Integral Operators with Singular Canonical Relations

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Abstract

We derive certain estimates and asymptotics for oscillatory integral operators with degenerate phase functions, concentrating our attention on the situation when one of the projections from the associated canonical relation is a Whitney fold. We discuss related topics: convexity, higher order degeneracy of canonical relations, and almost orthogonality.

In the second part of the paper, we apply the developed technique to the Radon Transform of Melrose-Taylor, a particular degenerate Fourier integral operator which arises in the theory of scattering. We obtain its regularity properties for the scattering on a convex compact domain with a smooth boundary. The result is formulated in terms of the highest order of contact of lines with the boundary.

1 Oscillatory integral operators with singularities

1.1 Introduction

Let us consider the oscillatory integral operator T_λ, given by

$$T_\lambda u(x) = \int_{\mathbb{R}^n_R} e^{i\lambda S(x,\vartheta)}\psi(x,\vartheta)u(\vartheta)\,d\vartheta, \qquad x \in \mathbb{R}^n_L, \quad \vartheta \in \mathbb{R}^n_R, \tag{1}$$

where ψ is some smooth compactly supported function. We study the rate of high-frequency decay of the operator norm $\|T_\lambda\|_{L^2\to L^2}$ for large λ. This problem is intrinsically related to the regularity properties of Fourier integral operators and generalized

1991 *Mathematics Subject Classification.* Primary 43A32. Secondary 35L05, 35S30.

Radon transforms; see D.H. Phong and E.M. Stein [1991], [1994], D.H. Phong [1994], and A. Greenleaf and A. Seeger [1994]. We will discuss two particular cases in Section 1.2.

The classical result by L. Hörmander [1971] is the estimate for the case when the mixed Hessian $S_{x\vartheta} = \left[\partial_{x_i}\partial_{\vartheta_j} S\right]$, $i, j = 1, ..., n$, is non-degenerate:

$$\det S_{x\vartheta} \neq 0. \tag{2}$$

Then T_λ is bounded from L^2 to L^2, with the norm

$$\|T_\lambda\| \leq const\, \lambda^{-\frac{n}{2}}. \tag{3}$$

According to K. Asada and D. Fujiwara [1978], this estimate is also valid without the assumption that $\psi(x, \vartheta)$ is compactly supported.

The canonical relation associated to the operator (1) is given by

$$\mathcal{C} \equiv \{(x, S_x;\, \vartheta, S_\vartheta)\} \subset T^*\mathbb{R}^n_L \times T^*\mathbb{R}^n_R. \tag{4}$$

The condition (2) implies that the projections π_L, π_R from $\mathcal{C}$ onto the first and second factor of $T^*\mathbb{R}^n_L \times T^*\mathbb{R}^n_R$ are local diffeomorphisms, so that $\mathcal{C}$ is locally a graph of a symplectomorphism from $T^*\mathbb{R}^n_L$ to $T^*\mathbb{R}^n_R$.

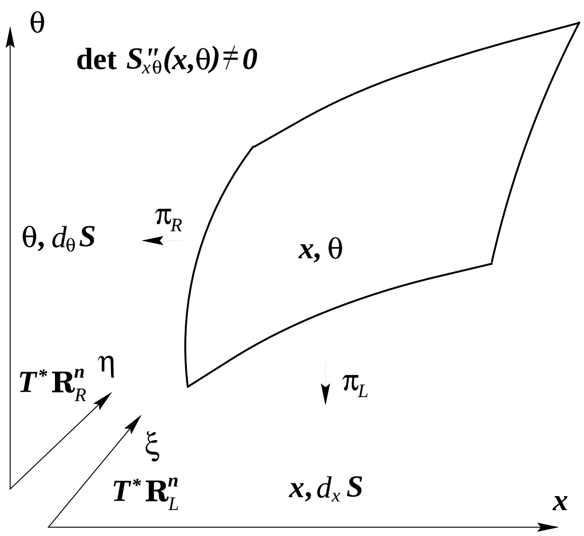

Fig. 1. *Canonical relation which is locally a graph of a symplectomorphism.*

In general, there is no guarantee that the inequality (2) is satisfied. We will be interested in the situation when the projections from the canonical relation are degenerate.

Using the parameterization

$$(x,\vartheta) \overset{\imath}{\longmapsto} (x, S_x;\, \vartheta, S_\vartheta)$$

of the canonical relation $\mathcal{C}$, we will consider $\pi_{_L}$, $\pi_{_R}$ as lifted by $\imath^*$ onto $\mathbb{R}^n_{_L} \times \mathbb{R}^n_{_R}$:

$$\pi_{_L} : (x,\vartheta) \mapsto (x, S_x), \qquad \pi_{_R} : (x,\vartheta) \mapsto (\vartheta, S_\vartheta).$$

The differentials of these maps,

$$d\pi_{_L} : (dx, d\vartheta) \mapsto (dx,\, S_{xx}dx + S_{x\vartheta}d\vartheta), \tag{5}$$

$$d\pi_{_R} : (dx, d\vartheta) \mapsto (d\vartheta, S_{\vartheta x}dx + S_{\vartheta\vartheta}d\vartheta), \tag{6}$$

are both degenerate on the critical variety

$$\Sigma^0 \equiv \{\, (x,\vartheta) \mid h(x,\vartheta) = 0\}\, ; \tag{7}$$

$h(x,\vartheta)$ stays for the determinant of the mixed Hessian, $h(x,\vartheta) \equiv \det S_{x\vartheta}$.

We will be assuming that one of the projections (for definiteness, $\pi_{_L}$) is a Whitney fold: the kernel of $d\pi_{_L}$ is one-dimensional and transversal to the critical variety Σ^0, and the determinant $h(x,\vartheta) = \det S_{x\vartheta}(x,\vartheta)$ of the mixed Hessian vanishes of order 1 in the direction of $\operatorname{Ker} d\pi_{_L}$.

According to Y. Pan and C. Sogge [1990], who used the results of R.B. Melrose and M.E. Taylor [1985], if *both* projections from the canonical relation $\mathcal{C}$ are Whitney folds, then the decay of the norm of T_λ is given by

$$\|T_\lambda\| \le const\, \lambda^{-\frac{n}{2}+\frac{1}{6}}. \tag{8}$$

An independent analytical proof of this result was obtained by S. Cuccagna in [1995].

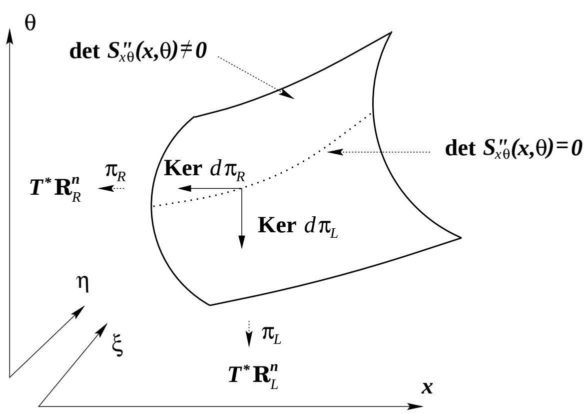

Fig. 2. *Both projections from the canonical relation are Whitney folds.*

The singular behavior of π_R could be quite different from that of π_L. In general, $d_x h(x,\vartheta)$ could vanish somewhere on the critical variety, or $\operatorname{Ker} d\pi_R$ may be not transversal to Σ^0. A. Greenleaf and A. Seeger [1994] gave the *a priori* estimate for the case when only one of the projections is a Whitney fold; then T_λ is a continuous operator from L^2 to L^2 with the norm

$$\|T_\lambda\| \le const\, \lambda^{-\frac{n}{2}+\frac{1}{4}}. \tag{9}$$

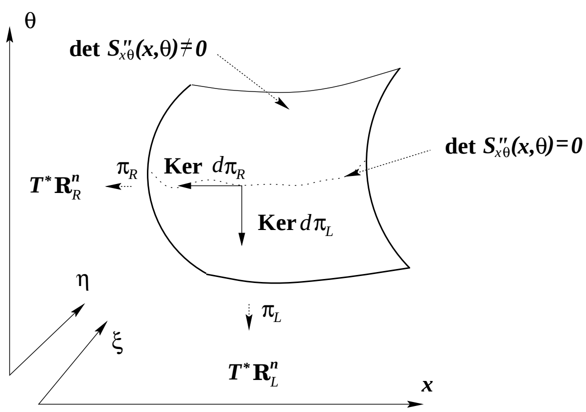

Fig. 3. *The projection π_L is a Whitney fold; no assumptions on π_R.*

We will concentrate our attention on the intermediate situation, when one of the projections (π_L) is a Whitney fold, and when something is also known about the other projection.

Dyadic decomposition with respect to $h(x,\vartheta)$. We will use the dyadic decomposition with respect to the values of the determinant of the mixed Hessian, which could be viewed as the distance to the critical variety $\Sigma^0 \equiv \{h(x,\vartheta)=0\}$.

We choose the symmetric function $\bar\beta \in C^\infty_{\mathrm{comp}}([-2,2])$ such that

$$\begin{cases} \bar\beta(t)=1, & |t|\le 1,\\ \bar\beta(t)=0, & |t|\ge 2. \end{cases} \tag{10}$$

Let $\beta_+(t) \in C^\infty_{\mathrm{comp}}([1/2,2])$ be given by

$$\begin{cases} \beta_+(t)=\bar\beta(t)-\bar\beta(2t), & t\ge 0,\\ \beta_+(t)=0, & t\le 0, \end{cases}$$

and let $\beta_-(t)$ be its reflection, $\beta_-(t)\equiv\beta_+(-t)\in C^\infty_{\mathrm{comp}}([-2,-1/2])$.

We use the partition of 1,

$$1 = \sum_{\pm} \sum_{N<N_o} \beta_{\pm}(2^N h(x,\vartheta)) + \bar{\beta}(2^{N_o} h(x,\vartheta)), \qquad N,\, N_o \in \mathbf{Z}, \tag{11}$$

to define

$$T_{\pm}^{\hbar} u(x) = \int e^{i\lambda S(x,\vartheta)}\, \psi(x,\vartheta)\, \beta_{\pm}(\hbar^{-1} h(x,\vartheta))\, u(\vartheta)\, d\vartheta, \tag{12}$$

$$\bar{T}^{\hbar} u(x) = \int e^{i\lambda S(x,\vartheta)}\, \psi(x,\vartheta)\, \bar{\beta}(\hbar^{-1} h(x,\vartheta))\, u(\vartheta)\, d\vartheta, \tag{13}$$

where $\hbar = 2^{-N}$, and decompose the operator (1) into a finite sum

$$T_\lambda = \sum_{\pm} \sum_{\hbar>\hbar_o}^{\hbar\le\Lambda} T_{\pm}^{\hbar} + \bar{T}^{\hbar_o}, \qquad \hbar = 2^{-N}, \quad \hbar_o = 2^{-N_o}, \quad N,\, N_o \in \mathbf{Z}. \tag{14}$$

Here $\Lambda \equiv 2^{N_1} \ge \sup |h(x,\vartheta)|$. The cut-off value $\hbar_o = 2^{-N_o}$ is to be chosen later.

We will not write $\pm$-subscripts of the functions $\beta_\pm$ and of the operators $T_{\pm}^{\hbar}$.

The operators $T^{\hbar}$ are localized away from the critical variety Σ^0, and therefore Hörmander's estimate is applicable: $\left\|T^{\hbar}\right\| \le C(\hbar)\lambda^{-n/2}$. Apparently, this estimate blows up when $\hbar$ tends to zero. From the informal rescaling argument one can see that $\|T\| \sim |\det S_{x\vartheta}|^{-1/2}$, hence the best we could count on is $C(\hbar) \le \mathit{const}\, \hbar^{-1/2}$, so that

$$\left\|T^{\hbar}\right\| \le \mathit{const}\, \lambda^{-\frac{n}{2}} \hbar^{-\frac{1}{2}}. \tag{15}$$

We will justify (15) in the case when one of the projections from the associated canonical relation is a Whitney fold, while the other satisfies some very weak *finite type* condition.

The concept of *pseudoconvexity* is introduced in Section 1.3, and finite type conditions are discussed in Section 1.4. The proof of (15) is contained in Section 1.6 (Theorem 1.6.1).

The estimate (15) also suggests that if the integral kernel of T_λ vanishes on the critical variety Σ^0, then $\|T^{\hbar}\| \to 0$ for small values of $\hbar$, and $\|T_\lambda\|$ satisfies the estimate $\|T_\lambda\| \le \mathit{const}\, \lambda^{-\frac{n}{2}}$ on operators with non-degenerate phase functions. This is the statement of Theorem 1.6.2.

The support of the operator $\bar{T}^{\hbar_o}$ contains the critical variety Σ^0. Since the rank of the mixed Hessian could only drop to $n-1$, we can use Hörmander's estimate (3) in the lower dimension (choosing the axes $x_1, \ldots, x_{n-1}$ and *theta*s $\vartheta_1, \ldots, \vartheta_{n-1}$ in the non-critical directions of the mixed Hessian): $\left\|\bar{T}^{\hbar_o}\right\| \le c(\hbar_o)\lambda^{-(n-1)/2}$.

We expect that the support of $\bar{T}^{\hbar_o}$ is asymptotically small in the *critical* directions, x_n and ϑ_n. If one of these projections is a Whitney fold, then the support of $\bar{T}^{\hbar_o}$ in the corresponding critical direction (ϑ_n for the projection π_L) is of size $\Delta\vartheta_n \sim \hbar_o$.

This could be seen on Fig. 2 and Fig. 3. According to Young's inequality, we expect that this contributes a factor $\hbar_o^{1/2}$ to $c(\hbar_o)$:

$$\left\|\bar{T}^{\hbar_o}\right\| \leq const\, \lambda^{-\frac{n-1}{2}} \hbar_o^{\frac{1}{2}}. \tag{16}$$

If the other projection from the canonical relation is a Whitney fold, too, then $\Delta x_n \sim \hbar_o$ (see Fig. 2), and $c(\hbar_o)$ contains one more factor $\hbar_o^{1/2}$:

$$\left\|\bar{T}^{\hbar_o}\right\| \leq const\, \lambda^{-\frac{n-1}{2}} \hbar_o^{\frac{1}{2}} \hbar_o^{\frac{1}{2}}. \tag{17}$$

We can find the value of $\hbar_o$ when this estimate meets the estimate (15) on $T^{\hbar}$; it happens when $\hbar_o = \lambda^{-1/3}$. If we substitute this value of $\hbar_o$ into (14) and then use the estimates (15) and (17), we arrive at the estimate (8):

$$\|T_\lambda\| \leq const\, \lambda^{-\frac{n}{2}+\frac{1}{6}}.$$

The rigorous proof of this estimate can be obtained via the almost orthogonal decomposition (S. Cuccagna [1995]). See Theorem 1.6.1 and Remarks 3 and 4 in Section 1.6.

In the second part of the paper, Sections 2.1 − 2.6, we will approach a particular degenerate integral operator from the paper of R.B. Melrose and M.E. Taylor [1985], which we call *the Radon Transform of Melrose-Taylor*:

$$F : \mathcal{E}'(\mathbb{R}\times \mathbf{S}^n) \to \mathcal{D}'(\mathbb{R}\times B),$$

$$Fu(t,\mathbf{r}) = \int\limits_{\mathbb{R}\times\mathbf{S}^n} \delta(t-s-\mathbf{r}\cdot\omega)\, u(s,\omega)\, ds\, d\Omega_\omega. \tag{18}$$

Here $\mathbf{r}$ is a vector in $\mathbb{R}^{n+1}$, pointing to the boundary B of a compact domain (*obstacle*) $K \subset \mathbb{R}^{n+1}$. This is an excellent example of a Fourier integral operator associated to a canonical relation which is not a local graph.

We will derive the regularity properties of (18) for the scattering on an arbitrary convex compact domain with a smooth boundary. The smoothness becomes worse when the principal curvatures of the boundary vanish of higher order.

Roughly, if the principal curvature vanishes of order $k-1$, then there is a loss of $\frac{1}{2}\frac{k}{2k+1}$ derivatives in the regularity properties (comparing to non-degenerate Fourier integral operators). This expression plays the role of the interpolation between $1/6$ for two-sided Whitney folds and $1/4$ for one-sided Whitney folds.

Indeed, one of the projections from the canonical relation determined by the phase function $S = \mathbf{r}\cdot\omega$ is basically the orthogonal projection from a sphere, and thus is a Whitney fold.

The other projection is basically the orthogonal projection from the boundary of an obstacle. The value $k=1$ corresponds to non-vanishing principal curvature, when the orthogonal projection from the boundary is a Whitney fold. This was the case originally considered by Melrose and Taylor.

The limit $k \to \infty$ means that the second projection from the canonical relation has a singularity of an arbitrarily high order. According to Greenleaf and Seeger [1994], there is a loss of up to $1/4$ derivatives, which is the limit of $\frac{1}{2}\frac{k}{2k+1}$.

The loss of $\frac{1}{2}\frac{k}{2k+1}$ derivatives in the regularity properties seems to be a general result for the Fourier integral operators such that one of the projections from the associated canonical relation is a Whitney fold and the other has *degeneracies of order not greater than* k (see Definition 1.4.1).

Let us now mention two results which we do not prove in the paper.

Asymptotic estimates for oscillatory integral operators. It is useful to consider the asymptotic situation, when both $\pi_{_L}$ and $\pi_{_R}$ are Whitney folds, but one of them ($\pi_{_R}$) is asymptotically degenerating (in the sense that in proper local coordinates it is given by $\eta_{i'} = x_{i'}$ for $i' = 1, \ldots, n-1$, and $\eta_n = \sigma x_n^2/2$, with $0 < \sigma \ll 1$). We would like to state the corresponding asymptotics of the estimates on the oscillatory integral operators, together with the asymptotics corresponding to infinitesimally small localization of the integral kernel.

We consider the oscillatory integral operator localized near some point (x_o, ϑ_o):

$$T_{\lambda\delta\varepsilon} u(x) = \int\limits_{\mathbb{R}^n_R} \rho((x-x_o)/\delta)\, \varrho((\vartheta-\vartheta_o)/\varepsilon)\, e^{i\lambda S(x,\vartheta)}\, \psi(x,\vartheta)\, u(\vartheta)\, d\vartheta, \tag{19}$$

here $\rho(x)$ and $\varrho(\vartheta)$ are smooth functions supported near the origin; δ and ε are two asymptotically small parameters. For simplicity, we assume that (x_o, ϑ_o) is shifted to the origin in $\mathbb{R}^n_L \times \mathbb{R}^n_R$. We will use the notation $W_{\delta\varepsilon} \equiv \operatorname{supp} \rho((x-x_o)/\delta)\varrho((\vartheta-\vartheta_o)/\varepsilon)$.

The local coordinates $x = (x', x_n)$ and $\vartheta = (\vartheta', \vartheta_n)$ are chosen so that the vectors ∂_{ϑ_n} and ∂_{x_n} are transversal to Σ^0, while $S_{x'\vartheta'}$ is a non-degenerate matrix.

Theorem 1.1.1 *Let $S(x,\vartheta) \in C^{2n+3}(W_{\delta\varepsilon})$ and let $\psi(x,\vartheta) \in C^{2n+1}(W_{\delta\varepsilon})$.*

Let the mixed Hessian $S_{x\vartheta}$ be of rank at least $n-1$, with its non-degenerate part given by $S_{x'\vartheta'}$, $|\det S_{x'\vartheta'}| \ge const > 0$. There is the following estimate on the norm of $T_{\lambda\delta\varepsilon}$ in (19):

$$\|T_{\lambda\delta\varepsilon}\| \le const\, \lambda^{-\frac{n-1}{2}} (\delta\varepsilon)^{\frac{1}{2}}. \tag{20}$$

If $\pi_{_L}$ is a Whitney fold, so that $\det S_{x\vartheta}$ vanishes of the first order in ϑ_n,

$$|\partial_{\vartheta_n} \det S_{x\vartheta}| \ge const > 0, \tag{21}$$

then there is also the following estimate on the norm of $T_{\lambda\delta\varepsilon}$:

$$\|T_{\lambda\delta\varepsilon}\| \le const\, \lambda^{-\frac{n}{2}+\frac{1}{4}} \delta^{\frac{1}{4}}. \tag{22}$$

If, in addition, π_R *is a degenerating Whitney fold,*

$$\sigma \equiv \inf_{W_{\delta\varepsilon}} |\partial_{x_n} \det S_{x\vartheta}| > 0, \qquad \sigma \ll 1, \tag{23}$$

and if

$$\sup_{W_{\delta\varepsilon}} \left(\|S^{-1}_{x'\vartheta'}\| \, \|S_{x_n\vartheta'}\|\right) \frac{\sup_{W_{\delta\varepsilon}} \|\nabla_{x'} \det S_{x\vartheta}\|}{\inf_{W_{\delta\varepsilon}} |\partial_{x_n} \det S_{x\vartheta}|} \le const < 1, \tag{24}$$

then also

$$\|T_{\lambda\delta\varepsilon}\| \le const\, \lambda^{-\frac{n}{2}+\frac{1}{6}} \sigma^{-\frac{1}{6}}. \tag{25}$$

The constants in (20), (22), *and* (25) *do not depend on* λ, δ, ε, *and* σ.

The estimate (20) is apparent: the factor $\lambda^{-\frac{n-1}{2}}$ is L. Hörmander's estimate (3) in $n-1$ dimensions, and the factor $(\delta\varepsilon)^{\frac{1}{2}}$ appears from Young's inequality, due to the size of the support of $T_{\lambda\delta\varepsilon}$ in the critical directions.

The estimate (22) is a counterpart of the estimate (9) due to A. Greenleaf and A. Seeger for the case when π_L is a Whitney fold. No assumption on π_R is needed. The asymptotically small support in x contributes $\delta^{\frac{1}{4}}$ to the estimate.

The estimate (25) is a counterpart of the estimate (8), and it gives the asymptote corresponding to infinitesimally small principal curvature (23) of the Whitney fold π_R. The geometrical meaning of the condition (24) is that the kernel of $d\pi_R$ is within a certain conic neighborhood of the direction of $\nabla_x \det S_{x\vartheta}$. This estimate can be derived with the methods similar to those in Section 1.6. We refer to the paper [1997] of the author.

Operators associated to canonical relations with stable singularities. This is still an open question how to relate the rate of decay of oscillatory integral operators with the singularities of the projections from the canonical relation, for topologically stable classes of singularities. Similar problem for oscillatory integrals was solved by A.N. Varchenko in [1976]; see the book of V.I. Arnold, S.M. Gusein-Zade, and A.N. Varchenko [1985].

So far, we know the estimate (8) for the operators associated to two-sided Whitney folds. We can also prove the following result for the topologically stable canonical relations with one of the projections being a Whitney fold and the other one being a cusp:

Theorem 1.1.2 (Operators associated to fold & cusp) *Let* T_λ *be an oscillatory integral operator associated to the canonical relation with the fold × cusp singularity. Then there is the following estimate on* T_λ*:*

$$\|T_\lambda\| \le const\, \lambda^{-\frac{n}{2}+\frac{1}{5}}. \tag{26}$$

The proof basically uses the methods described in the paper; the details will appear elsewhere.

We believe that, in the general case, the operators associated to canonical relations with topologically stable singularities could be considered by combining the theory of V.P. Maslov (see, for example, V.P. Maslov and V.E. Nazaikinskii [1988]) with the Catastrophe Theory of V.I. Arnold. We also hope that this could give an alternative proof of the results stated in this paper.

1.2 Relation with Fourier integral operators

Littlewood-Paley theory allows to relate the rate of decay of oscillatory integral operators to the regularity properties of Fourier integral operators. The details for the general case can be found in the papers of A. Seeger [1993] and A. Greenleaf and A. Seeger [1994]. We are going to illustrate this relation in two particular situations.

Let $\mathcal{F}$ be a compactly supported Fourier integral operator

$$\mathcal{F}u(x) = \int\limits_{\mathbb{R}^N \times \mathbb{R}^n} e^{i\phi(x,\theta,y)} a(x,\theta) u(y)\, d\theta\, dy, \tag{27}$$

x, $y \in \mathbb{R}^n$, $\theta \in \mathbb{R}^N$, $N = n$, with the symbol $a(x,\theta) \in S^d_{1,0}$ (compactly supported in x), and with the phase function

$$\phi(x,\theta,y) = S(x,\theta) - \theta \cdot y, \tag{28}$$

where S is homogeneous of degree 1 in θ. We assume that $d_x\phi = d_x S \neq 0$ on the support of $a(x,\vartheta)$ as long as $\theta \neq 0$.

The associated canonical relation

$$\mathcal{C} = \{(x, d_x\phi) \times (y, -d_y\phi) \mid d_\theta\phi = 0\} = \{(x, d_x S) \times (d_\theta S, \theta)\}$$

is parameterized by x and θ.

Proposition 1.2.1 (Continuity of Fourier integral operators) *If for any density $\psi \in C^\infty_{\text{comp}}(\mathbb{R}^n \times \mathbb{R}^n)$ the oscillatory integral operator*

$$T_\lambda u(x) = \int_{\mathbb{R}^n} e^{i\lambda S(x,\vartheta)} \psi(x,\vartheta) u(\vartheta) d\vartheta \tag{29}$$

is continuous from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$ with the norm $\|T_\lambda\| \leq C_\psi \lambda^{-\frac{n}{2}+r}$, with r independent of ψ, and if $d_x S(x,\vartheta) \neq 0$ as long as $\vartheta \neq 0$, then for any real s the operator $\mathcal{F}$ in (27) *defines a continuous map*

$$\mathcal{F} :\, H^s(\mathbb{R}^n) \to H^{s-d-r}(\mathbb{R}^n). \tag{30}$$

Proof. It suffices to prove that for $d+r\le 0$ the operator $\mathcal{F}$ is continuous from L^2 to L^2.

We employ the dyadic partition

$$1=\sum_{\lambda=2^N,\,N\in\mathbb{N}}\beta\left(\frac{\|\theta\|}{\lambda}\right)+\beta_0(\|\theta\|),\quad \beta_0\in C^\infty_{\rm comp}([-2,2]),\quad \beta\in C^\infty_{\rm comp}([1/2,2]) \tag{31}$$

to decompose $\mathcal{F}$ into

$$\sum_{\lambda=2^N,\ N\in\mathbb{N}}\mathcal{F}_\lambda+\mathcal{F}_0, \tag{32}$$

with $\mathcal{F}_0$ being infinitely smoothing.

According to the assumptions of the Proposition, the L^2-norm of the function

$$\begin{aligned}\mathcal{F}_\lambda u(x)&=\int e^{iS(x,\theta)}a(x,\theta)\beta(\|\theta\|/\lambda)\hat u(\theta)d\theta\\ &=\lambda^{n+d}\int e^{i\lambda S(x,\vartheta)}\left[\frac{a(x,\lambda\vartheta)}{\lambda^d}\beta(\|\vartheta\|)\right]\hat u(\lambda\vartheta)d\vartheta\end{aligned} \tag{33}$$

is bounded by

$$\|\mathcal{F}_\lambda u(x)\|_{L^2}\le const\,\lambda^{n+d}\lambda^{-\frac{n}{2}+r}\|\hat u(\lambda\vartheta)\|_{L^2}\le const\,\lambda^{d+r}\|\hat u(\theta)\|_{L^2}. \tag{34}$$

This shows that the operators $\mathcal{F}_\lambda$ are uniformly bounded from L^2 to L^2 if $d+r$ is less than or equal to zero.

Now it suffices to prove that the operators with different *lambdas*, $\lambda_1=2^{N_1}$ and $\lambda_2=2^{N_2}$, are *almost orthogonal* (see Section 1.5). The composition $\mathcal{F}_{\lambda_1}\mathcal{F}^*_{\lambda_2}$ is simply equal to zero (if $|N_1-N_2|\ge 2$), since the functions $\beta(\theta/\lambda_1)$ and $\beta(\theta/\lambda_2)$ have no common support. When considering the composition $\mathcal{F}^*_{\lambda_1}\mathcal{F}_{\lambda_2}$, we concentrate the attention on the exponential in the expression for the integral kernel of $\mathcal{F}^*_{\lambda_1}\mathcal{F}_{\lambda_2}$:

$$K(\mathcal{F}^*_{\lambda_1}\mathcal{F}_{\lambda_2})(\vartheta,w)=\int_{\mathbb{R}^n}dx\,e^{-i\lambda(S(x,\vartheta)-S(x,w))}\times\dots. \tag{35}$$

We have:

$$\|S_x(x,\theta)-S_x(x,w)\|\ge const\,|\lambda_1-\lambda_2|, \tag{36}$$

if $|N_1-N_2|$ is larger than some constant (here we need the condition $d_xS(x,\vartheta)\ne 0$ when $\vartheta\ne 0$). Therefore, we can integrate by parts in (35), gaining arbitrarily large negative powers of $|\lambda_1-\lambda_2|$. This proves the almost orthogonality of the operators $\mathcal{F}_\lambda$, $\lambda=2^N$, and completes the argument. □

Another theme is the regularity properties of generalized Radon Transforms. We follow the idea of D.H. Phong and E.M. Stein [1991].

Let us consider the Radon Transform

$$\mathcal{R}_\psi : \; C^\infty_{\mathrm{comp}}(\mathbb{R}\times Y) \to C^\infty(\mathbb{R}\times X),$$

such that $\mathcal{R}_\psi u(t,x)$ is obtained by integrating the function $u(s,\vartheta)$ with some density ψ over the hypersurface $\mathcal{M}_{t,x}\subset \mathbb{R}\times Y$, described by the condition $t-s-S(x,\vartheta)=0$.

Thus, let the Radon Transform $\mathcal{R}_\psi$ be given by

$$\mathcal{R}_\psi u(t,x) = \int\limits_{\mathbb{R}\times Y} \delta(t-s-S(x,\vartheta))\,\psi(x,\vartheta)\,u(s,\vartheta)\,ds\,d\vartheta. \tag{37}$$

The density $\psi\in C^\infty_{\mathrm{comp}}(X\times Y)$ is assumed to have a compact support; then X and Y can be viewed as Euclidean spaces (possibly, of different dimensions n_X and n_Y).

We rewrite the integral transform (37) as a Fourier integral operator with $N=1$,

$$\mathcal{R}_\psi u(t,x) = \int\limits_{\mathbb{R}\times\mathbb{R}\times Y} e^{-i\lambda(t-s-S(x,\vartheta))}\,\psi(x,\vartheta)\,u(s,\vartheta)\,\frac{d\lambda}{2\pi}\,ds\,d\vartheta. \tag{38}$$

Let us interpret (38) as a pseudodifferential operator, in the following way: One first performs the Fourier transform with respect to s, $u(s,\vartheta)\to\hat{u}(\lambda,\vartheta)$, then applies the symbol $a_\psi(\lambda)$ which acts according to

$$a_\psi(\lambda)\,\hat{u}\,(\lambda,x) = \int\limits_{Y} e^{i\lambda S(x,\vartheta)}\,\psi(x,\vartheta)\,\hat{u}(\lambda,\vartheta)\,d\vartheta, \tag{39}$$

and then one takes the inverse Fourier transform (from λ to t).

In other words, we can write the partial Fourier transform of (38) as

$$F_{t\mapsto\lambda}\left(\mathcal{R}_\psi u\right)(\lambda,x) = \int\limits_{Y} e^{i\lambda(S(x,\vartheta))}\,\psi(x,\vartheta)\,\hat{u}(\lambda,\vartheta)\,d\vartheta = a_\psi(\lambda)\,\hat{u}\,(\lambda,x).$$

The following lemma is transparent:

Lemma 1.2.1 *If there is the estimate* $\|a_\psi(\lambda)\|\le C_\psi\lambda^{-d}$ *when* $\lambda>1$, *for some* $d>0$, *on the* $L^2(Y)\to L^2(X)$*-action of the operator-valued symbol* $a_\psi(\lambda)$*, then the Radon Transform* $\mathcal{R}_\psi$ *in* (37) *is continuous from* $H^s(\mathbb{R})\otimes L^2(Y)$ *to* $H^{s+d}(\mathbb{R})\otimes L^2(X)$, $s\in\mathbb{R}$*, with the norm bounded by* $\|\mathcal{R}_\psi\|\le C_\psi$.

We would like to study the resulting regularity properties of $\mathcal{R}_\psi$ in more detail. Let us note that differentiation of (37) with respect to x can be represented, in a certain way, as differentiation with respect to t, allowing one to take off the derivatives with respect to x, thus "improving" Sobolev properties of $\mathcal{R}_\psi$ in x-directions. This approach can be used for a wider class of averaging operators. E.M. Stein has proposed for this the term "trading the derivatives".

Lemma 1.2.2 *Let the Radon Transform $\mathcal{R}_\psi$ in* (37) *be continuous from some Banach space $B(\mathbb{R}\times Y)$ to $H^a(\mathbb{R})\otimes H^b(X)$, for any density $\psi\in C^\infty_{\rm comp}(X\times Y)$, with the norm $\|\mathcal{R}_\psi\|\le C_\psi$.*

Then

$$\mathcal{R}_\psi:\,B(\mathbb{R}\times Y)\to H^{a-\mu}(\mathbb{R})\otimes H^{b+\mu}(X),$$

for any $\mu\ge 0$, with the norm $\|\mathcal{R}_\psi\|\le C_{\psi,\mu}$.

Proof. Let us consider the identity

$$\mathcal{R}_\psi=(1-\Delta_x)^{-1}\circ(\mathcal{R}_\psi-\Delta_x\circ\mathcal{R}_\psi), \tag{40}$$

where the operator $\Delta_x\circ\mathcal{R}_\psi$ acts on functions according to

$$\Delta_x\left(\mathcal{R}_\psi u(t,x)\right)=\int\limits_{\mathbb{R}\times Y}\Delta_x\left(\delta(t-s-S(x,\vartheta))\,\psi(x,\vartheta)\right)\,u(s,\vartheta)\,ds\,d\vartheta. \tag{41}$$

We rewrite the integral in the right-hand side as

$$\int\limits_{\mathbb{R}\times Y}\left[\{\psi\,S_x^2\}\,\delta''-\{\psi\,\Delta_x S+2\,\psi_x\cdot S_x\}\,\delta'+\,\{\Delta_x\psi\}\,\delta\right]\,u(s,\vartheta)\,ds\,d\vartheta$$

$$=\partial_t^2\left(\mathcal{R}_1 u(t,x)\right)-\partial_t\left(\mathcal{R}_2 u(t,x)\right)+\mathcal{R}_3 u(t,x), \tag{42}$$

where $\mathcal{R}_1\equiv\mathcal{R}_{\psi S_x^2}$, $\mathcal{R}_2\equiv\mathcal{R}_{\psi\Delta_x S+2\psi_x\cdot S_x}$, and $\mathcal{R}_3\equiv\mathcal{R}_{\Delta_x\psi}$ are the Radon Transforms of the form (37), with the specified densities. Therefore, according to the assumption of the Lemma, all these Radon Transforms are continuous from $B(\mathbb{R}\times Y)$ to $H^a(\mathbb{R})\otimes H^b(X)$. Hence, the operator

$$\mathcal{R}_\psi-\Delta_x\circ\mathcal{R}_\psi=\mathcal{R}_\psi-\partial_t^2\circ\mathcal{R}_1+\partial_t\circ\mathcal{R}_2-\mathcal{R}_3$$

is continuous from $B(\mathbb{R}\times Y)$ to $H^{a-2}(\mathbb{R})\otimes H^b(X)$, and then

$$\mathcal{R}_\psi=(1-\Delta_x)^{-1}\circ(\mathcal{R}_\psi-\Delta_x\circ\mathcal{R}_\psi)$$

is continuous from $B(\mathbb{R}\times Y)$ to $H^{a-2}(\mathbb{R})\otimes H^{b+2}(X)$.

In the same way one proves that $\mathcal{R}_\psi$ is continuous from $B(\mathbb{R}\times Y)$ to $H^{a-2k}(\mathbb{R})\otimes H^{b+2k}(X)$, for any natural k, with the norm depending on k.

This means that for any function $v(t,x)\in\mathcal{R}_\psi(B(\mathbb{R}\times Y))$, its Fourier transform $\hat v(\tau,\xi)$ has a bounded L^2-norm when integrated with the weight

$$(1+\tau^2)^{a-2k}(1+\|\xi\|^2)^{b+2k}, \tag{43}$$

for any integer k. Then $\hat v(\tau,\xi)$ also has a bounded L^2-norm when integrated with the weight $(1+\tau^2)^{a-\mu}(1+\|\xi\|^2)^{b+\mu}$, for any non-negative μ, and therefore

$$v(t,x)\in H^{a-\mu}(\mathbb{R})\otimes H^{b+\mu}(X).\qquad\square$$

Note that the norm on the action from $B(\mathbb{R}\times X)$ to $H^{a-\mu}(\mathbb{R})\otimes H^{b+\mu}(Y)$ depends on μ. This dependence is due to the appearance of the derivatives of $S(x,\vartheta)$ and of $\psi(x,\vartheta)$ in (41). Of course, this is inevitable if we want the image of $\mathcal{R}_\psi$ to be smoother in x.

Proposition 1.2.2 *Let the Radon Transform $\mathcal{R}_\psi$ in* (37) *define the continuous action*

$$H^s(\mathbb{R})\otimes L^2(Y) \xrightarrow{\mathcal{R}_\psi} H^{s+d}(\mathbb{R})\otimes L^2(X), \qquad s\in\mathbb{R}, \tag{44}$$

for any density $\psi\in C^\infty_{\text{comp}}(X\times Y)$, with the norm $\|\mathcal{R}_\psi\|\le C_\psi<\infty$.

Then the following action is also continuous:

$$H^{s+\nu}(\mathbb{R})\otimes H^{-\nu}(Y) \xrightarrow{\mathcal{R}_\psi} H^{s+d-\mu}(\mathbb{R})\otimes H^{\mu}(X), \tag{45}$$

for any non-negative μ and ν, with the norm $\|\mathcal{R}_\psi\|=C_{\psi,\mu,\nu}$ depending on the values of μ and ν.

The statement of this Proposition follows from the application of Lemma 1.2.2 to $\mathcal{R}$ and then also to its adjoint,

$$H^{-s}(\mathbb{R})\otimes L^2(Y) \xleftarrow{\mathcal{R}^*_\psi} H^{-s-d+\mu}(\mathbb{R})\otimes H^{-\mu}(X). \tag{46}$$

Corollary 1.2.1 *Under the assumptions of Proposition* 1.2.2*, the following action of $\mathcal{R}_\psi$ is continuous:*

$$H^s(\mathbb{R}\times Y) \xrightarrow{\mathcal{R}_\psi} H^{s+d}(\mathbb{R}\times X), \tag{47}$$

with the norm $\|\mathcal{R}_\psi\|\le C_{\psi,s}$ depending on the value of s.

This Corollary follows from the inclusion

$$\big\{H^s(\mathbb{R})\otimes H^0(X)\big\}\cap\big\{H^0(\mathbb{R})\otimes H^s(X)\big\}\subset H^s(\mathbb{R}\times X). \tag{48}$$

1.3 Pseudoconvexity

Most approaches to the integral operators of Fourier type are based on the following procedure. Given the operator T_λ,

$$T_\lambda u(x) = \int\limits_{\mathbb{R}^n} e^{i\lambda S(x,\vartheta)}\psi(x,\vartheta)\,u(\vartheta)\,d\vartheta,$$

we consider the integral kernel of the convolution $T_\lambda T^*_\lambda$,

$$K(T_\lambda T^*_\lambda)(x,y) = \int\limits_{\mathbb{R}^n} e^{i\lambda(S(x,\vartheta)-S(y,\vartheta))}\psi(x,\vartheta)\psi(y,\vartheta)\,d\vartheta,$$

and integrate by parts with the aid of the operator

$$L_\vartheta \equiv -\frac{1}{i\lambda} \cdot \frac{(S_\vartheta(y,\vartheta) - S_\vartheta(x,\vartheta)) \cdot \nabla_\vartheta}{|S_\vartheta(y,\vartheta) - S_\vartheta(x,\vartheta)|^2};$$

this gives certain bounds on $|K(T_\lambda T_\lambda^*)(x,y)|$, and we may apply Young's inequality.

The main ingredient of this approach is the estimate from below for the expression in the denominator of L_ϑ. The usual approximation

$$|S_\vartheta(y,\vartheta) - S_\vartheta(x,\vartheta)| = S_{x\vartheta}(x,\vartheta) \cdot (y-x) + o\left(\|y-x\|\right) \approx S_{x\vartheta}(x,\vartheta) \cdot (y-x)$$

is applicable if we use the fine localization of the integral kernel of T_λ with the step $\sim |\det S_{x\vartheta}|$, so that $\|y-x\| \le \mathit{const}\ |\det S_{x\vartheta}|$. In the non-degenerate case, this approximation gives

$$|S_\vartheta(y,\vartheta) - S_\vartheta(x,\vartheta)| \ge \mathit{const}\ |\det S_{x\vartheta}|\,\|y-x\| \ge \mathit{const}\ \|y-x\|\,. \tag{49}$$

If $S_{x\vartheta}$ becomes degenerate, then we can not count on (49) any more. Still, we could continue with the argument if we had something like

$$|S_\vartheta(y,\vartheta) - S_\vartheta(x,\vartheta)| \ge \mathit{const}\ \|y-x\|\ \inf_U\ |\det S_{x\vartheta}|\,, \tag{50}$$

for any connected set $U \subset \mathbb{R}^n \times \mathbb{R}^n$ and for any (x,ϑ), $(y,\vartheta) \in U$.

Unfortunately, if we use the fine localization of the integral kernel of the operator $T^\hbar$ from Section 1.1 with the step $\Delta x \sim |\det S_{x\vartheta}| \approx \hbar$, then the number of pieces of $T^\hbar$ grows up like some power of $\hbar^{-1}$ when $\hbar \to 0$. Each of these pieces satisfies certain estimates, but the almost orthogonality of these operators occurs only if π_R is a Whitney fold (see the paper of S. Cuccagna [1995]).

The inequality (50) could be viewed as a certain convexity property (*pseudoconvexity*) of the map $\pi_R|_\vartheta : x \mapsto S_\vartheta(x,\vartheta)$. To legitimate this idea, let us define the notion of a *pseudoconvex map.*

First, for a real $N \times N$-matrix A, we define the bound from below on its action:

$$\mathit{min}(A) \equiv \inf_{\|\mathbf{u}\|=1} \|A\mathbf{u}\|\,. \tag{51}$$

If A is degenerate, then $\mathit{min}(A) = 0$; otherwise, $\mathit{min}(A) = \left\|A^{-1}\right\|^{-1}$.

There is the inequality

$$\mathit{min}(A) \ge m(A) \equiv \frac{|\det A|}{\|A\|^{N-1}_{\mathrm{End}(\mathbb{R}^N)}}, \tag{52}$$

which gives a useful estimate for $\mathit{min}(A)$ if A is of rank not less than $N-1$.

Let μ be a map which acts from (a subset of) $\mathbb{R}^N$ to $\mathbb{R}^N$. Then the Mean Value Theorem states that for any two points x, $y \in \mathbb{R}^N$,

$$\|\mu(y) - \mu(x)\| \le \|y-x\| \cdot \sup_{z\in l}\ \|J_z(\mu)\|_{\mathrm{End}(\mathbb{R}^N)}\,, \tag{53}$$

here $J_z(\mu)$ is the Jacobian $\nabla\mu$ of the change of variables $x \mapsto \mu(x)$ at the point $x = z$, $z \in l$; l is the line segment from x to y (which is assumed to be in the domain of μ).

We would like to have a similar bound *from below* for $\|\mu(y) - \mu(x)\|$.

If the map μ is a bijection and if its range is convex, then we can apply the Mean Value Theorem (53) to the function μ^{-1}:

$$\|y - x\| \le \|\mu(y) - \mu(x)\| \cdot \sup_{w \in L} \left\|J_w(\mu^{-1})\right\|_{\mathrm{End}(\mathbb{R}^N)}.$$

Here L is a line segment from $\mu(x)$ to $\mu(y)$.

Since $\left\|J(\mu^{-1})\right\| = \left\|J(\mu)^{-1}\right\| = min(J(\mu))^{-1}$, we can rewrite the previous inequality as

$$\|\mu(y) - \mu(x)\| \ge \|y - x\| \cdot \inf_{z \in \gamma} min(J_z(\mu)), \tag{54}$$

here the path γ is the preimage of the line segment L.

We use the inequality (54) to define the notion of a *pseudoconvex map*:

Definition 1.3.1 (c-pseudoconvexity) *Let μ be a map from $\mathbb{R}^N$ to $\mathbb{R}^N$. Given $c > 0$, the map μ is called c-pseudoconvex on a path connected subset $U \subset \mathbb{R}^N$ if for any x, $y \in U$*

$$\|\mu(y) - \mu(x)\| \ge c\, \|y - x\| \cdot \inf_{z \in \gamma} min(J_z(\mu)), \tag{55}$$

for any path $\gamma \in U$ from x to y.

If (55) *is satisfied for some constant $c > 0$, we will simply say that the map μ is pseudoconvex.*

Any function $f : \mathbb{R} \to \mathbb{R}$ is automatically pseudoconvex, with $c = 1$.

The property of maps to be *pseudoconvex* is preserved under compositions.

A change of coordinates can be considered as a bijection from $U \subset \mathbb{R}^N$ to $U' \subset \mathbb{R}^N$ (which is pseudoconvex). Therefore, the property of a map to be *pseudoconvex* is retained under the changes of local coordinates in U and V (only a particular value of c changes).

The most apparent example of a non-pseudoconvex map is a local diffeomorphism which is not a bijection. Still, if U is compact and if $min(J(\mu)) \ge const$, then for any $c < 1$ there is a finite covering $U = \cup U_j$, such that μ is c-pseudoconvex on each U_j. We will consider such maps as being *pseudoconvex*, not specifying each time that we must have used some finite localization.

The condition (55) is non-trivial if $J(\mu)$ becomes degenerate. For example, a map with the fold singularity is pseudoconvex, while a map with the cusp singularity is not.

Remark. An important example of a pseudoconvex map (which also explains the origin of the definition) is the orthogonal projection from the boundary ∂K of a convex compact domain $K \subset \mathbb{R}^{N+1}$ onto a hyperplane. An elementary geometrical argument shows that this map satisfies the pseudoconvexity condition (55) (see Section 2.3

below). Moreover, the domain K is convex if and only if the orthogonal projections from its boundary onto all hyperplanes are pseudoconvex.

In the following section we will show that the projections from the canonical relation could be considered as being *pseudoconvex* in a very general situation, when they only satisfy certain *finite type* condition.

1.4 Geometry of the canonical relation

Let us show the way to exploit the geometrical properties of the canonical relation.

We assume that on the critical variety

$$\Sigma^0 = \{(x,\vartheta) \mid h(x,\vartheta) = 0\},$$

where $h(x,\vartheta) = \det S_{x\vartheta}$, the rank of maps

$$\pi_L : (x,\vartheta) \mapsto (x,\, S_x), \qquad \pi_R : (x,\vartheta) \mapsto (\vartheta,\, S_\vartheta)$$

is at least $n-1$, and choose local coordinates so that the directions $x_{i'}$ and $\vartheta_{i'}$, $i' = 1, \ldots, n-1$, are non-degenerate:

$$\det S_{x'\vartheta'} \geq c_0 > 0. \tag{56}$$

The kernel of the differentials $d\pi_L$ and $d\pi_R$ are then generated by the vectors

$$\mathbf{k}_L = \partial_{\vartheta_n} - S^{\vartheta' x'} S_{x'\vartheta_n} \cdot \nabla_{\vartheta'}, \qquad \mathbf{k}_R = \partial_{x_n} - S^{\vartheta' x'} S_{x_n \vartheta'} \nabla_{x'}, \qquad (x,\vartheta) \in \Sigma^0. \tag{57}$$

Here $S^{\vartheta' x'}$ is the inverse matrix to $S_{x'\vartheta'}$. The summation with respect to the indices of repeating lower and upper variables is assumed.

We define the vector fields $\mathbf{K}_L$ and $\mathbf{K}_R$ on $\mathbb{R}^n_L \times \mathbb{R}^n_R$ by the same right-hand sides:

$$\mathbf{K}_L = \partial_{\vartheta_n} - S^{\vartheta' x'} S_{x'\vartheta_n} \cdot \nabla_{\vartheta'}, \qquad \mathbf{K}_R = \partial_{x_n} - S^{\vartheta' x'} S_{x_n \vartheta'} \nabla_{x'}. \tag{58}$$

The capital letters are used to emphasize that the vectors are defined everywhere.

This definition depends on the choice of local coordinates. In different local coordinates (y,w) the vector fields

$$\mathbf{K}'_L = \partial_{w_n} - S^{w' y'} S_{y' w_n} \cdot \nabla_{w'}, \qquad \mathbf{K}'_R = \partial_{y_n} - S^{w' y'} S_{y_n w'} \nabla_{y'} \tag{59}$$

are different from $\mathbf{K}_L$ and $\mathbf{K}_R$. Note that $S^{w'y'}$, the inverse to the matrix $S_{y'w'}$, is not expressed in terms of $S^{\vartheta' x'}$.

Since we know that on the critical variety $\Sigma^0 = \{h(x,\vartheta) = 0\}$ the directions of $\mathbf{K}_L$ and $\mathbf{K}_R$ are determined by $\operatorname{Ker} d\pi_L$ and $\operatorname{Ker} d\pi_R$, we expect that locally

$$\mathbf{K}'_L = a_L(x,\vartheta)\, \mathbf{K}_L + h(x,\vartheta)\, b^i_L(x,\vartheta)\, \partial_{\vartheta_i}, \tag{60}$$

$$\mathbf{K}'_R = a_R(x,\vartheta)\, \mathbf{K}_R + h(x,\vartheta)\, b^i_R(x,\vartheta)\, \partial_{x_i}, \tag{61}$$

where $a_L(x,\vartheta)$ and $a_R(x,\vartheta)$ are different from zero. Fortunately, we will not need the precise relations, which look as follows:

$$\mathbf{K}'_L = \left(\frac{\partial\vartheta_n}{\partial w_n} - S^{w'y'}S_{y'w_n}\partial_{w'}\vartheta_n\right)\left[\mathbf{K}_L + \left(\frac{\det S_{y\vartheta}}{\det S_{y'\vartheta'}}\right)S^{\vartheta'x'}\partial_{x'}y_n\,\partial_{\vartheta'}\right],$$

$$\mathbf{K}'_R = \left(\frac{\partial x_n}{\partial y_n} - S^{w'y'}S_{y_n w'}\partial_{y'}x_n\right)\left[\mathbf{K}_R + \left(\frac{\det S_{xw}}{\det S_{x'w'}}\right)S^{\vartheta'x'}\partial_{\vartheta'}w_n\,\partial_{x'}\right].$$

Note that $\det S_{y\vartheta} \sim \det S_{xw} \sim \det S_{x\vartheta} = h(x,\vartheta)$.

The vector fields $\mathbf{K}_L$ and $\mathbf{K}_R$ have a very simple realization. Let us fix x and consider the map $\pi_L|_x : \vartheta \mapsto \eta \equiv S_x(x,\vartheta)$ as the composition

$$\vartheta \overset{\pi'_L}{\longmapsto} (\xi' \equiv S_{x'},\, \vartheta_n) \overset{\pi^s_L}{\longmapsto} (\xi',\, \xi_n \equiv S_{x_n}).$$

The map π'_L is a diffeomorphism, since $\det S_{x'\vartheta'} \neq 0$; hence, the singular behavior of π_L is encoded in the map π^s_L.

The kernel of the differential $d\pi^s_L$ is generated by $\left(\frac{\partial}{\partial\vartheta_n}\right)_{\xi'}$. The relation

$$\left(\frac{\partial}{\partial\vartheta_n}\right)_{\xi'} = \mathbf{K}_L = \partial_{\vartheta_n} - S^{\vartheta'x'}S_{x'\vartheta_n}\nabla_{\vartheta'} \tag{62}$$

is valid not only on the critical variety. Indeed, $\mathbf{K}_L\vartheta_n = 1$, and

$$\mathbf{K}_L\xi' = \mathbf{K}_L S_{x'} = S_{x'\vartheta_n} - S^{\vartheta_{i'}x_{j'}}S_{x_{j'}\vartheta_n}S_{x'\vartheta_{i'}} = S_{x'\vartheta_n} - S_{x'\vartheta_n} = 0.$$

Analogously,

$$\left(\frac{\partial}{\partial x_n}\right)_{\eta'} = \mathbf{K}_R, \tag{63}$$

where $\eta' = S_{\vartheta'}(x,\vartheta)$.

Whitney folds. The map $\mu : \mathbb{R}^N \to \mathbb{R}^N$ is said to be a Whitney fold (a map with the singularity of the type $\Sigma^{1,0}$) if in proper local coordinates it is represented by

$$y_{i'} = x_{i'}, \quad i' = 1,\, \dots,\, N-1,$$

$$y_N = \frac{x_N^2}{2}, \tag{64}$$

with the determinant of the Jacobian being equal to x_N and with the kernel of the differential generated by $\mathbf{k} = \partial/\partial x_N$, so that $\mathbf{k}\det J(\mu) \neq 0$.

The map π_L is thus a Whitney fold in an open neighborhood W of a point $p_o \in \Sigma^0$ if

$$\mathbf{k}_L h(x,\vartheta) \neq 0 \qquad \text{in} \quad \Sigma^0 \cap W.$$

We substitute $\mathbf{k}_L$ by $\mathbf{K}_L$ and exercise the condition that π_L is a Whitney fold near p_o assuming that

$$|\mathbf{K}_L h(x,\vartheta)| \geq c_L > 0, \tag{65}$$

for some positive constant c_L, as long as $(x,\vartheta) \in W$. Due to (60), this condition does not depend on the choice of local coordinates (only a particular value of the constant c_L depends).

If π_R is also a Whitney fold, then we can assume that there is some constant $c_R > 0$ such that $\mathbf{K}_R h(x,\vartheta) \geq c_R$. The estimate on the associated oscillatory integral operators is given by (8).

Cusps. Another stable class of singularities of differentiable maps is a cusp (type $\Sigma^{1,1,0}$). The map $\mu : \mathbb{R}^N \to \mathbb{R}^N$ has a stable singularity of this type if in proper local coordinates it can be represented by

$$y_{i'} = x_{i'}, \quad i' = 1, \ldots, N-1,$$

$$y_N = \frac{x_N^3}{3} - x_N x_{N-1}. \tag{66}$$

The critical variety is determined by the condition

$$\det J(\mu) = x_N^2 - x_{N-1} = 0.$$

The kernel of this map is generated by the vector $\mathbf{k} = \partial/\partial x_N$, which has a contact of order at most $k = 2$ with the graph of the map μ in $\mathbb{R}^N \times \mathbb{R}^N$:

$$\mathbf{k}^2 \det J(\mu) \neq 0.$$

Thus, if π_R has a cusp singularity at p_o, then

$$h(p_o) = 0, \qquad \mathbf{K}_R|_{p_o} h = 0, \qquad \mathbf{K}_R^2|_{p_o} h \neq 0. \tag{67}$$

The first two conditions do not depend on the choice of local coordinates.

Let us show that the third condition is also coordinate-independent. Indeed, in different local coordinates (y,w), we need to check that

$$\left(\mathbf{K}'_R\right)^2 h|_{p_o} = \left(a_R \mathbf{K}_R + h\, b_R^i \partial_{x_i}\right)^2 h|_{p_o} \tag{68}$$

is different from zero. Recalling that at the cusp point $h = 0$ and $\mathbf{K}_R h = 0$, we obtain:

$$\left(\mathbf{K}'_R\right)^2 h|_{p_o} = a_R^2 \mathbf{K}_R^2 h|_{p_o} \neq 0. \tag{69}$$

When we consider π_R to be a cusp, we can assume that in an open neighborhood W of a cusp point p_o we have:

$$\left|\mathbf{K}_R^2 h(x,\vartheta)\right| \geq c_R^{(2)} > 0, \tag{70}$$

with the value of $c^{(2)}_{R}$ depending on the choice of local coordinates. This condition implies that the varieties

$$\Gamma_{R}(\sigma) \equiv \{\mathbf{K}_{R} h = \sigma\}$$

are smooth hypersurfaces in $\mathbb{R}^n_L \times \mathbb{R}^n_R$, with the vector field $\mathbf{K}_R$ being transversal to all of them (as long as σ is small).

Together with $\mathbf{K}_R$, these hypersurfaces depend on the choice of local coordinates.

Since at the cusp points $\mathbf{K}_R \in T\Sigma^0$, we know that Σ^0 and $\Gamma_R(\sigma = 0)$ intersect transversally, and thus the cusp variety

$$\Sigma^1_R = \Sigma^0 \cap \Gamma_R(0)$$

is a smooth submanifold of $\mathbb{R}^n_L \times \mathbb{R}^n_R$ (of codimension 2). This submanifold could also be characterized as a subset of Σ^0 where the restriction $\pi_R|_{\Sigma^0}$ becomes degenerate. See D.H. Phong and E.M. Stein [1991] and D.H. Phong [1994] for further generalizations.

Since π_R is a cusp, the restriction of π_R onto Σ^1_R is a diffeomorphism onto its image (in an agreement with (70)).

Degeneracies of higher order. Let us generalize what we have learned about Whitney folds and cusps.

Definition 1.4.1 (Degeneracy of order k) *We will say that the map π_R has degeneracy of order $k \in \mathbb{N}$ at the point $p_o \in \Sigma^0$ if*

$$\mathbf{K}^{k'}_R h(x,\vartheta)|_{p_o} = 0, \qquad \forall\ k' \in \mathbb{N}, \quad k' < k, \tag{71}$$

$$\mathbf{K}^{k}_R h(x,\vartheta)|_{p_o} \neq 0. \tag{72}$$

We will say that the determinant of the mixed Hessian vanishes of order not greater than k in the direction of $\operatorname{Ker} d\pi_R$ if the map π_R has degeneracies of order less than or equal to k, for some $k < \infty$, at all points of Σ^0. Then we will also say that the map π_R is of finite type.

Similarly for π_L.

This definition does not depend on the choice of local coordinates. The argument is the same as for the cusp: the relation (69) now becomes

$$\left(\mathbf{K}'_R\right)^k h|_{p_o} = a^k_R \left(\mathbf{K}_R\right)^k h|_{p_o} \neq 0.$$

In general, the orders of degeneracies are not preserved under the deformations of the canonical relation, but they are well-defined for the projections with stable singularities: for the Whitney fold the order of degeneracy is equal to $k = 1$, for the cusp $k = 2$ at the cusp point and $k = 1$ otherwise.

Formally, we also assume that at non-singular points ($p \notin \Sigma^0$) the map has degeneracy of order $k = 0$, since $\mathbf{K}^0_R h(x,\vartheta)|_p = h(x,\vartheta)|_p \neq 0$.

Pseudoconvexity of projections from the canonical relation. It turns out that π_R being of finite type is a sufficient condition for π_R to be treated as pseudoconvex.

We recall the dyadic decomposition (14), $T_\lambda = \sum_{\hbar > \hbar_o} T^\hbar + \bar{T}^{\hbar_o}$, where

$$T^\hbar u(x) = \int e^{i\lambda S(x,\vartheta)}\,\psi(x,\vartheta)\,\beta(\hbar^{-1}h(x,\vartheta))\,u(\vartheta)\,d\vartheta, \qquad \beta \in C^\infty_{\mathrm{comp}}([1/2,2]).$$

We will show that the support of the integral kernel of each operator $T^\hbar$, which we denote by

$$\operatorname{supp} K(T^\hbar) \equiv \operatorname{supp}\psi(x,\vartheta) \cap \operatorname{supp}\beta(\hbar^{-1}h(x,\vartheta)),$$

can be split into several pieces so that π_R is pseudoconvex on each piece.

Proposition 1.4.1 *If the projection π_R is of finite type, then there is a finite partition of 1, $1 = \sum_\sigma \rho^\hbar_\sigma(x,\vartheta)$, with the functions $\rho^\hbar_\sigma$ satisfying the uniform estimates*

$$\left|\partial_x^\alpha \partial_\vartheta^\beta \rho^\hbar_\sigma(x,\vartheta)\right| \le c_{\alpha\beta}\hbar^{-|\alpha|-|\beta|}, \tag{73}$$

with $c_{\alpha\beta}$ independent of $\hbar$, so that π_R restricted to

$$\operatorname{supp} K(T^\hbar) \cap \operatorname{supp}\rho^\hbar_j(x,\vartheta)$$

is pseudoconvex.

The same is true for π_L.

It suffices to show that such a partition of 1 exists in some open neighborhood of a point p_o where the projection π_R has degeneracy of order k, where we thus have

$$\mathbf{K}_R^k h(x,\vartheta)\big|_{p_o} \ne 0. \tag{74}$$

We can assume that

$$\mathbf{K}_R^k h(x,\vartheta)\big|_{p_o} \ge 0 \tag{75}$$

in some open neighborhood W of this point (later we will impose some additional restrictions on W and on its size).

Choose $\rho \in C^\infty(\mathbb{R})$ so that $\rho(t) = 1$ for $t \ge 1$, $\rho(t) = 0$ for $t \le -1$, and

$$\rho(t) + \rho(-t) = 1.$$

The mentioned partition of 1 is given for $k > 1$ by

$$1 = \sum_\sigma \left(\prod_{k'=1}^{k-1} \rho(\sigma_{k'}\hbar^{-1}\mathbf{K}_R^{k'} h(x,\vartheta))\right) \equiv \sum_\sigma \rho^\hbar_\sigma(x,\vartheta), \tag{76}$$

where $\sigma = (\sigma_1, \ldots, \sigma_{k-1})$, $\sigma_{k'} = \pm 1$.

For $k=1$ (Whitney fold), we do not need any partition, and we claim that the map $\pi_{_R}$ is pseudoconvex on the entire $\operatorname{supp} K(T^\hbar)\cap W$.

Let us denote

$$W_\sigma^\hbar \equiv \operatorname{supp} K(T^\hbar)\cap W\cap \operatorname{supp}\rho_\sigma^\hbar(x,\vartheta). \tag{77}$$

We claim that $\pi_{_R}$ is pseudoconvex on $W_\sigma^\hbar$ for each $\sigma=(\sigma_1,\,\dots,\,\sigma_{k-1})$.

Let us fix ϑ, and let (x,ϑ) and (y,ϑ) be two points in $W_\sigma^\hbar$. We need to show that

$$\|\eta(y)-\eta(x)\|\geq \mathit{const}\,\hbar\,\|y-x\|\,, \tag{78}$$

here $\eta(x)=S_\vartheta(x,\vartheta)$.

We consider the images of x and y in the (η',x_n)-space: $\pi'_{_R}(x)=(\eta'(x),x_n)$, $\pi'_{_R}(y)=(\eta'(y),y_n)$. We also denote by $\Omega_\sigma^\hbar$ the image of $W_\sigma^\hbar$.

Let $\mathcal{A}$ be the line segment between the points $\pi'_{_R}(x)$ and $\pi'_{_R}(y)$. Its length is bounded from below by

$$\|\mathcal{A}\|\geq \mathit{const}\;\|y-x\|\,, \tag{79}$$

for some $\mathit{const}>0$.

We treat separately two cases:

• If $\mathcal{A}$ is outside of the conic neighborhood of magnitude $c\hbar$ of either of the directions of the axis x_n in (η',x_n)-space, (the value of the constant c is to be chosen later), then we are done:

$$\|\eta(y)-\eta(x)\|\geq\|\eta'(y)-\eta'(x)\|\geq\|y-x\|\sin c\hbar\approx c\hbar\,\|y-x\|\,. \tag{80}$$

• Now let $\mathcal{A}$ be inside the conic neighborhood of magnitude $c\hbar$ of the axis x_n:

$$\|\eta'(y)-\eta'(x)\|\leq|y_n-x_n|\tan c\hbar. \tag{81}$$

We have:

$$\begin{aligned}|\eta_n(y)-\eta_n(x)|&\geq|y_n-x_n|\ \inf_{\mathcal{A}}\left|\left(\frac{\partial}{\partial x_n}\right)_{\eta'}\eta_n\right|-\|\eta'(y)-\eta'(x)\|\ \sup_{\mathcal{A}}\ \|\nabla_{\eta'}\eta_n\|\\ &\geq|y_n-x_n|\left(\inf_{\mathcal{A}}\left|\left(\frac{\partial}{\partial x_n}\right)_{\eta'}\eta_n\right|-\tan c\hbar\ \sup_{\mathcal{A}}\ \|\nabla_{\eta'}\eta_n\|\right).\end{aligned} \tag{82}$$

Since $\|y-x\|\sim\|\mathcal{A}\|\approx|y_n-x_n|$, we only need to show that the factor at $|y_n-x_n|$ in the right-hand side of (82) is of magnitude $\hbar$, and then the inequality (78) follows.

We will show below (Lemmas 1.4.1 and 1.4.2) that $\left|\left(\frac{\partial}{\partial x_n}\right)_{\eta'}\eta_n\right|\sim\hbar$, as long as (η',x_n) is on the line segment $\mathcal{A}$. Therefore, (82) is greater than $\mathit{const}\,\hbar\,|y_n-x_n|$ if we choose the constant c small enough.

This justifies (78) and concludes the proof of the Proposition.

Lemma 1.4.1 *There is the relation* $\left(\frac{\partial}{\partial x_n}\right)_{\eta'}\eta_n = \frac{h(x,\vartheta)}{\det S_{x'\vartheta'}}$.

The value of the derivative $\left(\frac{\partial}{\partial x_n}\right)_{\eta'}\eta_n$ can be determined from the decomposition $\pi_{_R}|_{\vartheta} = \pi_{_R}^{s} \circ \pi_{_R}'$,

$$x \overset{\pi_{_R}'}{\longmapsto} (\eta' \equiv S_{\vartheta'}, x_n) \overset{\pi_{_R}^{s}}{\longmapsto} (\eta', \eta_n \equiv S_{x_n}),$$

which leads to the following relation of the Jacoby matrices:

$$J(\pi_{_R}^{s}) \cdot J(\pi_{_R}') = J(\pi_{_R}|_{\vartheta}). \tag{83}$$

This relation is written explicitly as

$$\begin{bmatrix} 1 & & & 0 \\ & \ddots & & \vdots \\ & & 1 & 0 \\ & \nabla_{\eta'}\eta_n & & \left(\frac{\partial}{\partial x_n}\right)_{\eta'}\eta_n \end{bmatrix} \cdot \begin{bmatrix} & \nabla_{x'}\eta' & & \partial_{x_n}\eta' \\ & & & \\ 0 & \cdots & 0 & 1 \end{bmatrix} = S_{x\vartheta}, \tag{84}$$

where $\nabla_{x'}\eta' = S_{x'\vartheta'}$. Therefore, $\left(\frac{\partial}{\partial x_n}\right)_{\eta'}\eta_n \cdot \det S_{x'\vartheta'} = h(x,\vartheta)$, i.e.,

$$\left(\frac{\partial}{\partial x_n}\right)_{\eta'}\eta_n = \frac{h(x,\vartheta)}{\det S_{x'\vartheta'}}. \qquad \square \tag{85}$$

Now we need to check that on the line segment $\mathcal{A}$ the value of $h(x,\vartheta)$ is of magnitude $\hbar$.

Lemma 1.4.2 *If* $\|\mathcal{A}\| \le 1/12$ *and if* c *is small, then everywhere on* $\mathcal{A}$

$$|h(x,\vartheta)| \ge \frac{\hbar}{4}. \tag{86}$$

Here we assume that $\mathcal{A}$ could be not entirely on the support of the integral kernel of $T^{\hbar}$, where $|h| \ge \hbar/2$.

Proof. Since both $\pi_{_R}'(x)$ and $\pi_{_R}'(y)$ are on $\Omega_{\sigma}^{\hbar} \equiv \pi_{_R}'(W_{\sigma}^{\hbar})$, we know that at both these points

$$|h| \ge \hbar/2, \tag{87}$$

$$\sigma_{k'} \cdot \left(\frac{\partial}{\partial x_n}\right)_{\eta'}^{k'} h \ge -\hbar, \qquad \forall\, k' \in \mathbb{N}, \quad k' < k. \tag{88}$$

We can also assume that

$$\left(\frac{\partial}{\partial x_n}\right)_{\eta'}^{k} h \geq 0 \tag{89}$$

everywhere on $\mathcal{A}$, since the neighborhood W can be chosen so that $\pi'_{_R}(W)$ is convex (so that $\mathcal{A} \subset \pi'_{_R}(W)$), while

$$\left(\frac{\partial}{\partial x_n}\right)_{\eta'}^{k} h|_{\pi'_{_R}(W)} = \mathbf{K}_{_R}^{k} h|_W \geq 0.$$

Let t be a parameter on the line segment $\mathcal{A}$, changing from $t = 0$ at the point $\pi'_{_R}(x)$ to $t = \|\mathcal{A}\|$ at the point $\pi'_{_R}(y)$. We can consider $h(x, \vartheta)$ as a function of t.

Since $\mathcal{A}$ is within the $c\hbar$-cone of $(\frac{\partial}{\partial x_n})_{\eta'}$, we know that

$$\left|\left(\frac{d}{dt}\right)^{j} h - \left(\frac{\partial}{\partial x_n}\right)_{\eta'}^{j} h\right| \leq c\hbar\, Const\,, \qquad \forall\ j \leq k,$$

where $Const$ is determined by the bounds on partial derivatives of $h(x, \vartheta)$, up to order k. Choosing c small enough we can rewrite the condition (88) as

$$\sigma_{k'} \cdot \left(\frac{d}{dt}\right)^{k'} h(t) \geq -2\hbar \qquad \text{for } t = 0 \text{ or } t = \|\mathcal{A}\|\,, \tag{90}$$

and the condition (89) as

$$\left(\frac{d}{dt}\right)^{k} h(t) \geq -3\hbar \qquad \text{for all } t \text{ between } 0 \text{ and } \|\mathcal{A}\|\,. \tag{91}$$

We choose the right-hand side to be $-3\hbar$ for the sake of convenience.

Let us analyze (90) and (91). The latter inequality says that $\left(\frac{d}{dt}\right)^{k-1} h(t)$ is almost monotone. An elementary calculation shows that since at $t = 0$ and at $t = \|\mathcal{A}\|$ the values of $\sigma_{k-1} \cdot \left(\frac{d}{dt}\right)^{k-1} h$ are not less than $-2\hbar$, the values of this "almost monotone" function do not drop below $-3\hbar$ on $\mathcal{A}$, as long as $\|\mathcal{A}\|$ is bounded by some constant:

$$\sigma_{k-1} \left(\frac{d}{dt}\right)^{k-1} h(t) \geq -3\hbar \qquad \text{for all } t \text{ between } 0 \text{ and } \|\mathcal{A}\|\,. \tag{92}$$

Precisely, since $\left(\frac{d}{dt}\right)^{k} h(t) \geq -3\hbar$, the smallest value of $\sigma_{k-1} \cdot \left(\frac{d}{dt}\right)^{k-1} h(t)$ is not less than $-2\hbar - \|\mathcal{A}\| \cdot 3\hbar$; we thus need $\|\mathcal{A}\| \leq 1/3$.

Analogously, $\sigma_{k-2} \cdot \left(\frac{d}{dt}\right)^{k-2} h$ is not less than $-3\hbar$ on the entire $\mathcal{A}$; we descend to $\sigma_1 \cdot \left(\frac{d}{dt}\right) h$ and conclude that it is also not less than $-3\hbar$ on the entire $\mathcal{A}$, so that $h(t)$ is "almost monotone" in t.

Since $h(x,\vartheta)$ and $h(y,\vartheta)$ are both between $\hbar/2$ and $2\hbar$, the absolute value of h on $\mathcal{A}$ is between $\frac{\hbar}{2}-\frac{\hbar}{4}$ and $2\hbar+\frac{\hbar}{4}$ (again, if $\|\mathcal{A}\|$ is bounded from above; now the requirement is $\|\mathcal{A}\|\le 1/12$). □

Weak finite type condition. The main assumption which we needed in the proof of Proposition 1.4.1 was the inequality (75). Hence, we can prove the statement of the Proposition if the map π_R satisfies the following *weak finite type condition*:

Definition 1.4.2 *We will say that the map π_R is of a weak finite type if in an open neighborhood W_{p_o} of each point $p_o\in\Sigma^0$ we can choose some local coordinates (x',x_n) and (ϑ',ϑ_n) so that $S_{x'\vartheta'}$ is non-degenerate and*

$$\mathbf{K}_R^k h(x,\vartheta)|_{W_{p_o}}\ge 0 \qquad (\textit{or instead } \le 0), \qquad \textit{for some} \quad k\in\mathbb{N}. \tag{93}$$

Similarly for π_L.

We do not even need to assume anything about lower order derivatives (except the uniform boundedness).

The condition (93) is useful as long as in the neighborhood of each point we can choose proper local coordinates. Unlike the system of inequalities (71) – (72), this inequality is not coordinate-independent.

A finite type condition (93) is almost tautological and holds for a general type of a canonical relation (except maybe some sophisticated counterexamples).

Let us also mention that the support of $\psi(x,\vartheta)$ is compact, so that there is a finite covering of $\operatorname{supp} K(T)$ with appropriate neighborhoods.

1.5 Cotlar-Stein Almost Orthogonality

When deriving the estimates on integral operators one often uses the Almost Orthogonality principle of M. Cotlar and E.M. Stein, first formulated by M. Cotlar in [1955]. This result is fairly classical; our excuse for formulating it once again is its special weighted form which sometimes allows to reduce the number of integrations by parts in half (hereby weakening smoothness requirements).

Let E and F be Hilbert spaces, and let T be a linear operator which acts from E to F. An often situation is that one can decompose the operator T into an infinite sum of operators $T=\sum_i T_i$, which satisfy certain estimates, and the question is, under which assumptions on T_i one can deduce an adequate estimate on T.

Definition 1.5.1 (Almost orthogonal operators) *We will call a family of continuous operators*

$$\{T_i \mid i\in\mathbf{Z}\}$$

almost orthogonal *if they satisfy the following conditions:*

$$\|T_i^*T_j\|\le a(i,j), \qquad \left\|T_iT_j^*\right\|\le b(i,j), \tag{94}$$

where $a(i,j)$ and $b(i,j)$ are non-negative symmetric functions on $\mathbf{Z}\times\mathbf{Z}$ which satisfy

$$\sup_i \sum_j a^{1/2}(i,j) \equiv A < \infty, \qquad \sup_i \sum_j b^{1/2}(i,j) \equiv B < \infty, \tag{95}$$

or, more generally,

$$\sup_i \sum_j a^{\mu}(i,j) \equiv A < \infty, \qquad \sup_i \sum_j b^{\nu}(i,j) \equiv B < \infty, \tag{96}$$

with some non-negative weights $\quad 0 \le \mu,\, \nu \le 1, \quad \mu+\nu = 1.$

Lemma 1.5.1 (Cotlar-Stein) *Let $\{T_i : E \to F \mid i \in \mathbf{Z}\}$ be a family of almost orthogonal operators satisfying* (94) *and* (95)*, or, more generally,* (94) *and* (96)*.*

Then the formal sum $\sum_i T_i$ converges weakly to a continuous linear operator

$$T:\, E \to F,$$

which is bounded by

$$\|T\| \le A^{1/2}B^{1/2} = \left(\sup_i \sum_j a^{\mu}(i,j)\right)^{\frac{1}{2}} \left(\sup_i \sum_j b^{\nu}(i,j)\right)^{\frac{1}{2}}. \tag{97}$$

The proof is based on two propositions below.

Of course, a finite sum of continuous operators defines a continuous operator from E to F. The important result is that the sum of *any* finite number of operators has the same bound (97).

Proposition 1.5.1 *Let $\{T_i \mid i \in \mathbf{Z}\}$ be a family of almost orthogonal operators, with some constants A and B in* (95) *or in* (96)*. For any finite subset $I \subset \mathbf{Z}$, the operator given by the finite sum $T_I \equiv \sum_{i\in I} T_i$ is bounded by $\|T_I\| \le A^{1/2}B^{1/2}$.*

One can easily adapt the elegant proof from the book of E.M. Stein [1993].

Now we need to know in what sense we can draw the conclusion about the norm of T which consists of infinite number of almost orthogonal pieces T_i.

Proposition 1.5.2 *Let $\{T_i : E \to F \mid i \in \mathbf{Z}\}$ be a family of continuous operators such that any finite sum of them is bounded by C. Then the series $\sum_{i\in\mathbf{Z}} T_i$ weakly converges to a continuous linear operator $T:\, E \to F$, also bounded by C.*

Proof of Proposition 1.5.2**.** We notice that for any $u \in E$, $v \in F$, the formal series of numbers (real or complex)

$$\sum_{i\in\mathbf{Z}} \langle v, T_i u\rangle \tag{98}$$

converges.

If it were not the case, we would be able to pick some finite subset $I \subset \mathbf{Z}$, so that $|\sum_{i\in I}\langle v, T_i u\rangle|$ would be arbitrarily large. But then the finite sum of operators $\sum_{i\in I} T_i$ would not be bounded, contradicting our assumptions.

In the same manner one can prove that the limit of the series (98) is bounded in the absolute value by $C\,\|u\|\,\|v\|$.

The convergence of the series (98), for any $u \in E$, $v \in F$, implies that the operator series $\sum_{i\in\mathbf{Z}} T_i$ converges in the weak operator topology. □

This Proposition concludes the proof of Lemma 1.5.1.

1.6 Operators with one-sided Whitney fold

To give a self-contained derivation of the regularity properties of the Radon Transform of Melrose-Taylor (see below) and to illustrate the technical methods, we will derive the asymptotics (15) for the oscillatory integral operators such that one of the projections from the canonical relation is a Whitney fold, while the other is *pseudoconvex* (or of *finite type*).

Let β be a smooth function supported on the interval $[1/2, 2]$, and let us localize the operator T_λ to the variety where the determinant $h(x,\vartheta)$ of the mixed Hessian $S_{x\vartheta}$ is of some infinitesimally small magnitude $\sim \hbar$, as in (12):

$$T^\hbar u(x) \equiv \int\limits_{\mathbb{R}^n_R} e^{i\lambda S(x,\vartheta)}\,\psi(x,\vartheta)\,\beta(\hbar^{-1}h(x,\vartheta))\,u(\vartheta)\,d\vartheta; \quad x \in \mathbb{R}^n_L, \quad \vartheta \in \mathbb{R}^n_R. \tag{99}$$

Theorem 1.6.1 (Operators with asymptotically degenerate phase) *If one of the projections from the canonical relation is a Whitney fold and if the other one is pseudoconvex, then, as long as*

$$\hbar \geq \lambda^{-\frac{1}{2}+\varepsilon}, \tag{100}$$

for some $\varepsilon > 0$, there is the following estimate:

$$\left\|T^\hbar\right\| \leq const\,\lambda^{-\frac{n}{2}}\hbar^{-\frac{1}{2}}. \tag{101}$$

We assume that $\pi_{_L}$ is a Whitney fold and $\pi_{_R}$ is pseudoconvex.

Remark 1. According to Proposition 1.4.1, the statement of the theorem is valid if, instead of being *pseudoconvex*, the map $\pi_{_R}$ is of *finite type* (Definition 1.4.1), or even of *weak finite type* (Definition 1.4.2). In this case, we use the finite partition of 1 given by (76) and continue the proof for each piece of $T^\hbar$. We need to keep in mind that the functions $\rho^\hbar_\sigma(x,\vartheta)$ from the partition satisfy the bounds (73).

For example, the theorem is applicable when $\pi_{_R}$ is a cusp.

Remark 2. When we use this Theorem to derive estimates of the type

$$\|T_\lambda\| \leq const\,\lambda^{-\frac{n}{2}+r},$$

we would not allow $\hbar$ to drop below λ^{-2r}. Since the loss of $r = 1/4$ is the *a priori* result of A. Greenleaf and A. Seeger [1994] (for oscillatory integral operators with one-sided Whitney folds), we will only consider $\hbar \geq \lambda^{-2r}$ with $r < 1/4$. Hence, the condition (100) is not restrictive.

Let us choose some local coordinates $(x_1, \ldots, x_n)$ and $(\vartheta_1, \ldots, \vartheta_n)$, so that $S_{x'\vartheta'}$ is non-degenerate.

We will use the almost orthogonal decomposition with respect to ϑ: we take sets of integers, $\Theta \in \mathbf{Z}^n$, and localize the integral kernel of $T^\hbar$, multiplying it by

$$\chi(\hbar^{-1}\vartheta - \Theta) \equiv \prod_{i=1}^{n} \chi(\hbar^{-1}\vartheta_i - \Theta_i).$$

We decompose $T^\hbar$ into $T^\hbar = \sum_{\Theta \in \mathbf{Z}^n} T^\hbar_\Theta$, where $T^\hbar_\Theta = T^\hbar \circ \chi(\hbar^{-1}\vartheta - \Theta)$.

To prove the Theorem formulated above, we need to prove that (a) each $T^\hbar_\Theta$ is bounded by the right-hand side of (101), and that (b) the operators corresponding to different Θ's are almost orthogonal. These statements are proved in two propositions below.

Proposition 1.6.1 *The estimate* (101) *holds for each individual* $T^\hbar_\Theta$.

Proof. For the sake of convenience, we will write τ for $T^\hbar_\Theta$:

$$\tau u(x) = \int e^{i\lambda S(x,\vartheta)}\, \psi(x,\vartheta)\, \beta(\hbar^{-1}h)\, \chi(\hbar^{-1}\vartheta - \Theta)\, u(\vartheta)\, d\vartheta.$$

Let us consider the integral kernel of $\tau\tau^*$:

$$K(\tau\tau^*)(x,y) = \int d^n\vartheta\, e^{i\lambda(S(x,\vartheta)-S(y,\vartheta))}\psi^2\chi^2(\hbar^{-1}\vartheta - \Theta)\beta^2(\hbar^{-1}h(x,\vartheta)). \qquad (102)$$

We insert into (102) the operator

$$L_\vartheta = -\frac{1}{i\lambda} \cdot \frac{(S_\vartheta(y,\vartheta) - S_\vartheta(x,\vartheta)) \cdot \nabla_\vartheta}{\|S_\vartheta(y,\vartheta) - S_\vartheta(x,\vartheta)\|^2}, \qquad (103)$$

which is the identity when acting on the exponential, and integrate by parts.

When acting on cut-offs, ∇_ϑ gives a factor bounded by $\hbar^{-1}$. The derivative ∇_ϑ can act on the denominator of L_ϑ itself, also giving a factor bounded by $const\,\hbar^{-1}$:

$$\frac{\|S_{x\vartheta}(x,\vartheta) - S_{x\vartheta}(y,\vartheta)\|}{\|S_\vartheta(x,\vartheta) - S_\vartheta(y,\vartheta)\|} \leq const\, \frac{\|x-y\|}{\hbar\, \|x-y\|} \leq const\, \hbar^{-1}. \qquad (104)$$

The bound $\|S_\vartheta(x,\vartheta) - S_\vartheta(y,\vartheta)\| \ge const\,\hbar\ \|x-y\|$ which we have used in (104) is due to the assumption that the map $\pi_{_R}$ (and hence $\pi_{_R}|_\vartheta : x \mapsto S_\vartheta$) is pseudoconvex, while on the support of $T^\hbar$

$$min(J(\pi_{_R})|_\vartheta) = min(S_{x\vartheta}) \ge \frac{\det S_{x\vartheta}}{\|S_{x'\vartheta'}\|^{n-1}} \sim \hbar.$$

Integration by parts $n+1$ times is thus equivalent to adding the following factor to the integral kernel (102) of $\tau\tau^*$:

$$const\,\frac{1}{1+|\lambda\hbar\,\|S_\vartheta(x,\vartheta)-S_\vartheta(y,\vartheta)\||^{n+1}}. \tag{105}$$

To apply Young's inequality, we integrate the absolute value of (102), with the extra factor (105), with respect to x. We change the order of integration with respect to x and ϑ and also change the variables, $x \mapsto \eta = S_\vartheta(x,\vartheta)$:

$$\begin{aligned}
\int dx\,|K(\tau\tau^*)(x,y)| \;&\le\; \int d\vartheta \cdot \frac{d\eta}{|\det S_{x\vartheta}|}\cdot\frac{\chi^2(\hbar^{-1}\vartheta-\Theta)h^2(\hbar^{-1}h)}{1+|\lambda\hbar\,\|\eta - S_\vartheta(y,\vartheta)\||}\\
&\le\; const\int d\vartheta\,\chi^2(\hbar^{-1}\vartheta-\Theta)\cdot\frac{1}{\hbar}\cdot\frac{1}{\lambda^n\hbar^n}\\
&\le\; const\,\lambda^{-n}\hbar^{-1}.
\end{aligned} \tag{106}$$

We conclude that $\|\tau\|^2 = \|\tau\tau^*\| \le const\,\lambda^{-n}\hbar^{-1}$. □

Proposition 1.6.2 *The operators $T^\hbar_\Theta$, $\Theta \in \mathbf{Z}^n$, are almost orthogonal.*

Proof. First, let us consider the composition $T^\hbar_\Theta T^{\hbar}_W{}^*$. If $\|W-\Theta\| \ge 2\sqrt{n}$, then

$$\chi(\hbar^{-1}\vartheta-\Theta)\,\chi(\hbar^{-1}\vartheta - W) \equiv 0,$$

and therefore $T^\hbar_\Theta T^\hbar_W{}^* \equiv 0$. (Hence, we can take $\nu = 0$ in (96), so that $\mu = 1-\nu = 1$).

Now let us consider the composition $T^{\hbar *}_\Theta T^\hbar_W$. Its integral kernel is given by

$$\begin{aligned}
K(T^{\hbar *}_\Theta T^\hbar_W)(\vartheta,w) = \int d^n x\, e^{-i\lambda(S(x,\vartheta)-S(x,w))}\psi(x,\vartheta)\psi(x,w)\\
\times\beta(\hbar^{-1}h(x,\vartheta))\,\beta(\hbar^{-1}h(x,w))\,\chi(\hbar^{-1}\vartheta-\Theta)\,\chi(\hbar^{-1}w-W).
\end{aligned} \tag{107}$$

We consider the decomposition of the map $\pi_{_L}|_x$,

$$\vartheta \overset{\pi_L'}{\longmapsto} (\xi' \equiv S_{x'},\,\vartheta_n) \overset{\pi_L^s}{\longmapsto} (\xi',\,\xi_n \equiv S_{x_n}).$$

Since $\det S_{x'\vartheta'} \neq 0$, the map π'_{L} is a diffeomorphism, and we can assume that the length of the line segment $\mathcal{A}$ from $(\xi'(x,\vartheta),\vartheta_n)$ to $(\xi'(x,w),w_n)$, in (ξ',ϑ_n)-space, is not less than $const\,\|w-\vartheta\| \approx const\,\hbar\,\|W-\Theta\|$, for some $const > 0$.

Let us consider two cases.

• If the line segment $\mathcal{A}$ is within the conic neighborhood of magnitude α (to be chosen) of the directions $\pm\left(\frac{\partial}{\partial\vartheta_n}\right)_{\xi'}$, then

$$
\begin{aligned}
|h(x,w)-h(x,\vartheta)| &\geq \|\mathcal{A}\|\cos\alpha\left(\frac{\partial}{\partial\vartheta_n}\right)_{\xi'} h - \|\mathcal{A}\|\sin\alpha\,\|\nabla_{\xi'} h\| \\
&\approx \|\mathcal{A}\|\left(\frac{\partial}{\partial\vartheta_n}\right)_{\xi'} h - Const\,\cdot\alpha\,\|\mathcal{A}\|\,,
\end{aligned}
\tag{108}
$$

here $\|\mathcal{A}\| \geq const\,\hbar\,\|W-\Theta\|$ is the length of the line segment $\mathcal{A}$.

Since π_{L} is a Whitney fold, $\left(\frac{\partial}{\partial\vartheta_n}\right)_{\xi'} = \mathbf{K}_{L} h \geq c_{L}$, for some $c_{L} > 0$. Therefore, the right-hand side of (108) is greater than $4\hbar$ if α is chosen sufficiently small and if $\|W-\Theta\|$ is large enough. Then

$$
\beta(\hbar^{-1}h(x,\vartheta))\,\beta(\hbar^{-1}h(x,w)) \equiv 0,
$$

and therefore $K(T^{\hbar *}_{\Theta} T^{\hbar}_{W})(\vartheta,w) \equiv 0$.

• Thus, we conclude that if $\|W-\Theta\|$ is sufficiently large, then the line segment $\mathcal{A}$ is outside α-cone of $\pm\left(\frac{\partial}{\partial\vartheta_n}\right)_{\xi'}$:

$$
\|\xi'(x,w)-\xi'(x,\vartheta)\| \geq \sin\alpha\,\|\mathcal{A}\| \geq \sin\alpha\,const\,\hbar\,\|W-\Theta\|\,. \tag{109}
$$

Each integration by parts in the expression (107) for the integral kernel of $T^{\hbar *}_{\Theta} T^{\hbar}_{W}$, with the aid of the operator

$$
L_x = \frac{1}{i\lambda}\cdot\frac{(S_x(x,w)-S_x(x,\vartheta))\cdot\nabla_x}{\|S_x(x,w)-S_x(x,\vartheta)\|^2}, \tag{110}
$$

adds the factor bounded by

$$
const\,\left|\lambda\hbar^2\,\|W-\Theta\|\right|^{-N}.
$$

Indeed, each derivative ∇_x contributes at most $\hbar^{-1}$, and due to (109),

$$
\|S_x(x,w)-S_x(x,\vartheta)\| \geq \sin\alpha\,const\,\|w-\vartheta\| \approx \alpha\,const\,\hbar\,\|W-\Theta\|\,,
$$

giving the necessary bound from below for the denominator of (110).

Since, according to (103), we are only interested in the values of $\hbar$ not less than $\lambda^{-\frac{1}{2}+\varepsilon}$, we have: $\lambda\hbar^2 \geq \lambda^{2\varepsilon}$. Young's inequality yields

$$
\left\|T^{\hbar *}_{\Theta} T^{\hbar}_{W}\right\| \leq \int d\vartheta\,\left|K(T^{\hbar *}_{\Theta} T^{\hbar}_{W})(\vartheta,w)\right| \leq const\,\hbar^n\left(\lambda^{2\varepsilon}\,\|W-\Theta\|\right)^{-N}, \tag{111}
$$

here the factor $\hbar^n$ is due to integration in ϑ.

The required almost orthogonality follows from (111) if we only choose N large enough, so that $N \geq n+1$ and $\hbar^n \lambda^{-2\varepsilon N} \leq \lambda^{-n}\hbar$. This proves Proposition 1.6.2 and concludes the proof of Theorem 1.6.1. □

Remark 3. In the *Almost Orthogonality* part of the proof (Proposition 1.6.2) we did not need the non-degeneracy of the mixed Hessian (the condition $|h(x,\vartheta)| \geq \hbar/2$, which is satisfied on the support of $T^\hbar$). Therefore, the same fine localization is applicable to the operator $\bar{T}^\hbar$: the operators $\bar{T}^\hbar_\Theta \equiv \bar{T}^\hbar \circ \chi(\hbar^{-1}\vartheta - \Theta)$ are almost orthogonal (again, as long as $\hbar \geq \lambda^{-\frac{1}{2}+\varepsilon}$).

It is easy to show that the estimate for each $\bar{T}^\hbar_\Theta$ is $const\,\lambda^{-\frac{n-1}{2}}\hbar^{\frac{1}{2}}$ (where $\lambda^{-\frac{n-1}{2}}$ is Hörmander's estimate in x' and ϑ', and $\hbar^{\frac{1}{2}}$ is due to the size of the support in ϑ_n). This justifies the estimate (16) on $\bar{T}^\hbar$.

Remark 4. If the associated canonical relation is a two-sided Whitney fold, then one can also use the spatial localization with respect to x, with the same step $\hbar$. The resulting pieces

$$\bar{T}^\hbar_{X\Theta} \equiv \chi(\hbar^{-1}x - X) \circ \bar{T}^\hbar \circ \chi(\hbar^{-1}\vartheta - \Theta), \qquad X \in \mathbf{Z}^n, \quad \Theta \in \mathbf{Z}^n,$$

are almost orthogonal (we now have to take non-zero weights μ and ν in (96)), and each piece is bounded by $const\,\lambda^{-\frac{n-1}{2}}\hbar^{\frac{1}{2}}\hbar^{\frac{1}{2}}$. This proves the estimate (17) on $\bar{T}^\hbar$, and hence the estimate $\|T_\lambda\| \leq const\,\lambda^{-\frac{n}{2}+\frac{1}{6}}$ for the oscillatory integral operators associated to canonical relations with two-sided Whitney folds.

Theorem 1.6.1 gives the estimate on $T^\hbar$ if the operator T_λ is of a rather general type. Still, to draw a conclusion about the norm of T_λ, we need to know the asymptotic behavior of the operator $\bar{T}^\hbar$ which is responsible for the very small neighborhood of the critical variety Σ^0.

In fact, we expect that if the integral kernel of T_λ vanishes at Σ^0, then the contribution of $\bar{T}^\hbar$ into the estimate on T_λ becomes negligible. Let us formulate this result.

Let U_λ be the oscillatory integral operator with its symbol vanishing on the critical variety Σ^0 as the first (or higher) power of $h(x,\vartheta) = \det S_{x\vartheta}$:

$$U_\lambda u(x) = \int_{\mathbb{R}^n_R} e^{i\lambda S(x,\vartheta)}\,\psi(x,\vartheta)\,h(x,\vartheta)\,u(\vartheta)\,d\vartheta, \quad x \in \mathbb{R}^n_L, \quad \vartheta \in \mathbb{R}^n_R, \tag{112}$$

where $\psi \in C^\infty_{\mathrm{comp}}(\mathbb{R}^n_L \times \mathbb{R}^n_R)$.

Theorem 1.6.2 (Operators with symbol vanishing on Σ^0) *If one of the projections from the canonical relation is a Whitney fold and the other one is pseudoconvex or at least of finite type, then*

$$\|U_\lambda\| \leq const\,\lambda^{-\frac{n}{2}}. \tag{113}$$

According to C.D. Sogge and E.M. Stein [1986], the estimate (113) is also valid in the case when no assumptions on π_L and π_R are made, if the integral kernel of U_λ vanishes at Σ^0 as $|h|^{\frac{5n}{2}}$.

Let us also mention the paper of D.H. Phong and E.M. Stein [1994], where the estimate $const\,\lambda^{-\frac{1}{2}}$ is given for the case $n=1$ for the operators with polynomial phases, under the assumption that the density vanishes on Σ^0 as $|h|^{\frac{1}{2}}$.

We can not consider the densities vanishing as fractional powers of $|h(x,\vartheta)|$ because of the smoothness restrictions.

Remark 5. Together with the results from Section 1.2, Theorem 1.6.2 implies that Fourier integral operators associated to the canonical relations with one-sided Whitney fold, with the symbol vanishing on the critical variety, have the same smoothness properties as the operators associated to local graphs.

Proof. We use the dyadic decomposition,

$$U_\lambda = \sum_{\pm} \sum_{\hbar>\hbar_o}^{\hbar\le\Lambda} U_\pm^\hbar + \bar{U}^{\hbar_o}, \qquad \hbar = 2^{-N}, \quad \hbar_o = 2^{-N_o}, \quad N,\, N_o \in \mathbf{Z}, \tag{114}$$

where $\Lambda \equiv 2^{N_1} \ge \sup|h(x,\vartheta)|$, and $\hbar_o = 2^{-N_o}$ is to be chosen appropriately.

The operators $U_\pm^\hbar$ and $\bar{U}^\hbar$ are given by

$$U_\pm^\hbar u(x) = \int e^{i\lambda S(x,\vartheta)}\,\psi(x,\vartheta)\,h(x,\vartheta)\,\beta_\pm(\hbar^{-1}h(x,\vartheta))\,u(\vartheta)\,d\vartheta, \tag{115}$$

$$\bar{U}^\hbar u(x) = \int e^{i\lambda S(x,\vartheta)}\,\psi(x,\vartheta)\,h(x,\vartheta)\,\bar{\beta}(\hbar^{-1}h(x,\vartheta))\,u(\vartheta)\,d\vartheta. \tag{116}$$

According to Theorem 1.6.1,

$$\left\|U^\hbar\right\| \le const\,\lambda^{-\frac{n}{2}}\hbar^{-\frac{1}{2}}\cdot\hbar. \tag{117}$$

The extra factor $\hbar$ is due to the presence of $h(x,\vartheta)\sim\hbar$ in the integral kernel of $U^\hbar$.

The estimate (117) only becomes better when $\hbar$ tends to zero. Still, we need to remember the restriction $\hbar \ge \lambda^{-\frac{1}{2}+\varepsilon}$ from the conditions of Theorem 1.6.1.

Recalling the estimate (16) from Section 1.1, we conclude that the operator $\bar{U}^\hbar$ is bounded by

$$\left\|\bar{U}^\hbar\right\| \le const\,\lambda^{-\frac{n-1}{2}}\hbar^{\frac{1}{2}}\cdot\hbar. \tag{118}$$

Again, the extra factor of $\hbar$ is due to the vanishing of the integral kernel on Σ^0. Since we want to prove the bound $\|U_\lambda\| \le const\,\lambda^{-\frac{n}{2}}$, we can take $\hbar_o = \lambda^{-\frac{1}{3}}$ (satisfying the condition $\hbar_o \ge \lambda^{-\frac{1}{2}+\varepsilon}$). □

2 Radon Transform of Melrose-Taylor

2.1 Background from Scattering Theory

We are going to apply the developed technique to a particular integral operator which arises in the scattering theory: the Radon Transform of Melrose and Taylor. Let us first give the necessary background from the scattering theory. We follow the fundamental paper of R.B. Melrose and M.E. Taylor [1985] and the book of R.B. Melrose [1995].

We consider the wave equation in the exterior of the convex compact domain $K \subset \mathbb{R}^{n+1}$, with the Dirichlet conditions on $B = \partial K$:

$$\left(\frac{\partial^2}{\partial t^2} - \Delta\right)\hat{u} = 0 \qquad \text{on} \quad \mathbb{R}\times\Omega, \quad \Omega = \mathbb{R}^{n+1}\backslash K, \tag{119}$$

$$\hat{u}|_{\mathbb{R}\times B} = 0. \tag{120}$$

We specify that for $t \ll 0$ the solution has the form of the plane wave,

$$\hat{u}(t,x) = \delta(t - \mathbf{r}\cdot\omega) \qquad \text{for} \quad t \ll 0, \tag{121}$$

with ω being a unit vector in $\mathbb{R}^{n+1}$. We decompose the solution $\hat{u}$ as

$$\hat{u}(t,\mathbf{r}) = \delta(t - \mathbf{r}\cdot\omega) + \hat{w}(t,\mathbf{r}),$$

and define the Fourier transforms of $\hat{u}$ and $\hat{w}$ with respect to t:

$$u(\lambda,\mathbf{r},\omega) = \int e^{i\lambda t}\hat{u}(t,\mathbf{r})\,dt, \qquad w(\lambda,\mathbf{r},\omega) = \int e^{i\lambda t}\hat{w}(t,\mathbf{r})\,dt.$$

Since $u(\lambda,\mathbf{r},\omega) = e^{i\lambda\mathbf{r}\cdot\omega} + w(\lambda,\mathbf{r},\omega)$, w satisfies the following system:

$$\left(\Delta + \lambda^2\right) w = 0 \qquad \text{on} \quad \Omega, \tag{122}$$

$$w|_{\mathbb{R}\times B} = -e^{i\lambda\mathbf{r}\cdot\omega}. \tag{123}$$

The initial data (121) gives rise to the *Sommerfeld radiation condition*:

$$w = O(r^{-n/2}), \qquad \left(\frac{\partial}{\partial r} - i\lambda\right) w = o(r^{-n/2}), \qquad \text{as} \quad r\to\infty. \tag{124}$$

The function w has the following asymptotic behavior for large values of r:

$$w = a_s(\theta,\omega,\lambda) r^{-n/2} e^{i\lambda r} + o(r^{-n/2}), \tag{125}$$

here θ is a unit vector in the direction of $\mathbf{r}$.

The coefficient $a_s(\theta,\omega,\lambda)$ represents the *scattering amplitude*.

We consider the inverse Fourier transform of a_s with respect to λ,

$$k_A(\theta,\omega,t) = \int e^{-i\lambda t} a_s(\theta,\omega,\lambda)\,\frac{d\lambda}{2\pi}. \tag{126}$$

The *scattering operator* of Lax and Phillips [1971] is given by

$$S = \mathrm{Id} + D_t^{n/2} A, \tag{127}$$

where $A : \mathcal{E}'(\mathbb{R}\times\mathbf{S}^n) \to \mathcal{D}'(\mathbb{R}\times\mathbf{S}^n)$ has Schwartz kernel $k_A(\theta,\omega,t-t')$.

The Kirchhoff integral formula expresses w in terms of its behavior on the boundary:

$$w(\mathbf{r}) = \int_B \left\{ w(\mathbf{r}')\,\frac{\partial G_\lambda}{\partial \mathbf{n}_{\mathbf{r}'}}(\mathbf{r}-\mathbf{r}') - \frac{\partial w}{\partial \mathbf{n}_{\mathbf{r}'}}(\mathbf{r}')\,G_\lambda(\mathbf{r}-\mathbf{r}') \right\} dS_{\mathbf{r}'}. \tag{128}$$

Here $G_\lambda(\mathbf{r})$ is the fundamental solution to the wave equation,

$$(\Delta + \lambda^2)\,G_\lambda(\mathbf{r}) = -\delta(\mathbf{r}).$$

G_λ is chosen so that it satisfies the Sommerfeld radiation condition. For the scattering in three dimensions ($n=2$), $G_\lambda(\mathbf{r}) = (4\pi r)^{-1} e^{i\lambda r}$.

We can use the representation (128) as long as we have a sufficient knowledge of the normal derivative of w on the boundary B.

The Kirchhoff approximation is

$$\frac{\partial w}{\partial \mathbf{n}_{\mathbf{r}}} \approx -i\lambda\,|\mathbf{n}_{\mathbf{r}}\cdot\omega|\,e^{i\lambda \mathbf{r}\cdot\omega}. \tag{129}$$

The idea behind (129) is to approximate the scattering at a point in the illuminated region of B by the scattering on the tangent plane at this point, and to assume that in the shadow region behind the obstacle $e^{i\lambda\mathbf{r}\cdot\omega} + w(\lambda,\mathbf{r},\omega) \approx 0$.

The approximation (129) is not adequate near the shadow boundary, $\mathbf{n}_{\mathbf{r}}\cdot\omega = 0$.

The value of the normal derivative on the boundary $\frac{\partial \hat{w}}{\partial \mathbf{n}_{\mathbf{r}}}$ corresponding to the system

$$(\partial_t^2 - \Delta)\hat{w}(t,\mathbf{r}) = 0,$$

$$\hat{w}(t,\mathbf{r})|_{\mathbb{R}\times B} = f(t,\mathbf{r}), \qquad \hat{w}(t,\mathbf{r}) = 0 \qquad \text{for} \quad t \ll 0 \tag{130}$$

is denoted by the action of the *forward Neumann operator* N_+ on $f(t,\mathbf{r})$:

$$N_+ : \mathcal{E}'(\mathbb{R}\times B) \to \mathcal{D}'(\mathbb{R}\times B),$$

$$N_+ f(t,\mathbf{r}) \equiv \frac{\partial \hat{w}}{\partial \mathbf{n}_{\mathbf{r}}}(t,\mathbf{r}), \qquad \mathbf{r}\in B. \tag{131}$$

We are going to rewrite the integral kernel of A in (127) in terms of the Neumann operator N_+.

Given the boundary value problem (130) with

$$f(t,\mathbf{r}) = -\delta(t-\mathbf{r}\cdot\omega),$$

we consider the partial Fourier transform of $\hat{w}$, which is $w(\lambda,\mathbf{r},\omega)$, with the corresponding boundary value problem given by (122)-(124). We express the normal derivative in terms of the Neumann operator,

$$\frac{\partial w}{\partial \mathbf{n}_{\mathbf{r}}}(\lambda,\mathbf{r},\omega) = \int dt\, e^{i\lambda t} N_+(-\delta(t-\mathbf{r}\cdot\omega)), \tag{132}$$

and substitute this expression into the Kirchhoff integral formula (128):

$$w(\lambda,\mathbf{r},\omega) = \int_{\mathbb{R}\times B} ds\, dS_{\mathbf{r}'}\, e^{i\lambda s}\left\{-\frac{\partial G_\lambda}{\partial \mathbf{n}_{\mathbf{r}'}} + G_\lambda(\mathbf{r}-\mathbf{r}')\, N_+\right\}\delta(s-\mathbf{r}'\cdot\omega). \tag{133}$$

Now we express the scattering amplitude $a_s(\theta,\omega,\lambda)$ from the relation (125) and obtain $k_A(\theta,\omega,t)$:

$$k_A(\theta,\omega,t) = \int \frac{d\lambda}{2\pi}\, e^{-i\lambda t} a_s(\theta,\omega,\lambda) = \lim_{r\to\infty}\frac{1}{2\pi}\int\limits_{\mathbb{R}\times\mathbb{R}\times B} d\lambda\, ds\ dS_{\mathbf{r}'}$$

$$\times e^{-i\lambda(t-s)} r^{n/2} e^{-i\lambda r}\left\{-\frac{\partial G_\lambda}{\partial \mathbf{n}_{\mathbf{r}'}} + G_\lambda(\mathbf{r}-\mathbf{r}')\, N_+\right\}\delta(s-\mathbf{r}'\cdot\omega). \tag{134}$$

For our convenience, we restrict our attention to the classical scattering in $n+1=3$ dimensions:

$$G_\lambda(\mathbf{r}) = \frac{e^{i\lambda r}}{4\pi r}, \qquad \frac{\partial G_\lambda}{\partial \mathbf{n}_{\mathbf{r}}} = i\lambda\frac{\mathbf{r}\cdot\mathbf{n}_{\mathbf{r}}}{r}\,\frac{e^{i\lambda r}}{4\pi r} + o\left(\frac{1}{r}\right). \tag{135}$$

Then we can rewrite (134) as

$$\lim_{r\to\infty}\frac{1}{8\pi^2}\int\limits_{\mathbb{R}\times\mathbb{R}\times B} d\lambda\, ds\ dS_{\mathbf{r}'}\, e^{-i\lambda(r+t-s)} e^{i\lambda\|\mathbf{r}-\mathbf{r}'\|}\left\{i\lambda\frac{(\mathbf{r}-\mathbf{r}')\cdot\mathbf{n}_{\mathbf{r}'}}{\|\mathbf{r}-\mathbf{r}'\|} + N_+\right\}\delta(s-\mathbf{r}'\cdot\omega).$$

Now, since $\mathbf{r}=r\theta$ and $r\gg$, we substitute $(\mathbf{r}-\mathbf{r}')\cdot\mathbf{n}_{\mathbf{r}'}/\,\|\mathbf{r}-\mathbf{r}'\| \approx \mathbf{n}_{\mathbf{r}'}\cdot\theta$ and $r-\|\mathbf{r}-\mathbf{r}'\|\approx \mathbf{r}'\cdot\theta$, obtaining

$$\begin{aligned} &\frac{1}{8\pi^2}\int\limits_{\mathbb{R}\times\mathbb{R}\times B} d\lambda\, ds\ dS_{\mathbf{r}'}\, e^{-i\lambda(t-s)-i\lambda\mathbf{r}'\cdot\theta}\left\{i\lambda\mathbf{n}_{\mathbf{r}'}\cdot\theta + N_+\right\}\delta(s-\mathbf{r}'\cdot\omega) \\ =\ &\frac{1}{8\pi^2}\int\limits_{\mathbb{R}\times\mathbb{R}\times B} d\lambda\, ds\ dS_{\mathbf{r}'}\, e^{-i\lambda(t-s)-i\lambda\mathbf{r}'\cdot\theta}\left\{-i\lambda\mathbf{n}_{\mathbf{r}'}\cdot\omega + N_+\right\}\delta(s-\mathbf{r}'\cdot\omega). \end{aligned} \tag{136}$$

Shifting the variable s, we have:

$$\begin{aligned} & k_A(\theta,\omega,t-t') \\ & = \frac{1}{8\pi^2}\int\limits_{\mathrm{I\!R}\times\mathrm{I\!R}\times B} d\lambda\, ds\;\; dS_{\mathbf{r}'}\, e^{-i\lambda(t-s)-i\lambda\mathbf{r}'\cdot\theta}\left\{-i\lambda\mathbf{n}_{\mathbf{r}'}\cdot\omega+N_+\right\}\delta(s-t'-\mathbf{r}'\cdot\omega) \\ & = \frac{1}{4\pi}\int\limits_{\mathrm{I\!R}\times B} ds\;\; dS_{\mathbf{r}'}\,\delta(t-s+\mathbf{r}'\cdot\theta)\left\{(\mathbf{n}_{\mathbf{r}'}\cdot\omega)D_s+N_+\right\}\delta(s-t'-\mathbf{r}'\cdot\omega). \quad (137)\end{aligned}$$

In larger dimensions, $n \geq 3$, the integral kernel of A in (127) has the same form as (137), with some factor c_n instead of $1/4\pi$.

The integral kernel $\delta(t-s-\mathbf{r}\cdot\omega)$ defines the following Radon Transform:

$$F : \mathcal{E}'(\mathrm{I\!R}\times\mathbf{S}^n) \to \mathcal{D}'(\mathrm{I\!R}\times B),$$

$$Fu(t,\mathbf{r}) = \int\limits_{\mathrm{I\!R}\times\mathbf{S}^n} \delta(t-s-\mathbf{r}\cdot\omega)\,u(s,\omega)\,ds\,d\Omega_\omega. \quad (138)$$

Here $\mathbf{r}$ is a vector in $\mathrm{I\!R}^{n+1}$, pointing to the boundary B of an obstacle $K \subset \mathrm{I\!R}^{n+1}$. We call (138) *the Radon Transform of Melrose and Taylor*, denoting F by $\mathcal{R}_{MT}$.

We rewrite (137) as

$$A = c_n\,\mathcal{R}^*_{MT}\cdot(N_+\cdot\mathcal{R}_{MT}+\partial_t\tilde{F}), \quad (139)$$

where the operator $\tilde{F} : \mathcal{E}'(\mathrm{I\!R}\times\mathbf{S}^n) \to \mathcal{D}'(\mathrm{I\!R}\times B)$ has the kernel

$$k_{\tilde{F}}(\mathbf{r},\omega,t-s) = (\mathbf{n}_\mathbf{r}\cdot\omega)\,\delta(t-s-\mathbf{r}\cdot\omega). \quad (140)$$

The expression (139) of the scattering operator A in terms of $\mathcal{R}_{MT}$ allows to relate important characteristics of scattering with the properties of $\mathcal{R}_{MT}$ (we refer the reader to the original paper of Melrose and Taylor [1985]).

2.2 Regularity properties of $\mathcal{R}_{MT}$

In this section, we are going to formulate the continuity properties of the Radon Transform of Melrose and Taylor.

According to R.B. Melrose and M.E. Taylor, $\mathcal{R}_{MT}$ is a Fourier integral operator with a degenerate canonical relation. In [1985], they showed that if the boundary B is strictly convex (with strictly positive Gaussian curvature), then there is a reduction of the canonical relation associated to $\mathcal{R}_{MT}$ to the normal form, and the problem reduces to finding estimates on Fourier integral operators of Airy type. Then one can show that there is a loss of $\mu = 1/6$ derivatives in

$$\mathcal{R}_{MT} : \quad H^s(\mathbb{R}\times\mathbf{S}^n) \to H^{s+\frac{n}{2}-\mu}(\mathbb{R}\times B).$$

Let us formulate the following general result.

Theorem 2.2.1 *Let K be a compact convex domain in $\mathbb{R}^{n+1}$, with the smooth boundary B. If B admits tangent lines with contact of order not greater than k, then the Radon Transform* (138) *is smoothing of order $\frac{n}{2}-\frac{1}{2}\frac{k}{2k+1}$ with respect to t:*

$$\mathcal{R}_{MT} : H^s(\mathbb{R})\otimes L^2(\mathbf{S}^n) \to H^{s+\frac{n}{2}-\frac{1}{2}\frac{k}{2k+1}}(\mathbb{R})\otimes L^2(B), \qquad \text{for any real} \quad s. \tag{141}$$

Using the particular form of the operator (138), one can show (see Corollary 1.2.1) that the smoothing with respect to *time*-variables implies the same smoothing with respect to *space*-variables. This results in the following continuity of $\mathcal{R}_{MT}$ in the Sobolev spaces:

Corollary 2.2.1 *Under the assumptions of Theorem* 2.2.1*, the operator $\mathcal{R}_{MT}$ also extends to a continuous operator*

$$H^s(\mathbb{R}\times\mathbf{S}^n) \stackrel{\mathcal{R}_{MT}}{\longrightarrow} H^{s+\frac{n}{2}-\frac{1}{2}\frac{k}{2k+1}}(\mathbb{R}\times B), \qquad \text{for any real} \quad s.$$

Contact of order not greater than k. Let the boundary B of an obstacle near the point $\mathbf{r}\in B$ be a graph of some function $b_{\mathbf{r}}$. We consider $b_{\mathbf{r}}$ as defined on the tangent plane $T_{\mathbf{r}}B$, and taking values in the direction of the inner normal $\mathbf{n}_{\mathbf{r}}\in\mathbb{R}^{n+1}$.

The tangent plane $T_{\mathbf{r}}B$ is the Euclidean space, with the metric endowed from $\mathbb{R}^{n+1}$, and with the origin at the point $\mathbf{r}$.

We say that at some point $\mathbf{r}$ the boundary B admits tangent lines with contact of order not greater than k, if there is a constant κ such that for any $\mathbf{V}\in T_{\mathbf{r}}B$, $\|\mathbf{V}\|=1$, and small values of $t\in\mathbb{R}$,

$$b_{\mathbf{r}}(t\mathbf{V}) \ge \kappa\,|t|^{k+1}. \tag{142}$$

When we say that B admits tangent lines with contact of order not greater than k, we assume that for all points $\mathbf{r}\in B$ the constants $\kappa_{\mathbf{r}}$ are uniformly bounded from below:

$$\kappa_{\mathbf{r}} \ge \kappa > 0.$$

Let us illustrate this in one-dimensional case.

Lemma 2.2.1 *Let D be an interval in $\mathbb{R}$. If the graph of a concave (convex down) function $b\in C^1(D)$ admits tangent lines with contact of order not greater than k, $k>0$, with the constants $\kappa_{\mathbf{r}}$ in* (142) *being bounded from below by some $\kappa>0$, then the following inequalities hold in D:*

$$b(y)-b(x)-(y-x)b'(x) \ge \kappa\,|y-x|^{k+1}, \tag{143}$$

$$|b'(y) - b'(x)| \geq 2\kappa\, |y - x|^k\,. \tag{144}$$

Here x and y are any two points in D.

The case of non-vanishing curvature corresponds to $k = 1$.

The inequality (143) follows immediately from the definition (142) of contact of order not greater than k (at the point x).

To obtain (144), we rewrite (143), interchanging x and y:

$$b(x) - b(y) - (x - y)b'(y) \geq \kappa\, |y - x|^{k+1}\,. \tag{145}$$

Then the inequality (144) is the sum of (143) and (145). □

Now let us formulate a similar result in the multi-dimensional case.

Lemma 2.2.2 *Let B be a boundary of a convex compact domain in $\mathbb{R}^{n+1}$, with the function $b_{\mathbf{r}_o} \in C^1(T_{\mathbf{r}_o}B)$ defining the boundary B in the vicinity of the point $\mathbf{r}_o \in B$.*

If B admits tangent lines with contact of order not greater than k, with the constants $\kappa_{\mathbf{r}}$ in (142) being bounded from below by some $\kappa > 0$, then for any two points $x,\, y \in T_{\mathbf{r}_o}B$ in the domain of $b_{\mathbf{r}_o}$ the following inequalities hold:

$$b_{\mathbf{r}_o}(y) - b_{\mathbf{r}_o}(x) - \|y - x\|\,\mathbf{V} b_{\mathbf{r}_o}(x) \geq \kappa\, \|y - x\|^{k+1}\,, \tag{146}$$

$$|\mathbf{V} b_{\mathbf{r}_o}(y) - \mathbf{V} b_{\mathbf{r}_o}(x)| \geq 2\kappa\, \|y - x\|^k\,. \tag{147}$$

Here $\mathbf{V} = \frac{y-x}{\|y-x\|} \in T_{\mathbf{r}_o}B, \quad \mathbf{V} b_{\mathbf{r}_o} = db_{\mathbf{r}_o}(\mathbf{V})$.

The inequality (147) contains all the information about the geometry of B which will be important for our purposes (this inequality enables us to control the size of a certain neighborhood of the critical variety).

Let us formulate the regularity properties of the operator

$$\tilde{F} : \mathcal{E}'(\mathbb{R} \times \mathbf{S}^n) \to \mathcal{D}'(\mathbb{R} \times B)$$

with the kernel (140): $k_{\tilde{F}}(\mathbf{r}, \omega, t - s) = (\mathbf{n}_{\mathbf{r}} \cdot \omega)\, \delta(t - s - \mathbf{r} \cdot \omega)$. It turns out that this operator behaves similarly to a non-degenerate Fourier integral operator:

Theorem 2.2.2 (Continuity properties of $\tilde{F}$) *Under the assumptions of Theorem 2.2.1, the operator $\tilde{F}$ in (139) extends to a continuous operator*

$$H^s(\mathbb{R} \times \mathbf{S}^n) \xrightarrow{\tilde{F}} H^{s+\frac{n}{2}}(\mathbb{R} \times B), \qquad \text{for any real} \quad s. \tag{148}$$

There is no dependence on a particular value of the maximal order of contact, k.

2.3 Associated canonical relation

According to the results in Section 1.2, Theorem 2.2.1 would follow from the following proposition:

Proposition 2.3.1 *Under the assumptions of Theorem* 2.2.1*, the oscillatory integral operator* $T_\lambda : \mathcal{D}'(\mathbf{S}^n) \to \mathcal{D}'(B)$,

$$T_\lambda u(\mathbf{r}) = \int\limits_{\mathbf{S}^n} e^{i\lambda\langle \mathbf{r},\omega\rangle} u(\omega) \tag{149}$$

has the following decay of its L^2 *operator norm:*

$$\|T_\lambda\| \le const\, \lambda^{-\frac{n}{2}+\frac{1}{2}\frac{k}{2k+1}}. \tag{150}$$

Let us analyze the canonical relation corresponding to the phase function in (149), $S(x,\vartheta) = \langle \mathbf{r}(x), \omega(\vartheta)\rangle$ (here x and ϑ are some local coordinates on B and $\mathbf{S}^n$, respectively). The cotangent space $T^*_{\mathbf{r}}B$ can be identified (via the standard scalar product) with the tangent plane at the same point $\mathbf{r}$. This shows that the differential $d_x\langle \mathbf{r}(x), \omega(\vartheta)\rangle$ is represented by the orthogonal projection of $\omega(\vartheta)$ onto $T_{\mathbf{r}(x)}B$. Therefore, the singular part of the map

$$\pi_{_L} : (x,\, \vartheta) \mapsto (x, d_x\langle \mathbf{r}(x),\, \omega(\vartheta)\rangle)$$

corresponds to the orthogonal projection from $\mathbf{S}^n$ onto the hyperplane $T_{\mathbf{r}}B$. This projection has a Whitney fold at the points ω such that $\langle \mathbf{n}_{\mathbf{r}}, \omega\rangle = 0$, here $\mathbf{n}_{\mathbf{r}}$ is the unit normal to B at the point $\mathbf{r}$. We conclude that both $d\pi_{_L}$ and $d\pi_{_R}$ are degenerate on the critical variety

$$\Sigma^0 = \{(x,\vartheta) \mid \langle \mathbf{n}_{\mathbf{r}(x)}, \omega(\vartheta)\rangle = 0\}. \tag{151}$$

Considering the orthogonal projection from $T_{\mathbf{r}(x)}B$ onto $T_{\omega(\vartheta)}\mathbf{S}^n$, we conclude that

$$\langle \mathbf{n}_{\mathbf{r}(x)}, \omega(\vartheta)\rangle \sim h(x,\vartheta) \tag{152}$$

in an agreement with (151). Intuitively, if $T_{\mathbf{r}}B$ and $T_\omega\mathbf{S}^n$ are not orthogonal, then the oscillatory integral operator (149) is basically the Fourier transform and has no critical points: $\|T_\lambda\| \le const\, \lambda^{-\frac{n}{2}}$.

Now we need to consider the other projection from $\mathcal{C}$. The singular part of the map

$$\pi_{_R} : (x,\, \vartheta) \mapsto (\vartheta,\, -d_\vartheta\langle \mathbf{r}(x), \omega(\vartheta)\rangle)$$

corresponds to the orthogonal projection from B onto $T_{\omega(\vartheta)}\mathbf{S}^n$. This projection is a Whitney fold at (x,ϑ) if and only if the principal curvature at $\mathbf{r}(x) \in B$ in the direction of $\omega(\vartheta)$ is different from zero.

Convexity of the map π_R. Let us show that the map π_R is pseudoconvex. For this, we need to show the convexity of

$$\pi_R\big|_\vartheta : x \mapsto \nabla_\vartheta S(x,\vartheta) = \nabla_\vartheta \langle \mathbf{r}(x), \omega(\vartheta)\rangle.$$

If we use the standard scalar product $\langle\,,\rangle$ in $\mathbb{R}^{n+1}$ to identify the tangent and cotangent fibers, $T_{\omega(\vartheta)}\mathbf{S}^n \cong T^*_{\omega(\vartheta)}\mathbf{S}^n$, then π_R is given geometrically by the orthogonal projection $\Pi_{\omega(\vartheta)}$ from B in the direction of $\omega(\vartheta)$, onto $T_{\omega(\vartheta)}\mathbf{S}^n$:

$$B \overset{\Pi_{\omega(\vartheta)}}{\longrightarrow} T_{\omega(\vartheta)}\mathbf{S}^n, \qquad \mathbf{r}(x) \mapsto \mathbf{r}(x) - \omega(\vartheta)\langle \mathbf{r}(x), \omega(\vartheta)\rangle.$$

If the hypersurface B is convex (not necessarily strictly convex), then for any $\mathbf{r}_1$, $\mathbf{r}_2$,

$$\|\Pi_\omega(\mathbf{r}_2 - \mathbf{r}_1)\|_{T_\omega\mathbf{S}^n} \ge \|\mathbf{r}_2 - \mathbf{r}_1\|_{\mathbb{R}^{n+1}} \inf_{\mathbf{r}\in\gamma} |\langle \mathbf{n}_\mathbf{r}, \omega\rangle|\,, \tag{153}$$

here γ is a path from $\mathbf{r}_1$ to $\mathbf{r}_2$.

This is the key inequality. Since Π_ω is the orthogonal projection, $min(\Pi_\omega)$ is equal to $|\langle \mathbf{n}_\mathbf{r}, \omega\rangle|$, and (153) suggests that the map Π_ω is pseudoconvex.

This justifies pseudoconvexity of the map π_R.

Local coordinates. Now let us rewrite our results introducing appropriate local coordinates on B and on $\mathbf{S}^n$. We would like to restrict our attention to the vicinity of some critical point $(\mathbf{r}_o, \omega_o)$, such that

$$\langle \mathbf{n}_{\mathbf{r}_o}, \omega_o\rangle = 0. \tag{154}$$

We consider B and $\mathbf{S}^n$ as parameterized by the Euclidean coordinates $(x_1, \ldots, x_n)$ and $(\vartheta_1, \ldots, \vartheta_n)$ in $T_{\mathbf{r}_o}B$ and $T_{\omega_o}\mathbf{S}^n$. We choose the axis x_n in $T_{\mathbf{r}_o}B$ in the direction of the vector ω_o (this is possible due to the condition (154)). Then, we choose the axis ϑ_n in the direction of the inner normal $\mathbf{n}_{\mathbf{r}_o}$. The directions of the axes $x_{i'}$ are chosen to coincide with the directions of $\vartheta_{i'}$, $i' = 1, \ldots, n-1$. We will use the notations $x' = (x_1, \ldots, x_{n-1})$, $\vartheta' = (\vartheta_1, \ldots, \vartheta_{n-1})$.

We can assume that $\mathbf{r}_o = \mathbf{r}(0)$ and $\omega_o = \omega(0)$ are both in the origin of $\mathbb{R}^{n+1}$. Such a shift changes the value of the phase $S(x,\vartheta) = \langle \mathbf{r}(x), \omega(\vartheta)\rangle$; still, it is irrelevant, since the values of mixed derivatives of $S(x,\vartheta)$ (which are the only "observable" quantities) remain unchanged.

The phase function S in (149) is given by

$$S(x,\vartheta) = \langle \mathbf{r}(x), \omega(\vartheta)\rangle = \vartheta_n b_{\mathbf{r}_o}(x) - x_n g(\vartheta) + x'\cdot\vartheta', \tag{155}$$

modulo terms which depend only on either x or ϑ. We have used the function

$$g(\vartheta) = 1 - \sqrt{1 - \|\vartheta\|^2} \in C^\infty_{\mathrm{loc}}(T_{\omega_o}\mathbf{S}^n),$$

which locally represents the sphere $\mathbf{S}^n$.

Later on, we assume that the point $\mathbf{r}_o$ is fixed, and use the notation $b(x)$ for the function $b_{\mathbf{r}_o}(x)$ which defines the boundary near $\mathbf{r}_o$.

The mixed Hessian corresponding to the phase function (155) is given by

$$S_{x\vartheta} = \begin{bmatrix} \mathbb{I}_{n-1} & b_{x'}(x)^t \\ -g_{\vartheta'}(\vartheta) & b_{x_n}(x) - g_{\vartheta_n}(\vartheta) \end{bmatrix}, \tag{156}$$

with its determinant being equal to

$$h(x,\vartheta) = b_{x_n}(x) - g_{\vartheta_n}(\vartheta) + b_{x'}(x)\cdot g_{\vartheta'}(\vartheta). \tag{157}$$

Note that this expression vanishes at $(x,\vartheta) = (0,0)$, which corresponds to the critical point $(\mathbf{r}_o, \omega_o)$.

The non-degenerate part of the mixed Hessian is given by the unit matrix:

$$S_{x'\vartheta'}(x,\vartheta) = \mathbb{I}_{n-1}. \tag{158}$$

Hence, the rank of $S_{x\vartheta}$ is at least $n-1$, and the kernels of the differentials $\pi_{_L}$ and $\pi_{_R}$ are at most one-dimensional.

The kernel of the differential $d\pi_{_L}$ is generated by the vector

$$\mathbf{K}_{_L}(x,\vartheta) = -b_{x'}(x)\cdot\nabla_{\vartheta'} + \partial_{\vartheta_n} \in T_x\mathbb{R}^n_{_L}\times T_\vartheta\mathbb{R}^n_{_R}, \qquad (x,\vartheta)\in\Sigma^0. \tag{159}$$

The vector $\mathbf{K}_{_L}(x,\vartheta)$ is also well-defined by (159) away from the critical variety. Moreover, we have a convenient relation

$$h(x,\vartheta) = b_{x_n}(x) - \mathbf{K}_{_L} g(\vartheta).$$

This yields

$$\mathbf{K}_{_L} h(x,\vartheta) = -{\mathbf{K}_{_L}}^2 g(\vartheta) \le -1,$$

and we conclude that the kernel of the differential $d\pi_{_L}$ is transversal to the critical variety Σ^0.

Hence, the projection $\pi_{_L}$ is a Whitney fold.

This is not necessarily true for the map $\pi_{_R}$, since the kernel of its differential is generated by

$$\mathbf{K}_{_R} = g_{\vartheta'}\cdot\nabla_{x'} + \partial_{x_n},$$

and therefore

$$\mathbf{K}_{_R} h(x,\vartheta) = \mathbf{K}_{_R}^2 b(x) \ge 0.$$

The latter expression is non-negative since we assume that the boundary B is convex. At the same time, $\mathbf{K}_{_R} h$ can vanish if the curvature of B is not strictly positive. In this case the projection $\pi_{_R}$ has a worse singularity than a Whitney fold does.

In the language of Section 1.4, the inequality $\mathbf{K}_{_R} h(x,\vartheta) = \mathbf{K}_{_R}^2 b(x) \ge 0$ shows that the projection $\pi_{_R}$ satisfies a weak finite type condition (Definition 1.4.2).

2.4 Isolated precise points

We are going to show that the asymptotics of the estimates for oscillatory integral operators (Theorem 1.1.1) easily yield the continuity properties of $\mathcal{R}_{MT}$ in a particular situation, when the curvature of a boundary vanishes at isolated points where the tangent planes have *precise* contacts of finite order with the boundary (in the sense specified after the theorem). This result and the derivation are taken from the paper of the author [1997].

Theorem 2.4.1 *Let B be a smooth hypersurface in $\mathbb{R}^{n+1}$, and let its Gaussian curvature vanish at an isolated point $\mathbf{r}_o$. If the tangent plane at $\mathbf{r}_o$ has a precise contact of order $k > 1$ with B, then the Radon Transform $\mathcal{R}_{MT}$ localized near $\mathbf{r}_o$ extends to a continuous operator*

$$H^s(\mathbb{R}) \otimes L^2(\mathbf{S}^n) \overset{\mathcal{R}_{MT}}{\longrightarrow} H^{s+\frac{n}{2}-\frac{1}{2}\frac{k}{2k+1}}(\mathbb{R}) \otimes L^2_{\mathrm{loc}}(B), \qquad \text{for any real} \quad s.$$

Precise contact of finite order. Let the hypersurface B in $\mathbb{R}^{n+1}$ be locally given by the graph $x_{n+1} = b(x_1, \ldots, x_n)$ of some smooth function $b \in C^\infty(\mathbb{R}^n)$, defined in the neighborhood of the origin in $\mathbb{R}^n$. We assume that $b(0) = 0$, $b'(0) = 0$.

Definition 2.4.1 (Precise contact of order k) *We say that the hyperplane given by $\{x_{n+1} = 0\}$ has a precise contact of order $k > 1$ with the graph $\{x_{n+1} = b(x)\}$ of the function $b \in C^2(\mathbb{R}^n)$, if $b(0) = 0$, $b'(0) = 0$, and if the Hessian $[\partial_{x_i}\partial_{x_j} b]$ is isotropic and vanishes uniformly of order $k-1$. That is, for any unit vectors $\mathbf{u}$, $\mathbf{v} \in \mathbb{R}^n$,*

$$u^i u^j \frac{\partial^2 b(x)}{\partial x_i \partial x_j} \geq c\, \|x\|^{k-1}, \qquad \left| u^i v^j \frac{\partial^2 b(x)}{\partial x_i \partial x_j} \right| \leq C\, \|x\|^{k-1}, \tag{160}$$

where c and C are some positive numbers, and $\|x\|^2 \equiv x_1^2 + \ldots + x_n^2$.

We will call the point $(0, \ldots, 0)$ a precise point of the hyperplane.

The first condition defines the contact of order *at most* k, while the second condition defines the contact of order *at least* k. Of course, the conditions (160) do not depend on the choice of Euclidean coordinates $(x_1, \ldots, x_n)$.

As an example, we can take a function $b(x) = f(\|x\|)$, for some smooth $f(r)$ satisfying $cr^{k-1} \leq f''(r) \leq Cr^{k-1}$; say, $b(x) = \|x\|^{k+1}$, for odd k. (k could also be any real number greater than or equal to $2n+1$, since in fact we only need $b \in C^{2n+2}(\mathbb{R}^n)$, and also $\psi \in C^{2n+1}(B \times \mathbf{S}^n)$.)

Proposition 2.4.1 *Let $T_\lambda : \mathcal{D}'(\mathbf{S}^n) \to \mathcal{D}'(B)$ be an oscillatory integral operator given by*

$$T_\lambda u(x) = \int_{\mathbf{S}^n} d\mathrm{vol}_\vartheta\, e^{i\lambda\langle \mathbf{r}, \omega\rangle}\, \psi(x, \vartheta)\, u(\vartheta), \qquad \psi \in C^\infty(B \times \mathbf{S}^n),$$

with B a smooth convex hypersurface in $\mathbb{R}^{n+1}$.

Let the Gaussian curvature of B vanish at an isolated point $\mathbf{r}_o$ on the support of ψ. If the tangent plane at $\mathbf{r}_o$ has a precise contact of order $k > 1$ with B, then T_λ acts continuously from $L^2(\mathbf{S}^n)$ to $L^2_{\mathrm{loc}}(B)$, with the norm

$$\|T_\lambda\| \le \mathit{const}\, \lambda^{-\frac{n}{2}+\frac{1}{2}\frac{k}{2k+1}}. \tag{161}$$

Proof. We use the same local coordinates x as in (160), so that the origin $x = 0$ in $\mathbb{R}^n_L$ corresponds to the critical point $\mathbf{r}_o \in B$.

We will need a fine spatial partition of unity, as x approaches the origin. Let us introduce a function $\rho(x) \in C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$, supported at the strip $1/2 \le \|x\| \le 2$, such that

$$\sum_{j\ge 0} \rho(2^j x) \equiv 1 \qquad \text{for} \quad 0 < \|x\| \le 1.$$

We split $\rho(x)$ into a sum of several functions $\rho(x) = \sum \rho_a(x)$, $a = 1 \dots A$, supported in balls of radius $1/10$ centered at some points $p_a \in \operatorname{supp}\rho$, $1/2 \le \|p_a\| \le 2$: $\rho_a \in C^\infty_{\mathrm{comp}}(\mathbb{B}^n_{1/10}(p_a))$. In each ball $\mathbb{B}^n_{1/10}(p_a)$, we have:

$$\|x\| \ge \frac{1}{2} - \frac{1}{10}, \qquad \frac{\inf\ \|x\|}{\sup\ \|x\|} \ge \frac{\frac{1}{2}-\frac{1}{10}}{\frac{1}{2}+\frac{1}{10}} = \frac{2}{3}.$$

Decompose T_λ as follows:

$$T_\lambda = \sum_{\delta\le 1}\sum_{a=1}^{A} \rho_a(x/\delta)\circ T_\lambda, \qquad \delta = 2^{-j}, \quad j \ge 0. \tag{162}$$

We will apply Theorem 1.1.1 to the operators $\rho_a(x/\delta)\circ T_\lambda$.

We need to rewrite the technical assumptions of Theorem 1.1.1 for the phase function (155).

Let us consider a hypersurface B, locally given by the graph of $b(x) \in C^2(\mathbb{R}^n_L)$.

The condition (24) amounts to

$$\sup\ \|\nabla_{\vartheta'} g\| \ \frac{\sup\ \|\nabla_{x'}(\partial_{x_n} b + \nabla_{x'} b\cdot\nabla_{\vartheta'} g)\|}{\inf\ |\partial_{x_n}(\partial_{x_n} b + \nabla_{x'} b\cdot\nabla_{\vartheta'} g)|} \le \frac{1}{2}. \tag{163}$$

Under a very weak assumption $\|\nabla_x b(x)\| \le \|x\|$, the inequality (163) is satisfied if

$$\|x\| \le 1/6, \tag{164}$$

and if

$$\|\vartheta\| \le \frac{1}{6}\frac{\inf \partial^2_{x_n} b}{\sup\ \|b''\|}. \tag{165}$$

Here $\|b''\|$ is the Hilbert-Schmidt norm of the Hessian of b.

The condition (160) gives $\partial^2_{x_n} b \geq c \left\|x\right\|^{k-1}$, $\|b''\| \leq C \left\|x\right\|^{k-1}$, and the restriction (165) on $\|\vartheta\|$ becomes

$$\|\vartheta\| \leq \frac{c \inf \left\|x\right\|^{k-1}}{6C \sup \left\|x\right\|^{k-1}}.$$

Since on the support of each $\rho_a(x/\delta) \circ T_\lambda$ we have $\inf \|x\| / \sup \|x\| \geq 2/3$, it suffices to require that

$$\|\vartheta\| \leq \varepsilon \equiv \frac{c}{6C} \cdot \left(\frac{2}{3}\right)^{k-1}.$$

Note that the value of ε is only finitely small, and we could cover the sphere by finitely many neighborhoods of this size.

As long as x and ϑ are in the limits specified by (164) and (165), we have:

$$|\partial_{x_n} h(x,\vartheta)| \approx \partial^2_{x_n} b(x) \geq const \left\|x\right\|^{k-1}, \tag{166}$$

and since on the support of the integral kernel of the operator $\rho_a(x/\delta) \circ T_\lambda$ we have $\|x\| \sim \delta$, we obtain the following value for the principal curvature of the fold π_R:

$$\sigma(\delta) \sim \delta^{k-1}. \tag{167}$$

Theorem 1.1.1 gives the following bounds on $\|\rho_a(x/\delta) \circ T_\lambda\|$:

$$const\, \lambda^{-\frac{n}{2}+\frac{1}{4}} \delta^{\frac{1}{4}} \qquad \text{and} \qquad const\, \lambda^{-\frac{n}{2}+\frac{1}{6}}\, \sigma(\delta)^{-\frac{1}{6}}. \tag{168}$$

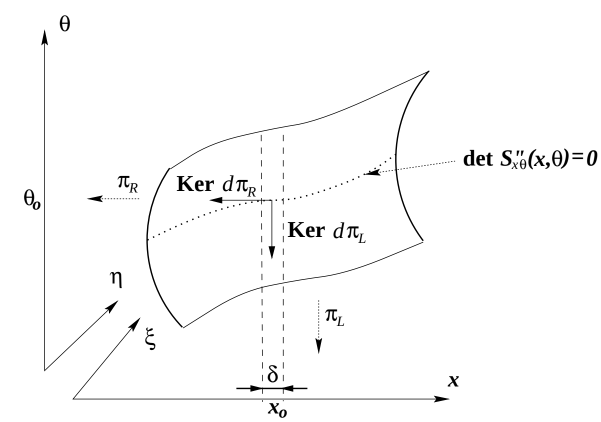

Fig. 4. *The projection π_R fails to be a Whitney fold at (x_o, ϑ_o).*
If we truncate the δ-neighborhood of x_o, then the principal curvature of π_R is bounded from below by some $\sigma = \sigma(\delta)$. We choose δ so that the estimate on the truncated T_λ coincide with the estimate on the localization of T_λ at the δ-neighborhood of x_o.

Now we can evaluate the operator norm of (162). Let us find the value of δ when the two estimates (168) on $\|\rho_a(x/\delta)\circ T_\lambda\|$ clutch together:

$$\lambda^{\frac{1}{4}}\delta^{\frac{1}{4}} = \lambda^{\frac{1}{6}}\delta^{-\frac{k-1}{6}}.$$

This gives $\delta_o = \lambda^{-\frac{1}{2k+1}}$.

Considering the sum (162) and using the first of the bounds (168) when $\delta \le \delta_o$ and the second bound otherwise, we conclude that

$$\|T_\lambda\| \le const\,\lambda^{-\frac{n}{2}+\frac{1}{2}\frac{k}{2k+1}}. \qquad \Box$$

This proves the estimate (161) in Proposition 2.4.1.

Theorem 2.4.1 follows from Proposition 2.4.1 and from the results of Section 1.2.

2.5 Proof for the general case

Now we would like to derive the regularity properties of $\mathcal{R}_{MT}$ in the most general situation.

The properties formulated in Theorem 2.2.1 would follow from the estimate (150) stated in Proposition 2.3.1

We split the operator T_λ in (149) with respect to the values of the determinant of the mixed Hessian,

$$T_\lambda = \sum_{\hbar>\hbar_o} T^\hbar + \bar{T}^{\hbar_o},$$

as in Section 1.1. According to Theorem 1.6.1,

$$\left\|T^\hbar\right\| \le const\,\lambda^{-\frac{n}{2}}\hbar^{-\frac{1}{2}}. \qquad (169)$$

To justify the estimate (150), we are only left to show that the operator $\bar{T}^\hbar$, which is responsible for the neighborhood of the critical variety where $|h(x,\vartheta)| \le 2\hbar$, is bounded by

$$\left\|\bar{T}^\hbar\right\| \le const\,\lambda^{-\frac{n-1}{2}}\hbar^{\frac{k+1}{2k}}. \qquad (170)$$

The estimates (169) and (170) clutch at $\hbar_o = \lambda^{-\frac{k}{2k+1}}$, yielding the estimate (150) on T_λ. Note that $\frac{k}{2k+1} < \frac{1}{2}$, so that the condition (100) of Theorem 1.6.1 is satisfied.

We already know from Remark 3 in Section 1.6 that the spatial decomposition of $\bar{T}^{\hbar_o}$ with respect to ϑ, with the step $\hbar_o$, is almost orthogonal. Therefore, the estimate on $\bar{T}^{\hbar_o}$ is entirely determined by the estimate on operators $\bar{T}^\hbar_\Theta$:

$$\bar{T}^\hbar_\Theta u(x) = \int e^{i\lambda S(x,\vartheta)}\,\psi(x,\vartheta)\,\bar\beta(\hbar^{-1}h(x,\vartheta))\,\chi(\hbar^{-1}\vartheta-\Theta)\,u(\vartheta)\,d\vartheta,$$

$$\bar\beta \in C^\infty_{\mathrm{comp}}([-2,2]), \quad \chi \in C^\infty_{\mathrm{comp}}(\mathbb{B}^n_1). \qquad (171)$$

Proposition 2.5.1 (Individual estimates for $\bar{T}^{\hbar}_{\Theta}$) *If the boundary B admits tangent lines with contact of order not greater than k, with some constant $\kappa > 0$ in* (142), *then*

$$\left\|\bar{T}^{\hbar}_{\Theta}\right\| \leq const\,\lambda^{-\frac{n-1}{2}}\hbar^{\frac{k+1}{2k}}. \tag{172}$$

We fix $\Theta \in \mathbf{Z}^n$ and introduce the notation

$$\tau \equiv \bar{T}^{\hbar}_{\Theta}.$$

Let ω_1 be the point on the sphere $\mathbf{S}^n$ which corresponds to the point $\Theta\hbar$ on $T_{\omega_0}\mathbf{S}^n$. We also fix some point $\mathbf{r}_1 \in B$ such that the normal at $\mathbf{r}_1$ is orthogonal to ω_1. (Then $(\mathbf{r}_1, \omega_1)$ is a point in the critical variety.)

We consider B and $\mathbf{S}^n$ as locally parameterized by Euclidean coordinates x^{new} and ϑ^{new} in $T_{\mathbf{r}_1}B$ and $T_{\omega_1}\mathbf{S}^n$, with the coordinate axes chosen as before: x^{new}_n is in the direction of ω_1, ϑ^{new}_n is in the direction of the inner normal $\mathbf{n}_{\mathbf{r}_1}$, and the directions of the axes $x^{new}_{i'}$ coincide with the directions of $\vartheta^{new}_{i'}$, $i' = 1, \ldots, n-1$.

The cut-off function $\chi(\hbar^{-1}\vartheta^{old} - \Theta)$ is equal to $\chi^{new}(\hbar^{-1}\vartheta^{new})$, with χ^{new} a smooth function supported inside a ball of some larger radius (< 2) and centered in the origin.

Later on, we do not use the superscript *new* for the new local coordinates and for the function χ^{new}.

As in (155), the phase function S in the new coordinates is given by

$$S(x, \vartheta) = \langle \mathbf{r}(x), \omega(\vartheta)\rangle = \vartheta_n b_{\mathbf{r}_1}(x) - x_n g(\vartheta) + x' \cdot \vartheta', \tag{173}$$

modulo some irrelevant terms.

The determinant of the mixed Hessian corresponding to the phase function (173) is given by

$$h(x, \vartheta) = \partial_{x_n} b_{\mathbf{r}_1}(x) - g_{\vartheta_n}(\vartheta) + \nabla_{x'} b_{\mathbf{r}_1}(x) \cdot g_{\vartheta'}(\vartheta). \tag{174}$$

The non-singular part $S_{x'\vartheta'}$ of the mixed Hessian is a unit matrix.

The kernel of the differential of the map $\pi_{R} : (x, \vartheta) \mapsto (\vartheta, S_\vartheta)$ is generated by the vector

$$\mathbf{K}_{R} = g_{\vartheta'} \cdot \nabla_{x'} + \partial_{x_n} \in T_x\mathbb{R}^n_{L} \times T_\vartheta\mathbb{R}^n_{R}, \qquad \mathbf{K}_{R} \in \operatorname{Ker} d\pi_{R} \quad \text{if} \quad (x, \vartheta) \in \Sigma^0.$$

We consider $\mathbf{K}_{R}$ as defined everywhere on the support of the integral kernel of τ.

Note that this vector does not depend on x, while $\|g_{\vartheta'}(\vartheta)\| \approx \|\vartheta'\| \leq 2\hbar$. This implies that on the support of τ, the vector $\mathbf{K}_{R}$ is always inside the conic neighborhood of magnitude $2\hbar$ of the direction ∂_{x_n}. (This was the motivation to change the coordinates.)

Now we introduce the fine localization with respect to x, with the step $\delta = \hbar^{1/k}$:

$$\tau = \sum_{X \in \mathbf{Z}^n} \tau_X, \qquad \tau_X = \chi(\delta^{-1}x - X) \circ \tau. \tag{175}$$

The mixed Hessian of the phase function S is of rank at least $n-1$, while the x-support of τ_X is of size $\hbar^{1/k}$, and ϑ-support is of size $\hbar$. From the easy part of Theorem 1.1.1 (the estimate (20)) it follows that

$$\|\tau_X\| \le const\,\lambda^{-\frac{n-1}{2}}(\hbar^{1/k}\hbar)^{\frac{1}{2}} \le const\,\lambda^{-\frac{n-1}{2}}\,\hbar^{\frac{k+1}{2k}}, \tag{176}$$

in an agreement with (172).

Before checking the almost orthogonality relations for the operators τ_X, let us substitute for $\hbar$ the particular value $\hbar_o = \lambda^{-\frac{k}{2k+1}}$ when the estimates (169) and (170) meet.

We claim that τ_X are almost orthogonal for different $X \in \mathbf{Z}^n$:

$$\|\tau_X^*\tau_Y\| \le const\,\lambda^{-n+1}\hbar_o^{\frac{k+1}{k}}\,a(X-Y), \qquad \sum_{X\in\mathbf{Z}^n}|a(X)|^{\mu} \le const\,, \tag{177}$$

$$\|\tau_X\tau_Y^*\| \le const\,\lambda^{-n+1}\hbar_o^{\frac{k+1}{k}}\,b(X-Y), \qquad \sum_{X\in\mathbf{Z}^n}|b(X)|^{\nu} \le const\,, \tag{178}$$

for some non-negative μ, ν, such that $\mu+\nu=1$. We will take $\mu=0$, $\nu=1$.

The inequalities (176)-(178) would prove Proposition 2.5.1.

The condition (177) is straightforward, since if $\|X-Y\|$ is large enough (greater than $2\sqrt{n}$), then

$$\tau_X^*\circ\tau_Y \equiv \tau^*\circ\,\chi(\delta^{-1}x-X)\,\chi(\delta^{-1}x-Y)\circ\tau \equiv 0.$$

We now need to justify (178).

The integral kernel of $\tau_X\tau_Y^*$ is given by

$$K(\tau_X\tau_Y^*)(x,y) = \int d\vartheta\, e^{i\lambda(S(x,\vartheta)-S(y,\vartheta))}\times\dots\,. \tag{179}$$

We consider two cases.

• When the vector $v = y-x \in \mathbb{R}^n_L$ is inside the conic neighborhood of magnitude $3\hbar_o$ of the directions $\pm\partial_{x_n}$, we start with

$$|h(y,\vartheta)-h(x,\vartheta)| \ge \left|h(x+\|v\|\,\tilde{\mathbf{K}}_R,\vartheta)-h(x,\vartheta)\right|$$
$$-\left|h(y,\vartheta)-h(x+\|v\|\,\tilde{\mathbf{K}}_R,\vartheta)\right|, \tag{180}$$

here $\tilde{\mathbf{K}}_R = \mathbf{K}_R/\,\|\mathbf{K}_R\|$.

Since

$$h(x,\vartheta) = \partial_{x_n}b_{\mathbf{r}_1}(x) - g_{\vartheta_n}(\vartheta) + \nabla_{x'}b_{\mathbf{r}_1}(x)\cdot g_{\vartheta'}(\vartheta)$$
$$= \|\mathbf{K}_R\|\,\tilde{\mathbf{K}}_R b_{\mathbf{r}_1}(x) - \partial_{\vartheta_n}g(\vartheta), \tag{181}$$

the first term in the right-hand side of (180) has a very special form:

$$\|\mathbf{K}_R\| \left|\tilde{\mathbf{K}}_R b_{\mathbf{r}_1}(x + \|v\| \tilde{\mathbf{K}}_R) - \tilde{\mathbf{K}}_R b_{\mathbf{r}_1}(x)\right|. \tag{182}$$

According to the assumption that B admits tangent lines with contact of order not greater than k and according to Lemma 2.2.2 (see the inequality (147)), the expression (182) is not less than $2\kappa \|v\|^k$.

From the other hand, since v is within $3\hbar_o$-cone of ∂_{x_n}, while $\tilde{\mathbf{K}}_R = \mathbf{K}_R / \|\mathbf{K}_R\|$ is within $2\hbar_o$-cone of ∂_{x_n}, we conclude that the angle between $\tilde{\mathbf{K}}_R$ and v is at most $5\hbar_o$. This results in

$$\left\|v - \tilde{\mathbf{K}}_R \|v\|\right\| \le 5\hbar_o \|v\|.$$

Since the left-hand side is the distance from $x + \|v\| \tilde{\mathbf{K}}_R$ to y, the second term in the right-hand side of (180) is bounded by

$$5\hbar_o \|v\| \|\nabla_x h\|.$$

Thus, (180) is bounded from below by

$$|h(y, \vartheta) - h(x, \vartheta)| \ge 2\kappa \|y - x\|^k - 5\hbar_o \|y - x\| \|\nabla_x h\|.$$

This expression is greater than $4\hbar_o$ whenever $\|Y - X\|$ is large enough (since $\|y - x\|$ is approximately equal to $\hbar_o^{1/k} \|Y - X\|$).

• When the vector $v = y - x$ is outside the conic neighborhood of magnitude $3\hbar_o$ of the directions $\pm\partial_{x_n}$, we have

$$\|v'\| \ge 3\hbar_o |v_n|. \tag{183}$$

Here $v' \in \mathbb{R}^n_L$ is a vector obtained from v by taking 0 for its nth component.

We employ the operator L_ϑ,

$$L_\vartheta = -\frac{1}{i\lambda} \cdot \frac{(S_\vartheta(y, \vartheta) - S_\vartheta(x, \vartheta)) \cdot \nabla_\vartheta}{\|S_\vartheta(y, \vartheta) - S_\vartheta(x, \vartheta)\|^2}. \tag{184}$$

We need to find the bound from below for the difference $S_\vartheta(y, \vartheta) - S_\vartheta(x, \vartheta)$, which is in the denominator of L_ϑ. Note that now we are right on the critical variety.

We have:

$$\begin{aligned}\|S_\vartheta(y, \vartheta) - S_\vartheta(x, \vartheta)\| &\ge \left\|\int_0^1 \frac{d}{dt} S_{\vartheta'}(x + vt, \vartheta)\, dt\right\| \\ &\ge \left|\int_0^1 \nabla_{x'} S_{\vartheta'}(x + vt, \vartheta) \cdot v'\, dt\right| - \int_0^1 \|\partial_{x_n} S_{\vartheta'}(x + vt, \vartheta)\| \, |v_n| \, dt \\ &\ge \|v'\| - 2\hbar_o |v_n|, \end{aligned} \tag{185}$$

since $S_{x'\vartheta'}$ is a unit matrix, and since

$$\|\partial_{x_n} S_{\vartheta'}\| = \|g_{\vartheta'}(\vartheta)\| \approx \|\vartheta'\| \le 2\hbar_o.$$

Using (183), $\|v'\| \ge 3\hbar_o\,|v_n|$, we derive the bound

$$\|S_\vartheta(y,\vartheta) - S_\vartheta(x,\vartheta)\| \ge \frac{1}{3}\,\|v'\| \ge \frac{1}{4}\,(\|v'\| + \hbar_o\,|v_n|)$$
$$\approx \frac{\delta}{4}\,(\|Y' - X'\| + \hbar_o\,|Y_n - X_n|)\,. \tag{186}$$

Now we integrate in (179) by parts, with the aid of the operator L_ϑ defined in (184). Each derivative ∇_ϑ contributes $\hbar_o^{-1}$, including the case when the derivative acts on the denominator of L_ϑ itself. We gain the factor

$$\frac{const}{\lambda\hbar_o\delta\,(\|Y' - X'\| + \hbar_o\,|Y_n - X_n|)}.$$

We repeat this procedure $N = N_1 + N_2$ times, obtaining the factor bounded by

$$const\,\frac{1}{(\lambda\hbar_o\delta\,\|Y' - X'\|)^{N_1}}\cdot\frac{1}{\left(\lambda\hbar_o^2\delta\,|Y_n - X_n|\right)^{N_2}}.$$

Since $\lambda\hbar_o\delta = \lambda^{\frac{k}{2k+1}}$, we will comply with (178) if we take N_1 sufficiently large and $N_2 \ge 2$.

This proves the required almost orthogonality relations (178) for the operators τ_X with different indices $X \in \mathbf{Z}^n$, and concludes the proof of Proposition 2.5.1. □

2.6 Regularity properties of the operator $\tilde F$

According to the results of Section 1.2, the regularity properties of the operator $\tilde F : \mathcal{E}'(\mathrm{I\!R}\times\mathbf{S}^n) \to \mathcal{D}'(\mathrm{I\!R}\times B)$ with the integral kernel (140),

$$k_{\tilde F}(\mathbf{r},\omega,t-s) = (\mathbf{n_r}\cdot\omega)\,\delta(t - s - \mathbf{r}\cdot\omega), \tag{187}$$

depend on the rate of decay of the oscillatory integral operator

$$U_\lambda u(x) = \int\limits_{\mathrm{I\!R}^n} e^{i\lambda S(x,\vartheta)}\,\langle \mathbf{n}_{\mathbf{r}(x)},\omega(\vartheta)\rangle\,\psi(x,\vartheta)\,u(\vartheta)\,d\vartheta, \quad \psi \in C^\infty_{\rm comp}(\mathrm{I\!R}^n\times\mathrm{I\!R}^n), \tag{188}$$

with the phase function as in (149), $S(x,\vartheta) = \langle \mathbf{r}(x),\omega(\vartheta)\rangle$, $x \in B$, $\vartheta \in \mathbf{S}^n$.

The key difference of this operator from the operator T_λ in (149) is the factor $\langle \mathbf{n}_{\mathbf{r}(x)},\omega(\vartheta)\rangle$ in the integral kernel. According to (152),

$$\langle \mathbf{n}_{\mathbf{r}(x)},\omega(\vartheta)\rangle \sim h(x,\vartheta) = \det S_{x\vartheta}.$$

Therefore, we apply Theorem 1.6.2 and conclude that

$$\|U_\lambda\| \leq const\, \lambda^{-\frac{n}{2}}. \tag{189}$$

Note that this estimate does not depend on the order of contact k.

Together with the results of Section 1.2, the estimate (189) yields the regularity properties of $\tilde{F}$ stated in Theorem 2.2.2:

$$H^s(\mathbb{R} \times \mathbf{S}^n) \xrightarrow{\tilde{F}} H^{s+\frac{n}{2}}(\mathbb{R} \times B).$$

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