

# OSCILLATORY INTEGRAL OPERATORS IN SCATTERING THEORY

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ABSTRACT. We consider a particular Fourier integral operator with folding canonical relations, which arises in scattering theory: the Radon Transform of Melrose and Taylor. We obtain the regularity properties of this operator when the obstacle admits tangent planes with contact of precise order  $k$  (Theorem 1.1 and its Corollary).

For these purposes, we derive asymptotic estimates for oscillatory integral operators in  $\mathbb{R}^n$  with folding canonical relations (Theorem 2.2). Asymptotics correspond to vanishing principal curvature of a fold of one of the projections from the canonical relation, and to small support of the localization of oscillatory integral operator.

## 1. INTRODUCTION AND RESULTS

From the fundamental paper of R.B. Melrose and M.E. Taylor [1985] we know that important characteristics of scattering, such as corrections to the Kirchhoff approximation and the behavior of the scattering amplitude for small angles of scattering, depend on the properties of the following Radon

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Transform:

$$\mathcal{R} : \mathcal{E}'(\mathbb{R} \times \mathbb{S}^n) \rightarrow \mathcal{D}'(\mathbb{R} \times B),$$

$$(1.1) \quad \mathcal{R}u(t, \mathbf{r}) = \int_{\mathbb{R} \times \mathbb{S}^n} \delta(t - s - \langle \mathbf{r}, \omega \rangle) u(s, \omega) ds d\Omega.$$

Here  $\mathbf{r}$  is a vector in  $\mathbb{R}^{n+1}$ , pointing to the boundary  $B$  of a scatterer  $K \subset \mathbb{R}^{n+1}$ . For the scattering in three dimensions we put  $n = 2$ . We always assume that  $K$  is a compact region with the boundary being a smooth strictly convex hypersurface. A unit vector  $\omega \in \mathbb{S}^n$  corresponds to a direction of light rays. The variables  $t$  and  $s$  refer to the moments of time.

We will call (1.1) *the Radon Transform of Melrose and Taylor*.

We would like to know the regularity properties of  $\mathcal{R}$ :

$$(1.2) \quad \mathcal{R} : H^s(\mathbb{R} \times \mathbb{S}^n) \rightarrow H^{s+\frac{n}{2}-\mu}(\mathbb{R} \times B).$$

If  $\mathcal{R}$  were a non-degenerate Fourier integral operator, the value of  $\mu$  in (1.2) would be 0. Instead, according to R.B. Melrose and M.E. Taylor,  $\mathcal{R}$  is a Fourier integral operator with degenerate canonical relation. Their argument in [1985] works for the case when the hypersurface  $B$  is strictly convex and its Gaussian curvature is strictly positive. In this case, the operator  $\mathcal{R}$  can be reduced to the normal form. Then one can show that there is a loss of  $\mu = 1/6$  derivatives in (1.2).

The analogous result for Fourier integral operators with folding canonical relations is proved by L. Hörmander [1985]. His proof was also based on the reduction to the normal form.

Due to the results of A. Greenleaf and A. Seeger [1994], there is a loss of at most  $\mu = 1/4$  derivatives in (1.2).

What is the optimal value of  $\mu$  if the curvature vanishes of some finite order? In this paper, we will develop a technique which will allow us to answer this question in a particular situation. Exactly, we will be able to consider the case when  $B$  admits a tangent plane with the contact of *precise* order  $k$  (in the sense specified after the theorem and its corollary):

**Theorem 1.1.** *Let  $B$  be a smooth hypersurface in  $\mathbb{R}^{n+1}$ , and let its Gaussian curvature vanish at an isolated point  $\mathbf{r}_o$ . If the tangent plane at  $\mathbf{r}_o$  has a contact of precise order  $k > 1$  with  $B$ , then the Radon Transform  $\mathcal{R}$  localized near  $\mathbf{r}_o$  extends to a continuous operator*

$$H^s(\mathbb{R}) \otimes L^2(\mathbb{S}^n) \xrightarrow{\mathcal{R}} H^{s+\frac{n}{2}-\frac{1}{2}-\frac{k}{2k+1}}(\mathbb{R}) \otimes L^2_{\text{loc}}(B), \quad \text{for any real } s.$$

Due to the particular form of the Radon Transform (1.1), the smoothing with respect to *time*-variables implies the same smoothing effect with respect to other variables. This results in the following continuity of  $\mathcal{R}$  in the Sobolev spaces:

**Corollary 1.2.** *Let  $K$  be a convex compact domain in  $\mathbb{R}^{n+1}$  with a smooth boundary  $B = \partial K$ . Let the Gaussian curvature of  $B$  vanishes only at isolated points. If at these points the boundary  $B$  admits tangent planes with contact of precise order not greater than  $k$ , then the operator  $\mathcal{R}$  extends to a continuous operator*

$$H^s(\mathbb{R} \times \mathbb{S}^n) \xrightarrow{\mathcal{R}} H^{s+\frac{n}{2}-\frac{1}{2}\frac{k}{2k+1}}(\mathbb{R} \times B), \quad \text{for any real } s.$$

*Contact of precise order.* Let the hypersurface  $B$  in  $\mathbb{R}^{n+1}$  be locally given by the graph  $x_{n+1} = b(x_1, \dots, x_n)$  of some smooth function  $b \in C^\infty(\mathbb{R}^n)$ , defined in the neighborhood of the origin in  $\mathbb{R}^n$ . We assume that  $b(0) = 0$ ,  $\nabla_x b(0) = 0$ .

**Definition 1.3** (*Contact of precise order  $k$* ).

We say that the hyperplane  $\{x_{n+1} = 0\}$  has a contact of precise order  $k$  with the graph  $\{x_{n+1} = b(x)\}$  of the function  $b \in C^\infty(\mathbb{R}^n)$ , if  $b(0) = 0$ ,  $\nabla_x b(0) = 0$ , and if the Hessian  $[\partial_{x_i} \partial_{x_j} b]$  is isotropic and vanishes uniformly of order  $k - 1$ . That is, for any unit vectors  $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ ,

$$(1.3) \quad u^i u^j \frac{\partial^2 b(x)}{\partial x_i \partial x_j} \geq c \|x\|^{k-1}, \quad \left| u^i v^j \frac{\partial^2 b(x)}{\partial x_i \partial x_j} \right| \leq C \|x\|^{k-1},$$

$$\text{where} \quad \|x\|^2 \equiv x_1^2 + \dots + x_n^2, \quad 0 < c < C < \infty.$$

The first condition defines the contact of order *at most*  $k$ , while the second condition defines the contact of order *at least*  $k$ . Of course, the conditions (1.3) do not depend on the choice of Euclidean coordinates  $(x_1, \dots, x_n)$ .

As an example, we can take a function  $b(x) = f(\|x\|)$ , where  $f \in C^\infty(\mathbb{R})$  is such that  $cr^{k-1} \leq f''(r) \leq Cr^{k-1}$ ; say,  $b(x) = \|x\|^{k+1}$ , for odd  $k$ .

Let us specify the required smoothness of  $b(x)$ . In the argument we use, we only need  $b \in C^{2n+2}(\mathbb{R}^n)$ . Hence, in the above example,  $k$  could also be any real number greater than or equal to  $2n + 1$ .

Notice that if the Gaussian curvature of  $B$  does not vanish, then  $k = 1$ , and formally the theorem gives the loss of  $\mu = \frac{1}{2} \frac{k}{2k+1} = 1/6$  derivatives. If, instead,  $k$  becomes large, then  $\mu$  approaches  $1/4$ . This gives an interpolation between the results of Melrose – Taylor and Greenleaf – Seeger.

*Relation to oscillatory integral operators.* The Radon Transform (1.1) can be written as a Fourier integral operator:

$$(1.4) \quad \mathcal{R}u(t, x) = \int_{\mathbb{R} \times \mathbb{R} \times \mathbb{S}^n} e^{-i\lambda(t-s-S(x, \vartheta))} \psi(x, \vartheta) u(s, \vartheta) \frac{d\lambda}{2\pi} ds d^n \vartheta,$$

here  $x$  and  $\vartheta$  are local coordinates on  $B$  and on  $\mathbb{S}^n$ , and the phase shift

$$(1.5) \quad S(x, \vartheta) = \langle \mathbf{r}(x), \omega(\vartheta) \rangle$$

is a dot product of  $n + 1$ -dimensional vectors pointing to  $x \in B$  and  $\vartheta \in \mathbb{S}^n$ . We introduced some smooth cut-off function  $\psi(x, \vartheta)$ , which localizes the integral operator to the region where the local coordinates  $x$  and  $\vartheta$  are defined.

As it was pointed out by R.B. Melrose and M.E. Taylor, the canonical relation associated with (1.4) has folding singularities. We will analyze these singularities in more detail after reducing the problem to oscillatory integral operators.

Following the paper of D.H. Phong and E.M. Stein [1991], we rewrite (1.4) as a pseudo-differential operator in one dimension, with the “operator-valued symbol”:

$$\mathcal{R}u(t, x) = \int_{\mathbb{R}} \frac{d\lambda}{2\pi} e^{-i\lambda t} \int_{\mathbb{R}^n} d^n \vartheta e^{i\lambda S(x, \vartheta)} \psi(x, \vartheta) \int_{\mathbb{R}} ds e^{i\lambda s} u(s, \vartheta).$$

The symbol  $a(\lambda)$  of this pseudo-differential operator acts on the Fourier transform of  $u(s, \vartheta)$  (with respect to  $s$ ) as

$$(1.6) \quad a(\lambda) \hat{u}(\lambda) = \int_{\mathbb{R}^n} d^n \vartheta e^{i\lambda S(x, \vartheta)} \psi(x, \vartheta) \hat{u}(\lambda, \vartheta).$$

The standard methods of the theory of  $\Psi$ DO show that the smoothing effect of the operator  $\mathcal{R}$  (with respect to the *time*-variables) coincides with the rate of decay of the operator norm of  $a(\lambda)$ . Precisely,

$$(1.7) \quad \text{if} \quad \|a(\lambda)\|_{L^2_{\text{comp}}(\mathbb{R}^n) \rightarrow L^2_{\text{loc}}(\mathbb{R}^n)} \leq \text{const } \lambda^{-\frac{n}{2} + \mu},$$

$$\text{then} \quad H^s(\mathbb{R}) \otimes L^2_{\text{comp}}(\mathbb{R}^n) \xrightarrow{\mathcal{R}} H^{s+\frac{n}{2}-\mu}(\mathbb{R}) \otimes L^2_{\text{loc}}(\mathbb{R}^n).$$

Now we proceed to studying the estimates for oscillatory integral operators, like (1.6).

In Section 2, we discuss oscillatory integral operators with degenerate canonical relations and formulate our main technical result (Theorem 2.2). The idea of the proof is in Section 3, and the details are in Sections 4 and 5.

We approach the particular oscillatory integral operator with the phase function (1.5) in Section 6.

## 2. OSCILLATORY INTEGRAL OPERATORS WITH FOLDING CANONICAL RELATIONS

In this section, we formulate the asymptotic estimates for oscillatory integral operators with folding canonical relations.

We study oscillatory integral operators  $T_\lambda$ , given by

$$(2.1) \quad T_\lambda : u(\vartheta) \mapsto T_\lambda u(x) = \int_{\mathbb{R}_R^n} e^{i\lambda S(x, \vartheta)} \psi(x, \vartheta) u(\vartheta) d\vartheta, \\ x \in \mathbb{R}_L^n, \quad \vartheta \in \mathbb{R}_R^n,$$

where  $\psi$  is a smooth compactly supported function. We want to know the rate of high-frequency decay of the operator norm  $\|T_\lambda\|_{L^2 \rightarrow L^2}$  for large  $\lambda$ . The simplest example is the Fourier transform:  $S(x, \vartheta) = x \cdot \vartheta$ . Then this is easy to show that  $\|T_\lambda\| \leq \text{const } \lambda^{-\frac{n}{2}}$ .

The classical result by L. Hörmander [1971] is the estimate for the case when the mixed Hessian  $H = S_{x\vartheta} = [\partial_{x_i} \partial_{\vartheta_j} S]$ ,  $i, j = 1, \dots, n$ , is non-degenerate:  $\det H \neq 0$ . Then  $T_\lambda$  is bounded from  $L^2$  to  $L^2$ , with the norm

$$(2.2) \quad \|T_\lambda\|_{L^2 \rightarrow L^2} \leq \text{const } \lambda^{-\frac{n}{2}}.$$

Let us consider the diagram

$$(2.3) \quad \begin{array}{ccc} \mathbb{R}_L^n \times \mathbb{R}_R^n & \xrightarrow{\iota} & \mathcal{C} \hookrightarrow T^*\mathbb{R}_L^n \times T^*\mathbb{R}_R^n \\ & & \begin{array}{ccc} \pi_L \swarrow & & \searrow \pi_R \\ T^*\mathbb{R}_L^n & & T^*\mathbb{R}_R^n \end{array} \end{array}$$

where  $\iota : (x, \vartheta) \mapsto (x, S_x) \times (\vartheta, -S_\vartheta)$ . The image  $\mathcal{C} = \iota(\mathbb{R}_L^n \times \mathbb{R}_R^n)$  is the canonical relation associated with (2.1). The maps  $\pi_L$  and  $\pi_R$  are the projections from  $\mathcal{C}$  onto the first and second factors of  $T^*\mathbb{R}_L^n \times T^*\mathbb{R}_R^n$ .

The maps  $\hat{\pi}_L = \pi_L \circ \iota$  and  $\hat{\pi}_R = \pi_R \circ \iota$  from  $\mathbb{R}_L^n \times \mathbb{R}_R^n$  fail to be diffeomorphisms on the critical variety

$$(2.4) \quad \mathcal{L} \equiv \{ (x, \vartheta) \mid \det H(x, \vartheta) = 0 \},$$

where we thus have  $\text{Ker } d\hat{\pi}_L \neq 0$  and  $\text{Ker } d\hat{\pi}_R \neq 0$ .

This all means that the estimate (2.2) is valid when both projections in (2.3) are local diffeomorphisms.

We say that  $\hat{\pi}_L$  is a fold if the kernel of  $d\hat{\pi}_L$  is one-dimensional and transversal to the critical variety  $\mathcal{L}$ . Moreover,  $\hat{\pi}_L$  is a Whitney fold if the determinant of the mixed Hessian vanishes only of order one on  $\mathcal{L}$ :

$$(2.5) \quad d_{\vartheta} \det H|_{\mathcal{L}} \neq 0.$$

From now on, we always assume that  $\hat{\pi}_L$  is a Whitney fold.

Since  $\dim \text{Ker } d\hat{\pi}_L = 1$ , the rank of the mixed Hessian  $H$  on the critical variety  $\mathcal{L}$  is equal to  $n - 1$ . Therefore, the dimension of  $\text{Ker } d\hat{\pi}_R$  is also 1. Still, the projection  $\hat{\pi}_R$  is not necessarily a fold; in general, the differential  $d_x \det H$  can vanish at some points of the critical variety  $\mathcal{L}$ .

According to R.B. Melrose and M.E. Taylor (see also Y.B. Pan and C.D. Sogge [1990] and S. Cuccagna [1995]), if both projections from the associated canonical relation  $\mathcal{C}$  are at most Whitney folds, then the decay of the norm of  $T_\lambda$  is given by

$$(2.6) \quad \|T_\lambda\|_{L^2 \rightarrow L^2} \leq \text{const } \lambda^{-\frac{n}{2} + \frac{1}{8}}.$$

The estimate due to A. Greenleaf and A. Seeger [1994] holds when at least one of the projections is a Whitney fold (we always assume it is  $\pi_L$ ); then the norm of  $T_\lambda$  decays as

$$(2.7) \quad \|T_\lambda\|_{L^2 \rightarrow L^2} \leq \text{const } \lambda^{-\frac{n}{2} + \frac{1}{4}}.$$

Let us see what canonical relation corresponds to the phase function  $S(x, \vartheta)$  defined in (1.5). The cotangent space  $T_{\mathbf{r}}^*B$  can be identified (via the standard scalar product) with the tangent plane at the same point  $\mathbf{r}$ . This shows that the differential  $d_x \langle \mathbf{r}(x), \omega(\vartheta) \rangle$  is represented by the orthogonal projection of  $\omega(\vartheta)$  onto  $T_{\mathbf{r}(x)}B$ . Therefore, the singular part of the map

$$\hat{\pi}_L : (x, \vartheta) \mapsto (x, d_x \langle \mathbf{r}(x), \omega(\vartheta) \rangle)$$

corresponds to the orthogonal projection from  $\mathbb{S}^n$  onto the hyperplane  $T_{\mathbf{r}}B$ . This projection has a Whitney fold at the points  $\omega$  such that  $\langle \mathbf{n}_{\mathbf{r}}, \omega \rangle = 0$ , here  $\mathbf{n}_{\mathbf{r}}$  is the unit normal to  $B$  at the point  $\mathbf{r}$ . We conclude that both  $d\hat{\pi}_L$  and  $d\hat{\pi}_R$  are degenerate on the critical variety

$$(2.8) \quad \mathcal{L} = \{(x, \vartheta) \mid \langle \mathbf{n}_{\mathbf{r}(x)}, \omega(\vartheta) \rangle = 0\}.$$

Intuitively, if  $T_{\mathbf{r}}B$  and  $T_{\omega}\mathbb{S}^n$  are not orthogonal ( $\langle \mathbf{n}_{\mathbf{r}}, \omega \rangle \neq 0$ ), then, after proper localization, the oscillatory integral operator is basically the Fourier transform and has no critical points.

Let us consider the other projection from  $\mathcal{C}$ . The singular part of the map

$$\hat{\pi}_R : x, \vartheta \mapsto \vartheta, -d_{\vartheta}\langle \mathbf{r}(x), \omega(\vartheta) \rangle$$

corresponds to the orthogonal projection from  $B$  onto  $T_{\omega(\vartheta)}\mathbb{S}^n$ . As long as  $B$  is strictly convex,  $\hat{\pi}_R$  is a fold. It is also a Whitney fold at the points  $\mathbf{r} \in B$  where the principal curvature in the direction  $\omega$  is different from zero.

All this suggests that we consider an asymptotic situation, when  $\pi_R$  is a Whitney fold, but its principal curvature is asymptotically small. Let us explain what we call a principal curvature of a Whitney fold: locally, a singular part of a Whitney fold corresponds to the projection from the parabola  $y = \frac{1}{2}\sigma x^2$  onto the  $y$ -axis; then, roughly,  $\sigma$  is the principal curvature of this fold.

Let us try to translate this into analytical language, for the operator (2.1) with a general phase function  $S(x, \vartheta)$ . According to (2.5), we will assume that near the critical variety  $\mathcal{L} = \{\det H = 0\}$  we have the bounds  $\|\nabla_{\vartheta} \det H\| \geq \text{const}$  and  $\|\nabla_x \det H\| \geq \sigma$ . If  $\sigma$  becomes small, the behavior of the projection  $\hat{\pi}_R$  becomes more degenerate, and we expect that there is a certain asymptotic behavior for  $\|T_{\lambda}\|$ , corresponding to small values of  $\sigma$ . This behavior turns out to be

$$\|T_{\lambda}\| \sim \lambda^{-\frac{n}{2} + \frac{1}{6}} \sigma^{-\frac{1}{6}}.$$

In Theorem 2.2, we will derive this asymptote together with asymptotics corresponding to small support of the localization of  $T_{\lambda}$  near some point  $(x_o, \vartheta_o)$ :

$$T_{\lambda\delta\varepsilon} \equiv \rho((x - x_o)/\delta) \circ T_{\lambda} \circ \varrho((\vartheta - \vartheta_o)/\varepsilon).$$

*Pseudoconvex maps.* We need to require that  $\pi_R$  in (2.3) be convex, in a certain sense. First, for an  $N \times N$ -matrix  $A$ , we define the bound from below for its action:

$$(2.9) \quad \min(A) \equiv \inf_{\|\mathbf{u}\|=1} \|\mathbf{A}\mathbf{u}\|.$$

If  $A$  is non-degenerate, then  $\min(A) = \|A^{-1}\|^{-1}$ .

There is the following bound for  $\min(A)$ :

$$\min(A) \geq m(A) \equiv \frac{|\det A|}{\|A\|_{\text{End}(\mathbb{R}^N)}^{N-1}}.$$

To see this, it suffices to treat  $A$  as a linear endomorphism of  $\mathbb{C}^N$ , so that  $\det A$  would be a product of the eigenvalues.

**Definition 2.1** (*c-pseudoconvexity*).

Let  $U$  and  $V$  be two regions in  $\mathbb{R}^N$ . Given  $c > 0$ , the map  $\Phi : U \rightarrow V$  is called *c-pseudoconvex* on  $U$ , if on any connected subset  $\mathcal{O} \subset U$  it satisfies the condition

$$(2.10) \quad \|\Phi(y) - \Phi(x)\| \geq c \|y - x\| \inf_{z \in \mathcal{O}} \min(\nabla\Phi(z)).$$

Here  $\nabla\Phi(x)$  is the Jacoby matrix corresponding to the change of variables  $x \mapsto \Phi(x)$ . The value of  $\min(*)$  at a given point is defined in (2.9).

Note that any function  $f \in C^1(\mathbb{R})$  is then pseudoconvex with  $c = 1$ .

The typical example when (2.10) does not hold is a map  $\Phi$  which is a local diffeomorphism but not a bijection. Still, if  $U$  is compact and  $\min(\nabla\Phi) \geq \text{const}$ , then for any  $c < 1$  there is a finite partition  $U = \cup U_j$ , such that  $\Phi$  is *c-pseudoconvex* on each  $U_j$ .

Condition (2.10) is non-trivial if  $\nabla\Phi$  is degenerate at some points of  $U$ .

*Asymptotic estimates for oscillatory integral operators.* Now we formulate our main result about oscillatory integral operators. We consider an operator (2.1) localized near some point  $(x_o, \vartheta_o)$ :

$$(2.11) \quad T_{\lambda\delta\varepsilon}u(x) = \int_{\mathbb{R}^n} \rho((x - x_o)/\delta) \varrho((\vartheta - \vartheta_o)/\varepsilon) e^{i\lambda S(x, \vartheta)} \psi(x, \vartheta) u(\vartheta) d\vartheta,$$

here  $\rho(x)$  and  $\varrho(\vartheta)$  are smooth functions supported in a unit ball centered in the origin;  $\delta$  and  $\varepsilon$  are two small parameters (less than 1). The integral kernel of  $T_{\lambda\delta\varepsilon}$  is thus supported in  $\mathbb{B}_\delta^n(x_o) \times \mathbb{B}_\varepsilon^n(\vartheta_o)$ , where  $\mathbb{B}_r^n(p)$  is a ball in  $\mathbb{R}^n$  of radius  $r$ , centered at  $p$ . For simplicity, we assume that  $(x_o, \vartheta_o)$  is shifted to the origin in  $\mathbb{R}_L^n \times \mathbb{R}_R^n$ . We will use the notation  $D_{\delta\varepsilon} \equiv \mathbb{B}_\delta^n(0) \times \mathbb{B}_\varepsilon^n(0)$ .

In the Theorem we want to formulate, both  $\hat{\pi}_L$  and  $\hat{\pi}_R$  are assumed to be Whitney folds. The local coordinates  $x = (x', x_n)$  and  $\vartheta = (\vartheta', \vartheta_n)$  are chosen so that the vectors  $\partial_{\vartheta_n}$  and  $\partial_{x_n}$  are transversal to  $\mathcal{L} = \{(x, \vartheta) \mid \det H = 0\}$ , while  $H' = \nabla_{x'} \nabla_{\vartheta'} S$  is a non-degenerate matrix.

**Theorem 2.2** (*Asymptotic estimates for oscillatory integral operators*).

Let  $S(x, \vartheta) \in C^{2n+3}(D_{\delta\varepsilon})$  and  $\psi(x, \vartheta) \in C^{2n+1}(D_{\delta\varepsilon})$ .

Let the mixed Hessian  $H = S_{x\vartheta}$  be of rank at least  $n - 1$ , with its non-degenerate part given by  $H' = \nabla_{x'} \nabla_{\vartheta'} S$ ,  $|\det H'| \geq \text{const} > 0$ .

If  $\det H$  vanishes of the first order in  $x_n$  and  $\vartheta_n$ , so that

$$(2.12) \quad |\partial_{\vartheta_n} \det H| \geq \text{const} > 0 \quad \text{and} \quad \sigma \equiv \inf_{D_{\delta\varepsilon}} |\partial_{x_n} \det H| > 0,$$



the map  $\hat{\pi}_R|_{\vartheta} : x \mapsto S_{\vartheta}(x, \vartheta)$  satisfies the pseudoconvexity condition (2.10) with some  $c \geq \text{const} > 0$ , and if the following inequalities hold:

$$(2.13) \quad \sup_{D_{\delta\varepsilon}} \left( \|H'^{-1}\| \|\nabla_{x'} \partial_{\vartheta_n} S\| \right) \frac{\sup_{D_{\delta\varepsilon}} \|\nabla_{\vartheta'} \det H\|}{\inf_{D_{\delta\varepsilon}} |\partial_{\vartheta_n} \det H|} \leq \frac{1}{2},$$

$$(2.14) \quad \sup_{D_{\delta\varepsilon}} \left( \|H'^{-1}\| \|\nabla_{\vartheta'} \partial_{x_n} S\| \right) \frac{\sup_{D_{\delta\varepsilon}} \|\nabla_{x'} \det H\|}{\inf_{D_{\delta\varepsilon}} |\partial_{x_n} \det H|} \leq \frac{1}{2},$$

then there are the following estimates on the norm of  $T_{\lambda\delta\varepsilon}$  in (2.11):

$$(2.15) \quad \|T_{\lambda\delta\varepsilon}\| \leq \text{const} \lambda^{-\frac{n-1}{2}} \delta^{\frac{1}{2}} \varepsilon^{\frac{1}{2}},$$

$$(2.16) \quad \|T_{\lambda\delta\varepsilon}\| \leq \text{const} \lambda^{-\frac{n}{2} + \frac{1}{4}} \delta^{\frac{1}{4}},$$

$$(2.17) \quad \|T_{\lambda\delta\varepsilon}\| \leq \text{const} \lambda^{-\frac{n}{2} + \frac{1}{6}} \sigma^{-\frac{1}{6}},$$

$$(2.18) \quad \|T_{\lambda\delta\varepsilon}\| \leq \text{const} \lambda^{-\frac{n}{2} + \frac{1}{4}} \sigma^{-\frac{1}{4}} \varepsilon^{\frac{1}{4}}.$$

The constants in (2.15)-(2.18) do not depend on  $\lambda$ ,  $\delta$ ,  $\varepsilon$ , and  $\sigma$ .

Let us say a couple of words about the conditions entering Theorem 2.2, and about the estimates stated there.

The conditions (2.12) simply say that  $\hat{\pi}_L$  is a Whitney fold and that  $\hat{\pi}_R$  is a degenerating Whitney fold (if  $\sigma$  is asymptotically small).

The left-hand sides of both (2.13) and (2.14) are zeroes if we assume that  $S(x, \vartheta)$  splits into purely non-degenerate and degenerate parts,  $S(x, \vartheta) = S_1(x', \vartheta') + S_2(x_n, \vartheta_n)$ . The problem then reduces to one-dimensional oscillatory integral operators with folding canonical relations, when everything is much simpler. The geometrical meaning of the condition (2.13) is that the kernel of  $d\hat{\pi}_L$  is within a certain cone about the direction of  $\nabla_{\vartheta'} \det H$ . The condition (2.14) requires the same of  $\text{Ker } \hat{\pi}_R$  and  $\nabla_x \det H$ . We will need these conditions when proving the almost orthogonality relations (Section 5).

The requirement that the map  $x \mapsto S_{\vartheta}$  is convex is in fact guaranteed by (2.14); intuitively, “any fold is convex”. Not to overload the paper, we only give a simple geometric argument which proves the pseudoconvexity of the map  $\hat{\pi}_R|_{\vartheta}$  for the phase function (1.5) (see Section 6).

The estimate (2.15) on  $T_{\lambda\delta\varepsilon}$  is straightforward. The factor  $\lambda^{-\frac{n-1}{2}}$  is L. Hörmander’s estimate (2.2) in the dimension  $n - 1$ , since  $H$  has the rank not less than  $n - 1$ . Such estimates are considered by Y.B. Pan [1991]. The factor  $\delta^{\frac{1}{2}} \varepsilon^{\frac{1}{2}}$  appears due to the size of the support of the integral kernel of  $T_{\lambda\delta\varepsilon}$  in

the variables  $x_n$  and  $\vartheta_n$ . No assumptions on the behavior of  $\hat{\pi}_L$  and  $\hat{\pi}_R$  are needed.

The estimate (2.16) holds when  $\hat{\pi}_L$  is a Whitney fold, and actually no assumption on  $\hat{\pi}_R$  is needed. This is a counterpart of the estimate (2.7) due to A. Greenleaf and A. Seeger.

Other estimates on the norm of  $T_{\lambda\delta\varepsilon}$  hold when  $\hat{\pi}_R$  is also a Whitney fold. The estimate (2.17) is a counterpart of (2.6), and it gives an asymptote corresponding to a small principal curvature of the Whitney fold  $\hat{\pi}_R$ . The last estimate (2.18) is weak and “does not have a classical analog”. Roughly, this estimate follows from (2.16), since the size of  $x$ -support in the critical direction is controlled by the ratio  $\varepsilon/\sigma$ .

### 3. IDEA OF PROOF OF THEOREM 2.2

We follow the papers of D.H. Phong and E.M. Stein [1991], [1994], and D.H. Phong [1994], where the estimates for oscillatory integral operators in one dimension were considered.

We use the dyadic decomposition with respect to a distance from the critical variety  $\{\det H = 0\}$ . Take a function  $\bar{\beta} \in C^\infty(\mathbb{R})$ , such that  $\bar{\beta}(t) = 1$  for  $|t| \leq 1$ ,  $\bar{\beta}(t) = 0$  for  $|t| \geq 2$ . We define  $\beta(t) \in C_0^\infty([1/2, 2])$  by  $\beta(t) = \bar{\beta}(t) - \bar{\beta}(2t)$  for  $t \geq 0$ ,  $\beta(t) = 0$  for  $t \leq 0$ . We also define  $\beta_\pm(t) = \beta(\pm t)$ . Then we have the following partition of 1:

$$1 = \bar{\beta}(2^{N_0}t) + \sum_{\pm} \sum_{N < N_0} \beta_\pm(2^N t), \quad t \in \mathbb{R}.$$

We will apply this partition to the operator  $T_{\lambda\delta\varepsilon}$ , decomposing it with respect to  $\det H(x, \vartheta)$  (“the distance from the critical variety”). We define:

$$(3.1) \quad \bar{T}^{\hbar} u(x) = \int \rho(x/\delta) \varrho(\vartheta/\varepsilon) e^{i\lambda S(x, \vartheta)} \psi(x, \vartheta) \bar{\beta}(\hbar^{-1} \det H) u(\vartheta) d\vartheta,$$

$$(3.2) \quad T_{\pm}^{\hbar} u(x) = \int \rho(x/\delta) \varrho(\vartheta/\varepsilon) e^{i\lambda S(x, \vartheta)} \psi(x, \vartheta) \beta_{\pm}(\hbar^{-1} \det H) u(\vartheta) d\vartheta,$$

here  $\hbar = 2^{-N}$ ,  $N \in \mathbb{Z}$ .

We rewrite the operator (2.11) as a finite sum of operators (3.1) and (3.2),

$$(3.3) \quad T_{\lambda\delta\varepsilon} = \bar{T}^{\hbar_0} + \sum_{\pm} \sum_{\substack{\hbar \leq \Lambda \\ \hbar > \hbar_0}} T_{\pm}^{\hbar}, \quad \hbar = 2^{-N}, \quad N \in \mathbb{Z},$$

where  $\hbar_0 = 2^{-N_0}$  is chosen up to our convenience;  $\Lambda = 2^{N_1} \geq \sup |\det H|$ .

The crucial property of  $T_{\pm}^{\hbar}$  is that on the support of its integral kernel the determinant of the mixed Hessian is non-degenerate:  $|\det H| \sim \hbar$  (precisely,  $|\det H| \geq \hbar/2$ , in accordance with the Uncertainty Principle). Then we can try to use a variant of L. Hörmander's estimate (2.2). Of course, this estimate becomes worse for smaller values of  $\hbar$  (see (3.14) below).

Later we will not write  $\pm$ -indexes for the operators  $T_{\pm}^{\hbar}$ .

The operator  $\bar{T}^{\hbar_0}$  is responsible for a tiny neighborhood of the critical variety, of size  $\hbar_0$ . We expect that its norm decreases as we take smaller values of  $\hbar_0$  (see (3.13) below).

We will obtain the estimates on  $\|\bar{T}^{\hbar}\|$  and  $\|T^{\hbar}\|$  using a spatial localization and then the almost orthogonality argument. We follow the preprint of S. Cuccagna [1995], who used a similar localization in his analytic proof of the  $L^2$ -estimates on the oscillatory integral operators with Whitney folds. The idea of almost orthogonality argument (*the Cotlar-Stein Lemma*) can be read from the book of E.M. Stein [1993].

We need a spatial partition of 1:

$$(3.4) \quad 1 = \sum_{n \in \mathbb{Z}} \chi(t - n), \quad t \in \mathbb{R}, \quad \text{supp } \chi \subset [-1, 1].$$

We take two sets of integers,  $X = \{X', X_n\} = \{X_i\} \in \mathbb{Z}^n$  and  $\Theta \in \mathbb{Z}^n$ , and localize the integral kernels of  $\bar{T}^{\hbar}$  and  $T^{\hbar}$ , multiplying them by

$$\chi(\Sigma \hbar^{-1} x' - X') \equiv \prod_{i'=1}^{n-1} \chi(\Sigma \hbar^{-1} x_{i'} - X_{i'}), \quad \chi(\sigma \hbar^{-1} x_n - X_n),$$

and

$$\chi(\hbar^{-1} \vartheta - \Theta) \equiv \prod_{i=1}^n \chi(\hbar^{-1} \vartheta_i - \Theta_i),$$

where

$$(3.5) \quad \sigma \equiv \inf_{D_{\delta\varepsilon}} |\partial_{x_n} \det H|, \quad \Sigma \equiv \sup_{D_{\delta\varepsilon}} \|\nabla_x \det H\|, \quad \Sigma \geq \sigma > 0.$$

We decompose the operators (3.1) and (3.2):

$$(3.6) \quad \bar{T}^{\hbar} = \sum_{X \in \mathbb{Z}^n, \Theta \in \mathbb{Z}^n} \bar{T}_{X\Theta}^{\hbar}, \quad T^{\hbar} = \sum_{X \in \mathbb{Z}^n, \Theta \in \mathbb{Z}^n} T_{X\Theta}^{\hbar},$$

where  $T_{X\Theta}^{\hbar}$  has the following integral kernel:

$$(3.7) \quad \begin{aligned} K(T_{X\Theta}^{\hbar})(x, \vartheta) &= \rho(x/\delta) \varrho(\vartheta/\varepsilon) e^{i\lambda S(x, \vartheta)} \psi(x, \vartheta) \beta(\hbar^{-1} \det H(x, \vartheta)) \\ &\times \chi(\Sigma \hbar^{-1} x' - X') \chi(\sigma \hbar^{-1} x_n - X_n) \chi(\hbar^{-1} \vartheta - \Theta). \end{aligned}$$

The integral kernel of  $\bar{T}_{x\Theta}^{\hbar}$  is given by the same expression, with the function  $\bar{\beta}$  instead of  $\beta$ .

Referring to sizes of  $x$ - and  $\vartheta$ - support of the integral kernels of  $T_{x\Theta}^{\hbar}$  and  $\bar{T}_{x\Theta}^{\hbar}$ , we will use the notations

$$(3.8) \quad \Delta_{\sigma} \equiv \min\left\{\delta, \frac{\hbar}{\sigma}\right\}, \quad \Delta_{\Sigma} \equiv \min\left\{\delta, \frac{\hbar}{\Sigma}\right\} \leq \Delta_{\sigma}, \quad E \equiv \min\{\varepsilon, \hbar\}.$$

The estimates on the norms of  $\bar{T}^{\hbar}$  and  $T^{\hbar}$  are due to the following lemma:

**Lemma 3.1.** *There are the following estimates on  $\bar{T}_{x\Theta}^{\hbar}$  and  $T_{x\Theta}^{\hbar}$ :*

$$(3.9) \quad \|\bar{T}_{x\Theta}^{\hbar}\|^2 \leq \text{const } \lambda^{-n+1} \Delta_{\sigma} E,$$

$$(3.10) \quad \|T_{x\Theta}^{\hbar}\|^2 \leq \text{const } \lambda^{-n} \hbar^{-1}.$$

As long as  $\hbar$  is large enough, so that the right-hand side of (3.10) is not larger than that of (3.9),

$$(3.11) \quad \hbar^{-1} \leq \lambda \Delta_{\sigma} E,$$

the operators  $\bar{T}_{x\Theta}^{\hbar}$  and  $T_{x\Theta}^{\hbar}$  satisfy the almost orthogonality relations.

Precisely,  $\|\bar{T}_{x\Theta}^{\hbar} \bar{T}_{yW}^{\hbar*}\|$ ,  $\|\bar{T}_{x\Theta}^{\hbar*} \bar{T}_{yW}^{\hbar}\|$ ,  $\|T_{x\Theta}^{\hbar} T_{yW}^{\hbar*}\|$ , and  $\|T_{x\Theta}^{\hbar*} T_{yW}^{\hbar}\|$  are all bounded by

$$(3.12) \quad \text{const } \lambda^{-n} \hbar^{-1} \gamma^2(Y - X, W - \Theta).$$

Here  $\gamma$  is a function on  $\mathbb{Z}^n \times \mathbb{Z}^n$  such that  $\sum_{X \in \mathbb{Z}^n, \Theta \in \mathbb{Z}^n} |\gamma(X, \Theta)| \leq \text{const}$ .

We will prove this lemma in Section 4 (the estimates (3.9) and (3.10)) and in Section 5 (almost orthogonality relations).

The application of the Cotlar-Stein Lemma proves the following corollary:

**Corollary 3.2.** *There are the following estimates on  $\bar{T}^{\hbar}$  and  $T_{\pm}^{\hbar}$ :*

$$(3.13) \quad \|\bar{T}^{\hbar}\| \leq \text{const } \lambda^{-\frac{n-1}{2}} \left( \min\left\{\delta, \frac{\hbar}{\sigma}\right\} \min\{\varepsilon, \hbar\} \right)^{\frac{1}{2}},$$

$$(3.14) \quad \|T_{\pm}^{\hbar}\| \leq \text{const } \lambda^{-\frac{n}{2}} \hbar^{-\frac{1}{2}}.$$

We take  $\hbar_0$  in (3.3) such that the bounds (3.13) and (3.14) coincide:

$$(3.15) \quad \hbar_0^{-1} = \lambda \min\left\{\delta, \frac{\hbar_0}{\sigma}\right\} \min\{\varepsilon, \hbar_0\} \equiv \lambda \Delta_{\sigma} E.$$

Then the estimate  $\|T_{\lambda\delta\varepsilon}\| \leq \text{const } \lambda^{-\frac{n}{2}} \hbar_0^{-\frac{1}{2}}$  gives all the estimates (2.15)-(2.18) stated in Theorem 2.2. ( $\square$ )

For example, to obtain the estimate (2.17), we take  $\hbar_0^{-1} = \lambda \cdot \frac{\hbar_0}{\sigma} \cdot \hbar_0$ .

#### 4. PROOF OF LEMMA 3.1: INDIVIDUAL ESTIMATES

We are going to derive the individual estimates (3.9) and (3.10) stated in Lemma 3.1.

In the argument, we usually omit the function  $\psi$ , and occasionally other functions when they are irrelevant for the argument. We keep in mind that their derivatives arise each time when we integrate by parts, and therefore the resulting estimates depend on the bounds on these functions and on their derivatives (up to some finite order).

*Estimate*  $\|\bar{T}_{x\Theta}^{\hbar}\|^2 \leq \text{const } \lambda^{-n+1} \Delta_\sigma E$ .

For the convenience, we will write  $\bar{\tau}$  for the operator  $\bar{T}_{x\Theta}^{\hbar}$ :

$$\begin{aligned} \bar{\tau}u(x) &= \int \chi(\Sigma\hbar^{-1}x' - X') \chi(\sigma\hbar^{-1}x_n - X_n) \rho(x/\delta) \varrho(\vartheta/\varepsilon) \\ &\quad \times e^{i\lambda S(x,\vartheta)} \psi(x, \vartheta) \bar{\beta}(\hbar^{-1} \det H) \chi(\hbar^{-1}\vartheta - \Theta) u(\vartheta) d\vartheta. \end{aligned}$$

First of all, we note that for the operator  $\bar{\tau}_{x_n\vartheta_n} : \mathcal{E}'(\mathbb{R}^{n-1}) \rightarrow \mathcal{D}'(\mathbb{R}^{n-1})$  with the integral kernel given by  $K(\bar{\tau})(x, \vartheta)$  with  $x_n, \vartheta_n$  fixed, there is the estimate

$$(4.1) \quad \|\bar{\tau}_{x_n\vartheta_n}\|_{L^2(\mathbb{R}^{n-1}) \rightarrow L^2(\mathbb{R}^{n-1})} \leq \text{const } \lambda^{-\frac{n-1}{2}},$$

since  $|\det(\nabla_{x'} \nabla_{\vartheta'} S)| = |\det H'| \geq \text{const} > 0$ , and the estimate (2.2) for oscillatory integral operators with non-degenerate canonical relations applies. This estimate is uniform in  $x_n$  and  $\vartheta_n$ . It is also uniform in  $\delta, \varepsilon$ , and  $\hbar$ .

The size of the integral kernel of  $\bar{\tau}$  in the variable  $\vartheta_n$  is  $\min\{\varepsilon, \hbar\} \equiv E$ . Therefore,

$$|\bar{\tau}u(x)|^2 = \left| \int d\vartheta_n \bar{\tau}_{x_n\vartheta_n} [u(*, \vartheta_n)](x') \right|^2 \leq E \int d\vartheta_n |\bar{\tau}_{x_n\vartheta_n} [u(*, \vartheta_n)](x')|^2,$$

here we have written  $\bar{\tau}_{x_n\vartheta_n} [u(*, \vartheta_n)](x') = \int K(\bar{\tau})(x, \vartheta) u(\vartheta', \vartheta_n) d^{n-1}\vartheta'$ .

Now let us integrate  $|\bar{\tau}u(x)|^2$  with respect to  $x$ , recalling that the size of the support of the integral kernel of  $\bar{\tau}$  in  $x_n$  is  $\min\{\delta, \frac{\hbar}{\sigma}\} = \Delta_\sigma$ :

$$\begin{aligned} \int dx |\bar{\tau}u(x)|^2 &\leq \Delta_\sigma \int dx' E \int d\vartheta_n |\bar{\tau}_{x_n\vartheta_n} [u(*, \vartheta_n)](x')|^2 \\ &\leq \int dx |\bar{\tau}u(x)|^2 \leq \text{const } \lambda^{-(n-1)} \Delta_\sigma E \int d^n\vartheta |u(\vartheta)|^2. \end{aligned}$$

We conclude that  $\|\bar{\tau}\| \leq \text{const } \lambda^{-\frac{n-1}{2}} \Delta_\sigma^{\frac{1}{2}} E^{\frac{1}{2}}$ . This proves the estimate (3.9) of Lemma 3.1.

This estimate is analogous to (2.15), since the support of the operator  $\bar{\tau}$  in the variables  $x_n$  and  $\vartheta_n$  is bounded by  $\Delta_\sigma$  and  $E$ , respectively.

*Estimate*  $\|T_{x\Theta}^{\hbar}\|^2 \leq \text{const } \lambda^{-n} \hbar^{-1}$ . For this estimate, it is crucial that the integral kernel of  $T_{x\Theta}^{\hbar}$  is localized away from the critical variety, so that  $|\det H| \geq \hbar/2$ .

We will write  $\tau$  for  $T_{x\Theta}^{\hbar}$ :

$$\begin{aligned} \tau u(x) = & \int \chi(\Sigma \hbar^{-1} x' - X') \chi(\sigma \hbar^{-1} x_n - X_n) \rho(x/\delta) \varrho(\vartheta/\varepsilon) \\ & \times e^{i\lambda S(x, \vartheta)} \psi(x, \vartheta) \beta(\hbar^{-1} \det H) \chi(\hbar^{-1} \vartheta - \Theta) u(\vartheta) d\vartheta. \end{aligned}$$

Let us consider the integral kernel of  $\tau\tau^*$ :

$$(4.2) \quad K(\tau\tau^*)(x, y) = \int d^n \vartheta e^{i\lambda(S(x, \vartheta) - S(y, \vartheta))} \times \dots$$

We insert into (4.2) the operator

$$L_\vartheta = \frac{1}{i\lambda} \cdot \frac{(S_\vartheta(x, \vartheta) - S_\vartheta(y, \vartheta)) \cdot \nabla_\vartheta}{\|S_\vartheta(x, \vartheta) - S_\vartheta(y, \vartheta)\|^2},$$

which is the identity when acting on the exponential, and integrate by parts. When acting on cut-offs,  $\nabla_\vartheta$  gives a factor bounded by  $\max\{\varepsilon^{-1}, \hbar^{-1}\} \equiv E^{-1}$ . The derivative  $\nabla_\vartheta$  can act on the denominator of  $L_\vartheta$  itself, also giving a factor bounded by  $\text{const } \hbar^{-1}$ :

$$\frac{\|S_{x\vartheta}(x, \vartheta) - S_{x\vartheta}(y, \vartheta)\|}{\|S_\vartheta(x, \vartheta) - S_\vartheta(y, \vartheta)\|} \leq \text{const} \frac{\|x - y\|}{\hbar \|x - y\|} \leq \text{const } \hbar^{-1}.$$

The bound  $\|S_\vartheta(x, \vartheta) - S_\vartheta(y, \vartheta)\| \geq \text{const } \hbar \|x - y\|$  which we have used is guaranteed by the assumption that the map  $\hat{\pi}_R|_\vartheta$  is pseudoconvex. (If the support of  $\beta(\hbar^{-1} \det H)$  is not simply connected, we need a separate treatment for each simply connected part.)

Integration by parts  $n+1$  times is thus equivalent to adding the following factor to the integral kernel (4.2) of  $\tau\tau^*$ :

$$(4.3) \quad \text{const} \frac{1}{1 + |\lambda E \|S_\vartheta(x, \vartheta) - S_\vartheta(y, \vartheta)\||^{n+1}}.$$

To apply the Schur lemma, we need to integrate the absolute value of (4.2), with the extra factor (4.3), with respect to  $x$ . To do so, we integrate

with respect to  $\eta = S_\vartheta(x, \vartheta)$  and divide by the determinant of the Jacoby matrix. The Jacoby matrix is given by

$$J = \frac{\partial(\eta_i)}{\partial(x_j)} = \frac{\partial(\partial_{\vartheta_i} S(x, \vartheta))}{\partial(x_j)} = H(x, \vartheta),$$

and therefore  $|\det J| = |\det H| \geq \frac{\hbar}{2}$ .

The integration of (4.3) gives  $\text{const} \frac{1}{|\det J|} \cdot \frac{1}{\lambda^n E^n} \leq \text{const} \frac{1}{\hbar} \cdot \frac{1}{\lambda^n E^n}$ .

The factor  $E^{-n}$  goes away after (postponed) integration with respect to  $\vartheta$  in (4.2), since the size of  $\vartheta_i$ -support of the integral kernel of  $\tau$  is bounded by  $E \equiv \min\{\varepsilon, \hbar\}$ . Therefore,  $L^1$ -norm of  $K(\tau\tau^*)(x, y)$  with respect to  $x$  or  $y$  is bounded by  $\text{const} \lambda^{-n} \hbar^{-1}$ .

This results in the estimate  $\|\tau\|^2 = \|\tau\tau^*\| \leq \text{const} \lambda^{-n} \hbar^{-1}$ , proving (3.10).

## 5. PROOF OF LEMMA 3.1: ALMOST ORTHOGONALITY RELATIONS

Let us check the almost orthogonality relations for operators localized near different lattice points in  $\mathbb{R}^n \times \mathbb{R}^n$ . We write  $T^\hbar$  for both  $\bar{T}^\hbar$  and  $T^\hbar$  and do not distinguish them, since the argument stays the same. (We will not use the key feature of  $T^\hbar$  that on its support  $|\det H| \geq \hbar/2$ .)

*Almost orthogonality relations:*  $\|T_{X\Theta}^{\hbar*} T_{YW}^{\hbar}\|$ .

The integral kernel  $K(T_{X\Theta}^{\hbar*} T_{YW}^{\hbar})(\vartheta, w)$  of  $T_{X\Theta}^{\hbar*} T_{YW}^{\hbar}$  is given by

$$(5.1) \quad \begin{aligned} & \int d^n x \rho^2(x/\delta) \varrho(\vartheta/\varepsilon) \varrho(w/\varepsilon) e^{-i\lambda(S(x, \vartheta) - S(x, w))} \\ & \times \beta(\hbar^{-1} \det H(x, \vartheta)) \beta(\hbar^{-1} \det H(x, w)) \\ & \times \chi(\Sigma \hbar^{-1} x' - X') \chi(\sigma \hbar^{-1} x_n - X_n) \chi(\Sigma \hbar^{-1} x' - Y') \chi(\sigma \hbar^{-1} x_n - Y_n) \\ & \times \chi(\hbar^{-1} \vartheta - \Theta) \chi(\hbar^{-1} w - W). \end{aligned}$$

If  $\|Y - X\| \geq 2\sqrt{n}$ , then (5.1) has empty  $x$ -support, and  $T_{X\Theta}^{\hbar*} T_{YW}^{\hbar} \equiv 0$ .

Let us consider the case when  $|W_n - \Theta_n|$  dominates. We will show that for sufficiently large values of  $|W_n - \Theta_n|$

$$|\det H(x, w) - \det H(x, \vartheta)| \geq 4\hbar,$$

then  $\beta(\hbar^{-1} \det H(x, \vartheta)) \beta(\hbar^{-1} \det H(x, w)) = 0$ , and also  $\bar{T}_{X\Theta}^{\hbar*} \bar{T}_{YW}^{\hbar} = 0$ .

Let us put  $v = w - \vartheta$ . For the difference  $|\det H(x, w) - \det H(x, \vartheta)|$  we have:

$$(5.2) \quad \int_0^1 dt \frac{d}{dt} \det H(x, \vartheta + tv) \geq \int_0^1 dt (|v_n| |\partial_{\vartheta_n} \det H| - \|v'\| \|\nabla_{\vartheta'} \det H\|).$$

Note that the arc given by  $(x, \vartheta + t(w - \vartheta))$ ,  $0 \leq t \leq 1$ , is inside the convex region  $D_{\delta\varepsilon}$ .

On the support of (5.1) the following inequalities are satisfied:

$$\|w' - \vartheta'\| \leq \hbar \|W' - \Theta'\| + 2\hbar\sqrt{n-1}, \quad |w_n - \vartheta_n| \geq \hbar |W_n - \Theta_n| - 2\hbar.$$

Therefore, the expression under the integral sign in the right-hand side of (5.2) is greater than  $4\hbar$ , if  $|W_n - \Theta_n|$  is greater than some constant and if

$$(5.3) \quad \alpha\hbar |W_n - \Theta_n| \geq \hbar \|W' - \Theta'\| \sup_{D_{\delta\varepsilon}} \frac{\|\nabla_{\vartheta'} \det H\|}{|\partial_{\vartheta_n} \det H|}, \quad \alpha = 2^{-\frac{1}{4}}.$$

This proves the almost orthogonality when  $|W_n - \Theta_n|$  is relatively large. If (5.3) does not hold, then

$$(5.4) \quad \hbar \|W' - \Theta'\| \sup_{D_{\delta\varepsilon}} \frac{\|\nabla_{\vartheta'} \det H\|}{|\partial_{\vartheta_n} \det H|} > \alpha\hbar |W_n - \Theta_n|,$$

and due to the condition (2.13),

$$(2.13') \quad \frac{1}{2} \geq \sup_{D_{\delta\varepsilon}} \left( \|H'^{-1}\| \|\nabla_{x'} \partial_{\vartheta_n} S\| \right) \frac{\sup_{D_{\delta\varepsilon}} \|\nabla_{\vartheta'} \det H\|}{\inf_{D_{\delta\varepsilon}} |\partial_{\vartheta_n} \det H|},$$

we have:

$$\frac{1}{2} \hbar \|W' - \Theta'\| \geq \alpha\hbar |W_n - \Theta_n| \sup_{D_{\delta\varepsilon}} \left( \|H'^{-1}\|^{-1} \|\nabla_{x'} \partial_{\vartheta_n} S\| \right).$$

For large enough  $\|W - \Theta\|$ , we can rewrite this inequality as

$$(5.5) \quad \frac{1}{2} \|w' - \vartheta'\| \geq \alpha^2 |w_n - \vartheta_n| \sup_{D_{\delta\varepsilon}} \left( \|H'^{-1}\|^{-1} \|\nabla_{x'} \partial_{\vartheta_n} S\| \right).$$

We have spared one power of  $\alpha$  for the errors in the relations

$$\|w' - \vartheta'\| \approx \hbar \|W' - \Theta'\|, \quad |w_n - \vartheta_n| \approx \hbar |W_n - \Theta_n|.$$

Now we employ the operator  $L_x$ , given by

$$L_x = \frac{1}{i\lambda} \cdot \frac{(S_x(x, w) - S_x(x, \vartheta)) \cdot \nabla_x}{\|S_x(x, w) - S_x(x, \vartheta)\|^2}.$$



We will show that for relatively large values of  $\|W' - \Theta'\|$  the denominator of  $L_x$  is not less than  $const \|w' - \vartheta'\|$  (squared). It suffices to consider the first  $n - 1$  components of the vector  $S_x(x, \vartheta) - S_x(x, w)$ .

$$(5.6) \quad \begin{aligned} \|\nabla_{x'} S(x, w) - \nabla_{x'} S(x, \vartheta)\| &= \left\| \int_0^1 dt \frac{d}{dt} \nabla_{x'} S(x, \vartheta + vt) \right\| \\ &= \left\| \int_0^1 \nabla_{x'} \nabla_{\vartheta'} S(x, \vartheta + vt) \cdot v' + \nabla_{x'} \partial_{\vartheta_n} S(x, \vartheta + vt) v_n dt \right\|, \end{aligned}$$

here  $v = w - \vartheta$ . Let us mention that the integral in the right-hand side is entirely in  $D_{\delta\varepsilon}$ .

We concentrate our attention on the first term in the integral in the right-hand side. The vector  $v'$  does not depend on  $t$ , while the matrix  $\nabla_{x'} \nabla_{\vartheta'} S(x, \vartheta + vt) \equiv H'_t$  is a continuous function of  $t$ . The magnitude of the vector  $H'_t v'$  is not less than  $\|v'\| \inf_{D_{\delta\varepsilon}} \min(H')$ . We can provide (possibly, splitting the support of  $\psi(x, \vartheta)$  into several smaller pieces) that everywhere in  $D_{\delta\varepsilon}$  the vector  $H'_t v'$  remains in a small cone of magnitude  $\phi$ , such that  $\cos \phi \geq \alpha \equiv 2^{-1/4}$ . Then the right-hand side of (5.6) is bounded from below by

$$\begin{aligned} &\int_0^1 \alpha \|v'\| \min(H'_t) - |v_n| \|\nabla_{x'} \partial_{\vartheta_n} S(x, \vartheta + vt)\| dt \\ &\geq \int_0^1 \alpha(1 - \alpha) \|v'\| \min(H'_t) dt \\ &+ \int_0^1 (\alpha^2 \|v'\| \min(H'_t) - |v_n| \|\nabla_{x'} \partial_{\vartheta_n} S(x, \vartheta + vt)\|) dt. \end{aligned}$$

This gives the required bound from below for (5.6), since the first integral in the right-hand side is greater than  $const \|v'\|$ , and the second integral is non-negative due to (5.5). When using (5.5), we need to recall that  $\min(H'(x, \vartheta))$  defined in (2.9) is equal to  $\|H'(x, \vartheta)^{-1}\|^{-1}$ .

We insert  $L_x^{2n+1}$  into the integral kernel of  $T_{X\Theta}^{\hbar*} T_{YW}^{\hbar}$  and integrate by parts. Each derivative  $\nabla_x$  gives at most  $\max\{\delta^{-1}, \sigma\hbar^{-1}\} = \Delta_{\Sigma}^{-1}$ . This adds the following factor to the integral kernel of  $T_{X\Theta}^{\hbar*} T_{YW}^{\hbar}$ :

$$const \frac{1}{|\lambda \Delta_{\Sigma} \|w' - \vartheta'\|^{2n+1}}.$$

Note that when the derivative  $\nabla_x$  acts on the denominator of  $L_x$ , the appearing factor is bounded by

$$\frac{const \|w - \vartheta\|}{\|S_x(x, \vartheta) - S_x(x, w)\|} \leq \frac{const \hbar \|W - \Theta\|}{\hbar \|W' - \Theta'\|} \leq const,$$

since the relation (5.4) implies that  $\|W' - \Theta'\| \geq \text{const} \|W - \Theta\|$ .

We have for the  $L^1$ -norm of (5.1) with respect to  $\vartheta$  (or  $w$ ):

$$(5.7) \quad \int d\vartheta |K(T_{X\Theta}^{\hbar*} T_{YW}^{\hbar})(\vartheta, w)| \leq \int_{\text{supp } T_{X\Theta}^{\hbar} T_{YW}^{\hbar*}} dx d\vartheta \frac{\text{const}}{|\lambda \Delta_{\Sigma} \|w' - \vartheta'\|^{2n+1}}$$

$$\leq \text{const} \Delta_{\Sigma}^{n-1} \Delta_{\sigma} E^n (\lambda \Delta_{\Sigma} \hbar \|W - \Theta\|)^{-n-1}.$$

We used the relation  $\|w' - \vartheta'\| \approx \hbar \|W' - \Theta'\| \geq \text{const} \hbar \|W - \Theta\|$ , which follows from (5.4).

Since  $E \leq \hbar$ , we may rewrite the factor before  $\|W - \Theta\|^{-2n-1}$  as

$$\frac{\Delta_{\Sigma}^{n-1} \Delta_{\sigma} E^n}{(\lambda \Delta_{\Sigma} \hbar)^{2n+1}} \leq \text{const} \frac{1}{\lambda^n} \frac{\Delta_{\sigma}}{\Delta_{\Sigma}} \frac{1}{(\lambda \Delta_{\Sigma} \hbar)^{n+1}} \leq \text{const} \lambda^{-n} \hbar^{-1}.$$

Here we used the inequality  $\Delta_{\Sigma} \geq \text{const} \min\{\delta, \frac{\hbar}{\Sigma}\} \geq \text{const} \delta \hbar \geq \text{const} \Delta_{\sigma} \hbar$ . Note that, according to (3.8) and (3.11),  $\lambda \Delta_{\Sigma} \hbar \geq \lambda \Delta_{\sigma} \hbar^2 \geq \lambda \Delta_{\sigma} E \hbar \geq 1$ .

Therefore, (5.7) is bounded by  $\text{const} \lambda^{-n} \hbar^{-1} \|W - \Theta\|^{-2n-1}$ . This proves the estimate (3.12) for the operators  $\bar{T}_{X\Theta}^{\hbar*} \bar{T}_{YW}^{\hbar}$  and  $T_{X\Theta}^{\hbar*} T_{YW}^{\hbar}$  when  $\|W' - \Theta'\|$  dominates.

*Almost orthogonality relations:*  $\|T_{X\Theta}^{\hbar} T_{YW}^{\hbar*}\|$ . All the ideas are the same. We consider the integral kernel  $K(T_{X\Theta}^{\hbar} T_{YW}^{\hbar*})(x, y)$  of the operator  $T_{X\Theta}^{\hbar} T_{YW}^{\hbar*}$ :

$$(5.8) \quad \int d^n \vartheta \rho(x/\delta) \rho(y/\delta) \varrho^2(\vartheta/\varepsilon) e^{i\lambda(S(x,\vartheta) - S(y,\vartheta))}$$

$$\times \beta(\hbar^{-1} \det H(x, \vartheta)) \beta(\hbar^{-1} \det H(y, \vartheta))$$

$$\times \chi(\Sigma \hbar^{-1} x' - X') \chi(\Sigma \hbar^{-1} y' - Y') \chi(\sigma \hbar^{-1} x_n - X_n) \chi(\sigma \hbar^{-1} y_n - Y_n)$$

$$\times \chi(\hbar^{-1} \vartheta - \Theta) \chi(\hbar^{-1} \vartheta - W).$$

If  $\|W - \Theta\| > 2\sqrt{n}$ , then (5.8) has empty  $\vartheta$ -support, and  $T_{X\Theta}^{\hbar} T_{YW}^{\hbar*} \equiv 0$ .

If  $|X_n - Y_n|$  is large, we want to find the condition for the difference

$$(5.9) \quad |\det H(y, \vartheta) - \det H(x, \vartheta)|$$

to be greater than  $4\hbar$ ; then  $\beta(\hbar^{-1} \det H(x, \vartheta))$  and  $\beta(\hbar^{-1} \det H(y, \vartheta))$  have no common support, and (5.8) is identically zero.

We rewrite (5.9) as

$$\int_0^1 dt \frac{d}{dt} \det H(x + tv, \vartheta) \geq \int_0^1 dt (|v_n| |\partial_{x_n} \det H| - \|v'\| \|\nabla_{x'} \det H\|).$$

On the support of (5.8), we can write:

$$|v_n| \geq \sigma^{-1} \hbar (|Y_n - X_n| - 2), \quad \|v'\| \leq \Sigma^{-1} \hbar (\|Y' - X'\| + 2\sqrt{n-1}).$$

Since we defined  $\sigma \equiv \inf_{D_{\delta\varepsilon}} |\partial_{x_n} \det H|$ ,  $\Sigma \equiv \sup_{D_{\delta\varepsilon}} \|\nabla_x \det H\|$ , the expression (5.9) is greater than  $4\hbar$  if

$$(5.10) \quad \alpha |X_n - Y_n| \geq \|X' - Y'\|.$$

This is all for the case when  $|Y_n - X_n|$  is relatively large.

If (5.10) does not hold, then we have:

$$(5.11) \quad \|X' - Y'\| \geq \alpha |X_n - Y_n|,$$

and hence

$$(5.12) \quad \|X - Y\| \leq (1 + \alpha^{-1}) \|X' - Y'\|.$$

We multiply (5.11) by (2.14),

$$(2.14') \quad \frac{1}{2} \geq \sup_{D_{\delta\varepsilon}} \left( \|H'^{-1}\| \|\nabla_{\vartheta'} \partial_{x_n} S\| \right) \frac{\sup_{D_{\delta\varepsilon}} \|\nabla_{x'} \det H\|}{\inf_{D_{\delta\varepsilon}} |\partial_{x_n} \det H|},$$

and obtain the relation

$$\frac{1}{2} \Sigma^{-1} \hbar \|Y' - X'\| \geq \alpha \sigma^{-1} \hbar |Y_n - X_n| \sup_{D_{\delta\varepsilon}} \left( \|H'^{-1}\| \|\nabla_{\vartheta'} \partial_{x_n} S\| \right).$$

We recall that on the support of (5.8)

$$\|v'\| \approx \Sigma^{-1} \hbar \|Y' - X'\|, \quad |v_n| \approx \sigma^{-1} \hbar |Y_n - X_n|,$$

and for sufficiently large  $\|Y - X\|$

$$(5.13) \quad \frac{1}{2} \|v'\| \geq \alpha^2 |v_n| \sup_{D_{\delta\varepsilon}} \left( \|H'^{-1}\| \|\nabla_{\vartheta'} \partial_{x_n} S\| \right).$$

We sacrificed one power of  $\alpha$  to the errors in the approximations.

We are going to employ the operator

$$L_{\vartheta} = \frac{1}{i\lambda} \cdot \frac{(S_{\vartheta}(x, \vartheta) - S_{\vartheta}(y, \vartheta)) \cdot \nabla_{\vartheta}}{\|S_{\vartheta}(x, \vartheta) - S_{\vartheta}(y, \vartheta)\|^2},$$

which gives 1 when acting on the exponential in the integral kernel of (5.8). We claim that for sufficiently large values of  $\|X' - Y'\|$  the denominator of  $L_\vartheta$  is bounded from below by  $\text{const} \|x' - y'\|$  (squared). We can prove it using the relation (5.13) and repeating the argument after (5.6) (interchanging  $x$  and  $\vartheta$ ). We conclude:

$$\|S_\vartheta(x, \vartheta) - S_\vartheta(y, \vartheta)\| \geq (\alpha - \alpha^2) \|x' - y'\| \inf_{D_{\delta\varepsilon}} \min(H'),$$

and this gives the desired bound from below for the denominator of  $L_\vartheta$ .

Now we insert  $L_\vartheta^{2n+1}$  into the integral kernel of  $T_{X\Theta}^{\hbar} T_{Y\mathcal{W}}^{\hbar*}$  and integrate by parts. Each derivative  $\nabla_\vartheta$  gives at most  $\text{const} \max\{\varepsilon^{-1}, \hbar^{-1}\} \equiv \text{const} E^{-1}$ . When the derivative  $\nabla_\vartheta$  falls on the denominator of  $L_\vartheta$ , the factor which appears is also bounded by  $\text{const} \hbar^{-1}$ :

$$\text{const} \frac{\|x - y\|}{\|x' - y'\|} \leq \text{const} \frac{\delta}{\Sigma^{-1} \hbar \|X' - Y'\|} \leq \text{const} \delta \Sigma \hbar^{-1}.$$

Integration by parts in (5.8) yields the factor bounded by

$$\text{const} \frac{1}{|\lambda E \|x' - y'\||^{2n+1}}.$$

We integrate the absolute value of (5.8) with respect to  $x$  (or  $y$ ), obtaining the  $L^1$ -norm of  $K(T_{X\Theta}^{\hbar} T_{Y\mathcal{W}}^{\hbar*})$ :

$$(5.14) \quad \int dx |K(T_{X\Theta}^{\hbar} T_{Y\mathcal{W}}^{\hbar*})(x, y)| \leq \int_{\text{supp } T_{X\Theta}^{\hbar} T_{Y\mathcal{W}}^{\hbar*}} dx d\vartheta \frac{\text{const}}{|\lambda E \|x' - y'\||^{2n+1}} \\ \leq \text{const} \Delta_\Sigma^{n-1} \Delta_\sigma E^n (\lambda E \Sigma^{-1} \hbar \|X - Y\|)^{-2n-1}.$$

Note that  $\|x' - y'\| \approx \Sigma^{-1} \hbar \|X' - Y'\| \geq \text{const} \Sigma^{-1} \hbar \|X - Y\|$ , due to the inequality (5.12).

Since  $\Delta_\Sigma \geq \Delta_\sigma \hbar$  and  $\lambda E \Sigma^{-1} \hbar \geq \lambda E \Delta_\Sigma \geq \lambda E \Delta_\sigma \hbar \geq 1$ , we see that

$$\frac{\Delta_\Sigma^{n-1} \Delta_\sigma E^n}{(\lambda E \Sigma^{-1} \hbar)^{2n+1}} \leq \text{const} \frac{\Delta_\Sigma^{n-1} \Delta_\sigma E^n}{(\lambda E \Delta_\Sigma)^n} \leq \text{const} \lambda^{-n} \hbar^{-1}.$$

This results in the bound  $\text{const} \lambda^{-n} \hbar^{-1} \|X - Y\|^{-2n-1}$  for (5.14), proving the estimate (3.12) for the operators  $\bar{T}_{X\Theta}^{\hbar} \bar{T}_{Y\mathcal{W}}^{\hbar*}$  and  $T_{X\Theta}^{\hbar} T_{Y\mathcal{W}}^{\hbar*}$  when  $\|X' - Y'\|$  dominates.

This concludes the proof of Lemma 3.1 and of Theorem 2.2.  $\square$

## 6. APPLICATIONS TO SCATTERING THEORY

Now we return to the oscillatory integral operator (1.6) with the phase function (1.5):

$$(6.1) \quad T_\lambda u(x) = \int_{\mathbb{S}^n} d\text{vol}_\vartheta e^{i\lambda\langle \mathbf{r}(x), \omega(\vartheta) \rangle} \psi(x, \vartheta) u(\vartheta).$$

The variables  $x$  and  $\vartheta$  are some local coordinates on  $B$  and  $\mathbb{S}^n$ . The vector  $\mathbf{r} \in \mathbb{R}^{n+1}$  has its end in the point  $x$  on the hypersurface  $B$ , and  $\omega$  is a vector pointing to  $\vartheta$  on  $\mathbb{S}^n$ .

*Local coordinates on  $B$  and on  $\mathbb{S}^n$ .* Let us localize the integral operator (6.1) near some critical point  $(x_o, \vartheta_o)$ . We use the orthogonal projection  $p_L : B \rightarrow T_{\mathbf{r}_o}B$  to parameterize  $B$ . We take a Euclidean coordinate system  $(x_1, \dots, x_n, x_{n+1})$  centered at  $\mathbf{r}_o$ , so that the direction of the axis  $x_{n+1}$  coincides with the direction of the inner normal to  $B$  at  $\mathbf{r}_o$ . The axes  $x_i$ ,  $i = 1, \dots, n$  are thus in the tangent plane  $T_{\mathbf{r}_o}B$ . The hypersurface  $B$  is locally parameterized by  $x = (x_1, \dots, x_n) \in T_{\mathbf{r}_o}B \equiv \mathbb{R}_L^n$ :

$$x_{n+1} = b(x) \geq 0, \quad x = (x_1, \dots, x_n) \in U \subset \mathbb{R}_L^n.$$

In the same way, we use the orthogonal projection  $p_R : \mathbb{S}^n \rightarrow T_{\omega_o}\mathbb{S}^n \equiv \mathbb{R}_R^n$  to parameterize the sphere. We take a Euclidean coordinate system  $(\vartheta_1, \dots, \vartheta_n, \vartheta_{n+1})$  centered at  $\omega_o$ , with the axis  $\vartheta_{n+1}$  being the inner normal to  $\mathbb{S}^n$  at the point  $\omega_o$ . The sphere is then given locally by

$$\vartheta_{n+1} = g(\vartheta) \equiv 1 - \sqrt{1 - \|\vartheta\|^2} \simeq \frac{\|\vartheta\|^2}{2}, \quad \text{here } \|\vartheta\|^2 = \vartheta_1^2 + \dots + \vartheta_n^2.$$

Since  $(\mathbf{r}_o, \omega_o)$  is a critical point, the planes  $T_{\mathbf{r}_o}B$  and  $T_{\omega_o}\mathbb{S}^n$  are mutually orthogonal. We choose the directions of the axes  $x_i$  and  $\vartheta_i$  so that the direction of  $\vartheta_n$  is the same as that of  $x_{n+1}$ ,  $x_n$  is opposite to  $\vartheta_{n+1}$ , and the directions  $x_{i'}$  and  $\vartheta_{i'}$  coincide ( $i' = 1, \dots, n-1$ ).

The dot product  $\langle \mathbf{r}, \omega \rangle$  is then given by  $x' \cdot \vartheta' + \vartheta_n x_{n+1} - x_n \vartheta_{n+1}$ , and therefore

$$(6.2) \quad S(x, \vartheta) = \langle \mathbf{r}, \omega \rangle = x' \cdot \vartheta' + \vartheta_n b(x) - x_n g(\vartheta).$$

The determinant of the mixed Hessian of this phase function is given by

$$(6.3) \quad \det H(x, \vartheta) = b_{x_n}(x) - g_{\vartheta_n}(\vartheta) + b_{x'}(x) \cdot g_{\vartheta'}(\vartheta).$$

The non-singular part of the mixed Hessian is a unit matrix:

$$H' = \nabla_{x'} \nabla_{\vartheta'} S = \mathbb{I}_{\mathbb{R}^{n-1}},$$

hence in (2.9)  $\min(H) = 1$ .

*Convexity of the map  $\hat{\pi}_R$ .* Let us show that the map  $\hat{\pi}_R|_{\vartheta}$  corresponding to the phase function (6.2) is pseudoconvex.

This map is defined as

$$\hat{\pi}_R|_{\vartheta} : x \mapsto S_{\vartheta}(x, \vartheta) = \nabla_{\vartheta} \langle \mathbf{r}(x), \omega(\vartheta) \rangle.$$

If we use the standard scalar product  $\langle \cdot, \cdot \rangle$  in  $\mathbb{R}^{n+1}$  to identify the tangent and cotangent fibers,  $T_{\omega(\vartheta)}\mathbb{S}^n \cong T_{\omega(\vartheta)}^*\mathbb{S}^n$ , then  $\hat{\pi}_R$  is given geometrically by the orthogonal projection  $\Pi_{\omega(\vartheta)}$  from  $B$  onto  $T_{\omega(\vartheta)}\mathbb{S}^n$ :

$$B \xrightarrow{\Pi_{\omega(\vartheta)}} T_{\omega(\vartheta)}\mathbb{S}^n, \quad \mathbf{r}(x) \mapsto \mathbf{r}(x) - \omega(\vartheta) \langle \mathbf{r}(x), \omega(\vartheta) \rangle.$$

If the hypersurface  $B$  is convex (not necessarily strictly convex), then on each connected set  $O \subset B$ , for any  $\mathbf{r}_1, \mathbf{r}_2 \in O$ ,

$$(6.4) \quad \|\Pi_{\omega}(\mathbf{r}_2 - \mathbf{r}_1)\|_{T_{\omega}\mathbb{S}^n} \geq \|\mathbf{r}_2 - \mathbf{r}_1\|_{\mathbb{R}^{n+1}} \inf_{\mathbf{r} \in O} |\langle \mathbf{n}_{\mathbf{r}}, \omega \rangle|.$$

This is the key inequality. Since  $\langle \mathbf{n}_{\mathbf{r}}, \omega \rangle$  is approximately the determinant of the Jacobian of  $\Pi_{\omega}$ , (6.4) suggests that the map  $\Pi_{\omega}$  is pseudoconvex.

Using the orthogonal projections  $p_L$  and  $p_R$ ,

$$p_L^{-1} : \mathbb{R}_L^n \equiv T_{\mathbf{r}_o}B \rightarrow B \quad \text{and} \quad p_R : \mathbb{S}^n \rightarrow T_{\omega_o}\mathbb{S}^n \equiv \mathbb{R}_R^n,$$

we write the map  $\hat{\pi}_R|_{\vartheta} : \mathbb{R}_L^n \rightarrow T_{\vartheta}^*\mathbb{R}_R^n \cong T_{\vartheta}\mathbb{R}_R^n$  as

$$\hat{\pi}_R|_{\vartheta} : x \xrightarrow{p_L^{-1}} \mathbf{r}(x) \xrightarrow{\Pi_{\omega(\vartheta)}} \mathbf{r}(x) - \omega(\vartheta) \langle \mathbf{r}(x), \omega(\vartheta) \rangle \xrightarrow{dp_R} S_{\vartheta}(x, \vartheta) \in T_{\vartheta}\mathbb{R}_R^n.$$

The maps  $p_L^{-1}$  and  $dp_R$  are diffeomorphisms, and we expect that  $\hat{\pi}_R|_{\vartheta}$  is pseudoconvex as a composition of pseudoconvex maps.

Let us write this in more detail.

We substitute into the relation (6.4)  $\mathbf{r}_1 = \mathbf{r}(x)$ ,  $\mathbf{r}_2 = \mathbf{r}(y)$ ,  $\omega = \omega(\vartheta)$ :

$$\|\Pi_{\omega(\vartheta)}(\mathbf{r}(y) - \mathbf{r}(x))\|_{T_{\omega(\vartheta)}\mathbb{S}^n} \geq \|\mathbf{r}(y) - \mathbf{r}(x)\|_{\mathbb{R}^{n+1}} \inf_{z \in \mathcal{O}} |\langle \mathbf{n}_{\mathbf{r}(z)}, \omega \rangle|.$$

Here  $\mathcal{O} = p_L(O) \subset \mathbb{R}_L^n$ .

Then we take into account the relations

$$\|dp_R(*)\|_{\mathbb{R}_R^n} \geq |\langle \omega, \omega_o \rangle| \|*\|_{T_{\omega}\mathbb{S}^n} \quad \text{and} \quad \|\mathbf{r}(y) - \mathbf{r}(x)\|_{\mathbb{R}^{n+1}} \geq \|y - x\|_{\mathbb{R}_L^n}$$

and obtain:

$$(6.5) \quad \|S_\vartheta(y, \vartheta) - S_\vartheta(x, \vartheta)\|_{T_\vartheta^* \mathbb{R}_R^n} \geq \|y - x\|_{\mathbb{R}_L^n} |\langle \omega, \omega_o \rangle| \inf_{z \in \mathcal{O}} |\langle \mathbf{n}_r(z), \omega \rangle|.$$

Considering the projections  $\mathbb{R}_L^n \rightarrow T_{\mathbf{r}(x)}B \rightarrow T_{\omega(\vartheta)}\mathbb{S}^n \rightarrow \mathbb{R}_R^n$ , we conclude that

$$(6.6) \quad \det H(x, \vartheta) = \frac{\langle \omega(\vartheta), \omega_o \rangle}{\langle \mathbf{n}_r(x), \mathbf{n}_{r_o} \rangle} \langle \mathbf{n}_r(x), \omega(\vartheta) \rangle.$$

We have localized (6.1) near the point  $(x_o, \vartheta_o) = (0, 0)$ , and therefore

$$\langle \omega(\vartheta), \omega_o \rangle = \langle \omega(\vartheta), \omega(\vartheta_o) \rangle \simeq 1, \quad \langle \mathbf{n}_r(x), \mathbf{n}_{r_o} \rangle \simeq 1.$$

We rewrite (6.5) as

$$(6.5') \quad \|S_\vartheta(x, \vartheta) - S_\vartheta(y, \vartheta)\|_{T_\vartheta^* \mathbb{R}_R^n} \geq \frac{1}{2} \|x - y\|_{\mathbb{R}_L^n} \inf_{z \in \mathcal{O}} |\det H(z, \vartheta)|.$$

Note that since  $|\det H'| \geq \text{const}$ ,  $\min(H) \sim |\det H|$ .

This proves pseudoconvexity of the map  $\hat{\pi}_R|_\vartheta$ .  $\square$

*Conditions (2.12)-(2.14) of Theorem 2.2.* We want to rewrite the technical assumptions of Theorem 2.2 for the phase function (6.2).

Let us consider a hypersurface  $B$ , locally given by  $x_{n+1} = b(x)$ ,  $x = (x_1, \dots, x_n)$ ,  $b \in C^2(\mathbb{R}_L^n)$ .

The condition (2.13) is given by

$$(6.7) \quad \sup |\nabla_{x'} b| \frac{\sup \|\nabla_{\vartheta'} (-\partial_{\vartheta_n} g + \nabla_{x'} b \cdot \nabla_{\vartheta'} g)\|}{\inf |\partial_{\vartheta_n} (-\partial_{\vartheta_n} g + \nabla_{x'} b \cdot \nabla_{\vartheta'} g)|} \leq \frac{1}{2}.$$

Note that in the denominator  $\partial_{\vartheta_n}^2 g \approx 1$ .

Under a very weak assumption

$$(6.8) \quad \|\nabla_x b(x)\| \leq \|x\|,$$

the inequality (6.7) is satisfied if  $\|x\|$  and  $\|\vartheta\|$  are reasonably small:

$$(6.9) \quad \|x\| \leq 1/6, \quad \|\vartheta\| \leq 1/6.$$

The main restriction on  $\|\vartheta\|$  arises from the condition (2.14). This condition amounts to

$$(6.10) \quad \sup \|\nabla_{\vartheta'} g\| \frac{\sup \|\nabla_{x'} (\partial_{x_n} b + \nabla_{x'} b \cdot \nabla_{\vartheta'} g)\|}{\inf |\partial_{x_n} (\partial_{x_n} b + \nabla_{x'} b \cdot \nabla_{\vartheta'} g)|} \leq \frac{1}{2}.$$

Calculation gives the following sufficient restriction:

$$(6.11) \quad \|\vartheta\| \leq \frac{1}{6} \frac{\inf \partial_{x_n}^2 b}{\sup \|\nabla_x \nabla_x b\|}.$$

Here  $\|\nabla_x \nabla_x b\|$  is the norm of the Hessian of  $b$ , which is considered as a quadratic form on  $\mathbb{R}^n$ .

At last, as long as  $x$  and  $\vartheta$  are in the limits specified by (6.9) and (6.11), we have:

$$(6.12) \quad |\partial_{\vartheta_n} \det H| > 1/2, \quad |\partial_{x_n} \det H| > \partial_{x_n}^2 b(x)/2.$$

We have reformulated all the conditions of Theorem 2.2, to adapt them to the oscillatory integral operator (6.1). Now we are ready to proceed to deriving the estimates on its norm.

*Hypersurface with asymptotically small principal curvatures.* We start with the case when the principal curvatures of the hypersurface  $B$  are asymptotically small.

Let  $B$  be given locally by the graph  $x_{n+1} = b(x_1, \dots, x_n) \geq 0$ ,  $b(0) = 0$ , and let the Hessian  $[\partial_{x_i} \partial_{x_j} b]$  be isotropic:

$$(6.13) \quad u^i u^j \frac{\partial^2 b(x)}{\partial x_i \partial x_j} \geq \sigma, \quad \left| u^i v^j \frac{\partial^2 b(x)}{\partial x_i \partial x_j} \right| \leq C\sigma, \quad x \in U \subset \mathbb{R}^n,$$

for any unit vectors  $\mathbf{u}, \mathbf{v}$  in  $\mathbb{R}^n$ .  $C > 1$  is some constant. These conditions say that all sectional curvatures are of magnitude  $\sigma$ .

We would like to know the asymptotic behavior of the operator (6.1) when  $\sigma$  is asymptotically small. Of course,  $\sigma$  in (6.13) is going to represent the principal curvature of the fold  $\hat{\pi}_R$  in (2.12).

Without loss of generality, we assume that  $C\sigma < 1$ .

**Proposition 6.1** (*Asymptotically small sectional curvature*).

Consider an integral operator  $T_\lambda : \mathcal{E}'(\mathbb{S}^n) \rightarrow \mathcal{D}'(B)$ ,

$$T_\lambda u(x) = \int_{\mathbb{S}^n} d\text{vol}_\vartheta e^{i\lambda \langle \mathbf{r}(x), \omega(\vartheta) \rangle} \psi(x, \vartheta) u(\vartheta), \quad \psi \in C^{2n+1}(B \times \mathbb{S}^n),$$

with  $B$  a smooth strictly convex hypersurface in  $\mathbb{R}^{n+1}$  (locally a graph of  $C^{2n+2}$ -function).

Let all sectional curvatures of  $B$  be of the same magnitude  $\sigma > 0$ ,  $\sigma \ll 1$ .

Then  $T_\lambda$  is continuous from  $L^2(\mathbb{S}^n)$  to  $L_{\text{loc}}^2(B)$  with the norm

$$(6.14) \quad \|T_\lambda\| \leq \text{const } \lambda^{-\frac{n}{2} + \frac{1}{6}} \sigma^{-\frac{1}{6}}.$$



**Corollary 6.2.** *Let  $K$  be a strictly convex compact domain in  $\mathbb{R}^{n+1}$  with a smooth boundary  $B$ . Then the Radon Transform  $\mathcal{R}$  in (1.1), localized in the region where the sectional curvatures of  $B$  are of magnitude  $\sigma$ , acts continuously from  $H^s(\mathbb{R}) \otimes L^2(\mathbb{S}^n)$  to  $H^{s+\frac{n}{2}-\frac{1}{6}}(\mathbb{R}) \otimes L^2_{\text{loc}}(B)$ , with the norm  $\|\mathcal{R}\| \leq \text{const } \sigma^{-\frac{1}{6}}$ .*

*$\mathcal{R}$  also acts continuously from  $H^s(\mathbb{R} \times \mathbb{S}^n)$  to  $H^{s+\frac{n}{2}-\frac{1}{6}}(\mathbb{R} \times B)$ , with the norm  $\text{const}(s)\sigma^{-\frac{1}{6}}$ .*

The assumptions of Proposition 6.1 imply that locally the hypersurface  $B$  is a graph of a smooth function  $b$ , which satisfies (6.13). As a matter of fact, we do not need to require that all sectional curvatures of  $B$  be of the same magnitude, and the second condition in (6.13) is not necessary.

*Example: ball of radius  $R = \sigma^{-1}$ .* Let us first give an example of such a situation. We take  $B = \mathbb{S}^n_R$  and consider the estimate on the oscillatory integral operator

$$T_\lambda u(\mathbf{r}) = \int_{\mathbb{S}^n_1} d\text{vol}_\vartheta e^{i\lambda\langle \mathbf{r}, \omega \rangle} u(\vartheta), \quad \mathbf{r} \in \mathbb{S}^n_R.$$

The operator  $T_\lambda$ , considered as an operator to the space of functions defined on a *unit* sphere, is written as

$$T_\lambda u(\mathbf{n}_r) = \int_{\mathbb{S}^n_1} d\text{vol}_\vartheta e^{i\lambda R\langle \mathbf{n}_r, \omega \rangle} u(\vartheta), \quad \mathbf{n}_r = \frac{\mathbf{r}}{R} \in \mathbb{S}^n_1,$$

and the estimate (1.7) yields  $\|T_\lambda\|_{L^2(\mathbb{S}^n_1) \rightarrow L^2(\mathbb{S}^n_1)} \leq \text{const } (\lambda R)^{-\frac{n}{2}+\frac{1}{6}}$ .

Using this estimate, together with  $\int_{\mathbb{S}^n_R} d\text{vol} = R^n \int_{\mathbb{S}^n_1} d\text{vol}$ , we arrive at

$$\int_{\mathbb{S}^n_R} d\text{vol} |T_\lambda u(\mathbf{r})|^2 = R^n \int_{\mathbb{S}^n_1} d\text{vol} |T_\lambda u(\mathbf{n}_r)|^2 \leq R^n \text{const } (\lambda R)^{-n+\frac{1}{3}} \|u\|_{L^2(\mathbb{S}^n)}^2.$$

Thus,

$$\|T_\lambda\|_{L^2(\mathbb{S}^n_1) \rightarrow L^2(\mathbb{S}^n_R)} \leq \text{const } \lambda^{-\frac{n}{2}+\frac{1}{6}} R^{\frac{1}{6}},$$

in an agreement with (6.14).

The increase in the norm could be connected with both the “more critical behavior” at critical points (as the curvature gets smaller) and the increase of the area of  $B = \mathbb{S}^n_R$ . As a matter of fact, the increase of  $\|T_\lambda\|$  as  $R \rightarrow \infty$  is a sheer consequence of the diminishing of a curvature.

*Proof of Proposition 6.1.* We are going to apply Theorem 2.2 to the operator  $T_\lambda$  in (6.1), localized near a critical point  $(x_o, \vartheta_o)$ ,  $\langle \mathbf{n}_{\mathbf{r}(x_o)}, \omega(\vartheta_o) \rangle = 0$ .

Since we assume that  $b(x)$  in (6.2) satisfies the condition (6.13), the relations (6.12) yield

$$(6.15) \quad |\partial_{\vartheta_n} \det H| > 1/2, \quad |\partial_{x_n} \det H| > \sigma/2.$$

The restrictions (6.9) and (6.11) on  $\|x\|$  and  $\|\vartheta\|$  are

$$(6.16) \quad \|x\| \leq 1/6, \quad \|\vartheta\| \leq 1/6C,$$

with the constant  $C$  from (6.13). Formally, we take  $\delta = 1/6$ ,  $\varepsilon = 1/6C$ .

According to Theorem 2.2, we have:

$$\|T_\lambda\| \leq \text{const } \lambda^{-\frac{n}{2} + \frac{1}{6}} \sigma^{-\frac{1}{6}}. \quad \square$$

*Hypersurface with curvature vanishing at a point.* Now let us consider the case when the hypersurface  $B$  admits a tangent plane with a contact of precise order  $k > 1$ . Then  $B$  is given locally by

$$x_{n+1} = b(x_1, \dots, x_n) \sim \|x\|^{k+1}, \quad \|x\|^2 = x_1^2 + \dots + x_n^2, \quad x \in U \subset \mathbb{R}^n.$$

The principal curvatures behave like  $\sigma \sim \|x\|^{k-1}$  as  $\|x\|$  approaches 0. The Gaussian curvature of the hypersurface  $B$  vanishes  $\sim \|x\|^{n(k-1)}$ .

The definition of a contact of precise order  $k$  is given by (1.3): for any unit vectors  $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ , the Hessian of the function  $b(x)$  satisfies

$$(1.3') \quad u^i u^j \frac{\partial^2 b(x)}{\partial x_i \partial x_j} \geq c \|x\|^{k-1}, \quad \left| u^i v^j \frac{\partial^2 b(x)}{\partial x_i \partial x_j} \right| \leq C \|x\|^{k-1}.$$

**Proposition 6.3** (*Curvature uniformly vanishing at a point*).

Consider an integral operator  $T_\lambda : \mathcal{E}'(\mathbb{S}^n) \rightarrow \mathcal{D}'(B)$ ,

$$T_\lambda u(x) = \int_{\mathbb{S}^n} d\text{vol}_\vartheta e^{i\lambda \langle \mathbf{r}, \omega \rangle} \psi(x, \vartheta) u(\vartheta), \quad \psi \in C^{2n+1}(B \times \mathbb{S}^n),$$

with  $B$  a smooth convex hypersurface in  $\mathbb{R}^{n+1}$  (locally a graph of  $C^{2n+2}$ -function).

Let the Gaussian curvature of  $B$  vanish at an isolated point  $\mathbf{r}_o$  on the support of  $\psi$ . If the tangent plane at  $\mathbf{r}_o$  has a contact of precise order  $k > 1$  with  $B$ , then  $T_\lambda$  acts continuously from  $L^2(\mathbb{S}^n)$  to  $L^2_{\text{loc}}(B)$ , with the norm

$$(6.17) \quad \|T_\lambda\| \leq \text{const } \lambda^{-\frac{n}{2} + \frac{1}{2} \frac{k}{2k+1}}.$$

*Proof of Proposition 6.3.* We use the same local coordinates  $x$  as in (1.3), so that the origin  $x = 0$  in  $\mathbb{R}_L^n$  corresponds to the critical point  $\mathbf{r}_o \in B$ .

We will need a fine spatial partition of unity, as  $x$  approaches the origin. Let us introduce a function  $\rho(x) \in C_o^\infty(\mathbb{R}^n)$ , supported for  $2 \leq \|x\| \leq 8$ , such that

$$\sum_{j \geq 0} \rho(2^j x) \equiv 1 \quad \text{for} \quad \|x\| \leq 2.$$

We split  $\rho(x)$  into a sum of several functions  $\rho(x) = \sum \rho_a(x)$ ,  $a = 1 \dots A$ , supported in unit balls centered at some points  $p_a \in \text{supp } \rho$ ,  $2 \leq \|p_a\| \leq 8$ :  $\rho_a \in C_o^\infty(\mathbb{B}_1^n(p_a))$ . In each ball  $\mathbb{B}_1^n(p_a)$ , we have:

$$\|x\| \geq 1, \quad \sup \|x\| / \inf \|x\| \leq 3.$$

Decompose  $T_\lambda$  as follows:

$$(6.19) \quad T_\lambda = \sum_{\delta \leq 1} \sum_{a=1}^A \rho_a(x/\delta) \circ T_\lambda, \quad \delta = 2^{-j}, \quad j \geq 0.$$

We will apply Theorem 2.2 for the operators  $\rho_a(x/\delta) \circ T_\lambda$ .

The condition (1.3) gives  $\partial_{x_n}^2 b \geq c \|x\|^{k-1}$ ,  $\|\nabla_x \nabla_x b\| \leq C \|x\|^{k-1}$ , and the restriction (6.11) on  $\|\vartheta\|$  becomes

$$\|\vartheta\| \leq \varepsilon \equiv \frac{c}{6C} \cdot \frac{1}{3^{k-1}}.$$

Note that the value of  $\varepsilon$  is only finitely small, so that we could cover the sphere by finitely many neighborhoods of this size.

For each individual operator  $\rho_a(x/\delta) \circ T_\lambda$ , we need to consider the set  $D_{\delta\varepsilon}(p_a/\delta, 0) \equiv \mathbb{B}_\delta^n(p_a/\delta) \times \mathbb{B}_\varepsilon^n(0)$ , with  $\varepsilon$  as above. On this set,  $\|x\| \geq \delta$ .

The principal curvature of the fold  $\hat{\pi}_R$  on  $D_{\delta\varepsilon}(p_a/\delta, 0)$  is given by (6.12):

$$(6.20) \quad \sigma(\delta) > \frac{1}{2} c \delta^{k-1}.$$

Theorem 2.2 gives the following bounds on  $\|\rho_a(x/\delta) \circ T_\lambda\|$ :

$$(6.21) \quad \text{const } \lambda^{-\frac{n}{2} + \frac{1}{4}} \delta^{\frac{1}{4}} \quad \text{and} \quad \text{const } \lambda^{-\frac{n}{2} + \frac{1}{6}} \sigma(\delta)^{-\frac{1}{6}}.$$

Now we can evaluate the operator norm of (6.19). Let us find the value of  $\delta$  when the two estimates (6.21) on  $\|\rho_a(x/\delta) \circ T_\lambda\|$  clutch together:

$$\lambda^{\frac{1}{4}} \delta^{\frac{1}{4}} = \lambda^{\frac{1}{6}} \delta^{-\frac{k-1}{6}}.$$

This gives  $\delta = \lambda^{-\frac{1}{2k+1}}$ , and we conclude that  $\|T_\lambda\| \leq \text{const } \lambda^{-\frac{n}{2} + \frac{1}{2} \frac{k}{2k+1}}$ .  $\square$

Theorem 1.1 formulated in the Introduction is a consequence of Proposition 6.3 and of (1.7).

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