VARIETIES BIRATIONALLY ISOMORPHIC TO AFFINE G-VARIETIES

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1. Introduction

Let a linear algebraic group G act on an algebraic variety X. Classification of all these actions, in particular birational classification, is of great interest. A complete classification related to Galois cohomologies of the group G was established in [1]. Another important question is reducibility, in some sense, of this action to an action of G on an affine variety. In [2] it was shown that if the stabilizer of a typical point under the action of a reductive group G on a variety X is reductive then X is birationally isomorphic to an affine variety $\bar{X}$ with stable action of G. In this paper I show that if a typical orbit of the action of G is quasiaffine then the variety X is birationally isomorphic to an affine variety $\bar{X}$.

In essence, the classification of all actions amounts to the classification of families of subgroups which are stabilizers of points along a quasisection; in particular, I investigate families of subgroups corresponding to quasiaffine quotients, observable subgroups. The paper [3] contains an extensive study of families of subgroups although in an aspect different from what is required here: I am interested in a subgroup which contains in some sense all subgroups of the family to some extent, and the above paper deals with subgroups contained in all subgroups of the family. The whole paper is devoted to the proof of the following theorem:

Theorem 1. *Suppose there exist an algebraic group G and a G-variety X such that an orbit in general position for the action of G on this variety is quasiaffine. Then X is birationally isomorphic to an affine G-variety.*

2. Notation and agreements

For a group H which is a subgroup of a group G we introduce the following notation:
$\mathrm{R}(H)$ is the radical of H;
$\mathrm{R}_u(H)$ is the unipotent radical of H;
H^0 is the connected component of the identity element in H;
L_H is some maximal connected reductive subgroup of H;
$\mathrm{Ob}(G)$ is the set of observable subgroups of G: these are just the subgroups such that G/H is quasiaffine.

Definition 1. *A subgroup H of a group G is bound by a pair of subgroups (H_1, H_2) if $H_1 \subset H^0 \subset H_2$ and $\mathrm{R}_u(H) \subset \mathrm{R}_u(H_2)$*

Definition 2. *Let G be an algebraic group. A subgroup H of G is regularly embedded in it if $\mathrm{R}_u(H) \subset \mathrm{R}_u(G)$.*

Definition 3. *A family with base S of subgroups of an algebraic group G is a closed subvariety $Y \subset S\times G$ such that the image $(\cdot,\cdot) : Y \times_S Y \to S\times G$ lies in Y as well as the image $\sigma : Y \to S\times G$ where $(\cdot,\cdot) : G \times G \to G$ is a group operation, and $\sigma : G \to G$ is the inversion.*

Definition 4. *Let $\varphi : X \to Y$ be a dominant morphism of algebraic varieties. Then a quasisection S for this action is a closed subvariety X such that $\varphi|_S : S \to Y$ is a rational covering.*

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The general scheme of the proof can be described as follows.

Suppose a group G acts on a variety X and all orbits are quasiaffine. Then stabilizers of points in general position are mutually 'similar': their unipotent radicals have equal dimension, their reductive components are conjugate, and some subgroups conjugate to stabilizers of points in general position are bound by pair (L_H, H) where $H = H^0 \in Ob(G)$. This condition is equivalent to the statement that an orbit in general position for the action of G on X is quasiaffine.

To each family $Y \subset S \times G$ of subgroups of group G there corresponds a G-variety K containing a section for the action of G on K, isomorphic to S and such that the family with base S of stabilizers of points along S coincides with Y. This fact is proven in [4]. Moreover any two varieties M and M' having the above properties are birationally isomorphic. We will say that the variety M is generated by the family $Y \subset S \times G$. Furthermore any G-variety can be obtained as a quotient of some F-variety M by F where F is a discrete group (actions of F and G commute in this case) and M is generated by a family of subgroups $Y \subset S \times G$. I follow here the approach of [1].

Now we have two powerful facts at hand: first, each action is described by the family of its stabilizers along some quasisection [4], and the second fact describes these stabilizers in the conditions of Theorem 1. We may assume that the quasisection S for the action of G on X is chosen so that $\forall x \in S$ the subgroup St_x is bound by pair (L_H, H) for a fixed connected observable subgroup H. Furthermore the family of subgroups $Y_{\mathrm{R}_u(H)} \subset S \times \mathrm{R}_u(H) \subset X \times G$ $(su = s)$ generates a quasiaffine variety $M_{R_u(H)}$, and we endow it with the structure of an H-variety. The variety $M_G = G *_H M_H$ is quasiaffine as well and has a section S such that the family of stabilizers along it coincides with $Y_G \subset S \times G \subset X \times G$ $(g \in H \cap St_x = St_x^0)$. Thus there exists a map from M_G to X, finite on an open set. This implies quasiaffinity of this open G-invariant subset in X.

The ground field for all objects in question is $\mathbb{C}$ (an uncountable algebraically closed field of characteristic 0), all schemes are separable.

3. Binding of stabilizers

Definition 5. *A subset of an algebraic variety is called A-measurable if it is a union of at most countable set of constructive sets.*

Proposition 1. *If an irreducible constructive set X is embedded into a set $\cup_{i\in\mathbb{N}} V_i \subset S$ where all V_i are constructive sets, S is an algebraic variety, then there exists an open subset X' in X, embedded in V_i for some i.*

Proof. If $\forall i$ $\overline{V_i \cap X}$ does not coincide with $\overline{X}$ then $\overline{X}$ is a union of a countable set of its subvarieties having smaller dimension — a contradiction. Hence one of these sets is dense in X. This implies our assertion. □

Lemma 1. *At least one subgroup conjugate to an observable subgroup H of a group G is bound by one of pairs $\{(\mathrm{L}_{H_i}, H_i)\}_{i\in\mathbb{N}}$ where H_i are observable subgroups and L_{H_i} are their Levy subgroups.*

Proof. A subgroup conjugate to a subgroup H is regularly embedded into some quasiparabolic subgroup Q from the set $\{Q_i\}_{i\in\mathbb{N}}$(the set of dominant weights is countable). This quasiparabolic subgroup Q contains only a countable set (up to conjugation) of reductive subgroups R, one of which coincides with the reductive component of H. The set of pairs $(R, R{\cdot}\mathrm{R}_u Q)$ determines the required set of pairs (L_{H_i}, H_i). □

Theorem 2. *The set of subalgebras m of a Lie algebra $\mathfrak{g}$, having a given dimension n, is a closed subvariety in* $\mathrm{Gr}(n, \mathfrak{g})$ *(Grassmanian, the variety of n-dimensional subspaces). We will denote it by* $\mathrm{Lie}(n, \mathfrak{g})$.

Proof. Let v be a decomposable element of $\Lambda^n\mathfrak{g}$ corresponding to a point of $\mathrm{Gr}(n, \mathfrak{g})$. Then the corresponding subspace is a Lie algebra iff $[\langle v, x\rangle, \langle v, y\rangle] \wedge v = 0$ $\forall x, y \in (V^*)^{\otimes(n-1)}$ ($\langle u, w\rangle$ is a complete componentwise convolution of the tensor $u \otimes w$) □

Remark 1. *Let G be a unipotent group. Then to each subalgebra of its Lie algebra there corresponds some its algebraic subgroup.*

Remark 2. *The Lie subalgebras of all groups conjugate to a given one form an algebraic subvariety in the Grassmannian.*

Proof. These are just those subspaces of a given dimension in the Lie algebra which lie in the same orbit as the vector corresponding to this algebra. □

Definition 6. *A family with base S of subalgebras of dimension n in a Lie algebra $\mathfrak{g}$ is a map $S \to \mathrm{Lie}(n, \mathfrak{g})$.*

Lemma 2. *To each family of subgroups $Y \subset X \times G$ of a fixed dimension there corresponds a family of Lie subalgebras.*

Proof. The tangent bundle for the family of subgroups Y embeds into the tangent bundle $\mathrm{T}(G \times X)$ as a closed subvariety. Similarly, $X \times \mathrm{lie}\ G$ maps into the tangent bundle $\mathrm{T}(G \times X)$ (a reduction of the action of a group to the action of its algebra). The inverse image of the tangent bundle for the family determines a variety N. Consider the variety $P \subset S \times \mathfrak{g} \times \mathrm{Gr}(n, \mathfrak{g}) = \{(s, g, \overline{\mathfrak{g}}) | g \in \overline{\mathfrak{g}}\}$, the intersection $U = N \times \mathrm{Lie}(n, \mathfrak{g}) \cap P$ and the image of the projection $U \to X \times \mathrm{Lie}(n, \mathfrak{g})$, which is some variety A. Observe that the projection map $A \to X$ is a pointwise isomorphism, thus there exists a birational map $X \to \mathrm{Lie}(n, \mathfrak{g})$. □

Theorem 3. *The set of subgroups of a group G, having dimension n and bound by a pair (L_H, H) where L_H is a Levy subgroup in a subgroup H, determines a closed subset in $\mathrm{Lie}(n, \mathrm{lie}\ G)$ by its Lie algebras.*

Proof. A reductive connected group L_H and a unipotent group U are a Levy subgroup and the radical of the same group iff L_H preserves U under conjugation. In other words, $[\mathrm{lie}\ \mathrm{L}_H, \mathrm{lie}\ U] \subset \mathrm{lie}\ U$. In terms of corresponding decomposable vectors, this is formulated as follows:

$$\forall x \in \Lambda^{dim\ \mathrm{L}_H - 1}[\mathrm{lie}\ H]^* \ \forall y \in \Lambda^{dim U - 1}[\mathrm{lie}\ \mathrm{R}_u H]^* \ [\langle v_L, x\rangle, \langle v_U, y\rangle] \wedge v_U = 0.$$

This condition determines a closed set v_U in $\mathrm{Gr}(\mathrm{dim} U, \mathrm{lie}\ \mathrm{R}_u H)$. Hence the image of the map $(v_L, v_U) \to v_L \wedge v_U$, which is the required set of tangent algebras, is closed. □

Corollary 1. *The set of Lie algebras of all subgroups in a group G, such that their conjugates are regularly embedded into a given subgroup H, is an A-measurable set.*

Proof. Each subgroup S regularly embedded into H is up to conjugation bound by one of pairs (L_{H_i}, H_i) where $\{H_i\}_{i \in \mathbb{N}}$ is the set of subgoups G such that $\mathrm{R}_u H = \mathrm{R}_u H_i$. Thus the set of these Lie algebras is obtained from the set of Lie algebras of subgroups bound by pairs $\{(\mathrm{L}_{H_i}, H_i)\}_{i \in \mathbb{N}}$, under the action of the group. □

Theorem 4. *Suppose G is an algebraic group and there exists an irreducible equidimensional family of its algebraic subgroups $Y \subset X \times G$, such that each subgroup from the family is observable in G. Then there exists an open subset X' in X and a connected observable subgroup H in G such that the pair (L_H, H) bounds some conjugate subgroups for all subgroups in the family $Y \cap (X' \times G)$.*

Proof. To a family of subgroups there corresponds a family of algebras with the same base. The set of Lie algebras of subgroups such that their conjugates are bound by a pair (L_{H_i}, H_i) for some observable subgroup $\{H_i\}_{i \in \mathbb{N}}$, is an A-measurable set S. The family of Lie algebras $\varphi : X' \to \mathrm{Lie}(n, \mathfrak{g})$ where n is the dimension of subalgebras in the family and X' is an open set in X, lies in S. Then by Proposition 1 there exists an open set $X'' \subset X$ such that $\exists i \in \mathbb{N} : \forall x \in X'' \exists g \in G | St_{gx}$ is bound by pair (L_{H_i}, H_i). □

Theorem 5. *Suppose an algebraic group G acts on a variety X and a typical orbit is quasiaffine. Then there exists a connected observable subgroup $H \subset G$ such that for all points x from an open set X' in X some subgroup conjugate to St_x is bound by pair (L_H, H).*

Proof. The inverse image of the diagonal in $X \times X$ under the map $\varphi : X \times G \to X \times X$ of the form $(x, g) \mapsto (gx, x)$ is the family of stabilizers of points. By Theorem 4 there exists an observable subgroup H and accordingly a pair (L_H, H) such that some open set X' in X has the following property: $\forall x \in X' \exists g \in G | St_{gx}$ is bound by the pair (L_H, H). □

4. Some auxiliary facts

Theorem 6. *Let $H \subset G$ be algebraic groups. If G/H is quasiaffine then for every quasiaffine H-variety X the variety $G *_H X$ is quasiaffine.*

Proof. The scheme X is a subvariety of some H-module V_H which is by quasiaffinity of G/H a submodule of some G-module V_G. Hence $G *_H X$ embeds into $G *_H V_G$ but $G *_H V_G \cong G/H \times V_G$ where the isomorphism is given by the map $(g, v) \mapsto (gH, gv)$. But $G/H \times V_G$ is quasiaffine, hence $G *_H X$ is quasiaffine as well. □

Remark 3. *A similar argument can be found in* [1].

Lemma 3. *Suppose a finite group G acts on a quasiaffine variety X. Then there exists a quasiaffine geometric quotient $X \to Y$.*

Proof. The variety X is an open subvariety in an affine G-variety $\tilde{X}$ for which the geometric quotient exists and is affine. □

Lemma 4. *Let $\varphi : X \to Y$ be a finite surjective map of normal varieties. Then if X is affine (quasiaffine) then the same is true for Y.*

Proof. Let n be the degree of φ. Then $\varphi_n : X \times_Y X \times_Y ... \times_Y X(n \text{ copies}) \cong X_\varphi^{(n)} \to Y$ is a finite map. In $X_\varphi^{(n)}$ there exists a subvariety $\tilde{X}$ which is the closure of a set whose elements are ordered n-tuples of pairwise distinct elements from X, which map into the same element from Y. Then $\tilde{X}$ is affine (quasiaffine). But $\tilde{X}/S_n \cong Y$ pointwise on an open set, and so finiteness of the map $\tilde{X}/S_n \to Y$ implies that it is an isomorphism. Hence Y is affine (quasiaffine) as well. □

Lemma 5. *Suppose $\varphi : X \to Y$ is a morphism of G-varieties and there exists an open set $Y' \subset Y$ such that $\forall y \in Y' |\{\varphi^{-1}(y)\}| < \infty$. Then there exists a G-invariant open set $Y'' \subset Y$ such that the map $\varphi|_{\varphi^{-1}(Y'')} : \varphi^{-1}(Y'') \to Y''$ is finite.*

Proof. It is well known that there exists an open affine set $\tilde{Y}$ such that $\varphi|_{\varphi^{-1}(\tilde{Y})} : \varphi^{-1}(\tilde{Y}) \to \tilde{Y}$ is finite. Then the map $\varphi|_{\varphi^{-1}(G\tilde{Y})} : \varphi^{-1}(G\tilde{Y}) \to G\tilde{Y}$ is finite as well (here $G\tilde{Y}$ is the minimal G-invariant open set containing $\tilde{Y}$) because it is finite on the atlas consisting of affine open sets $\{g\tilde{Y}\}_{g \in G}$. □

Lemma 6. *Let $\varphi : X \to Y$ be a dominant morphism of two algebraic varieties such that Y is irreducible. Then there exist a subvariety $S_X \subset X$ and an open subset $S_Y \subset Y$ such that the map $\varphi : S_X \to S_Y \subset Y$ is finite.*

Proof. We may assume that the variety X is affine. Let $k[X]$ be its algebra of regular functions, and $k(Y)$ be the field of rational functions on Y. Then by the homomorphism extension theorem there exists a homomorphism $\psi : k[X] \to k(\overline{Y})$ where $k(\overline{Y})$ is a finite extension of $k(Y)$. The variety $\bar{Y}$ is unique up to birational equivalence. Denote by S'_X the closure of the image of the birational morphism $\bar{\psi} : \bar{Y} \to X$ set corresponding to ψ. Obviously the morphism $\varphi|_{S'_X} : S'_X \to Y$ has a finite number of inverse images for any general point. Hence there exists an open set $S_Y \subset Y$ such that the morphism $\varphi|_{\varphi^{-1}(S_Y) \cap S'_X} : \varphi^{-1}(S_Y) \cap S'_X \to S_Y$ is finite. □

Remark 4. *In terms of quasisections, this result is proven in* [1].

5. The main result

Theorem 7. *For any family $Y \subset U \times S$ of subgroups of a unipotent group U there exist an affine U-variety X, an open subset $S' \subset S$ and a section for the action of U on it such that the family of stabilizers of points along this section coincides with $Y \cap U \times S'$.*

Proof. Consider an ascending filtration $\{U_i\}_{i\in\mathbb{N}}$ of the group U by normal subgroups, such that $U_0 = \{e\}, U_\infty = U, U_{i+1}/U_i \cong (\mathbb{C}^{k_i}, +)$. Let U' be a subgroup of U. Then the image $U' \cap U_{i+1}$ under the map $\{U' \cap U_{i+1} \to U_{i+1}/U_i\}$ is a subspace, and its complementary subspace has a section in U_{i+1} isomorphic to $\mathbb{C}^{k_i - dim\ ((U'\cap U_{i+1})/(U'\cap U_i))} = K_i$. Clearly various elementwise products $K = K_\infty \cdot ... \cdot K_1 \cdot K_0$ form a section for the left action of U' on U. The construction obviously implies that if U' is a part of a family then K is a section for an open affine set S' on the family. Conversely, by means of Campbell — Hausdorff formula we can construct an action of U on $S' \times K$ for an open part S' of a family S containing U', such that the family of stabilizers along S' coincides with (Y, S') □

Corollary 2. *Let H be a connected affine algebraic group with a Levy subgroup L, and $Y \subset S \times G$ is a family of subgroups bound by (L, H). Then there exist an open set S' in S, a quasiaffine H-variety X and a section for the action of H on it $t : X/H \to X$ such that the family of stabilizers along it coincides with $Y \cap (S' \times G)$.*

Proof. We may assume that the family is equidimensional. Each element of the family H_x is representable in the form $L \leftthreetimes \mathrm{R}_u H_x$. Consider the corresponding variety X for the family $x \to \mathrm{R}_u H_x$. It remains to observe that we can canonically introduce action of H on X. □

Theorem 8. *Suppose there are an algebraic group G and an irreducible G-variety X such that an orbit in general position under the action on this variety is quasiaffine. Then X is birationally isomorphic to an affine G-variety.*

Proof. By Theorem 5 there exist a connected observable subgroup H of G, an open invariant set

$X' \subset X : \forall x \in X' \exists g \in G | St_{gx}$ is bound by pair (L_H, H) where L_H is a Levy subgroup of H, and

$$\forall x, y \in X' \dim\ St_x = \dim\ St_y,$$

and a geometric quotient X'/G(Rosenlicht theorem). In the latter, there exists a closed set of points for which the dimension of the orbit of H is minimal, or equivalently the set of points $x \in X$ such that $St_x^0 \subset H$. It includes the set of points from X'' fixed under a fixed Levy subgroup L of H. The closed variety X'' meets each orbit for the action of G on X, hence the image of the map $\varphi : X'' \to X/G$ is dense. Thus there exists a quasisection $S \subset X''$ such that $\varphi|_S$ is finite over an open set $X_g/G \subset X/G$ where X_g is a G-invariant open subset in X. Then the variety $M = X_g \times_{X_g/G} S$ is a G-variety, and the projection map $pr : M \to X_g$ is finite. Moreover the projection map $M \to S$ is the quotient by the action of G, and the diagonal map $S \to S \times_{X_g/G} S \subset X_g \times_{X_g/G} S$ is a section. Stabilizers of its points are bound by pair (L, H).

Along the section S we have a family $Y \subset G \times S$ of subgroups of G with base S: $(g, s) \in Y \Leftrightarrow gs = s$ such that $g \in H; x \to St_x \cap H = St_x^0$. For this family of subgroups H there exists a quasiaffine (in fact affine) H-variety M_H with a section $S_H \cong S$ such that the family $Y_H \subset G \times S_H \to S_H$ of subgroups along it coincides with the above family of subgroups in H. The variety $M_G \cong G *_H M_H$ is quasiaffine, and the section $\{e\} * S_H$ determines a family $Y_G \subset G \times S \to S$ of subgroups in G, identical with the original one. There exist two natural maps: $\phi : S \times G \to M$ of the form $(s, g) \mapsto gs$ and $\psi : S \times G \to M_G$ of the form $(s, g) \mapsto gs$ where $\psi(x) = \psi(y) \Rightarrow \phi(x) = \phi(y)$. Hence there exists a G-equivariant birational morphism p from the quasiaffine variety M_G to M with a finite number of inverse images for any general point. Consequently it is a finite map on an invariant set $M'_G \subset M_G$. Thus M'_G is quasiaffine as well as $p(M'_G)$, and the latter is an open G-invariant subset in X. □

Theorem 1 is an obvious consequence of Theorem 8.

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