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THE MINIMUM ORDER OF COMPLETE CAPS IN $PG(4,4)$

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Abstract. It has been verified that in $PG(4,4)$ the smallest size of complete caps is 20 and that the values from 20 to 41 form the spectrum of possible sizes of complete caps. This result has been obtained by a computer-based proof helped by the non existence of some codes.

1. Introduction

In the projective space $PG(r,q)$ over the Galois Field F_q, an n-cap is a set of n points no three of which are collinear. An n-cap is called complete if it is not contained in an $(n+1)$-cap.

Let an $[n,k,d]_qR$ code be a linear q-ary code of length n, dimension k, minimum distance d, and covering radius R. This code is a k dimension subspace of the space of vectors of length n with components from F_q. The points of a complete n-cap in $PG(r,q)$ can be treated as columns of a parity check matrix of an $[n,n-(r+1),d]_q2$ linear code of distance $d=4$, with the exceptions of the complete 5-cap in $PG(3,2)$ and the complete 11-cap in $PG(4,3)$ corresponding to the binary $[5,1,5]_22$ code and to the Golay $[11,6,5]_32$ code respectively. The codes with $d=5$, $R=2$ are perfect. An $[n,n-(r+1),4]_q2$ code is quasi-perfect. Moreover, an $[n,n-3,4]_q2$ is a maximum distance separable (MDS) code and an $[n,n-4,4]_q2$ is a near MDS code. In the notation $[n,k,d]_qR$ we may omit R. For a more detailed presentation of coding theory refer to [5]. The main result of this paper is the following theorem, see Section 2.

Theorem 1.1. *The minimum size of complete caps in $PG(4,4)$ is 20.*

As a byproduct of the search for Theorem 1.1, in this work complete 21-caps in $PG(4,4)$ are obtained. The existence of 21-caps in $PG(4,4)$ together with Theorem 1.1 and the results collected (with the corresponding references) in [9, Table 2], give the following theorem:

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Theorem 1.2. *In $PG(4,4)$ a complete k-cap exists if and only if $20 \leq k \leq 41$.*

To determine the smallest size of complete caps in $PG(4,4)$, we start from caps $\mathcal{K}$, complete and incomplete, in $PG(3,4)$ of size $8 \leq |\mathcal{K}| \leq 17$, and then we extend them to complete caps in $PG(4,4)$. The smallest size of the caps in $PG(3,4)$ considered is 8, because of the non existence of particular linear codes and the following theorem, see [7] and [12, Theorem 4.1]:

Theorem 1.3. *The following are equivalent:*

1. *An $[n,k,d']_q$-code with $d' \geq d$.*
2. *A multiset $\mathcal{M}$ of points of the projective space $PG(k-1,q)$, which has cardinality n and satisfying the following: for every hyperplane $H \subset PG(k-1,q)$ there are at least d points of $\mathcal{M}$ outside H (in the multiset sense).*

Proposition 1. [11] *For $18 \leq n \leq 24$, linear $[n,5,d]_4$ codes with $d \geq n-7$ do not exist.*

Proof. In [11], it is noted (with the corresponding references) that, in the region $18 \leq n \leq 24$, for the best linear $[n,5,d]_4$ codes the equality $d = n-8$ holds. □

By Theorem 1.3 and Proposition 1, there exists a hyperplane containing at least 8 points of the corresponding n-caps. This consideration makes the search possible, because a size less then 8 is too expensive to handle with our tools. Table 1 presents the classification of the complete and incomplete k-caps in $PG(3,4)$ of size $k \geq 8$ (see [3]).

Table 1. Complete classification of non equivalent caps $\mathcal{K}$ in $PG(3,4)$

$\lvert\mathcal{K}\rvert$	8	9	10	11	12	13	14	15	16	17
# of complete caps $\mathcal{K}$	0	0	1	0	5	1	1	0	0	1
# of incomplete caps $\mathcal{K}$	16	19	22	15	8	3	1	1	1	0

A complete 20-cap in $PG(4,4)$ was obtained by T. Penttila and G. F. Royle (private communication) in 1995; see also [10, 14]. Theorems 1.1 and 1.2 are announced in [8]. Connections between the results of this work and quantum codes are noted in [1, 2] and [6].

Section 2 contains the results of our search, while Section 3 gives the description of three complete caps of minimal size.

2. Results

Using the exhaustive algorithm that exploits equivalence among arcs described in [13], we prove that no complete k-caps of size $k \leq 18$ exist in $PG(4,4)$. This search lasted 10 days on a computer with a 2.4 Ghz Intel Core 2. Then we searched for complete caps of size 19, extending k-caps in $PG(3,4)$ of size $k \geq 8$. In the extension process no points belonging to the hyperplane containing the starting cap are added and properties of equivalence among arcs are used to prune the search space. The algorithm is described in details in [4]. No complete 19-caps have been found, so Theorem 1.1 is proven. The execution time for the extension of the first 8-cap in $PG(3,4)$ has been about 4700 hours, while the total time for the extension of the sixteen 8-caps has been about 25000 hours, thanks to the pruning due to the equivalence properties. We used a computer with CPU Intel Quad-Core 8 Thread i7 920 and 12 Gb of memory. The extension processes of the caps in $PG(3,4)$ are independent each other, so they could be computed in different machines. In this way we realized a simple but efficient example of data parallelism.

We have also classified the minimal complete caps $\mathcal{K}$ in $PG(4,4)$, having maximal hyperplane intersection of h points, $h \geq 12$. The classification is summarized in Table 2, where the first row describes if the hyperplane intersection is complete (c) or not (i). All the caps in the table are non equivalent and no complete cap has both a complete and an incomplete hyperplane intersection of the same size.

TABLE 2. Minimal complete k-caps having hyperplane intersection of size $h \geq 12$

h	12	12	13	13	14	14	15	16	17
type of hyperplane intersection	i	c	i	c	i	c	i	i	c
size of minimal k-cap	20	20	21	21	25	25	26	26	26
# of non equivalent examples	1	2	1	1	1	3	13	5	1

3. COMPLETE 20-CAPS

In this Section we show three non equivalent examples of 20-complete caps in $PG(4,4)$. The first two examples are obtained from the same complete 12-cap in $PG(3,4)$, the last from an incomplete 12-cap. We represent the field as $F_4 = \{0,1,2,3\}$ where $2 = \alpha$, $3 = \alpha^2$, α is a root of the generating polynomial $x^2 + x + 1$.

CAP1

0 1 1 1 0 0 0 0 0 0 1 1 0 1 0 0 0 1 1 0
0 3 0 3 1 1 1 0 0 0 2 1 0 1 1 0 1 0 2 1
1 1 0 2 0 0 3 1 0 0 2 1 1 2 1 1 1 3 1 3
1 3 0 2 0 2 1 0 1 0 3 3 2 2 0 3 1 1 2 2
2 2 0 0 0 1 2 0 0 1 0 3 1 1 2 3 1 2 2 0

CAP2

0 1 0 0 0 0 0 0 1 1 0 1 0 0 0 1 1 1 0 1
0 0 1 1 0 1 0 0 2 1 0 1 1 0 1 0 2 3 1 3
1 0 0 0 1 3 0 0 2 1 1 2 1 1 1 3 1 0 3 3
1 0 0 2 0 1 1 0 3 3 2 2 0 3 1 1 2 1 2 0
2 0 0 1 0 2 0 1 0 3 1 1 2 3 1 2 2 1 0 3

CAP3

1 0 0 1 0 0 0 0 1 0 1 1 0 0 0 0 1 0 1 1
0 1 1 2 0 0 0 1 2 0 1 3 1 0 1 1 3 1 1 0
0 2 0 1 1 0 0 2 3 1 1 0 1 1 0 1 2 1 3 2
0 1 0 0 0 1 0 0 1 2 3 3 2 1 2 1 2 0 2 1
0 2 0 1 0 0 1 1 0 1 3 2 0 2 1 1 3 2 2 1

The weight enumerators of the generated code and the size of the stabilizer are listed in the following table:

	weight enumerators	size of the stabilizer
CAP1	$8^3 12^{117} 14^{432} 16^{312} 18^{144} 20^{15}$	48
CAP2	$8^3 12^{63} 13^{216} 14^{72} 15^{360} 16^{36} 17^{168} 18^{72} 19^{24} 20^9$	48
CAP3	$8^6 13^{192} 14^{432} 16^{57} 17^{192} 18^{144}$	384

CAP1 is an example of quantum cap (see [1, 2] and [6]), since the generated code has only even weights. We note that CAP1 and CAP2 have an hyperplane intersection of size 8.

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