CATEGORIES OF BOUNDED $(\mathfrak{sp}(\mathbf{S}^2V\oplus\mathbf{S}^2V^*),\mathfrak{gl}(V))$- AND $(\mathfrak{sp}(\Lambda^2V\oplus\Lambda^2V^*),\mathfrak{gl}(V))$-MODULES

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Аннотация. [Abstract]Let $\mathfrak{g}$ be a reductive Lie algebra over $\mathbb{C}$ and $\mathfrak{k}\subset\mathfrak{g}$ be a reductive in $\mathfrak{g}$ subalgebra. We call a $\mathfrak{g}$-module a $(\mathfrak{g},\mathfrak{k})$-*module* whenever M is a direct sum of a finite-dimensional $\mathfrak{k}$-modules. We call a $(\mathfrak{g},\mathfrak{k})$-*module* M *bounded* if there exists $C_M\in\mathbb{Z}_{\geq 0}$ such that for any simple finite-dimensional $\mathfrak{k}$-module E the dimension of the E-isotypic component is not more then $C_M\dim E$. Bounded $(\mathfrak{g},\mathfrak{k})$-modules is a subcategory of the category of $\mathfrak{g}$-modules. Let V be a finite-dimensional vector space. In this work we prove that the categories of bounded $(\mathfrak{sp}(\mathrm{S}^2V\oplus\mathrm{S}^2V^*),\mathfrak{gl}(V))$-modules and $(\mathfrak{sp}(\Lambda^2V\oplus\Lambda^2V^*),\mathfrak{gl}(V))$-modules are isomorphic to the direct sum of countably many copies of the category of representations of some explicitly described quiver with relations under some mild assumptions on the dimension of V.

1. Introduction and brief statements of results

Let $G_{\mathbb{R}}$ be a real semisimple Lie group and $\mathfrak{g}$ be a complexification of a Lie algebra of G, let $\mathfrak{g}^\sigma\subset\mathfrak{g}$ be a complex Lie subalgebra associated with G [VO]. Let $\mathfrak{k}\subset\mathfrak{g}$ be a complex reductive Lie subalgebra and $\mathrm{U}(\mathfrak{k})$ be the universal enveloping algebra of $\mathfrak{k}$.

Definition 1. A $(\mathfrak{g},\mathfrak{k})$-*module* is a $\mathfrak{g}$-module for which action of $\mathfrak{k}$ is *locally finite*, i.e. which is a direct sum of finite-dimensional $\mathfrak{k}$-modules.

The description of the unitary representations of $G_{\mathbb{R}}$ leads to the classification of simple infinite-dimensional $(\mathfrak{g},\mathfrak{g}^\sigma)$-modules [KV]. A $(\mathfrak{g},\mathfrak{g}^\sigma)$-module M is called *admissible* if for any simple $\mathfrak{g}^\sigma$-module E the dimension of the isotypic component of E in M is finite. In work [PSZ] I. Penkov, V. Serganova and G. Zuckerman note that admissibility (the essential property of $(\mathfrak{g},\mathfrak{g}^\sigma)$-modules of finite length) make sense for any pair $(\mathfrak{g},\mathfrak{k})$; they call such modules $(\mathfrak{g},\mathfrak{k})$-*modules of finite type*. In the same work [PSZ] they show that if $\mathfrak{g}\cong\mathfrak{sl}(V)$ and $\mathfrak{k}$ coincides with the self-normalizer in $\mathfrak{g}$, then there exists at least one simple infinite-dimensional $(\mathfrak{g},\mathfrak{k})$-module of finite type.

Definition 2. A $(\mathfrak{g},\mathfrak{k})$-module M is called *bounded*, if for some number C_M and for any simple $\mathfrak{k}$-module E the dimension of the isotypic component of E in M is less or equal to $C_M\dim E$.

Let $\mathfrak{h}\subset\mathfrak{g}$ be a Cartan subalgebra. In work [F] S. Fernando shows that the classification $(\mathfrak{g},\mathfrak{h})$-modules gives rise to a classification of bounded $(\mathfrak{g},\mathfrak{h})$-modules, the latter ones are classified by O. Mathieu [M]. We believe that a classification of generic $(\mathfrak{g},\mathfrak{k})$-modules of finite type gives rise to a classification of bounded $(\mathfrak{g},\mathfrak{k})$-modules; the latter classification problem seems to be much simpler (see [PS], [Pe], [Mi]).

Let V be a finite-dimensional space. Set $n_V:=\dim V$. Set $W=\mathrm{S}^2V(n_V\geq 3)$ or $W=\Lambda^2V(n_V\geq 5)$. In this work we show that any simple bounded $(\mathfrak{sp}(W\oplus W^*),\mathfrak{gl}(V))$-module is annihilated by a Joseph ideal [Jo2]; that is to say an ideal of maximal growth. We also show that the category of $(\mathfrak{sp}(W\oplus W^*),\mathfrak{k})$-modules annihilated by a given Joseph ideal is equivalent to the direct sum of two copies of the category of perverse sheaves of W, which are smooth along any $\mathrm{GL}(V)$-orbit. From the other hand, these categories (in particular the simple objects of these categories) have been described by T. Braden and M. Grinberg [BG].

2. Notation

All objects are defined over $\mathbb{C}$. All varieties considered are algebraic. By $\mathcal{T}X$ we denote the tangent bundle of a smooth variety X, by T_xX the tangent space to X at a point $x\in X$; by T^*_xX the dual to T_xX space. Let Y be a smooth subvariety of X. We denote by $\mathrm{N}_{Y/X}$ and $\mathrm{N}^*_{Y/X}$ the total spaces of the normal and conormal bundles to Y in X respectively. Let G be the adjoint group of $[\mathfrak{g},\mathfrak{g}]$, K be a connected subgroup of G with Lie algebra $\mathfrak{k}\cap[\mathfrak{g},\mathfrak{g}]$ and B be a Borel subgroup of K.

Definition 3. A K-variety X is called K-*spherical* if X is irreducible and has an open B-orbit.

Let X be a quasiaffine irreducible K-variety. Then X is spherical if and only if $\mathbb{C}[X]$ is a bounded $\mathfrak{k}$-module [VK].

2.1. **Associated varieties of $\mathfrak{g}$-modules.** The algebra $\mathrm{U}(\mathfrak{g})$ has a natural filtration by degree

$$0\subset\mathbb{C}\subset U_1\subset ...\subset \mathrm{U}(\mathfrak{g})=\cup_{i\in\mathbb{Z}_{\geq 0}}U_i.$$

The associated graded algebra

$$\mathrm{gr}\ \mathrm{U}(\mathfrak{g})=\oplus_{i\in\mathbb{Z}_{\geq 0}}(U_{i+1}/U_i)$$

is isomorphic to $\mathrm{S}(\mathfrak{g})$. The filtration $\{U_i\}_{i\in\mathbb{Z}_{\geq 0}}$ induces the filtration on any ideal I of $\mathrm{U}(\mathfrak{g})$, namely $\{I\cap U_i\}_{i\in\mathbb{Z}_{\geq 0}}$, hence $\mathrm{gr}I\subset\mathrm{S}(\mathfrak{g})$ is a well defined ideal. The ideal $\mathrm{gr}I$ of the commutative algebra $\mathrm{S}(\mathfrak{g})$ determines the variety

$$\mathrm{V}(\mathrm{gr}I):=\{x\in\mathfrak{g}^*\mid f(x)=0 \text{ for all } f\in\mathrm{gr}I\}.$$

In particular, if $I=\mathrm{Ann}M$ for a $\mathfrak{g}$-module M, we set $\mathrm{GV}(M):=\mathrm{V}(\mathrm{gr}I)$. We call an ideal I *Joseph ideal* whenever $\mathrm{V}(I)$ coincides with the closure of the nonzero nilpotent orbit of minimal dimension.

Theorem 1 ([Jo1])**.** For a simple $\mathfrak{g}$-module M the variety $\mathrm{GV}(M)$ is the closure of an orbit Gu and furthermore $0\in\overline{Gu}$.

Suppose that M is a $\mathrm{U}(\mathfrak{g})$-module and that a filtration

$$0\subset M_0\subset M_1\subset ...\subset M=\cup_{i\in\mathbb{Z}_{\geq 0}}M_i$$

of vector spaces is given. We say that this filtration is *good* if

(1) $U_iM_j=M_{i+j}$; (2) $\dim M_i<\infty$ for all $i\in\mathbb{Z}_{\geq 0}$.

Such a filtration arises from any finite-dimensional space of generators M_0. The corresponding associated graded object $\mathrm{gr}M=\oplus_{i\in\mathbb{Z}_{\geq 0}}M_{i+1}/M_i$ is a module over $\mathrm{gr}\ \mathrm{U}(\mathfrak{g})\cong\mathrm{S}(\mathfrak{g})$, and we set

$$\mathrm{J}_M:=\{s\in\mathrm{S}(\mathfrak{g})\mid \text{there exists } k\in\mathbb{Z}_{\geq 0} \text{ such that } s^km=0 \text{ for all } m\in\mathrm{gr}M\}.$$

In this way we associate to any $\mathfrak{g}$-module M the variety

$$\mathrm{V}(M):=\{x\in\mathfrak{g}^*\mid f(x)=0 \text{ for all } f\in\mathrm{J}_M\}.$$

It is easy to check that the module $\mathrm{gr}M$ depends on the choice of good filtration, but the ideal J_M and the variety $\mathrm{V}(M)$ does not.

Theorem 2 (Bernstein's theorem [KL, p. 118])**.** Let M be a finitely generated $\mathfrak{g}$-module. Then

$$\dim\ \mathrm{V}(M)\geq\tfrac{1}{2}\dim\ \mathrm{GV}(M).$$

Let $\mathrm{Z}(\mathfrak{g})$ be the center of $\mathrm{U}(\mathfrak{g})$.

Definition 4. a) We say that a $\mathfrak{g}$-module M *has a a character* if for some homomorphism of algebras $\chi:\mathrm{Z}(\mathfrak{g})\to\mathbb{C}$ we have $zm=\chi(z)m$ for all $z\in\mathrm{Z}(\mathfrak{g})$ and $m\in M$.
b) We say that a $\mathfrak{g}$-module M *has a generalized character* if for some homomorphism

$$\chi:\mathrm{Z}(\mathfrak{g})\to\mathbb{C}$$

and some $n\in\mathbb{Z}_{>0}$ we have

$$(z-\chi(z))^nm=0$$

for all $m\in M$ and $z\in\mathrm{Z}(\mathfrak{g})$.

Any simple module has a character [Dix]. Let M be a finitely generated $\mathfrak{g}$-module with a generalized character.

Theorem 3 (O. Gabber [Gab])**.** Let $\tilde{V}$ be an irreducible component of $\mathrm{V}(M)$ and $\mathcal{Z}$ be the unique open G-orbit of $G\tilde{V}$. Then $\dim\tilde{V}\geq\frac{1}{2}\dim\mathcal{Z}$.

Definition 5. A simple $\mathfrak{g}$-module M is called *holonomic* if

$$\dim\tilde{V}=\tfrac{1}{2}\dim G\tilde{V}$$

for any irreducible component $\tilde{V}$ of $\mathrm{V}(M)$.

Corollary 1 (S. Fernando [F])**.** The vector space

$$\mathrm{V}(M)^{\perp}=\{g\in\mathfrak{g}\mid v(g)=0 \text{ for all } v\in\mathrm{V}(M)\}$$

is a Lie algebra and $\mathrm{V}(M)$ is a $\mathrm{V}(M)^{\perp}$-variety.

Theorem 4 (S. Fernando [F, Cor. 2.7], V. Kac [Kac])**.** Set

$$\mathfrak{g}[M]:=\{g\in\mathfrak{g}\mid \dim(\mathrm{span}_{i\in\mathbb{Z}_{\geq 0}}\{g^im\})<\infty \text{ for all } m\in M\}.$$

Then $\mathfrak{g}[M]$ is a Lie algebra and $\mathfrak{g}[M]\subset \mathrm{V}(M)^{\perp}$.

Corollary 2. For a $(\mathfrak{g},\mathfrak{k})$-module M, $\mathrm{V}(M)\subset\mathfrak{k}^{\perp}$ and $\mathrm{V}(M)$ is a K-variety.

Let M be a $(\mathfrak{g},\mathfrak{k})$-module and M_0 be a $\mathfrak{k}$-stable finite-dimensional space of generators of M; J_M, $\mathrm{gr}M$ be the corresponding objects constructed as above. Consider the $\mathrm{S}(\mathfrak{g})$-modules

$$\mathrm{J}_M^{-i}\{0\}:=\{m\in\mathrm{gr}M\mid j_1...j_im=0 \text{ for all } j_1,...,j_i\in\mathrm{J}_M\}.$$

One can easily see that these modules form an ascending filtration

$$0\subset\mathrm{J}_M^{-1}\subset...\subset\mathrm{gr}M$$

such that

$$\cup_{i=1}^{\infty}\mathrm{J}_M^{-i}\{0\}=\mathrm{gr}M.$$

Since $\mathrm{S}(\mathfrak{g})$ is a Nötherian ring, the filtration stabilizes, i.e. $\mathrm{J}_M^{-i}\{0\}=\mathrm{gr}M$ for some i. By $\overline{\mathrm{gr}}M$ we denote the corresponding graded object. By definition, $\overline{\mathrm{gr}}M$ is an $\mathrm{S}(\mathfrak{g})/\mathrm{J}_M$-module. Suppose that $f\overline{\mathrm{gr}}M=0$ for some $f\in\mathrm{S}(\mathfrak{g})$. Then $f^i\mathrm{gr}M=0$ and hence $f\in\mathrm{J}_M$. This proves that the annihilator of $\overline{\mathrm{gr}}M$ in $\mathrm{S}(\mathfrak{g})/\mathrm{J}_M$ equals zero.

As M is a finitely generated $\mathfrak{g}$-module, the $\mathrm{S}(\mathfrak{g})$-modules $\mathrm{gr}M$ and $\overline{\mathrm{gr}}M$ are finitely generated. Let $\tilde{M}_0$ be a $\mathfrak{k}$-stable finite-dimensional space of generators of $\overline{\mathrm{gr}}M$. Then there is a surjective homomorphism

$$\psi:\tilde{M}_0\otimes_{\mathbb{C}}(\mathrm{S}(\mathfrak{g})/\mathrm{J}_M)\to\overline{\mathrm{gr}}M.$$

Set

$$\mathrm{Rad}M:=\{m\in\overline{\mathrm{gr}}M\mid \text{there exists } f\in\mathrm{S}(\mathfrak{g})/\mathrm{J}_M \text{ such that } fm=0 \text{ and } f\neq 0\}.$$

The space $\mathrm{Rad}M$ is a $\mathfrak{k}$-stable submodule of $\overline{\mathrm{gr}}M$ and $\tilde{M}_0\not\subset\mathrm{Rad}M$. The homomorphism ψ determines an injective homomorphism

$$\hat{\psi}:\mathrm{S}(\mathfrak{g})/\mathrm{J}_M\to\tilde{M}_0^*\otimes_{\mathbb{C}}\overline{\mathrm{gr}}M.$$

Proposition 1. The module M is $\mathfrak{k}$-bounded if and only if all irreducible components of $\mathrm{V}(M)$ are K-spherical.

Proof. Suppose that all irreducible components of $\mathrm{V}(M)$ are K-spherical. Then

$$\tilde{M}_0\otimes_{\mathbb{C}}(\mathrm{S}(\mathfrak{g})/\mathrm{J}_M)$$

is a bounded $\mathfrak{k}$-module. Therefore $\overline{\mathrm{gr}}M$ is bounded, which implies that M is a bounded $\mathfrak{k}$-module too.

Assume now that a $\mathfrak{g}$-module M is $\mathfrak{k}$-bounded. Then $\mathrm{S}(\mathfrak{g})/\mathrm{J}_M$ is a $\mathfrak{k}$-bounded module and all irreducible components of $\mathrm{V}(M)$ are K-spherical. This completes the proof of a). □

Corollary 3 ([PS]). Let M be a finitely generated bounded $(\mathfrak{g},\mathfrak{k})$-module. Then $\dim\mathfrak{b}_{\mathfrak{k}}\geq\frac{1}{2}\dim\ \mathrm{GV}(M)$.

Proof. As M is bounded, $\mathrm{V}(M)$ is K-spherical and therefore $\dim\mathfrak{b}_{\mathfrak{k}}\geq\dim\ \mathrm{V}(M)$. We have

$$\dim\ \mathrm{V}(M)\geq\frac{1}{2}\dim\ \mathrm{GV}(M).$$

□

2.2. **Bounded weight modules.** In the joint work [PS] I. Penkov and V. Serganova proved that 'boundness' is not a property of a module M but of the ideal $\mathrm{Ann}M$.

Theorem 5. Let M and N be simple $(\mathfrak{g},\mathfrak{k})$-modules such that $\mathrm{Ann}M=\mathrm{Ann}N$. Suppose that M is $\mathfrak{k}$-bounded. Then N is $\mathfrak{k}$-bounded.

We fix the following notation:
— $\{\mathrm{e}_i\}_{i\leq n_W}\subset W$ is a basis of W, $\{\mathrm{e}_i^*\}_{i\leq n_W}\subset W^*$ is the dual basis;
— $\{\mathrm{e}_{i,j}\}_{i\leq n_W,j\leq n_W}\subset\mathfrak{gl}(W)$ stands for the elementary matrix $\mathrm{e}_i\otimes\mathrm{e}_j^*$;
— $\varepsilon_i:=\mathrm{e}_{i,i}-\frac{1}{n_W}(\mathrm{e}_{1,1}+\mathrm{e}_{2,2}+...+\mathrm{e}_{n_W,n_W})$ for $i\leq n_W$;
— $\mathfrak{h}_W:=\mathrm{span}\langle\mathrm{e}_{i,i}\rangle_{i\leq n_W}$.

Consider the Weyl algebra of differential operators $\mathrm{D}(W)$ on W. This algebra is generated by e_i and ∂_{e_i} for $i\leq n_W$. Set

$$E:=\mathrm{e}_1\partial_{\mathrm{e}_1}+...+\mathrm{e}_{n_W}\partial_{\mathrm{e}_{n_W}}.$$

The operator $[E,\cdot]$: $\mathrm{D}(W)\to\mathrm{D}(W)$ is semisimple, and $\mathrm{D}(W)$ splits into a direct sum of eigenspaces

$$\{\mathrm{D}^i(W)\}_{i\in\mathbb{Z}}$$

with respect to this operator.

We note that $\mathfrak{h}_W$ is a Cartan subalgebra of $\mathfrak{sp}(W\oplus W^*)$. We identify the weights $\mathfrak{h}_W^*$ with n_W-tuples. We fix the system of positive roots of $\mathfrak{sp}(W\oplus W^*)$ in $\mathfrak{h}_W$ for which ρ is identified with $(n_W, ..., 1)$. Let $\bar{\mu}$ be an n_W-tuple and μ be the corresponding weight. We denote Ann L_μ by $\mathrm{I}(\mu)$. Let I be a primitive ideal of $\mathrm{U}(\mathfrak{sp}(W\oplus W^*))$. Then I=$\mathrm{I}(\mu)$ for some n_W-tuple $\bar{\mu}$ [Dix]. Set $\mu_0 := (n_W-\frac{1}{2}, n_W-\frac{3}{2}, ..., \frac{1}{2})$.

The spaces $\{\mathrm{D}^i(W)\}_{i\in\mathbb{Z}}$ define a $\mathbb{Z}$-grading of an algebra $\mathrm{D}(W)$. Therefore the spaces

$$\{\oplus_{i\in\mathbb{Z}}\mathrm{D}^{2i}(W), \oplus_{i\in\mathbb{Z}}\mathrm{D}^{2i+1}(W)\}$$

define a $\mathbb{Z}_2$-grading of the algebra $\mathrm{D}(W)$. We set

$$\mathrm{D}^{\bar{0}}(W) := \oplus_{i\in\mathbb{Z}}\mathrm{D}^{2i}(W), \quad \mathrm{D}^{\bar{1}}(W) := \oplus_{i\in\mathbb{Z}}\mathrm{D}^{2i+1}(W).$$

The space span $\langle \mathrm{e}_i\mathrm{e}_j, \mathrm{e}_i\partial_{\mathrm{e}_j}, \partial_{\mathrm{e}_i}\partial_{\mathrm{e}_j}, 1\rangle_{i,j\le n_W}$ is a Lie algebra with respect to commutator. This Lie algebra is isomorphic to $\mathfrak{sp}(W\oplus W^*)\oplus\mathbb{C}$. We note that span$\langle \mathrm{e}_i\partial_{\mathrm{e}_j}\rangle_{i,j\le n_W}$ is a Lie subalgebra and is isomorphic to $\mathfrak{gl}(W)$. This defines a homomorphism of associative algebras

$$\phi_{\mathfrak{sp}}:\mathrm{U}(\mathfrak{sp}(W\oplus W^*))\to\mathrm{D}(W),$$

and $\mathrm{I}(\mu_0)$=Ker$\phi_{\mathfrak{sp}}$, $\mathrm{D}^{\bar{0}}(W)$=Im$\phi_{\mathfrak{sp}}$.

2.3. **Regular singularities.** Let C be a smooth connected affine curve. We recall that $\mathbb{C}[C]$ is the algebra of regular functions of C and $\mathrm{D}(C)$ is the algebra of regular differential operators on C. Let $(x)\subset\mathbb{C}[C]$ be a maximal ideal and $x\in C$ be the corresponding point. Set

$$\mathrm{D}_x^{\geq 0}(C) := \{D\in\mathrm{D}(C) \mid Df\subset(x) \text{ for all } f\in(x)\}.$$

We say that a $\mathrm{D}(C)$-module F has *regular singularities at* x if for any finite-dimensional space $F_0\subset F$, $\mathrm{D}_x^{\geq 0}(C)F_0$ is a finitely generated $\mathbb{C}[C]$-module.

Let C^+ be a unique compact connected curve which contains C as an open subset. We say that a $\mathrm{D}(C)$-module F has *regular singularities* if for any point $x\in C^+$ there exists an affine open set $C(x)\subset C^+$ such that $x\in C(x)$ and $F\otimes_{\mathbb{C}[C]}\mathbb{C}[C\cap C(x)]$ considered as a $\mathrm{D}(C(x))$-module by extension from $C\cap C(x)$ to $C(x)$ has regular singularities at x.

Let X be a smooth variety. We denote by $\mathcal{D}(X)$ the sheaf of differential operators on X.

Definition 6. Let $\mathcal{F}$ be a coherent $\mathcal{D}(X)$-module. We say that $\mathcal{F}$ has *regular singularities* if $\mathcal{F}$ has regular singularities after restriction to any smooth connected affine curve.

3. Modules of small growth

Definition 7. Let M be a $\mathfrak{g}$-module. We say that M is $\mathrm{Z}(\mathfrak{g})$-*finite* if dim $\mathrm{Z}(\mathfrak{g})m<\infty$ for all $m\in M$, where

$$\mathrm{Z}(\mathfrak{g})m := \{m'\in M \mid m' = zm \text{ for some } m\in M \text{ and } z\in\mathrm{Z}(\mathfrak{g})\}.$$

Proposition 2. If M is an infinite-dimensional finitely generated $\mathfrak{sp}(W\oplus W^*)$-module then dim $\mathrm{V}(M)\geq n_W$.

Proof. The variety $\mathrm{GV}(M)$ is a union of several nilpotent $\mathrm{SP}(W\oplus W^*)$-orbits in $\mathfrak{sp}(W\oplus W^*)^*$. There is a unique nonzero nilpotent orbit of minimal dimension equal to $2n_W$. As M is infinite-dimensional, dim $\mathrm{V}(M)\geq\frac{1}{2}$dim $\mathrm{GV}(M)\geq n_W$ by Theorem 2. □

Definition 8. Let M be a finitely generated $\mathfrak{sp}(W\oplus W^*)$-module which is $\mathrm{Z}(\mathfrak{sp}(W\oplus W^*))$-finite. We say that M is *of small growth* if dim $\mathrm{V}(M)\leq n_W$, i.e. if dim $\mathrm{V}(M)$ equals either n_W or 0.

The $\mathfrak{sp}(W\oplus W^*)$-modules of small growth form a full subcategory of the category of $\mathfrak{sp}(W\oplus W^*)$-modules.

Let M be a finitely generated $\mathfrak{g}$-module with a good filtration $\{M_i\}_{i\in\mathbb{Z}_{\geq 0}}$. Set $\mathrm{b}(M) := n_W!\lim_{i\to\infty}\frac{\dim M_i}{i^{n_W}}$; for a module of small growth this number is always an integer [KL, p. 78] and and we call it *Bernstein's number.*

Proposition 3. Any $\mathfrak{sp}(W\oplus W^*)$-module M of small growth is of finite length.

Proof. As M is finitely generated, it is enough to check the descending chain condition for M. Let

$$...\subset M_{-i}\subset ...\subset M_{-0} = M$$

be a strictly descending chain of $\mathfrak{sp}(W \oplus W^*)$-submodules. These $\mathfrak{sp}(W \oplus W^*)$-submodules M_{-i} are finitely generated. We have dim $\mathrm{V}(M_{-i}) = n_W$ for all $i \in \mathbb{Z}_{\geq 0}$ or dim $\mathrm{V}(M_{-i}) = 0$ for some $i \in \mathbb{Z}_{\geq 0}$. In the second case dim $M_{-i} < \infty$ and therefore the chain $... \subset M_{-i}$ stabilizes.

Assume that dim $\mathrm{V}(M_{-i}) = n_W$ for all $i \in \mathbb{Z}_{\geq 0}$. If $i \geq j$ then $\mathrm{b}(M_{-i}) \leq \mathrm{b}(M_{-j})$, i.e. the sequence $\{\mathrm{b}(M_{-i})\}_{i \in \mathbb{Z}_{\geq 0}}$ is decreasing. Therefore there exists $i \in \mathbb{Z}_{\geq 0}$ such that $\mathrm{b}(M_{-i}) = \mathrm{b}(M_{-j})$ for all $j \geq i$. Therefore dim $\mathrm{V}(M_{-i}/M_{-j}) < n_W$ and M_{-i}/M_{-j} is a finite-dimensional $\mathfrak{sp}(W \oplus W^*)$-module for all $j \geq i$. We have an inclusion

$$M_{-i}/\cap_{j \geq i} M_{-j} \hookrightarrow \oplus_{j \geq i} M_{-i}/M_{-j}.$$

The right-hand side is a direct sum of a finite-dimensional $\mathfrak{sp}(W \oplus W^*)$-modules. As M_{-i} is $\mathrm{Z}(\mathfrak{sp}(W \oplus W^*))$-finite and finitely generated, only a finite number of simple $\mathfrak{sp}(W \oplus W^*)$-modules appear in this direct sum. Therefore $M_{-i}/\cap_{j \geq i} M_{-j}$ is a finite-dimensional $\mathfrak{sp}(W \oplus W^*)$-module. □

Proposition 4. Let I be the annihilator of a simple infinite-dimensional $\mathfrak{sp}(W \oplus W^*)$-module M of small growth. Then I is a Joseph ideal.

Proof. As dim $\mathrm{GV}(M) \leq 2$dim $\mathrm{V}(M)$ (Theorem 2),

$$\dim \mathrm{GV}(M) \leq 2n_W.$$

As M is infinite-dimensional, I is a Joseph ideal. □

4. Construction of $(\mathrm{D}(W), \mathfrak{k})$-modules

We recall that K is a connected reductive Lie group with Lie algebra $\mathfrak{k}$ and a Borel subgroup B. Let $\mathcal{G}$ be an associative algebra and ψ: $\mathrm{U}(\mathfrak{k}) \to \mathcal{G}$ be a homomorphism of associative algebras injective on $\mathfrak{k}$. We identify $\mathfrak{k}$ with its image. Any $\mathcal{G}$-module can be considered as a $\mathfrak{k}$-module.

Definition 9. (cf. Def. 1) A $(\mathcal{G}, \mathfrak{k})$-*module* is a $\mathcal{G}$-module which is locally finite as a $\mathfrak{k}$-module.

Definition 10. (cf. Def. 2) A *bounded* $(\mathcal{G}, \mathfrak{k})$-*module* is a $(\mathcal{G}, \mathfrak{k})$-module bounded as a $\mathfrak{k}$-module.

Let W be a spherical K-module. There is an obvious homomorphism $\mathrm{U}(\mathfrak{k}) \to \mathrm{D}(W)$. The algebra of differential operators $\mathrm{D}(W)$ has a natural filtration by degree

$$0 \subset \mathbb{C}[W] \subset D_1 \subset ... \subset \mathrm{D}(W) = \cup_{i \in \mathbb{Z}_{\geq 0}} D_i.$$

The associated graded algebra

$$\mathrm{gr}\ \mathrm{D}(W) = \oplus_{i \in \mathbb{Z}_{\geq 0}} (D_{i+1}/D_i)$$

is isomorphic to $\mathbb{C}[\mathrm{T}^*W]$. Let M be $\mathrm{D}(W)$-module with a finite-dimensional space of generators M_{gen}. This defines a filtration

$$\{D_i M_{gen}\}_{i \in \mathbb{Z}_{\geq 0}}$$

of M. The associated graded space

$$\mathrm{gr}M := \oplus_{i \in \mathbb{Z}_{\geq 0}} (D_{i+1} M_{gen}/D_i M_{gen})$$

is a finitely generated $\mathbb{C}[\mathrm{T}^*W]$-module. We denote the support of this $\mathbb{C}[\mathrm{T}^*W]$-module as $\mathcal{V}(\mathrm{Loc}M)$. As finitely generated $\mathrm{D}(W)$-modules are in a natural one-to-one correspondence with coherent $\mathcal{D}(W)$-modules, M corresponds to the $\mathcal{D}(W)$-module $\mathrm{Loc}M$. The variety $\mathcal{V}(\mathrm{Loc}M)$ is preserved by scalar action and is coisotropic in T^*W [Vi]. In what follows we assume that M is a finitely generated $(\mathrm{D}(W), \mathfrak{k})$-module.

Proposition 5. The module M is bounded if and only if all irreducible components of $\mathcal{V}(\mathrm{Loc}M)$ are K-spherical.

Proof. The proof repeats the proof of Proposition 1 verbatim. □

To a holonomic $\mathcal{D}(W)$-module $\mathcal{M}$ one assigns a pair (S, Y), where $S \subset W$ is an irreducible subvariety and Y is an $\mathcal{O}(S)$-coherent $\mathcal{D}(S)$-module [Bo]. Let K_{ss} be the connected simply-connected semisimple group with Lie algebra $[\mathfrak{k}, \mathfrak{k}]$, $A(K)$ be the center of K and $\mathfrak{a} \subset \mathfrak{k}$ be the Lie algebra of $A(K)$.

Lemma 1. The module M is $\mathfrak{k}$-bounded. If M is simple then M is $\mathfrak{k}$-multiplicity free. The $\mathcal{D}(W)$-module $\mathrm{Loc}M$ is holonomic and has regular singularities with respect to the stratification by K-orbits on W.

Proof. Let $\phi_{\mathfrak{k}}^{-1}(0) \subset \mathrm{T}^*W$ be the union of conormal bundles to all K-orbits in W. As $\mathfrak{k}$ acts locally finitely on M, $\mathcal{V}(M) \subset \phi_{\mathfrak{k}}^{-1}(0)$. Since the dimension of any irreducible component of $\phi_{\mathfrak{k}}^{-1}(0)$ is n_W and $\dim\mathcal{V}(M) \geq n_W$ (Theorem 2), and $\mathcal{V}(M)$ is the union of irreducible components of $\phi_{\mathfrak{k}}^{-1}(0)$. Therefore Loc$M$ is holonomic.

Let $\tilde{V}$ be an irreducible component of $\mathcal{V}(M)$. It is the total space of the conormal bundle to a K-orbit $S \subset W$. As W is a K-spherical variety, $\tilde{V}$ is a spherical variety [Pan]. Therefore M is a $\mathfrak{k}$-bounded module by Proposition 5. Assume M is simple. Since the algebra $\mathrm{D}(W)^{\mathfrak{k}}$ is commutative [Kn], M is multiplicity free by [PS2, Corollary 5.8].

We have to prove that Loc$M$ has regular singularities. Without loss of generality we assume that M is simple. Let (S, Y) be the pair corresponding to Loc$M$. Then S is K-stable. As W is K-spherical, S is a K-orbit. Since Loc$M$ is K_{ss}-equivariant, Y is also K_{ss}-equivariant. As $\mathfrak{a}$ acts locally finitely on M, $\mathfrak{a}$ acts locally finitely on $\Gamma(\mathrm{A}(K)s, M|_{\mathrm{A}(K)s})$ for any point $s \in S$. Since K is a quotient of $K_{ss}\times\mathrm{A}(K)$, Y has regular singularities. Therefore Loc$M$ has regular singularities (cf. [Bo, 12.11]). □

Lemma 2. If a holonomic $\mathrm{D}(W)$-module Loc$M$ has regular singularities with respect to the stratification of W by K-orbits, then M is $\mathfrak{k}$-locally finite.

Proof. Without loss of generality assume that M is simple. Let (S, Y) be the pair corresponding to Loc$M$. Then S is a K-orbit. As S is a homogeneous $K_{ss}\times\mathrm{A}(K)$-space, the sheaf Y is K_{ss}-equivariant [Bo, 12.11]. Hence Loc$M$ is K_{ss}-equivariant and therefore $[\mathfrak{k}, \mathfrak{k}]$ acts locally finitely on M. As $Y|_{\mathrm{A}(K)s}$ has regular singularities for any point $s \in S$, $\mathfrak{a}$ acts locally finitely on $\Gamma(\mathrm{A}(K)s, M|_{\mathrm{A}(K)s})$ for any point $s \in S$. Therefore $\mathfrak{a}$ acts locally finitely on M. As $\mathfrak{k} = [\mathfrak{k}, \mathfrak{k}] \oplus \mathfrak{a}$, $\mathfrak{k}$ acts locally finitely on M. □

By a famous theorem of M. Kashiwara [Ka], the category of holonomic $\mathrm{D}(W)$-modules with regular singularities with respect to a given stratification is equivalent to the category of perverse sheaves on W with respect to the same stratification.

Corollary 4. The category of bounded $(\mathrm{D}(W), \mathfrak{k})$-modules is equivalent to the category of perverse sheaves on W with respect to the stratification of W by K-orbits.

The categories of such perverse sheaves have been described explicitly for the stratifications of S^2V and Λ^2V by the $\mathrm{GL}(V)$-orbits [BG]. In particular, in [BG] the authors have described the simple objects of this categories.

Theorem 6. Assume $\mathrm{D}(W)^{\mathfrak{k}}$ is the image of $\mathrm{Z}(\mathfrak{k})$. Let E be a simple $\mathfrak{k}$-module. Then there exists not more then one simple $(\mathrm{D}(W), \mathfrak{k})$-module M such that $\mathrm{Hom}_{\mathfrak{k}}(E, M) \neq 0$.

To prove Theorem 6 we need the following preparatory lemma.

Lemma 3. Let E be a simple finite-dimensional $\mathfrak{k}$-module. Then for any $n \in \mathbb{Z}_{\geq 1}$ and $\phi \in \mathrm{Hom}(E, E \otimes \mathbb{C}^n)$ there exist $\phi_1, ..., \phi_s \in \mathrm{Hom}_{\mathfrak{k}}(E, E \otimes \mathbb{C}^n)$ such that $\phi_i \in \mathrm{U}(\mathfrak{k})\phi\mathrm{U}(\mathfrak{k})$ and $\phi \in +_i\mathrm{U}(\mathfrak{k})\phi_i\mathrm{U}(\mathfrak{k})$.

Proof. We omit the proof. □

Proof of Theorem 6. Let R be the isotypic component of E in $\mathrm{D}(W) \otimes_{\mathrm{U}(\mathfrak{k})} E$ and pr denote the projection $\mathrm{D}(W) \otimes_{\mathrm{U}(\mathfrak{k})} E \to R$. For any $g \in \mathrm{D}(W)$ there exists $g' \in \mathrm{D}(W)$ such that $\mathrm{pr}(gm) = g'm$ for any $m \in E$. Therefore g' induces a homomorphism ϕ from E to R. Let ϕ_i be as in Lemma 3. Then for any i there exist $g_i \in \mathrm{D}(W)$ such that $g_i m = \phi_i m$ for any $m \in E$. We have $[g_i, k]m = 0$ for any $k \in \mathfrak{k}$ and $m \in E$. As the adjoint action of $\mathfrak{k}$ on $\mathrm{D}(W)$ is semisimple, $g_i = c_i + a_i$ for some $c_i, a_i \in \mathrm{D}(W)$ such that $c_i \in \mathrm{D}(W)^{\mathfrak{k}}$ and $a_i m = 0$ for all $m \in E$. Hence $g_i E \subset \mathrm{U}(\mathfrak{k})E$ and therefore $R = E$.

Let $\tilde{M}$ be the unique maximal submodule of $\mathrm{D}(W) \otimes_{\mathrm{U}(\mathfrak{k})} E$ which does not contain $R = E$. Then the quotient $\mathrm{D}(W) \otimes_{\mathrm{U}(\mathfrak{k})} E/\tilde{M}$ is simple and, up to isomorphism, is the unique simple $(\mathrm{D}(W), \mathfrak{k})$-module for which $\dim \mathrm{Hom}_{\mathfrak{k}}(E, \cdot) \neq 0$. □

Let $\mathfrak{k} = \mathfrak{gl}(V)$ and let W equal Λ^2V or S^2V. Then $\mathrm{D}(W)^{\mathfrak{k}}$ is the image of $\mathrm{Z}(\mathfrak{k})$ [HU]. Therefore the conditions of Theorem 6 are satisfied.

4.1. **A useful algebraic trick.** We recall that E is the Euler operator and that $\{\mathrm{D}^i(W)\}_{i\in\mathbb{Z}}$ are the E-eigenspaces of $\mathrm{D}(W)$ (see discussion in Subsection 2.2). In the present section we relate the category of $(\mathrm{D}(W), E)$-modules to the categories of $(\mathrm{D}^{\bar{0}}(W)$, E)-modules (see also [PS2]). Let M be a locally finite E-module. Fix $t \in \mathbb{C}$. Set

$$M_t := \{m \in M \mid (E - t)^n m = 0 \text{ for some } n \in \mathbb{Z}_{\geq 0}\}.$$

Definition 11. We say that a $\mathrm{D}(W)$-module M is *a* $\mathrm{D}(W)$*-module with monodromy* $\mathrm{e}^{2\pi it}$ if $M = \oplus_{j\in\mathbb{Z}} M_{t+j}$.

Any simple $\mathrm{D}(W)$-module with a locally finite action of E is a module with a monodromy. Let δ_0 be the delta function at $0 \in W$. The functions 1 and δ_0 generate $\mathrm{D}(W)$-modules which we denote $\mathrm{D}(W)1$ and $\mathrm{D}(W)\delta_0$.

Lemma 4. Let M be a simple $\mathrm{D}(W)$-module with monodromy $\mathrm{e}^{2\pi it}$. Assume that

$$\dim M_t < \infty \text{ and } n_W \geq 2.$$

Then M is isomorphic to one of the following modules:

$$0,\ \mathrm{D}(W)1,\ \mathrm{D}(W)\delta_0.$$

Proof. Let $\dim M_t \neq 0$. Then the action of $\mathfrak{sl}(W)$ on M_t is locally finite and therefore M is a locally finite $\mathfrak{sl}(W)$-module. As W is a spherical $\mathrm{SL}(W)$-module, M is a holonomic $\mathrm{D}(W)$-module with regular singularities by Lemma 1. Let Sh be the corresponding to M simple perverse sheaf. Then Sh is constructible with respect to the stratification $\{\{0\}, W-\{0\}\}$ of W. As

$$\pi_1(\{0\}) = \pi_1(W - \{0\}) = 0,$$

Sh is the simple perverse sheaf corresponding to $\{0\}$ or $W-\{0\}$, in both cases the local system being trivial. The latter two perverse sheaves correspond to the simple $\mathrm{D}(W)$-modules $\mathrm{D}(W)\delta_0$ and $\mathrm{D}(W)1$, respectively. Therefore M is isomorphic to 0, to $\mathrm{D}(W)1$, or to $\mathrm{D}(W)\delta_0$.

Assume that $M_t = 0$. Then, for all $i \leq n_W$, the action of e_i on M is locally finite, or the action of ∂_{e_i} is locally finite for all i. In the first case M is isomorphic to $\mathrm{D}(W)\delta_0$ by Kashiwara's theorem. In the second case $\mathrm{D}(W)$ is isomorphic to $\mathrm{D}(W)1$ by Kashiwara's theorem (we can interchange the roles of e_i and ∂_{e_i}). □

We recall that $\mathrm{D}(W) = \mathrm{D}^{\bar{0}}(W) \oplus \mathrm{D}^{\bar{1}}(W)$ is a $\mathbb{Z}_2$-graded algebra. We say that a $\mathrm{D}^{\bar{0}}(W)$-module *has half-monodromy* $\mathrm{e}^{\pi it}$ if M is a direct sum $\oplus_{j\in\mathbb{Z}} M_{t+2j}$. We denote by

$$(\mathrm{D}(W), E)\text{-mod}$$

the category of $\mathrm{D}(W)$-modules with a locally finite action of E and by

$$(\mathrm{D}(W), E)\text{-mod}^{\mathrm{e}^{2\pi it}}$$

the subcategory of $(\mathrm{D}(W), E)$-mod of modules with monodromy $\mathrm{e}^{2\pi it}$ for any $t \in \mathbb{C}$. Similarly we define

$$(\mathrm{D}^{\bar{0}}(W), E)\text{-mod and } (\mathrm{D}^{\bar{0}}(W), E)\text{-mod}^{\mathrm{e}^{\pi it}}.$$

The sign '≅' stands for an equivalence of categories. For any $t \in \mathbb{C}$ we have two functors:

$$\begin{aligned}
\mathrm{Res}^{\mathrm{e}^{\pi it}}_{\mathrm{e}^{2\pi it}} &: \mathrm{D}(W)\text{-mod}^{\mathrm{e}^{2\pi it}} \to \mathrm{D}^{\bar{0}}(W)\text{-mod}^{\mathrm{e}^{\pi it}},\\
& M \mapsto \oplus_{j\in\mathbb{Z}} M_{t+2j};\\
\mathrm{Ind}^{\mathrm{e}^{2\pi it}}_{\mathrm{e}^{\pi it}} &: \mathrm{D}^{\bar{0}}(W)\text{-mod}^{\mathrm{e}^{\pi it}} \to \mathrm{D}(W)\text{-mod}^{\mathrm{e}^{2\pi it}},\\
& M \mapsto \mathrm{D}(W) \otimes_{\mathrm{D}^{\bar{0}}(W)} M.
\end{aligned}$$

Theorem 7. The functors Ind and Res are mutually inverse equivalences of the categories of $(\mathrm{D}(W), E)$-mod$^{\mathrm{e}^{2\pi it}}$ and $(\mathrm{D}^{\bar{0}}(W), E)$-mod$^{\mathrm{e}^{\pi it}}$.

Proof. For a $\mathrm{D}(W)$-module M with monodromy $\mathrm{e}^{2\pi it}$, set

$$M_{\bar{t}} := \oplus_{j\in\mathbb{Z}} M_{t+2j+1}.$$

Then $M = M_{\bar{t}} \oplus M_{\overline{t+1}}$, and this is a $\mathbb{Z}_2$-grading on M considered as a $\mathrm{D}(W)$-module.

Let $M_{\bar{t}}$ be a $\mathrm{D}^{\bar{0}}(W)$-module with half-monodromy $\mathrm{e}^{\pi it}$. The $\mathrm{D}(W)$-module

$$\mathrm{D}(W) \otimes_{\mathrm{D}^{\bar{0}}(W)} M_{\bar{t}}$$

is $\mathbb{Z}_2$-graded and has monodromy $\mathrm{e}^{2\pi it}$. Moreover,

$$(\mathrm{D}(W) \otimes_{\mathrm{D}^{\bar{0}}(W)} M_{\bar{t}})_{\bar{t}} = \mathrm{D}^{\bar{0}}(W) \otimes_{\mathrm{D}^{\bar{0}}(W)} M_{\bar{t}} = M_{\bar{t}}.$$

Therefore $\mathrm{Res}^{\mathrm{e}^{\pi it}}_{\mathrm{e}^{2\pi it}} \circ \mathrm{Ind}^{\mathrm{e}^{2\pi it}}_{\mathrm{e}^{\pi it}}$ is the identity functor.

Let M be a $\mathrm{D}(W)$-module with monodromy $\mathrm{e}^{2\pi it}$. If $M_t = 0$, then M has finite length and, up to isomorphism, the only simple constituents of it are $\mathrm{D}(W)1$ and $\mathrm{D}(W)\delta_0$. In both cases $M_{\bar{t}} \neq 0$. There is a natural homomorphism

$$\psi\colon \mathrm{D}(W) \otimes_{\mathrm{D}^{\bar{0}}(W)} M_{\bar{t}} \to M.$$

We have

$$(\mathrm{D}(W)\otimes_{\mathrm{D}^{\bar 0}(W)} M_{\bar t})_{\bar t} = \mathrm{D}(W)^{\bar 0}\otimes_{\mathrm{D}^{\bar 0}(W)} M_{\bar t} = M_{\bar t}.$$

Moreover,

$$\psi_{\bar t} : (\mathrm{D}(W)\otimes_{\mathrm{D}^{\bar 0}(W)} M_{\bar t})_{\bar t}\to M_{\bar t}$$

is an isomorphism. Therefore $(M/\mathrm{Im}\psi)_{\bar t}=0$ and hence $M/\mathrm{Im}\psi=0$. This shows that $\mathrm{Ind}^{\mathrm{e}^{2\pi it}}_{\mathrm{e}^{\pi it}}\circ\mathrm{Res}^{\mathrm{e}^{\pi it}}_{\mathrm{e}^{2\pi it}}$ is the identity functor. □

Corollary 5. We have

$$(\mathrm{D}^{\bar 0}(W),E)\text{-mod}\cong(\mathrm{D}(W),E)\text{-mod}\oplus(\mathrm{D}(W),E)\text{-mod}.$$

Proof. We have

$$(\mathrm{D}^{\bar 0}(W),E)\text{-mod}\cong\oplus_{t\in\mathbb{C}/\mathbb{Z}}(\mathrm{D}^{\bar 0}(W),E)\text{-mod}^{\mathrm{e}^{2\pi it}},$$
$$(\mathrm{D}(W),E)\text{-mod}\cong\oplus_{t\in\mathbb{C}/\mathbb{Z}}(\mathrm{D}(W),E)\text{-mod}^{\mathrm{e}^{2\pi it}}.$$

On the other hand $(\mathrm{D}^{\bar 0}(W),E)\text{-mod}^{\mathrm{e}^{\pi it}}\cong(\mathrm{D}(W),E)\text{-mod}^{\mathrm{e}^{2\pi it}}$. The two half-monodromies $\mathrm{e}^{\pi it}$, $-\mathrm{e}^{\pi it}=\mathrm{e}^{\pi i(t+1)}$ combine into monodromy $\mathrm{e}^{2\pi it}$, and

$$(\mathrm{D}^{\bar 0}(W),E)\text{-mod}\cong(\mathrm{D}(W),E)\text{-mod}\oplus(\mathrm{D}(W),E)\text{-mod}.$$

□

5. Categories of $(\mathfrak{sp}(W\oplus W^*),\mathfrak{gl}(V))$-modules

In this section

$$K:=\mathrm{GL}(V),\ \mathfrak{g}:=\mathfrak{sp}(W\oplus W^*)$$

where $W:=\Lambda^2V(n_V=2k,n_V\geq 5)$ or $W=\mathrm{S}^2V(n_V\geq 3)$.

Lemma 5. Let M be a simple bounded $(\mathfrak{sp}(W\oplus W^*),\mathfrak{gl}(V))$-module. Then $\mathrm{Ann}M$ is a Joseph ideal.

Proof. Let $\tilde{\mathbb{O}}$ be the image of the moment map $\phi:\mathrm{T}^*\mathbb{P}(W\oplus W^*)\to\mathfrak{sp}(W\oplus W^*)^*$ and $\mathbb{O}_{\mathfrak{sp}}$ be nonzero nilpotent orbit of minimal dimension. Assume that $\mathrm{GV}(M)$ does not equal to 0 or to $\bar{\mathbb{O}}_{\mathfrak{sp}}$. Then [CM, 6.2] $\tilde{\mathbb{O}}\subset\mathrm{GV}(M)$ and dim $\mathrm{GV}(M)\geq 2(2n_W-1)$. Therefore $\dim\mathfrak{b}_{\mathfrak{sl}(V)}\geq 2n_W-1$ by Corollary 3. This inequality is false. □

Theorem 8. Any simple bounded $(\mathfrak{sp}(W\oplus W^*),\mathfrak{gl}(V))$-module is of small growth.

Proof. As $\mathrm{Ann}M$ is a Joseph ideal, we have $\mathrm{V}(M)\subset\bar{\mathbb{O}}_{\mathfrak{sp}}\cap\mathfrak{k}^\perp$. As $\dim\bar{\mathbb{O}}_{\mathfrak{sp}}\cap\mathfrak{k}^\perp=n_W$, M is of small growth. □

To any n_W-tuple $\bar\mu:=(\mu_1,...,\mu_{n_W})$ one assigns a primitive ideal $\mathrm{I}(\lambda)\subset\mathrm{U}(\mathfrak{sp}(W\oplus W^*))$ and any primitive ideal of $\mathrm{U}(\mathfrak{sl}(W))$ arises in this way from some tuple $\bar\mu$.

Definition 12. We call an n_W-tuple a *Shale-Weil tuple* if $\mu_i>\mu_j$ for $i>j$, $\mu_{n_W-1}>|\mu_{n_W}|$ and $\mu_i\in\frac{1}{2}+\mathbb{Z}$ for all $i\in\{1,...,n_W\}$. We call $\bar\mu$ *positive* whenever $\mu_{n_W}>0$.

An ideal $I\subset\mathrm{U}(\mathfrak{sp}(W\oplus W^*))$ is a Joseph ideal if and only if $I=\mathrm{I}(\mu)$ for some Shale-Weil tuple $\bar\mu$.

Theorem 9. Let M be an infinite-dimensional simple bounded $(\mathfrak{sp}(W\oplus W^*),\mathfrak{gl}(V))$-module and $\bar\mu$ be an n_W-tuple such that $\mathrm{Ann}M=\mathrm{I}(\mu)$. Then $\bar\mu$ is a Shale-Weil tuple.

Proof. As M is a bounded module, $\mathrm{Ann}M$ is a Joseph ideal. Therefore $\mathrm{Ann}M=\mathrm{I}(\bar\mu)$ for some Shale-Weil tuple $\bar\lambda$. □

Theorem 10. Let M be a finitely generated $(\mathfrak{sp}(W\oplus W^*),\mathfrak{gl}(V))$-module and $\mathrm{I}(\mu)\subset\mathrm{Ann}M$ for some Shale-Weil tuple $\bar\mu$. Then M is $\mathfrak{gl}(V)$-bounded.

Proof. As M is an $(\mathfrak{sp}(W\oplus W^*),\mathfrak{gl}(V))$-module, we have $\mathrm{V}(M)\subset\mathfrak{gl}(V)^\perp\cap\mathbb{O}_{\mathfrak{sp}}$. Moreover, $\mathbb{O}_{\mathfrak{sp}}$ is a quotient of $W\oplus W^*$ by $\mathbb{Z}_2$, hence $\mathfrak{gl}(V)^\perp\cap\mathbb{O}_{\mathfrak{sp}}$ is a quotient by $\mathbb{Z}_2$ of the union of conormal bundles to $\mathrm{GL}(V)$-orbits. Since W is $\mathrm{GL}(V)$-spherical, these conormal bundles are $\mathrm{GL}(V)$-spherical. Therefore any irreducible component of $\mathrm{V}(M)$ is $\mathrm{GL}(V)$-spherical, and M is $\mathfrak{gl}(V)$-bounded by Proposition 1. □

We recall that $\bar\mu_0$ is an n_W-tuple $(n_W-\frac12, n_W-\frac32, ..., \frac12)$ and $\mu_0\subset\mathfrak{h}_W^*$ is the corresponding weight.

Theorem 11. a) For any two positive Shale-Weil tuples $\bar\mu_1,\bar\mu_2$ the categories of bounded $(\mathfrak{sp}(W\oplus W^*),\mathfrak{gl}(V))$-modules annihilated by $\mathrm{I}(\mu_1)$ and $\mathrm{I}(\mu_2)$ are equivalent.
b) In particular, the set of simple bounded $(\mathfrak{sp}(W\oplus W^*),\mathfrak{gl}(V))$-modules is naturally identified with the set of pairs $(\bar\mu, M)$, where $\bar\mu$ is a positive Shale-Weil n_W-tuple and M is a simple bounded $(\mathfrak{sp}(W\oplus W^*),\mathfrak{gl}(V))$-module annihilated by $\mathrm{I}((n_W-\frac12, n_W-\frac32, ..., \frac12))$.

Proof. Let $\bar\mu_1,\bar\mu_2$ be Shale-Weil n_W-tuples. The categories of bounded $(\mathfrak{sp}(W\oplus W^*),\mathfrak{gl}(V))$-modules annihilated by $\mathrm{I}(\mu_1)$ and $\mathrm{I}(\mu_2)$ are equivalent [BeG]. □

Theorem 12. Let $\bar\mu$ be a positive Shale-Weil n_W-tuple. Then the category of $(\mathfrak{sp}(W\oplus W^*),\mathfrak{gl}(V))$-modules annihilated by $\mathrm{I}(\mu)$ is equivalent to the direct sum of two copies of the category of perverse sheaves on W with respect to the stratification by $\mathrm{GL}(V)$-orbits.

Proof. The category of $(\mathfrak{sp}(W\oplus W^*),\mathfrak{gl}(V))$-modules annihilated by $\mathrm{I}(\mu)$ is equivalent, via the functors $\mathcal{H}_{\mu_0}^{\mu'}$ and $\mathcal{H}_{\mu'}^{\mu_0}$, to the category of $(\mathfrak{sp}(W\oplus W^*),\mathfrak{gl}(V))$-modules annihilated by $\mathrm{I}(\mu_0)$. The second category is equivalent to the direct sum of two copies of the category of perverse sheaves on W with respect to the stratification by $\mathrm{GL}(V)$-orbits (see Subsection 4.1). □

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6. Appendix: results of T. Braden and M. Grinberg

Let n be a positive integer and V be a $\mathbb{C}$-vector space of dimension $2n$. Then the category of perverse sheaves on Λ^2V with respect to the stratification by $\mathrm{GL}(V)$-orbits is equivalent to the category of representations of the following quiver A with relations:

$$\mathrm{A}_0 \underset{q_0}{\overset{p_0}{\rightleftarrows}} \mathrm{A}_1 \underset{q_1}{\overset{p_1}{\rightleftarrows}} \cdots \underset{q_{n-2}}{\overset{p_{n-2}}{\rightleftarrows}} \mathrm{A}_{n-1} \underset{q_{n-1}}{\overset{p_{n-1}}{\rightleftarrows}} \mathrm{A}_n,$$

$$\xi_i \text{ and } \nu_i \text{ are invertible for all } i,\ \xi_i=\nu_i \text{ for } i\in\{1,...,n-1\},$$

where $\xi_i := 1+q_{i-1}p_{i-1}$ for $i\in\{1,...,n\}$, and $\nu_i := 1+p_iq_i$ for $i\in\{0,...,n-1\}$. Let R be a representation of the quiver A. If R is simple, the invertible operators $\{\nu_0^n, \nu_1^{n-1}\xi^1, ..., \xi_n^n\}$ are proportional to the identity map with a fixed constant $c\in\mathbb{C}^*$. We call c *the monodromy of* R.

The set of eigenvalues of $1+p_iq_i$ is independent from i, and we call them *eigenvalues of* R. If R is simple, this set consists of one element λ. For a given eigenvalue $\lambda\neq 1$, there exists precisely one simple representation of A with eigenvalue λ. The simple representations of A with eigenvalue 1 are enumerated by the vertices of the quiver.

By definition, the *support of* R is the set of vertices corresponding to non-zero vector spaces. The corresponding to R perverse sheaf is supported at 0 if and only if R is supported at A_0. The corresponding to R perverse sheaf is smooth along W if and only if R is supported
at A_n.

Let now V be a $\mathbb{C}$-vector space of dimension n. Then the category of perverse sheaves on S^2V with respect to the stratification by $\mathrm{GL}(V)$-orbits is equivalent to the category of representations of the following quiver B with relations:

$$\mathrm{B}_0 \underset{q_0}{\overset{p_0}{\rightleftarrows}} \mathrm{B}_1 \underset{q_1}{\overset{p_1}{\rightleftarrows}} \cdots \underset{q_{n-2}}{\overset{p_{n-2}}{\rightleftarrows}} \mathrm{B}_{n-1} \underset{q_{n-1}}{\overset{p_{n-1}}{\rightleftarrows}} \mathrm{B}_n,$$

$$\xi_i \text{ and } \nu_i \text{ are invertible for all } i,\ \xi_i^2=\nu_i^2 \text{ for } i\in\{1,...,n-1\},$$

$$p_j\nu_{j+1}=-\nu_jp_j,\quad q_j\nu_j=-\nu_{j+1}q_j,\quad p_j\xi_{j+1}=-\xi_jp_j,\quad q_j\xi_j=-\xi_{j+1}q_j$$

whenever both sides of equalities are well-defined; here $\xi_i := 1+q_{i-1}p_{i-1}$ for $i\in\{1,...,n\}$, $\nu_i := 1+p_iq_i$ for $i\in\{0,...,n-1\}$. Let R be a representation of the quiver B. If R is simple, the invertible operators $\{\nu_0^n, \nu_1^{n-1}\xi^1, ..., \xi_n^n\}$ are proportional to the identity map with a fixed constant $c\in\mathbb{C}^*$. We call c *the monodromy of* R.

Let R be a simple representation of B. Then the set of eigenvalues of $(-1)^i\xi_i$ consists of one element $\bar\xi$; the set of eigenvalues of $(-1)^i\nu_i$ consists of one element $\bar\nu$. We call the pair $(\bar\xi,\bar\nu)$ the *spectrum of* R. We have $\bar\xi=\bar\nu=1$ or $\bar\xi=-\bar\nu$. The simple representations of B with the spectrum $(1,1)$ are enumerated by the inner vertices of the B-quiver. For $\lambda\neq\pm1$ there exists precisely one simple representation of B with the spectrum $(\lambda, -\lambda)$. The simple representations of B with the eigenvalues $(1,-1)$ and $(-1,1)$ are enumerated by the vertices of the quiver.

By definition, the *support of* R is the set of vertices corresponding to non-zero vector spaces. The corresponding to R perverse sheaf is supported at 0 if and only if R is supported at B_0. The corresponding to R perverse sheaf is smooth along W if and only if R is supported at B_n.

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