the infinite family of the codes V is "good" with $\overline{\mu}_q(R) = O(1)$.

3. New "small" saturating sets

3.1. Multifold strong blocking sets. In a finite projective space, a t-fold blocking set with respect to subspaces of some fixed dimension is a set of points that meets every such subspace in at least t points. To describe new constructions of small ρ-saturating sets in spaces $PG(v,q)$ with $q=(q')^{\rho+1}$ we introduce a new concept of t-fold **strong** blocking set.

Definition 3.1. Let $2 \le t \le v$. A point set B in a projective space $PG(v,q)$ is a *t-fold strong blocking set* if every $(t-1)$-dimensional subspace of $PG(v,q)$ is spanned by t points in B.

Let $(x_0, x_1, \ldots, x_v)$, where $x_i \in F_q$, be homogenous coordinates for a point P in $PG(v,q)$, and for any positive integer u let $P^u = (x_0^u, x_1^u, \ldots, x_v^u)$.

Theorem 3.2. *Let ρ be any positive integer. Let $q = (q')^{\rho+1}$. Let $v \ge \rho+1$. Then any $(\rho+1)$-fold strong blocking set in the subgeometry $PG(v,q') \subset PG(v,q)$ is a ρ-saturating set in $PG(v,q)$.*

Proof. Let B be a $(\rho+1)$-fold strong blocking set in $PG(v,q')$. Let P be a point in $PG(v,q)\setminus B$. By definition of $(\rho+1)$-fold strong blocking set we only need to show that there exists a ρ-dimensional subspace of $PG(v,q')$ passing through P. Consider the subspace $\Sigma(P)$ of $PG(v,q)$ generated by the point set $O(P) := \{P, P^{q'}, P^{(q')^2}, \ldots, P^{(q')^\rho}\}$. As $\left(P^{(q')^\rho}\right)^{q'} = P^q = P$, the Frobenius collineation $X \mapsto X^{q'}$ fixes $O(P)$. Therefore $\Sigma(P)$ is a subspace of $PG(v,q')$. Clearly $\Sigma(P)$ is contained in some ρ-dimensional subspace of $PG(v,q')$ (if the points in $O(P)$ are independent, then this subspace coincides with $\Sigma(P)$). As $P \in \Sigma(P)$, the assertion is proved. □

3.2. Small ρ-saturating sets in spaces $PG(\rho+1,(q')^{\rho+1})$.

Corollary 3.3. *Let $q=(q')^2$. Any 2-fold blocking set in the subplane $PG(2,\sqrt{q}) \subset PG(2,q)$ is a 1-saturating set in the plane $PG(2,q)$.*

Proof. As a line is spanned by any two its points, a 2-fold blocking set in a projective plane is always a 2-fold *strong* blocking set. Then we use Theorem 3.2. □

Note that Corollary 3.3 is also given in [47].

Theorem 3.4. *Let $q=(q')^4$. In $PG(2,q)$ there is a 1-saturating set of size $2\sqrt{q} + 2\sqrt[4]{q}+2$.*

Proof. The union of two disjoint Baer subplanes in $PG(2,\sqrt{q})$ is a 2-fold blocking set [4]. Then we use Corollary 3.3. □

In $PG(2,q)$, q not a square, 2-fold blocking sets of size $b \le 3q-2$ are not known in the literature [4, 5]. We give here some results for $q=p^3$, p prime.

Theorem 3.5. *Let $q = p^3$, p prime, $p \leq 73$. Then in $PG(2,q)$ there is a 2-fold blocking set of size* $2\left(q + \sqrt[3]{q^2} + \sqrt[3]{q} + 1\right)$.

Proof. By [42, Lem. 13.8 (iii)], the point set

$$B = \{(1, x, x^p) \mid x \in F_q\} \cup \{(0, 1, m) \mid m \in F_q, m^{p^2+p+1} = 1\}$$

is a 1-blocking set in $PG(2,q)$ of size $q+p^2+p+1$. We are looking for a projectivity γ for which $B\cap\gamma(B) = \emptyset$ holds. Then $B\cup\gamma(B)$ is a 2-fold blocking set in $PG(2,q)$.

Let H be the multiplicative subgroup of F_q^* consisting of the $(p-1)$th powers in F_q (equivalently, $H = \{y \in F_q \mid y^{p^2+p+1} = 1\}$). For $a, b \in F_q^*$, $b \notin H$, we consider the projectivity $\gamma_{a,b}(r, s, t) = (t - r, abr, as)$. Obviously, $\gamma_{a,b}(0, 1, m) = (m, 0, a) = (1, 0, a/m) \notin B$. Also, $\gamma_{a,b}(1, 1, 1) = (0, ab, a) = (0, b, 1) \notin B$ as $b^{p^2+p+1} \neq 1$. Finally, for $x \neq 1$, $\gamma_{a,b}(1, x, x^p) = (1, ab/(x^p - 1), ax/(x^p - 1)) \in B$ if and only if $a^{p-1}b^p = (x^p - 1)^{p-1}x$.

So, $B \cap \gamma_{a,b}(B) = \emptyset$ if and only if the equation $a^{p-1}b^p = (x^p - 1)^{p-1}x$ has no solution in F_q.

Now, note that any element $c \in F_q^* \setminus H$ can be expressed as a product $a^{p-1}b^p$ with $a, b \in F_q^*$, $b \notin H$. In fact, c belongs to some coset dH, $d \notin H$, and therefore $c = da^{p-1}$ for some $a \in F_q^*$. Let $b = d^{p^2} \notin H$, so that $b^p = d$.

Then the following claim is proved: if there is an element $c \in F_q^* \setminus H$ such that $c \neq (x^p - 1)^{p-1}x$ for any $x \in F_q$, then there exist a, b such that $B \cap \gamma_{a,b}(B) = \emptyset$.

The existence of such element c has been tested by computer for every prime $p \leq 73$. □

Corollary 3.6. *Let $q = (q')^6$, q' prime, $q' \leq 73$. In $PG(2,q)$ there is a 1-saturating set of size* $2\sqrt{q} + 2\sqrt[3]{q} + 2\sqrt[6]{q} + 2$.

Note that the smallest previously known 1-saturating sets in $PG(2,q)$, $q = (q')^2$, have size $3\sqrt{q} - 1$ [15, Th. 5.2], cf. Theorem 3.4 and Corollary 3.6.

Now we construct a 3-fold strong blocking set in $PG(3,q)$. Let l_1, l_2, l_3 be the lines with the following equations:

$$l_1 : x_0 = x_2 = 0;\ l_2 : x_1 = x_3 = 0;\ l_3 : x_0 = x_3,\ x_1 = x_2.$$

These lines are pairwise skew, and are all contained in the hyperbolic quadric $\mathcal{Q}$: $x_0x_1 = x_2x_3$. Let g be any line disjoint from $\mathcal{Q}$, and let

$$B = l_1 \cup l_2 \cup l_3 \cup g. \tag{3.1}$$

A possible choice for g is the following:

$$g : \begin{cases} x_0 = x_1,\ x_2 = kx_3,\ k \text{ non-square in } F_q, \text{ if } q \text{ odd;} \\ x_0 = x_1 + x_3,\ x_2 = kx_3,\ T^2 + T + k \text{ irreducible over } F_q, \text{ if } q \text{ even.} \end{cases}$$

Theorem 3.7. *The set B of (3.1) has size $4q + 4$ and it is a 3-fold strong blocking set in $PG(3,q)$.*

Proof. We need to show that any plane π of $PG(3,q)$ meets B in three non collinear points. If one line of B lies on π, then the assertion is trivial. Let $P_i = \pi \cap l_i$. Assume that P_1, P_2, P_3 are collinear. Then the line l through P_1, P_2 and P_3 is contained in $\mathcal{Q}$, by the "*three then all*" principle for quadrics in projective spaces. As $R = \pi \cap g \notin \mathcal{Q}$, we have that R is not collinear with P_1 and P_2. □

Remark 3.8. Any 3-fold strong blocking set B in $PG(3,q)$ has at least $3q+3$ points. Let l be any line such that $l \cap B = \emptyset$. Then each of the $q+1$ planes in the pencil through l must contain three points of B.

Corollary 3.9. *Let $q=(q')^3$. In $PG(3,q)$ there is a 2-saturating set of size $4q'+4$ consisting of four pairwise skew lines of $PG(3,q') \subset PG(3,q)$.*

Proof. We use Theorems 3.2 and 3.7. □

We now give an inductive construction of v-fold strong blocking sets in $PG(v,q)$.

Construction A. Let $H \cong PG(v,q)$ be a hyperplane in $PG(v+1,q)$, and let $B \subset H$ be a v-fold strong blocking set in H. Let $P_1, P_2, \ldots, P_{v+1}$ be $v+1$ independent points in H, and let $l_1, \ldots, l_{v+1}$ be concurrent lines in $PG(v,q)$ such that $l_i \cap H = P_i$ for each i. Let

$$B^\star = B \cup \bigcup_{i=1,\ldots,v+1} (l_i \setminus \{P_i\}) \,. \tag{3.2}$$

Theorem 3.10. *Let B be a v-fold strong blocking set in $PG(v,q)$ of size k. Then the set $B^\star$ of Construction A is a $(v+1)$-fold strong blocking set in $PG(v+1,q)$ of size $k+1+(v+1)(q-1)$.*

Proof. Let H be the hyperplane in $PG(v+1,q)$ as in Construction A. Let H_1 be any hyperplane in $PG(v+1,q)$. We need to show that H_1 is generated by $v+1$ points in $B^\star$. When $H=H_1$, this follows from the fact that B must contain $v+1$ independent points. Assume then that $H \neq H_1$, and let $\Sigma = H \cap H_1$. As Σ is a hyperplane in H, there exist v points $Q_1, \ldots, Q_v$ in B which generate Σ . Note that Σ does not pass through a point P_{i_0} for some $i_0 \in \{1,\ldots,v+1\}$, as otherwise Σ would coincide with H. Let $Q = H_1 \cap l_{i_0}$. As $Q \notin \Sigma$, and as Σ is a hyperplane of H_1, we have that $H_1 = <\Sigma, Q> = <Q_1,\ldots,Q_v,Q>$ with $\{Q_1,\ldots,Q_v,Q\} \subset B^*$. This proves that $B^\star$ is a $(v+1)$-fold blocking set. The size of $B^\star$ can be easily calculated from (3.2). □

Corollary 3.11. *In $PG(v,q)$, $v \geq 3$, there exists a v-fold strong blocking set of size*

$$(q-1)\left(\frac{v(v+1)}{2} - 2\right) + v + 5. \tag{3.3}$$

Proof. By Theorem 3.7, in $PG(3,q)$ there exists a 3-fold strong blocking set of size $4q+4$. Then the assertion follows by Theorem 3.10, taking into account that $4q+4+[1+4(q-1)]+[1+5(q-1)]+\ldots+[1+v(q-1)] = 4q+4+(v-3)+(q-1)\,(v(v+1)/2-6)\,.$ □

From Theorem 3.2 we deduce the following result.

Corollary 3.12. *Let $q=(q')^{\rho+1}$, $\rho \geq 2$. Then there exists a ρ-saturating set in $PG(\rho+1,q)$ of size*

$$(\sqrt[\rho+1]{q} - 1)\left(\frac{(\rho+1)(\rho+2)}{2} - 2\right) + \rho + 6. \tag{3.4}$$

Note that the smallest previously known ρ-saturating sets in $PG(\rho+1,\,(q')^{\rho+1})$, $\rho \geq 2$, have size $n = \frac{1}{2}(\sqrt[\rho+1]{q}-1)(\rho+1)(\rho+2)+\rho+2$ [30, Th. 6], e.g. $n=6q'-2$ for $\rho=2$ and $n=10q'-5$ for $\rho=3$; from (3.4) we obtain sizes $4q'+4$ and $8q'+1$, respectively.

Remark 3.13. The codes associated to the saturating sets of Corollaries 3.9 and 3.12 will be used as starting codes for q^m-concatenating constructions, see Sections 5 and 6. Therefore, we need to treat such codes as (R,ℓ)-objects with $\ell>0$ and to obtain the corresponding (R,ℓ)-partitions, see Definition 2.1. To this end, it is useful to represent some point P_i of a line l in $PG(v,q)$ as a linear combination with nonzero coefficients of u other points of l. We compute some of the admissible values of u. Let $l=\{P_0,P_1,\dots,P_q\}$. Without loss of generality we identify l with the projective line $PG(1,q)$, and assume that $P_0=(0,1)$, $P_1=(1,0)$, $P_i=(1,b_i)$, $i\geq 2$, where $\{b_2,\dots,b_q\}=F_q^*$.

i) Clearly, for each $i=0,1,\dots,\lfloor (q-2)/2\rfloor$, the point P_i can be written as $P_i=c_{2i+1}P_{2i+1}+c_{2i+2}P_{2i+2}$, for some $c_{2i+1},c_{2i+2}\in F_q^*$. So, $P_0=c_1P_1+c_2P_2=c_1c_3P_3+c_1c_4P_4+c_2P_2=c_1c_3P_3+c_1c_4P_4+c_2c_5P_5+c_2c_6P_6$, and so on. Therefore, each $u\in\{2,3,\dots,\lfloor (q+2)/2\rfloor\}$ is admissible.

ii) Note that $P_1=(1,0)=-\sum_{i=2}^{q}(1,b_i)$. Then $u=q-1$ is admissible.

iii) Let $q\geq u\geq 3$, $q\geq 4$. Then, for any $d_i\in F_q^*$, one can always choose a_0 and a_1 in F_q^* so that $a_0(0,1)+a_1(1,0)+\sum_{i=2}^{u-1}d_i(1,b_i)=a_2(1,b_q)$ with some $a_2\in F_q^*$.

3.3. Small ρ-saturating sets in spaces $PG(v,(q')^{\rho+1})$, $v=\rho+2,\rho+3,\dots,2\rho-1$.

Lemma 3.14. *Fix $1\leq k\leq v-1$. Let B_k be the subset of $PG(v,q)$ consisting of the points whose Hamming weight is at most $v-k+1$, i.e. B_k is the union of the $\binom{v+1}{k}$ subspaces of equation $x_{i_1}=\ldots=x_{i_k}=0$, where $0\leq i_1<i_2<\ldots<i_k\leq v$. Then B_k is a $(k+1)$-strong blocking set.*

Proof. Let W be any k-dimensional subspace of $PG(v,q)$. Let $w_1,\dots,w_{k+1}$ be $k+1$ independent points of W. Consider the matrix

$$A_W=\begin{bmatrix} ----w_1---- \\ ----w_2---- \\ \vdots \\ ----w_{k+1}---- \end{bmatrix}$$

whose rows are homogenous coordinates of points $w_1,\dots,w_{k+1}$. As the rank of A_W is equal to $k+1$, there exists a non singular $(k+1)\times(k+1)$ matrix $M=(m_{ij})$ such that MA_W contains a submatrix I_{k+1}. Note that the rows of MA_W are the coordinates of $(k+1)$ points of W; more precisely the i-th row of MA_W is $m_{i1}w_1+m_{i2}w_2+\ldots+m_{i(k+1)}w_{k+1}$. Clearly these points are independent, and they are contained in B_k as I_{k+1} is a submatrix of MA_W. □

Note that in the previous lemma

$$|B_k|=\frac{1}{q-1}\sum_{i=1}^{v-k+1}(q-1)^i\binom{v+1}{i}=\frac{V_q(v+1,v-k+1)-1}{q-1}, \tag{3.5}$$

see (1.1). Therefore the order of magnitude of the size of B_k is $\binom{v+1}{k}q^{v-k}$.

Theorem 3.15. *Let ρ be a positive integer. Let $q=(q')^{\rho+1}$ and $v>\rho+1$. Then in $PG(v,q)$ there exists a ρ-saturating set of size*

$$\frac{V_{q'}(v+1,v-\rho+1)-1}{q'-1}\sim\binom{v+1}{\rho}q^{(v-\rho)/(\rho+1)}. \tag{3.6}$$

Proof. By Theorem 3.2, the set $B_\rho \subset PG(v,q')$, where B_ρ is defined as in Lemma 3.14, is the desired ρ-saturating set. □

For some values of v and ρ, the coefficient $\binom{v+1}{\rho}$ can be improved. We show that this is possible for $v=4$, $\rho=2$.

Let $q=(q')^3$. Let $E_0=(1,0,0,0,0)$, $E_1=(0,1,0,0,0)$, $E_2=(0,0,1,0,0)$, $E_3=(0,0,0,1,0)$, and $E_4=(0,0,0,0,1)$ be points in $PG(4,q)$. For $k,i,j\in\{0,1,2,3,4\}$, $k<i<j$, let $\pi_{k,i,j}$ be the plane in $PG(4,q)$ generated by E_k, E_i and E_j. Let $\pi_1=\pi_{0,1,2}$, $\pi_2=\pi_{0,3,4}$, $\pi_3=\pi_{0,1,3}$, $\pi_4=\pi_{0,2,4}$, $\pi_5=\pi_{0,1,4}$, $\pi_6=\pi_{1,2,3}$, $\pi_7=\pi_{1,2,4}$, $\pi_8=\pi_{1,3,4}$, $\pi_9=\pi_{2,3,4}$. Let

$$S=\left(\bigcup_{s=1}^{9}\pi_s\right)\cap PG(4,q'). \tag{3.7}$$

The union S of the nine planes π_i consists of all points of $PG(4,q')$, apart from those belonging to the following three disjoint classes: points with all non-zero coordinates; points with precisely one zero coordinate; points $(x,0,y,z,0)$ with $xyz\neq 0$. Therefore,

$$|S|=\theta_{5,q'}-(q'-1)^4-5(q'-1)^3-(q'-1)^2=9\sqrt[3]{q^2}-8\sqrt[3]{q}+4.$$

Theorem 3.16. *Let $q=(q')^3$. The set S as in (3.7) has size $9\sqrt[3]{q^2}-8\sqrt[3]{q}+4$, and it is a 2-saturating set in $PG(4,q)$.*

Proof. Let P be a point in $PG(4,q)$. Let π be any plane of $PG(4,q')$ containing the subspace generated by $P,P^{q'},P^{(q')^2}$. Clearly π passes through P. Assume that π does not pass through E_0. Then among the points in $\{\pi\cap\pi_s \mid s=1,\ldots,5\}$ there are at least three non-collinear points of S. Assume that π passes through E_0. Let H_0 be the hyperplane generated by $E_1,\ldots,E_4$. Then $\pi\cap H_0$ consists of a line ℓ. Obviously, ℓ meets $\cup_{i=6}^{9}\pi_i$ in at least two non-collinear points. Then π passes through 3 non-collinear points in S. □

Open problem. Reduce the coefficient $\binom{v+1}{\rho}$ in (3.6), for generic v and ρ.

4. Codes with covering radius $R=2$

4.1. Tables of upper bounds on the length function $\ell_q(3,2)$. We remind that $\overline{\ell}_q(r,R)$ is the *smallest known* length of a q-ary linear code with codimension r and covering radius R. Clearly, $\ell_q(r,R)\leq\overline{\ell}_q(r,R)$. The dot "**.**" appears in a table when $\ell_q(r,R)=\overline{\ell}_q(r,R)$ holds. Subscripts indicate the minimum distance d of the $[\overline{\ell}_q(3,2),\overline{\ell}_q(3,2)-r,d]_qR$ codes. Multiple subscripts mean that the value of $\overline{\ell}_q(3,2)$ is provided by codes with distinct distances. Note that the distance $d=4$ occurs when the code arises from a complete arc in the plane $PG(2,q)$.

Table 1 gives values of $\overline{\ell}_q(3,2)$. We used [20, Tab. 3], [26, Tabs 2,4], [27, Tab. I], Theorem 3.4, Corollary 3.6, the relation $\ell_{(q')^2}(3,2)\leq 3q'-1$ [15, Th. 5.2], and computer search made in this work. It should be noted that for some q, including $q=601$ and 661, the complete arcs appearing in Table 1 arise from a recent investigation, both theoretical and computer-assisted, on point orbits of the projective plane under the action of some specific collineation groups [39].

From Table 1 the following result is obtained.

q	$\bar\ell_q$	q	$\bar\ell_q$	q	$\bar\ell_q$	q	$\bar\ell_q$	q	$\bar\ell_q$	q	$\bar\ell_q$
3	4_4 ▪	113	32_3	289	50_3	509	82_3	739	103_3	997	125_3
4	5_3 ▪	121	32_3	293	59_3	512	82_3	743	104_3	1009	124_3
5	$6_{3,4}$ ▪	125	34_3	307	60_3	521	84_3	751	105_3	1013	125_3
7	$6_{3,4}$ ▪	127	$35_{3,4}$	311	61_3	523	83_3	757	105_3	1019	126_3
8	6_4 ▪	128	$34_{3,4}$	313	61_3	529	68_3	761	105_3	1021	126_3
9	6_4 ▪	131	35_3	317	62_3	541	85_3	769	106_3	1024	95_3
11	7_4 ▪	137	36_3	331	63_3	547	86_3	773	106_3	1031	127_3
13	8_4 ▪	139	$37_{3,4}$	337	64_3	557	87_3	787	107_3	1033	127_3
16	$9_{3,4}$ ▪	149	$39_{3,4}$	343	64_3	563	87_3	797	108_3	1039	127_3
17	$10_{3,4}$	151	$39_{3,4}$	347	65_3	569	88_3	809	109_3	1049	128_3
19	$10_{3,4}$	157	$40_{3,4}$	349	65_3	571	88_3	811	110_3	1051	128_3
23	10_4 ▪	163	$41_{3,4}$	353	66_3	577	89_3	821	110_3	1061	129_3
25	$12_{3,4}$	167	$42_{3,4}$	359	66_3	587	90_3	823	110_3	1063	129_3
27	$12_{3,4}$	169	38_3	361	56_3	593	90_3	827	110_3	1069	129_3
29	$13_{3,4}$	173	42_3	367	67_3	599	91_3	829	110_3	1087	130_3
31	$14_{3,4}$	179	43_3	373	68_3	601	90_4	839	111_3	1091	131_3
32	13_3	181	43_3	379	69_3	607	91_3	853	113_3	1093	131_3
37	15_4	191	45_3	383	69_3	613	92_3	857	113_3	1097	131_3
41	16_4	193	45_3	389	70_3	617	92_3	859	113_3	1103	131_3
43	16_4	197	46_3	397	71_3	619	92_3	863	114_3	1109	132_3
47	$18_{3,4}$	199	46_3	401	71_3	625	62_3	877	115_3	1117	132_3
49	18_4	211	48_3	409	72_3	631	94_3	881	115_3	1123	133_3
53	18_4	223	49_3	419	73_3	641	95_3	883	115_3	1129	133_3
59	20_4	227	50_3	421	73_3	643	95_3	887	116_3	1151	135_3
61	20_4	229	50_3	431	75_3	647	95_3	907	117_3	1153	135_3
64	19_3	233	51_3	433	75_3	653	96_3	911	118_3	1163	135_3
67	$23_{3,4}$	239	51_3	439	75_3	659	96_3	919	118_3	1171	136_3
71	22_4	241	52_3	443	76_3	661	90_4	929	119_3	1181	137_3
73	24_4	243	52_3	449	76_3	673	98_3	937	120_3	1187	137_3
79	$26_{3,4}$	251	53_3	457	77_3	677	98_3	941	120_3	1193	138_3
81	$26_{3,4}$	256	42_3	461	77_3	683	99_3	947	121_3	1201	138_3
83	26_3	257	54_3	463	77_3	691	99_3	953	121_3	1213	139_3
89	$28_{3,4}$	263	55_3	467	78_3	701	100_3	961	92_3	1217	139_3
97	29_3	269	56_3	479	79_3	709	101_3	967	123_3		
101	$30_{3,4}$	271	56_3	487	80_3	719	102_3	971	123_3		
103	30_3	277	57_3	491	81_3	727	102_3	977	124_3		
107	31_3	281	57_3	499	81_3	729	80_3	983	124_3		
109	31_3	283	58_3	503	82_3	733	102_3	991	124_3		

TABLE 1. Upper bounds $\bar\ell_q = \bar\ell_q(3,2) < 4\sqrt{q}$ on the length function $\ell_q(3,2)$

Theorem 4.1. *For the length function* $\ell_q(3,2)$,

$$\ell_q(3,2) \le a_q\sqrt{q}, \text{ with} \tag{4.1}$$

$$a_q < 3 \text{ if } q \le 109,\ a_q < 3.5 \text{ if } q \le 349,\ a_q < 4 \text{ if } q \le 1217.$$

q	$\overline{\ell}_q(3,2)$	q	$\overline{\ell}_q(3,2)$	q	$\overline{\ell}_q(3,2)$	q	$\overline{\ell}_q(3,2)$	q	$\overline{\ell}_q(3,2)$
2^{11}	200_4	3^7	208_4	5^7	1877_3	11^5	3994_3	17^7	250599_3
2^{13}	453_4	3^9	764_3	5^9	9609_3	11^7	43947_3	19^5	20578_3
2^{15}	993_4	3^{11}	2771_3	7^5	1030_3	13^5	6592_3		
2^{17}	2576_3	3^{13}	8788_3	7^7	7205_3	13^7	85712_3		
2^{19}	5210_3	5^5	256_4	7^9	50947_3	17^5	14740_3		

Table 2. Upper bounds $\overline{\ell}_q(3,2)$ on the length function $\ell_q(3,2)$ for $q = p^{2t+1}$

In Table 2 we give a number of concrete sizes of 1-saturating sets and complete caps in $PG(2,q)$, $q = p^{2t+1}$, taken from [25, Tab. 1], [37, Tab. 2], and [38, Ap., Lem. 4.3]. The values for $q = 2^{11}, 2^{13}, 3^7, 5^5$ are obtained in [22]. These sizes are the values of $\overline{\ell}_q(3,2)$.

4.2. Direct sum (DS) and doubling constructions. The *direct sum* construction (DS) forms an $[n_1+n_2, n_1+n_2-(r_1+r_2)]_qR$ code V with $R = R_1+R_2$ from two codes: an $[n_1, n_1-r_1]_qR_1$ code V_1 and an $[n_2, n_2-r_2]_qR_2$ code V_2 [9, 11, 41]. Construction DS is denoted by $\oplus$, i.e. $V_1 \oplus V_2 = V$ or

$$[n_1, n_1-r_1]_qR_1 \oplus [n_2, n_2-r_2]_qR_2 = [n_1+n_2, n_1+n_2-(r_1+r_2)]_q(R_1+R_2). \tag{4.2}$$

In [16] Construction CP1 (“codimension plus one”) is proposed. The construction is similar to the construction in [45]. From an $[n, n-r]_q2$ code V_1 Construction CP1 forms an $[f_q(n), f_q(n)-(r+1)]_q2$ code V where $f_3(n) = 2n$, $f_4(n) = 3n-1$, $f_5(n) = 3n$. For $q = 3$ Construction CP1 is a *doubling construction*. In this case the parity-check matrix $\mathbf{H}$ of the new code V has the form

$$\mathbf{H} = \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{H}_1 & \mathbf{H}_1 \end{bmatrix}, \; q = 3, \tag{4.3}$$

where $\mathbf{0}$ is the all zeroes row, $\mathbf{1}$ is the all ones row, and $\mathbf{H}_1$ is a parity-check matrix of the starting code V_1. By (4.3), see also [53],

$$\ell_3(r+1, 2) \le 2\ell_3(r, 2). \tag{4.4}$$

4.3. Infinite code families of even codimension $r = 2t$. Let $q = 3$. By applying the doubling construction of (4.3) to the codes of [28, Th. 4] and by using the codes of [28, Th. 11] we obtain an infinite family of $[n, n-r]_32$ codes with the following parameters:

$$\mathcal{A}_{2,3}^{(0)} \quad : \quad R = 2, \; r = 2t \ge 4, \; q = 3, \; r \ne 8, \; \overline{\mu}_3(2) \approx \frac{25}{18}, \tag{4.5}$$

$$n = \frac{5}{2} \cdot 3^{(r-2)/2} - \frac{1}{2} + \begin{cases} 0 & \text{if } r = 4c+2; \\ \frac{1}{2} \cdot 3^{r/4} - \frac{1}{2} & \text{if } r = 8c+4; \\ \frac{1}{2} \cdot 3^{(r+4)/4} - \frac{1}{2} & \text{if } r = 8c. \end{cases}$$

For $r = 4$, from (4.5) we obtain an $[8,4]_32$ code. Note that by [3, Tab. II], $\ell_3(4,2) = 8$ holds.

Let $q \ge 4$. The geometrical constructions (named “oval plus line”) give $[2q+1, 2q-3]_q2$ codes, see [10, p. 104] for even q and [15, Th. 5.1] for arbitrary q. By computer, using the back-tracking algorithms [20, 52], we have proved the following proposition.

Proposition 4.2. $\ell_4(4,2) = 9$.

No examples of $[n, n-4]_q 2$ codes with $n < 2q + 1$, seem to be known.

Open problem. Determining whether $\ell_q(4,2) = 2q + 1$ for $q \geq 5$.

In [31] an infinite family of $[n, n-r]_q 2$ codes is obtained with the following parameters:

$$(4.6)\quad \mathcal{A}^{(0)}_{2,q} \quad : \quad \begin{array}{l} R = 2,\ r = 2t \geq 4,\ q \geq 7,\ q \neq 9,\ r \neq 8, 12, \\ n = 2q^{(r-2)/2} + q^{(r-4)/2},\ \ \overline{\mu}_q(2) < 2 - \dfrac{2}{q} - \dfrac{3}{2q^2} + \dfrac{1}{q^3} + \dfrac{1}{q^4}. \end{array}$$

Also, in [31] codes with $r = 8, 12$, $n = 2q^{(r-2)/2} + q^{(r-4)/2} + q^{(r-6)/2} + q^{(r-8)/2}$, $q \geq 7$, $q \neq 9$, are provided.

For $q = 4, 5, 9$ in [17, Ex. 5] an infinite family of $[n, n-r]_q 2$ codes is obtained with the following parameters:

$$(4.7)\quad \mathcal{A}^{(0)}_{2,q} \quad : \quad \begin{array}{l} R = 2,\ r = 2t \geq 4,\ q = 4, 5, 9,\ r \neq 8, 12 \text{ for all } q, \\ r \neq 14, 20 \text{ if } q = 4,\ n = 2q^{(r-2)/2} + q^{(r-4)/2} + \left\lfloor q^{(r-6)/2} \right\rfloor, \\ \overline{\mu}_q(2) < 2 - \dfrac{2}{q} + \dfrac{1}{2q^2} - \dfrac{2}{q^3} + \dfrac{2}{q^4}. \end{array}$$

Also, in [17] codes with $q = 4, 5, 9$, $n = 2q^3 + q^2 + 2q + 2$, $r = 8$, and $n = q^5 + \theta_{6,q}$, $r = 12$, are obtained.

4.4. INFINITE CODE FAMILIES OF ODD CODIMENSION $r = 2t + 1$. We recall here some of the known results on small 1-saturating sets in $PG(2, q)$. (For the new 1-saturating sets obtained in this paper we refer to Section 3 and Tables 1 and 2 of Section 4.1).

For large q the existence of 1-saturating sets in $PG(2, q)$ of size at most $5\sqrt{q \log q}$ was shown by means of *probabilistic methods* in [8, 48].

The following results were obtained by explicit constructions.

In $PG(2, q)$, $q = (q')^2$, a 1-saturating set of size $3\sqrt{q} - 1$ is provided in [15, Th. 5.2].

In the plane $PG(2, q)$, $q = (q')^m$, $m \geq 2$, projectively non-equivalent 1-saturating sets of size $2q^{(m-1)/m} + \sqrt[m]{q}$ are obtained in [30, Th. 2], [37, Th. 3.2].

In [8, 40], 1-saturating sets in $PG(2, q)$ of size approximately $cq^{\frac{3}{4}}$ with a constant c independent of q are constructed.

In [37], constructions of 1-saturating n-sets in $PG(2, q)$ of size n about $3q^{\frac{2}{3}}$ are proposed and the following bounds on n are obtained (here p is a prime):

$$(4.8)\quad \begin{array}{ll} n \leq & \frac{2q}{p^t} + \frac{(p^t-1)^2}{p-1} + 1,\ q = p^m,\ m \geq 2t; \\ n \leq & \frac{2}{p}\sqrt[3]{(qp)^2} + \frac{\sqrt[3]{(qp)^2} - 2\sqrt[3]{qp} + 1}{p-1} + 1,\ q = p^{3t-1}; \\ n \leq & \min\limits_{v=1,\dots,2t+1} \left\{ (v+1)p^{t+1} + \frac{(p^t-1)^{2v}}{(p-1)^v (p^{2t+1}-1)^{(v-1)}} + 2 \right\},\ q = p^{2t+1}. \end{array}$$

Several triples (t, p, v) such that $n < 5\sqrt{q \log q}$ are obtained in [37].

By [28, Thms. 4, 9], [29], and [28, Thms. 5, 10], there are infinite families $\mathcal{A}^{(1)}_{2,3}$, $\mathcal{A}^{(1)}_{2,4}$, and $\mathcal{A}^{(1)}_{2,5}$ of $[n, n-r]_q 2$ codes with the following parameters:

$$(4.9)\quad \mathcal{A}^{(1)}_{2,3} \;:\; R=2,\ r=2t+1\geq 5,\ q=3,\ r\neq 7,\ \overline{\mu}_3(2)\approx \frac{25}{24},$$
$$n=\frac{5}{4}\sqrt{3}\cdot 3^{(r-2)/2}-\frac{1}{4}+\begin{cases}0 & \text{if } r=4c+1;\\ \frac{3}{4}\cdot 3^{(r+1)/4}-\frac{3}{4} & \text{if } r=8c+3;\\ \frac{3}{4}\cdot 3^{(r+5)/4}-\frac{3}{4} & \text{if } r=8c+7.\end{cases}$$

$$(4.10)\quad \mathcal{A}^{(1)}_{2,4} \;:\; R=2,\ r=2t+1\geq 5,\ q=4,\ r\neq 7,11,13,19,$$
$$\overline{\mu}_4(2)\approx 1.587,\ n=2\cdot 4^{(r-2)/2}+\frac{3}{2}\cdot 4^{(r-4)/2}.$$

$$(4.11)\quad \mathcal{A}^{(1)}_{2,5} \;: R=2,\ r=2t+1\geq 7,\ q=5,\ r\neq 9,\ \overline{\mu}_5(2)\approx \frac{8}{5},$$
$$n=\sqrt{5}\cdot 5^{(r-2)/2}+\begin{cases}\overline{\ell}_5(\frac{r-1}{2},2) & \text{if } r=4c+3;\\ (\overline{\ell}_5(\frac{r-1}{4},2)+\frac{1}{4})\cdot 5^{(r-1)/4}-\frac{1}{4} & \text{if } r=8c+5;\\ (\overline{\ell}_5(\frac{r-5}{4},2)+\frac{1}{2})\cdot 5^{(r+3)/4}-\frac{1}{2} & \text{if } r=8c+1.\end{cases}$$

In [17, Ex. 6, Eq. (33)] an infinite family $\mathcal{A}^{(1)}_{2,q}$ of $[n, n-r]_q 2$ codes is constructed with parameters

$$(4.12)\quad \mathcal{A}^{(1)}_{2,q} \;:\; R=2,\ r=2t+1\geq 3,\ q=(q')^2\geq 16,$$
$$n=\frac{1}{\sqrt{q}}\,(3\sqrt{q}-1)\,q^{(r-2)/2}+\lfloor q^{(r-5)/2}\rfloor,$$
$$\overline{\mu}_q(2)<4.5-\frac{3}{\sqrt{q}}-\frac{17}{2q}+\frac{9}{q\sqrt{q}}+\frac{5}{2q^2}.$$

Now we will construct infinite code families by using the q^m-concatenating constructions in [17]. Terminology and notation of [17] will be used; in particular, we are going to consider 2^E-partitions, 2^+-partitions, and their cardinalities $h^E(H)$ and $h^+(H)$, see [17, Def. 1, Rem. 1]. The starting codes will be the codes associated to the 1-saturating sets given in Section 3 and in the beginning of Section 4.4.

Theorem 4.3. *For $q=(q')^4$ there is an infinite family $\mathcal{A}^{(1)}_{2,q}$ of $[n, n-r]_q 2$ codes with*

$$(4.13)\quad \mathcal{A}^{(1)}_{2,q} \;:\; R=2,\ r=2t+1\geq 3,\ q=(q')^4,$$
$$n=\frac{1}{\sqrt{q}}\,(2\sqrt{q}+2\sqrt[4]{q}+2)\,q^{(r-2)/2}+\lfloor q^{(r-5)/2}\rfloor,$$
$$\overline{\mu}_q(2)<2+\frac{4}{\sqrt[4]{q}}+\frac{6}{\sqrt{q}}+\frac{4}{\sqrt[4]{q^3}}-\frac{2}{q}-\frac{8}{q\sqrt[4]{q}}\,.$$

Proof. To obtain the code of (4.12), in [17, Ex. 6] the starting code V_0 (denoted as $\mathcal{W}$) is based on the previously mentioned 1-saturating $(3\sqrt{q}-1)$-set. It is noted in [17] that $h^E(H_{V_0})\leq 4$ and that this inequality allows us to obtain an effective iterative code chain. A similar situation arises if one takes as V_0 the $[n_0=2\sqrt{q}+2\sqrt[4]{q}+2, n_0-3]_q 2$ code based on Theorem 3.4. We partition the column set of the parity-check matrix into subsets $T_1,\dots,T_4$ so that $|T_1|=|T_3|=2$, $T_1\cup T_2=\pi_1$, $T_3\cup T_4=\pi_2$, where π_1,π_2 are the disjoint Baer subplanes in $PG(2,\sqrt{q})$. An arbitrary point of $PG(2,q)\setminus\{\pi_1\cup\pi_2\}$ lies on a line through two points belonging to the distinct subplanes. So, we obtain a 2-partition, see [17, Def. 1] and Definition 2.1. Moreover, every point of π_1 is a linear combination of two other points of π_1, one of which belongs to T_1, whereas the other is from T_2. A similar situation holds for π_2. So, the 2-partition considered is a 2^E-partition [17, Rem. 1], and $h^E(H_{V_0}) \le 4$. Now, by changing $3\sqrt{q}-1$ by $2\sqrt{q}+2\sqrt[4]{q}+2$, from (4.12) we obtain (4.13). □

Theorem 4.4. *Let $q \ge 7$. Assume that there exists an $[n_q, n_q-3]_q2$ code V_0 with $n_q < q$. Let a parity check matrix of V_0 admit a 2-partition into p_0 subsets. Then there exists an infinite family of $[n, n-r]_q2$ codes with the following parameters:*

$$\begin{aligned}(4.14)\qquad \mathcal{A}^{(1)}_{2,q}\ &: R=2,\ r=2t+1\ge 3,\ q\ge 7,\ r\ne 9,13,\ a_q=\frac{n_q}{\sqrt{q}},\\ n &= a_q\cdot q^{(r-2)/2}+2\lfloor q^{(r-5)/2}\rfloor+\begin{cases}0 & \text{if } 2p_0\le q+1;\\ \lfloor q^{(r-7)/2}\rfloor & \text{if } 2p_0> q+1,\end{cases}\\ \overline{\mu}_q(2) &\approx \frac{a_q^2}{2}-\frac{a_q^2}{q}+\frac{2a_q}{q\sqrt{q}}.\end{aligned}$$

For $r=9,13$, $n=a_q\cdot q^{(r-2)/2}+2\lfloor q^{(r-5)/2}\rfloor+q^{(r-7)/2}+q^{(r-9)/2}$ holds.

Proof. Take V_0 as the starting code for the constructions of [17]. Then, changing n_q by p_0, we use the same argument of [17, Ex. 6] on partition cardinalities $h^+(H_{\mathcal{W}})$ and $h^E(H_{\mathcal{W}})$. As a result, (4.14) is obtained, cf. [17, form. (32)]. □

Theorem 4.4 is the main tool to obtain infinite code families with growing odd codimension.

Theorem 4.5. *For $q=(q')^6$ there is an infinite family of $[n,n-r]_q2$ codes with*

$$\begin{aligned}(4.15)\qquad \mathcal{A}^{(1)}_{2,q} &: R=2,\ r=2t+1\ge 3,\ r\ne 9,13,\ q=(q')^6,\ q' \text{ prime},\\ q' &\le 73,\ n=\left(2+\frac{2}{\sqrt[6]{q}}+\frac{2}{\sqrt[3]{q}}+\frac{2}{\sqrt{q}}\right)q^{(r-2)/2}+2\lfloor q^{(r-5)/2}\rfloor,\\ \overline{\mu}_q(2) &\approx 2+\frac{4}{\sqrt[6]{q}}+\frac{6}{\sqrt[3]{q}}+\frac{8}{\sqrt{q}}+\frac{6}{\sqrt[3]{q^2}}+\frac{5}{\sqrt[3]{q^5}}.\end{aligned}$$

Proof. The assertion follows from Theorem 4.4 and Corollary 3.6. □

Remark 4.6. Comparing (4.12) with (4.13) and (4.15) we see that for $q=(q')^4$ and $q=(q')^6$ the new codes are essentially better than the known ones.

Lemma 4.7. *Let V_0 be an $[n_q, n_q-3, 3]_q2$ code. Then a parity check matrix of V_0 admits a 2-partition into $p_0 \le n_q-1$ subsets.*

Proof. In a parity-check matrix H of V_0 there are three linearly dependent columns. Let two of them form one subset of a partition $\mathcal{P}_0$ of H, while the other subsets of $\mathcal{P}_0$ contain precisely one column. By Definition 2.1, $\mathcal{P}_0$ is a 2-partition. □

Theorem 4.8. *For any $q \le 1217$, there exists an infinite family of $[n, n-r]_q 2$ codes with the following parameters:*

$$(4.16)\quad \mathcal{A}^{(1)}_{2,q}\ : R = 2,\ r = 2t+1 \ge 3,\ q \le 1217,\ r \ne 9, 13,\ a_q = \frac{\overline{\ell}_q(3,2)}{\sqrt{q}},$$

$$n = a_q \cdot q^{(r-2)/2} + 2\lfloor q^{(r-5)/2}\rfloor + \begin{cases} 0 & \text{if } 16 \le q \le 1217; \\ \lfloor q^{(r-7)/2}\rfloor & \text{if } 7 \le q \le 13, \end{cases}$$

$$\overline{\mu}_q(2) \approx \frac{a_q^2}{2} - \frac{a_q^2}{q} + \frac{2a_q}{q\sqrt{q}},\quad \overline{\mu}_q(2) < \begin{cases} 4.5, & \text{if } q \le 109; \\ 6.125, & \text{if } q \le 349; \\ 8, & \text{if } q \le 1217. \end{cases}$$

Proof. By Lemma 4.7 and Table 1, for $q = 16, 17$ we have $2p_0 \le q + 1$. Then the assertion follows from Table 1 and Theorems 4.1 and 4.4. □

For each of the infinite families (4.9)–(4.16) the covering density is bounded from above by a constant. If in (4.14) we take as V_0 a code with length $n_q \sim f(q)\sqrt{q}$, where $f(q)$ is some increasing function of q (see e.g. (4.8)), then the asymptotic covering density increases like $f^2(q)$. However, for concrete q's, new code families can be fairly good, see e.g. Table 2.

We end this section with Tables 3 and 4, which have been obtained from (4.2)–(4.7), (4.9)–(4.11), (4.16), Table 1, Proposition 4.2, and [3, Tab. II], [28, Tab. 1], [29, Tab. I], [31, Tab. I]. The dot "**.**" appears when $\ell_q(r,2) = \overline{\ell}_q(r,2)$.

5. Codes with covering radius $R = 3$

Nonbinary linear covering codes with covering radius 3 are considered in [3, 15, 21, 23, 24, 27, 30, 31, 32], see also the references therein. The recent paper [32] collects the most relevant previous results, and provides some new ones. In this section we quote some important results from [30, 31], which will be used in Sections 6 and 7. Moreover, in Theorems 5.1 and 5.2 we give two new constructions improving some known results. The constructions are based on the new saturating sets described in Section 3.

Let $q \ge 4$. The geometrical construction (named "two ovals plus line") [30, Th. 7] gives a $[3q+1, 3q-5]_q 3$ code. So,

$$(5.1)\qquad \ell_q(6,3) \le 3q+1 \text{ if } q \ge 4.$$

To our knowledge, no examples of $[n, n-6]_q 3$ code with $n < 3q+1$ are known.

Open problem. Determining whether $\ell_q(6,3) = 3q+1$ for $q \ge 4$.

In [31, Thm. 6] an infinite family of $[n, n-r]_q 3$ codes is obtained with parameters

$$(5.2)\quad \mathcal{A}^{(0)}_{3,q}\ :\ R = 3,\ r = 3t \ge 6,\ q \ge 5,\ n = 3q^{(r-3)/3} + q^{(r-6)/3},\ r \ne 9,$$

$$\overline{\mu}_q(3) < \frac{9}{2} - \frac{9}{q} + \frac{3}{2q^2} + \frac{14}{3q^3} - \frac{1}{2q^4}.$$

Also, in [31] it is shown that codes with parameters as in (5.2) exist for $r = 9$ if $q = 16$ or $q \ge 23$. For $q = 7, 8, 11, 13, 17, 19$, DS of the codes (4.6) and the $[\theta_{3,q}, \theta_{3,q} - 3]_q 1$ Hamming codes gives $[n = 3q^2 + 2q + 1, n - 9]_q 3$ codes. For $q = 5, 9$ and $r = 9$, by (4.7), codes with length $n = 3q^2 + 2q + 2$ are obtained.

r	$\overline{\ell}_3(r,2)$	$\overline{\ell}_4(r,2)$	$\overline{\ell}_5(r,2)$	$\overline{\ell}_7(r,2)$
3	4**.**	5**.**	6**.**	6**.**
4	8**.**	9**.**	11	15
5	11**.**	19	28	44
6	22	37	56	105
7	40	85	131	309
8	76	154	281	743
9	101	304	703	2164
10	202	592	1400	5145
11	323	1237	3153	15141
12	620	2389	7031	36407
13	911	4948	16406	106036
14	1822	9522	35000	252105
15	2915	19456	78256	741909
16	5588	37888	175000	1764735
17	8201	77824	410937	5193363
18	16402	151552	875000	12353145
19	24785	316672	1953828	36353541
20	49328	611328	4375000	86472015
21	73811	1245184	9853906	254474787
22	147622	2424832	21875000	605304105
23	223073	4980736	48831278	1781323509
24	443960	9699328	109375000	4237128735

TABLE 3. Upper bounds $\overline{\ell}_q(r,2)$ on the length function $\ell_q(r,2)$, $q = 3,4,5,7$, $r \leq 24$

q	$\gamma=0$	$\gamma=1$	q	$\gamma=0$	$\gamma=1$	q	$\gamma=0$	$\gamma=1$	q	$\gamma=0$	$\gamma=1$
3	1.389	1.042	7	1.687	2.087	11	1.807	1.943	17	1.878	2.665
4	1.504	1.587	8	1.729	1.880	13	1.838	2.183	19	1.891	2.410
5	1.606	1.600	9	1.782	1.707	16	1.870	2.287	23	1.910	2.023

TABLE 4. Covering densities $\overline{\mu}_q(2,\mathcal{A}_{2,q}^{(\gamma)})$ of infinite families $\mathcal{A}_{2,q}^{(\gamma)}$

Theorem 5.1. *For $q=(q')^3 \geq 64$ there exists an infinite family of $[n,n-r]_q3$ codes with the following parameters:*

$$\mathcal{A}_{3,q}^{(1)} \ : \ R=3,\ r=3t+1\geq 7,\ q=(q')^3\geq 64, \tag{5.3}$$
$$n=\left(4+\frac{4}{\sqrt[3]{q}}\right)q^{(r-3)/3},\ \overline{\mu}_q(3)<\frac{32}{3}+\frac{32}{\sqrt[3]{q}}+\frac{32}{\sqrt[3]{q^2}}-\frac{64}{3q}.$$

Proof. The 2-saturating $(4q'+4)$-set B of Corollary 3.9 consists of pairwise skew lines of $PG(3,q')$. As $q'\geq 4$, it can be shown that the related code C_B is a (3,3)-object, see (3.1), Definition 2.1, Lemma 2.2, and Remark 3.13. Let $n_0 = 4\sqrt[3]{q}+4$. We take C_B as the starting $[n_0, n_0-4, 3]_q 3, 3$ code V_0 for Construction QM$_3$ of Section 2. The trivial partition gives $p_0 = n_0 < q$. So, we take $m\geq 1$ and obtain a family of $[n=q^m n_0, n-(4+3m)]_q 3$ codes. □

Theorem 5.2. *For $q = (q')^3 \geq 27$ there exists an infinite family of $[n, n-r]_q 3$ codes with the following parameters:*

$$(5.4)\quad \mathcal{A}^{(2)}_{3,q} \;:\; R = 3,\ r = 3t+2 \geq 8,\ q = (q')^3 \geq 27,$$
$$n = \left(9 - \frac{8}{\sqrt[3]{q}} + \frac{4}{\sqrt[3]{q^2}}\right) q^{(r-3)/3},\ \ \overline{\mu}_q(3) < \frac{243}{2} - \frac{324}{\sqrt[3]{q}} + \frac{72}{\sqrt[3]{q^2}}.$$

Proof. Let S be as in (3.7). For any plane of S, let $\{P_1, P_2\}$, $\{P_3, P_4\}$, $\{P_5, P_6, \dots, P_{(q')^2+q'+1}\}$ be a partition of the set of its points such that $P_1, P_2 \notin l_{3,4}$ and $P_3, P_4 \notin l_{1,2}$, where $l_{i,j}$ is the line through the points P_i and P_j. It can be easily shown that if $u \in \{2, 3\}$, then every point of the plane is equal to a linear combination, with nonzero coefficients, of u other points belonging to distinct subsets of the partition. The corresponding partition of the columns of the parity-check matrix of the related code $\mathcal{C}_S$ is a (3,3)-partition whose number of subsets is at most $3 \cdot 9 = 27 \leq q$. Therefore we may take $\mathcal{C}_S$ as the starting $[n_0 = 9\sqrt[3]{q^2} - 8\sqrt[3]{q} + 4, n_0 - 4]_q 3, 3$ code V_0 for Construction QM_3 with $m \geq 1$. □

Remark 5.3. By (5.3), (5.4), the asymptotic covering densities of the families $\mathcal{A}^{(1)}_{3,q}$ and $\mathcal{A}^{(2)}_{3,q}$ are bounded from above by constants independent of q. To our knowledge, for $R = 3$, $r = 3t + 2$, no infinite families with density asymptotically independent on q are known.

6. Codes with covering radius $R \geq 4$

6.1. Infinite code families of codimension $r = Rt$ and arbitrary q. In this section we obtain a code V of covering radius $R \geq 4$ and codimension Rt from DS of g_2 codes V_2 with $R = 2$ and g_3 codes V_3 with $R = 3$. More precisely,

$$(6.1)\qquad V = \underbrace{V_2 \oplus \ldots \oplus V_2}_{g_2 \text{ times}} \oplus \underbrace{V_3 \oplus \ldots \oplus V_3}_{g_3 \text{ times}},\ g_2 = \begin{cases} 0 & \text{if } R \equiv 0 \pmod 3; \\ 1 & \text{if } R \equiv 2 \pmod 3; \\ 2 & \text{if } R \equiv 1 \pmod 3, \end{cases}$$
$$g_3 = \lceil R/3 \rceil - g_2,\ n = g_2 n_2 + g_3 n_3,\ 2g_2 + 3g_3 = R,$$

where V is an $[n, n-Rt]_q R$ code, V_2 is an $[n_2, n_2 - 2t]_q 2$ code, V_3 is an $[n_3, n_3 - 3t]_q 3$ code.

Theorem 6.1. *Let $R \geq 4$ and let $q \geq 4$. Then there exists an $[n = Rq + \lceil R/3 \rceil, n - 2R, 3]_q R, \ell$ code with $\ell \geq 1$.*

Proof. Geometrical constructions of a $[2q+1, 2q-3]_q 2$ code V_2 ("oval plus line") and of a $[3q+1, 3q-5]_q 3$ code V_3 ("two ovals plus line") are given in [10, p. 104], [15, Th. 5.1],[30, Th.7]. Using these codes in (6.1) with $t = 2$, we obtain an $[n = Rq + \lceil R/3 \rceil, n - 2R]_q R$ code V. Minimum distance $d = 3$ follows from the fact that the point sets associated to V_2 and V_3 contain triples of collinear points. The value $\ell \geq 1$ follows from Lemma 2.2. □

Open problem. Constructing $[n, n-2R]_q R$ codes with $R \geq 4$, $q \geq 4$, $n < Rq + \lceil R/3 \rceil$. In particular, for $R \geq 4$, generalizing the geometrical constructions "oval plus line" [10, p. 104], [15, Th. 5.1], and "two ovals plus line" [30, Th. 7].

Theorem 6.2. *There exist infinite families of $[n, n-r]_q R$ codes with parameters*

$$(6.2)\qquad \begin{aligned} \mathcal{A}_{R,q}^{(0)} \quad &: \quad R \geq 4,\ r = Rt \geq 5R,\ q \geq 7,\ q \neq 9,\ r \neq 6R, \\ & \quad n = Rq^{(r-R)/R} + \lceil R/3 \rceil\, q^{(r-2R)/R}. \\ \mathcal{A}_{R,q}^{(0)} \quad &: \quad R \geq 4,\ r = Rt \geq 2R,\ q = 5, 9,\ r \neq 3R, 4R, 6R, \\ & \quad n = Rq^{(r-R)/R} + \left(\left\lceil \frac{R}{3} \right\rceil + \frac{g_2}{q} \right) q^{(r-2R)/R}. \end{aligned}$$

Proof. We use (6.1) with the codes V_2 and V_3 taken from (4.6), (5.2) and (4.7), (5.2). □

It should be noted that the main term of the asymptotic covering density $\overline{\mu}_q(R, \mathcal{A}_{R,q}^{(0)})$ for the family of (6.2) is $\frac{R^R}{R!}$; it does not depend on q.

By the results on cases $r = 8, 12$ and $r = 9$ reported after (4.6), (4.7), and (5.2), one can easy fill up gaps in (6.2) for codes with $r = 3R$, $4R$, and $6R$.

6.2. Infinite code families of codimension $r = Rt + 1$, $q = (q')^R$.

Theorem 6.3. *Let $q = (q')^R$. Then there exists an infinite family of $[n, n-r]_q R$ codes with the following parameters:*

$$(6.3)\quad \begin{aligned} \mathcal{A}_{R,q}^{(1)} \quad : \quad & R \geq 4,\ r = Rt + 1,\ q = (q')^R,\ t = 1 \text{ and } t \geq t_0, \\ & q^{t_0 - 1} \geq n_{R,q}^{(1)},\ n_{R,q}^{(1)} = (\sqrt[R]{q} - 1) \left(\frac{R(R+1)}{2} - 2 \right) + R + 5, \\ & n = n_{R,q}^{(1)} \cdot q^{(r-R-1)/R} + w \frac{q^{(r-R-1)/R} - 1}{q - 1}, \\ & w = 0 \text{ if } q' \geq 4,\ w \in \{0, 1\} \text{ if } q' = 3. \end{aligned}$$

Proof. As the starting code V_0 for Constructions QM_2, QM_3 we take an $[n_{R,q}^{(1)}, n_{R,q}^{(1)} - (R+1), 3]_q R$ code $\mathcal{C}_K$ related to the $(R-1)$-saturating set $K \subset PG(R, q') \subset PG(R, q)$ of Corollary 3.12, see also (3.2), Construction A and Corollary 3.11. Note that K contains four pairwise skew lines of $PG(R, q')$, whereas for other $\frac{R(R+1)}{2} - 6 \geq 2R - 4$ lines all but one point belong to K. These latter lines are partitioned into $R - 3$ sets of concurrent lines. By Definition 2.1 and Remark 3.13, the code $\mathcal{C}_K$ is an (R, ℓ_0)-object with $\ell_0 = R$ if $q' \geq 4$ and $\ell_0 \geq R - 1$ if $q' = 3$. The trivial partition of its parity-check matrix is an (R, ℓ_0)-partition into $n_{R,q}^{(1)} \leq q^{t_0 - 1}$ subsets. Finally, we use formulas for Constructions OM_2 and QM_3 to get the assertion. □

It should be noted that the main term of the asymptotic covering density $\overline{\mu}_q(R, \mathcal{A}_{R,q}^{(1)})$ for the family of (6.3) is $\frac{(R^2+R)^R}{2^R R!}$; it does not depend on q.

6.3. Infinite code families of codimension $r = Rt + 2, \ldots, R(t+1) - 1$, $q = (q')^R$.

To our knowledge, for $R \geq 4$, $r = Rt + 2, \ldots, R(t+1) - 1$, no infinite families with density asymptotically independent on q are known.

Theorem 6.4. *Let $q=(q')^R$. Fix $\gamma\in\{2,3,\dots,R-1\}$. Then there exists an infinite family of $[n,n-r]_qR$ codes with*

$$(6.4)\quad \mathcal{A}_{R,q}^{(\gamma)}\ :\ \begin{aligned}&R\ge 4,\ r=Rt+\gamma,\ q=(q')^R,\ \gamma=2,3,\dots,R-1,\\ &t=1 \text{ and } t\ge t_0,\ q^{t_0-1}\ge n_{R,q}^{(\gamma)},\\ &n_{R,q}^{(\gamma)}=\frac{\sum_{i=1}^{\gamma+1}(\sqrt[R]{q}-1)^i\binom{R+\gamma}{i}}{\sqrt[R]{q}-1}\sim\binom{R+\gamma}{R-1}q^{\gamma/R},\\ &n=n_{R,q}^{(\gamma)}\cdot q^{(r-R-\gamma)/R}+w\frac{q^{(r-R-\gamma)/R}-1}{q-1},\ 0\le w\le R-3.\end{aligned}$$

Proof. As the starting code V_0 for Constructions QM$_2$, QM$_3$ we take an $[n_{R,q}^{(\gamma)},n_{R,q}^{(\gamma)}-(R+\gamma),3]_qR$ code $\mathcal{C}_{B_{R-1}}$ related to the $(R-1)$-saturating set $B_{R-1}\subset PG(R+\gamma-1,q')\subset PG(R+\gamma-1,q)$ of Lemma 3.14 and Theorem 3.15. In (3.5), (3.6) we put $k=\rho$, $v-\rho=\gamma\ge 2$, $\rho=R-1$. By Definition 2.1 and Remark 3.13, the code $\mathcal{C}_{B_{R-1}}$ is an (R,ℓ_0)-object with $\ell_0\ge 3$ as the set B_{R-1} contains lines. The trivial partition of a parity-check matrix of $\mathcal{C}_{B_{R-1}}$ is an (R,ℓ_0)-partition into $n_{R,q}^{(\gamma)}\le q^{t_0-1}$ subsets. Finally, we use formulas for Constructions QM$_2$ and QM$_3$. □

It should be noted that the main term of the asymptotic covering density $\overline{\mu}_q(R,\mathcal{A}_{R,q}^{(\gamma)})$ for the family of (6.4) is $\left(\frac{(R+\gamma)^{R-1}}{(R-1)!}\right)^R\cdot\frac{1}{R!}$, which does not depend on q.

7. Codes with nonprime covering radius $R=sR'$

We consider the case when covering radius R is nonprime, i.e. $R=sR'$ for some integers s and R'.

Lemma 7.1. *Let $R=sR'$. Assume that there exists an $[n',n'-(R't+t')]_qR'$ code $\mathcal{C}_0$ with $R'>t'$. Then there exists an $[n'\frac{R}{R'},n'\frac{R}{R'}-(Rt+\frac{R}{R'}t')]_qR$ code $\mathcal{C}$. Moreover, if the starting code $\mathcal{C}_0$ is short the new code $\mathcal{C}$ is short too.*

Proof. We take DS of s copies of $\mathcal{C}_0$. If the code $\mathcal{C}_0$ is short, then we have that $n'=O(q^{(R't+t'-R')/R'})$ or, in other words, $n'=cq^{(R't+t'-R')/R'}$ where c is a constant independent of q. Also, $(R't+t'-R')/R'=(Rt+st'-R)/R$. Therefore $n'\frac{R}{R'}=c\frac{R}{R'}q^{(Rt+\frac{R}{R'}t'-R)/R}$. Then the assertion is proved. □

Corollary 7.2. *For even $R\ge 4$ there exist infinite families $\mathcal{A}_{R,q}^{(R/2)}$ of $[n,n-r]_qR$ codes with codimension $r=Rt+\frac{R}{2}$ and the following parameters:*

$$(7.1)\quad q=(q')^2,\ t\ge 1,\ n=\frac{R}{2}\left(3-\frac{1}{\sqrt{q}}\right)q^{(r-R)/R}+\frac{R}{2}\left\lfloor\frac{1}{\sqrt{q}}q^{(r-2R)/R}\right\rfloor.$$

$$(7.2)\quad q=(q')^4,\ t\ge 1,\ n=R\left(1+\frac{1}{\sqrt[4]{q}}+\frac{1}{\sqrt{q}}\right)q^{(r-R)/R}+\frac{R}{2}\left\lfloor\frac{1}{\sqrt{q}}q^{(r-2R)/R}\right\rfloor.$$

$$(7.3)\quad \begin{aligned}&q=(q')^6,\ q' \text{ prime},\ q'\le 73,\ t\ge 1,\ t\ne 4,6,\\ &n=R\left(1+\frac{1}{\sqrt[6]{q}}+\frac{1}{\sqrt[3]{q}}+\frac{1}{\sqrt{q}}\right)q^{(r-R)/R}+R\left\lfloor\frac{1}{\sqrt{q}}q^{(r-2R)/R}\right\rfloor.\end{aligned}$$

Proof. Put $R'=2$ and use the codes of (4.12), (4.13), and (4.15) as the code $\mathcal{C}_0$ of Lemma 7.1. □

Corollary 7.3. *Let $q=(q')^3$ and assume that 3 divides R. Then there exist infinite families of $[n,n-r]_qR$ codes with the following parameters:*

$$(7.4)\qquad \mathcal{A}_{R,q}^{(R/3)} \ :\ R=3s,\ r=Rt+\frac{R}{3},\ q=(q')^3\geq 64,\ t\geq 1,$$
$$n=\frac{4R}{3}\left(1+\frac{1}{\sqrt[3]{q}}\right)q^{(r-R)/R}.$$

$$(7.5)\qquad \mathcal{A}_{R,q}^{(2R/3)} \ :\ R=3s,\ r=Rt+\frac{2R}{3},\ q=(q')^3\geq 27,\ t\geq 1,$$
$$n=\frac{R}{3}\left(9-\frac{8}{\sqrt[3]{q}}+\frac{4}{\sqrt[3]{q^2}}\right)q^{(r-R)/R}.$$

Proof. Put $R'=3$ and use the codes of (5.3) and (5.4) as the code $\mathcal{C}_0$ of Lemma 7.1. □

Corollary 7.4. *Let $R=sR'$. Let $q=(q')^{R'}$. Then there exist an infinite family of $[n,n-r]_qR$ codes with the following parameters:*

$$(7.6)\ \mathcal{A}_{R,q}^{(s)} \ :\ R=sR',\ R'\geq 4,\ r=Rt+\frac{R}{R'},\ q=(q')^{R'},\ t=1 \text{ and } t\geq t_0,$$
$$q^{t_0-1}\geq n_{R',q}^{(1)},\ n_{R',q}^{(1)}=(\sqrt[R']{q}-1)\left(\frac{R'(R'+1)}{2}-2\right)+R'+5,$$
$$n=\frac{R}{R'}\cdot n_{R',q}^{(1)}\cdot q^{(r-R-s)/R}+w\frac{R}{R'}\cdot\frac{q^{(r-R-s)/R}-1}{q-1},$$
$$w=0 \text{ if } q'\geq 4,\ w\in\{0,1\} \text{ if } q'=3.$$

Proof. We use the codes of (6.3) as the code $\mathcal{C}_0$ of Lemma 7.1. □

Corollary 7.5. *Let $R=sR'$. Let $q=(q')^{R'}$. Fix $\gamma\in\{2,3,\dots,R-1\}$. Then there exists an infinite family of $[n,n-r]_qR$ codes with the following parameters:*

$$(7.7)\qquad \mathcal{A}_{R,q}^{(s)} \ :\ R=sR',\ R'\geq 4,\ r=Rt+\frac{R}{R'}\gamma,\ q=(q')^{R'},$$
$$\gamma=2,3,\dots,R-1,\ t=1 \text{ and } t\geq t_0,\ q^{t_0-1}\geq n_{R',q}^{(\gamma)},$$
$$n_{R',q}^{(\gamma)}=\frac{\sum_{i=1}^{\gamma+1}(\sqrt[R']{q}-1)^i\binom{R'+\gamma}{i}}{\sqrt[R']{q}-1}\sim\binom{R'+\gamma}{R'-1}q^{\gamma/R'},$$
$$n=\frac{R}{R'}\cdot n_{R',q}^{(\gamma)}\cdot q^{(r-R-s\gamma)/R}+w\frac{q^{(r-R-s\gamma)/R}-1}{q-1},$$
$$0\leq w\leq R-3.$$

Proof. We use the codes of (6.4) as the code $\mathcal{C}_0$ of Lemma 7.1. □

It should be noted that for the infinite families (7.1)–(7.7), the main terms of the lower limits of covering density $\overline{\mu}_q(R,\mathcal{A}_{R,q}^{(\gamma)})$ are, respectively, $\frac{R^R}{R!}\left(\frac{3}{2}\right)^R$, $\frac{R^R}{R!}$, $\frac{R^R}{R!}$, $\frac{R^R}{R!}\left(\frac{4}{3}\right)^R$, $\frac{R^R}{R!}3^R$, $\frac{R^R}{R!}\left(\frac{R'+1}{2}\right)^R$, $\frac{R^R}{R!}\left(\frac{(R'+\gamma)^{R'-1}}{R'!}\right)^R$. All these terms do not depend on q.

Remark 7.6. It should be emphasized that codes of Corollaries 7.2-7.5 are "short" for $R = sR'$ though, in general, for these codes $q \neq (q')^R$. Previously, short codes had been constructed for $q = (q')^R$.

8. Conclusion

We considered infinite sequences $\mathcal{A}_{R,q}$ of linear nonbinary covering codes $\mathcal{C}_n$ of type $[n, n - r_n]_q R$. Without loss of generality, we assumed that the sequence of codimension r_n was not decreasing. For a given family $\mathcal{A}_{R,q}$, the covering radius R and the size q of the underlying Galois field are fixed. We considered also *infinite sets of families* $\mathcal{A}_{R,q}$, where R is fixed but q ranges over an infinite set of prime powers.

Each infinite family $\mathcal{A}_{R,q}$ consists of *supporting* and *filling* codes. The supporting codes are the codes $\mathcal{C}_n$ such that $r_n > r_{n+1}$. Non-supporting codes are called filling codes. This terminology is motivated by the fact that the parameters of the codes in a family are completely determined by those of its supporting codes. However, considering filling codes is necessary to investigate not only the lower limit (lim inf) of the covering densities of a family, but also its upper limit (lim sup).

Such lower and upper limits (denoted by $\overline{\mu}_q(R, \mathcal{A}_{R,q})$ and $\mu_q^*(R, \mathcal{A}_{R,q})$ respectively) are the most considerable asymptotic features of families $\mathcal{A}_{R,q}$. It is also relevant how these limits depend on q in infinite sets of families $\mathcal{A}_{R,q}$ with fixed R. We showed that for the *upper limit* the best possibility is $\mu_q^*(R, \mathcal{A}_{R,q}) = O(q)$. The problem of constructing infinite sets of families $\mathcal{A}_{R,q}$ with $\mu_q^*(R, \mathcal{A}_{R,q}) = O(q)$ is open in the general case. We call it *Open Problem 1*. In the literature, a solution to Open Problem 1 was known only for $R = 2$, q square.

We first showed in Introduction that Open Problem 1 for covering radius R is solved if a solution to the following *Open Problem 2* is achieved: construct R infinite code families $\mathcal{A}_{R,q}^{(\gamma)}$, $\gamma = 0, \dots, R-1$, such that $\overline{\mu}_q(R, \mathcal{A}_{R,q}^{(\gamma)}) = O(1)$ holds. Here $\mathcal{A}_{R,q}^{(\gamma)}$ is an infinite family such that its supporting codes are a sequence of $[n_u, n_u - r_u]_q R$ codes with codimension $r_u = Ru + \gamma$ and length $n_u = f_q^{(\gamma)}(r_u)$, where $u \geq u_0$; $f_q^{(\gamma)}$ is an increasing function for a fixed q.

The main achievement of the paper is a solution to Open Problem 2 (and, hence, to Open Problem 1) for an arbitrary covering radius $R \geq 2$. This solution consists of infinite sets of families $\mathcal{A}_{R,q}$ where $q = (q')^R$, q' is a prime power. The main tool was using codes related to certain saturating sets in projective spaces as starting points for q^m-concatenating constructions of covering codes. Combination of q^m-concatenating constructions and those saturating sets turned out to be very effective.

Note that by solving Open Problem 2 we have constructed, for all $1 \leq \gamma \leq R$, all $R \geq 2$, and $q = (q')^R$, infinite families $\mathcal{A}_{R,q}^{(\gamma)}$ with density $\overline{\mu}_q(R, \mathcal{A}_{R,q}^{(\gamma)}) = O(1)$, i.e. with *lower limit of their covering density bounded from above by a constant independent of* q.

In addition, the methods used for solving Open Problems 1 and 2 allowed us to obtain a number of results on covering codes of independent interest. In particular, we obtained many new upper bounds on the asymptotic covering density $\overline{\mu}_q(R, \mathcal{A}_{R,q}^{(\gamma)})$ for distinct R and γ. We obtained also several new asymptotic and finite upper bounds on the length function.

It was natural to analyze and survey the previously known results, as well as presenting the new ones. In particular, this was done for covering radius $R = 2$.

A survey of the most used q^m-concatenating constructions is also given. It should be noted that no surveys of nonbinary linear covering codes have been recently published.

We also point out that new upper bounds on the length function are also new upper bounds on the smallest possible sizes of saturating sets. More generally, the new results and methods concerning small saturating sets in projective spaces over finite fields that have been given in this paper, such as the new concept of multifold strong blocking sets, seem to be of independent interest.

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Received for publication September 2010.

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