SOBOLEV ESTIMATES FOR RADON TRANSFORM OF MELROSE AND TAYLOR

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Abstract. We derive the regularity properties of the Radon transform of Melrose and Taylor for the scattering on a compact convex obstacle with a smooth boundary.

The result is formulated in terms of the highest order of contact of tangent lines with the boundary of an obstacle.

The main ingredients of the proof are the estimates for degenerate oscillatory integral operators and almost orthogonal decompositions.

1. Introduction and Results

R.B. Melrose and M.E. Taylor, in their fundamental paper [MT], showed that the important properties of the scattering operator are encoded in the following integral transformation, which we call the Radon transform of Melrose – Taylor:

$$\begin{aligned} &F: \quad \mathcal{E}'(\mathbb{R}\times\mathbb{S}^n)\to\mathcal{D}'(\mathbb{R}\times B),\\ &Fu(t,\mathbf{r})=\int_{\mathbb{S}^n} d\mathrm{vol}_\omega\, u(t-(\mathbf{r}\cdot\omega),\,\omega). \end{aligned} \tag{1.1}$$

Here $\mathbf{r}$ is a vector pointing to the boundary B of a compact domain K in $\mathbb{R}^{n+1}$ ("obstacle"), the unit vector ω represents the direction of light, and $(\mathbf{r}\cdot\omega)$ stands for the standard scalar product of vectors in $\mathbb{R}^{n+1}$. The variable t refers to the moment of time. The scattering operator can be written as

$$S=\mathrm{Id}+D_t^{n/2}A,\qquad A=F^*\circ(N_+\circ F+\partial_t\tilde{F}), \tag{1.2}$$

where N_+ is the forward Neumann operator and $\tilde{F}$ is of form (1.1), with an extra factor $(\mathbf{n}_\mathbf{r}\cdot\omega)$ in the integral kernel. We write $\mathbf{n}_\mathbf{r}$ for the unit inner normal at $\mathbf{r}\in B$.

The Radon transform (1.1) is a Fourier integral operator of order $-n/2$ associated to a singular canonical relation; we recommend the paper of D.H. Phong [Ph] as a survey on such operators. If the boundary B of an obstacle is strictly convex, with non-vanishing Gaussian curvature, then both projections from the canonical relation have singularities of Whitney type. In this case, it was shown in [MT] that F is continuous from $H^s(\mathbb{R}\times\mathbb{S}^n)$ to $H^{s+\frac{n}{2}-\frac{1}{6}}(\mathbb{R}\times B)$, thus losing 1/6 against

The author is grateful to his advisor, Professor D.H. Phong, for drawing the author's attention to this problem and for his advice. The research was supported by the Faculty Fellowship from Columbia University.

the regularity properties of Fourier integral operators associated to local graphs. Note that the operator $\tilde{F}$ appearing in (1.2) is smoothing of order $-n/2$, due to a damping effect of the factor $(\mathbf{n_r}\cdot\omega)$.

In the paper of the author [Co] the regularity properties of (1.1) were derived for the case when the boundary of an obstacle admits tangent planes with the contact of *precise* order. In this paper, we systematically treat the Radon transform (1.1), proving

Main Theorem. *Let K be a compact convex domain in $\mathbb{R}^{n+1}$, with the smooth boundary B. If B admits tangent lines with contact of order not greater than k, then the Radon transform* (1.1) *is smoothing of order $\frac{n}{2}-1/(4+2k^{-1})$:*

$$F:\ H^s(\mathbb{R}\times\mathbb{S}^n)\longrightarrow H^{s+\frac{n}{2}-1/(4+2k^{-1})}(\mathbb{R}\times B),\qquad \textit{for any real}\quad s. \tag{1.3}$$

Contact of order not greater than k. Let the hypersurface B near the point $\mathbf{r}\in B$ be a graph of some function $b_{(\mathbf{r})}(x)$, $x\in T_{\mathbf{r}}B$. We consider $b_{(\mathbf{r})}$ as defined on the tangent plane $T_{\mathbf{r}}B$ (with the metric endowed from $\mathbb{R}^{n+1}$), and taking values in the direction of the inner normal $\mathbf{n_r}\in\mathbb{R}^{n+1}$, so that $b_{(\mathbf{r})}(x)\geq 0$ and $b_{(\mathbf{r})}|_{x=0}=0$.

Definition 1.1. We say that a boundary B of a convex obstacle admits tangent lines with contact of order not greater than k at some point $\mathbf{r}\in B$, if there is a constant $\varkappa_{\mathbf{r}}>0$ such that for any x in a small open neighborhood of $0\in T_{\mathbf{r}}B$,

$$b_{(\mathbf{r})}(x)\geq\varkappa_{\mathbf{r}}|x|^{k+1}. \tag{1.4}$$

We say that B admits tangent lines with contact of order not greater than k everywhere if for all points $\mathbf{r}\in B$ the constants $\varkappa_{\mathbf{r}}$ are uniformly bounded from below: $\varkappa_{\mathbf{r}}\geq\varkappa>0$.

The case of non-vanishing curvature corresponds to $k=1$.

Remark. The statement of the Main Theorem remains true if we drop off the assumption that the obstacle is convex. The corresponding definition of contact of order k at a point $\mathbf{r}\in B$ should be changed to the following:

Definition 1.2. We say that a hypersurface B admits tangent lines with contact of order not greater than k at some point $\mathbf{r}\in B$, if for any $v\in T_{\mathbf{r}}B$, $|v|=1$,

$$\mathbf{v}^{1+k(\mathbf{r},v)}b_{(\mathbf{r})}|_{x=0}\neq 0,$$

for some integer $k(\mathbf{r},v)\leq k$. Here $\mathbf{v}\in T(T_{\mathbf{r}}B)$ is the vector field generated by v.

To derive the regularity properties (1.3) stated in the Main Theorem for the Radon transform (1.1), we use the idea of D.H. Phong and E.M. Stein [PhSt] that such regularity properties are related to the rate of decay of a corresponding oscillatory integral operator:

$$T_\lambda:\ \mathcal{D}'(\mathbb{S}^n)\to\mathcal{D}'(B),\qquad T_\lambda u(\mathbf{r})=\int_{\mathbb{S}^n}d\mathrm{vol}_\omega\, e^{i\lambda(\mathbf{r}\cdot\omega)}\,u(\omega). \tag{1.5}$$

Here $\mathbf{r}$ is a vector in $\mathbb{R}^{n+1}$ pointing from the origin to B. According to Phong and Stein, it suffices to prove that, under the assumptions of the Main Theorem, there is the following decay of the L^2 operator norm of T_λ, for large values of λ:

$$\|T_\lambda\|_{L^2\to L^2}\leq const\,\lambda^{-\frac{n}{2}+1/(4+2k^{-1})}. \tag{1.6}$$

2. Canonical relation associated with Radon transform of Melrose and Taylor

Introducing some local coordinates x and ϑ on B and $\mathbb{S}^n$, we rewrite (1.5) as

$$T_\lambda u(x) = \int_{\mathbb{R}^n} e^{i\lambda(\mathbf{r}(x)\cdot\omega(\vartheta))}\,\psi(x,\vartheta)\,u(\vartheta)\,d\vartheta, \quad \psi\in C^\infty_{\mathrm{comp}}(\mathbb{R}^n\times\mathbb{R}^n). \tag{2.1}$$

Here $\mathbf{r}(x)$ is the vector in $\mathbb{R}^{n+1}$ pointing from the origin to a point on B with a local coordinate x, and $\omega(\vartheta)$ is the unit vector in $\mathbb{R}^{n+1}$ corresponding to a point ϑ on the unit sphere. The smooth function $\psi(x,\vartheta)$ localizes the integral kernel to a neighborhood where the local coordinates x and ϑ are defined. Let us consider the canonical relation $\mathcal{C}$ associated with (2.1):

$$\mathcal{C} = \{(x,S_x)\times(\vartheta,S_\vartheta)\mid x\in B,\,\vartheta\in\mathbb{S}^n\}\subset T^*_xB\times T^*_\vartheta\mathbb{S}^n.$$

Here S_x, S_ϑ are the components of the 1-forms d_xS and $d_\vartheta S$. Using the isomorphism $B\times\mathbb{S}^n\cong\mathcal{C}$, we write the projections from the canonical relation onto the first and second factors of $T^*_xB\times T^*_\vartheta\mathbb{S}^n$ in the following form:

$$\pi_L:\,(x,\vartheta)\mapsto(x,S_x),\qquad \pi_R:\,(x,\vartheta)\mapsto(\vartheta,S_\vartheta). \tag{2.2}$$

The singular part of the map π_L is given by $\vartheta\mapsto\omega(\vartheta)\mapsto d_x(\mathbf{r}(x)\cdot\omega(\vartheta))$. Since $\mathbf{r}(x)$ can only have increments in $T_{\mathbf{r}(x)}B$, the differential $d_x(\mathbf{r}(x)\cdot\omega(\vartheta))$ can be identified with the orthogonal projection of $\omega(\vartheta)$ onto $T_{\mathbf{r}(x)}B$ (via the isomorphism of tangent and cotangent spaces, which is determined by the Euclidean metric). We conclude that the singular part of the map π_L corresponds to the orthogonal projection from the sphere (in the direction specified by the particular value of x). Such a projection is of course a Whitney fold.

Analogously, the singular part of the map π_R corresponds to the orthogonal projection from the boundary B of an obstacle. It is a Whitney fold only as long as the curvature of the boundary is strictly positive.

The condition for a point $(\mathbf{r},\omega)\in B\times\mathbb{S}^n$ to be in the critical variety of the projections from $\mathcal{C}$ is that

$$(\mathbf{n}_\mathbf{r}\cdot\omega)=0, \tag{2.3}$$

here $\mathbf{n}_\mathbf{r}$ is the unit inner normal at the point $\mathbf{r}\in B$. Naively, if (2.3) does not hold, then the integral operator (1.5) localized near $(\mathbf{r},\omega)$ is basically the Fourier transform and hence has no critical points. According to L. Hörmander [Hö], the norm of such a localized operator is bounded by $const\,\lambda^{-\frac{n}{2}}$.

Coordinate representation. Let us restrict our attention to the vicinity of some critical point $(\mathbf{r}_o,\omega_o)$. We consider B and $\mathbb{S}^n$ as parameterized by the Euclidean coordinates $x=(x_1,\,\ldots,\,x_n)$ in $\mathbb{R}^n_L\equiv T_{\mathbf{r}_o}B$ and $\vartheta=(\vartheta_1,\,\ldots,\,\vartheta_n)$ in $\mathbb{R}^n_R\equiv T_{\omega_o}\mathbb{S}^n$. We choose the axis x_n in $T_{\mathbf{r}_o}B$ in the direction of the vector ω_o (this is possible due to the condition (2.3)). Then, we choose the axis ϑ_n in the direction of the inner normal $\mathbf{n}_{\mathbf{r}_o}$. The directions of the axes $x_{i'}$ are chosen to coincide with the directions of $\vartheta_{i'}$, $i'=1,\,\ldots,\,n-1$. We will write $x'=(x_1,\,\ldots,\,x_{n-1})$, $\vartheta'=(\vartheta_1,\,\ldots,\,\vartheta_{n-1})$. Later on, we assume that the point $\mathbf{r}_o$ is fixed, and use the notation $b(x)=b_{(\mathbf{r}_o)}(x)$ for the function which represents the boundary near the point $\mathbf{r}_o$.

In the above local coordinates, the phase function $S(x,\vartheta)$ is given by

$$S(x,\vartheta)=(\mathbf{r}(x)\cdot\omega(\vartheta))=\vartheta_n b(x)-x_n g(\vartheta)+x'\cdot\vartheta', \tag{2.4}$$

modulo terms which depend only on either x or ϑ. We have used the function

$$g(\vartheta)=1-\sqrt{1-|\vartheta|^2},$$

which locally represents the sphere $\mathbb{S}^n$. We will assume that $|\vartheta|<1/4$.

The mixed Hessian corresponding to the phase function (2.4) is given by

$$S_{x\vartheta}=\begin{bmatrix}\mathbb{I}_{n-1} & b_{x'}(x)^t\\ -g_{\vartheta'}(\vartheta) & b_{x_n}(x)-g_{\vartheta_n}(\vartheta)\end{bmatrix}, \tag{2.5}$$

with its determinant being equal to

$$h(x,\vartheta)=\det S_{x\vartheta}(x,\vartheta)=b_{x_n}(x)-g_{\vartheta_n}(\vartheta)+b_{x'}(x)\cdot g_{\vartheta'}(\vartheta). \tag{2.6}$$

The non-degenerate part of the mixed Hessian $S_{x\vartheta}$ is given by the unit matrix:

$$S_{x'\vartheta'}(x,\vartheta)=\mathbb{I}_{n-1}. \tag{2.7}$$

The differentials $d\pi_L=\begin{bmatrix}\mathbb{I}_n & S_{xx}\\ 0 & S_{x\vartheta}\end{bmatrix}$, $d\pi_R=\begin{bmatrix}0 & \mathbb{I}_n\\ S_{\vartheta\vartheta} & S_{x\vartheta}\end{bmatrix}$ become degenerate on the critical variety $\varSigma=\{\,(x,\vartheta)\mid h(x,\vartheta)=0\}$, where their coranks are equal to 1 (which is the corank of $S_{x\vartheta}$). Note that (2.6) vanishes at $(x,\vartheta)=(0,0)$, which corresponds to the critical point $(\mathbf{r}_o,\omega_o)$.

The kernel of the differential $d\pi_L$ is generated by the vector

$$\mathbf{k}_L(x,\vartheta)=b_{x'}(x)\cdot\nabla_{\vartheta'}-\partial_{\vartheta_n}\in T_x\mathbb{R}^n_L\times T_\vartheta\mathbb{R}^n_R,\qquad (x,\vartheta)\in\varSigma. \tag{2.8}$$

The vector field $\mathbf{k}_L(x,\vartheta)$ is also well-defined by (2.8) away from the critical variety. Moreover, we have a convenient relation $h(x,\vartheta)=b_{x_n}(x)+\mathbf{k}_L g(\vartheta)$. This yields

$$\mathbf{k}_L h(x,\vartheta)=\mathbf{k}_L^2 g(\vartheta)\ge 1,$$

and we conclude that (a) the determinant of $S_{x\vartheta}$ vanishes simply on $\varSigma$ and that (b) the kernel of the differential $d\pi_L$ is transversal to $\varSigma$. Hence, the projection π_L is a Whitney fold.

This is not necessarily true for the map π_R: the kernel of its differential is generated by $\mathbf{k}_R=g_{\vartheta'}\cdot\nabla_{x'}+\partial_{x_n}$, so that $h(x,\vartheta)=\mathbf{k}_R b(x)-g_{\vartheta_n}(\vartheta)$, and therefore

$$\mathbf{k}_R h(x,\vartheta)=\mathbf{k}_R^2 b(x).$$

The latter expression is non-negative, since we assume that the boundary B is convex. At the same time, $\mathbf{k}_R^2 b(x)$ can vanish if the curvature of B is not strictly positive. In this case the projection π_R has singularities worse than of Whitney type.

3. Almost orthogonal decompositions for oscillatory integral operators

We start with an arbitrary oscillatory integral operator T_λ, given by

$$T_\lambda : \mathcal{D}'(\mathbb{R}^n_R) \to \mathcal{D}'(\mathbb{R}^n_L),$$
$$T_\lambda : u(\vartheta) \mapsto T_\lambda u(x) = \int_{\mathbb{R}^n_R} e^{i\lambda S(x,\vartheta)}\, \psi(x,\vartheta)\, u(\vartheta)\, d\vartheta, \tag{3.1}$$

where $x \in \mathbb{R}^n_L$, $\vartheta \in \mathbb{R}^n_R$; $\psi \in C^\infty_{\mathrm{comp}}(\mathbb{R}^n_L \times \mathbb{R}^n_R)$ is a smooth compactly supported function, with the support assumed to be convex in each variable.

Let $\bar\beta \in C^\infty_{\mathrm{comp}}(\mathbb{R})$ be a symmetric function such that $\bar\beta(t) = 1$ for $|t| \le 1$, $\bar\beta(t) = 0$ for $|t| \ge 2$. We define $\beta(t) \in C^\infty_{\mathrm{comp}}([1/2, 2])$ by $\beta(t) = \bar\beta(t) - \bar\beta(2t)$ for $t \ge 0$, $\beta(t) = 0$ for $t \le 0$. Then we define $\beta_\pm(t) = \beta(\pm t)$. For each integer N_o, there is the following partition of 1:

$$1 = \sum_{\pm} \sum_{N\in\mathbb{Z},\, N<N_o} \beta_\pm(2^N t) + \bar\beta(2^{N_o} t), \quad \forall\, t \in \mathbb{R}.$$

We apply this partition to the operator T_λ, obtaining its dyadic decomposition with respect to the values of $h(x,\vartheta) = \det S_{x\vartheta}$. We define

$$T^\hbar_\pm u(x) = \int e^{i\lambda S(x,\vartheta)}\, \psi(x,\vartheta)\, \beta_\pm(\hbar^{-1} h(x,\vartheta))\, u(\vartheta)\, d\vartheta, \tag{3.2}$$

$$\bar T^\hbar u(x) = \int e^{i\lambda S(x,\vartheta)}\, \psi(x,\vartheta)\, \bar\beta(\hbar^{-1} h(x,\vartheta))\, u(\vartheta)\, d\vartheta, \tag{3.3}$$

(where $\hbar = 2^{-N}$), and decompose the operator (3.1) into a finite sum

$$T_\lambda = \sum_{\pm} \sum_{\hbar>\hbar_o}^{\hbar\le\Lambda} T^\hbar_\pm + \bar T^{\hbar_o}, \quad \hbar = 2^{-N}, \quad N \in \mathbb{Z}, \tag{3.4}$$

where $\Lambda = 2^{N_1} \ge \sup|\det S_{x\vartheta}|$, and $\hbar_o = 2^{-N_o}$ will be chosen later (see (3.7)).

We will not write $\pm$-indexes for operators $T^\hbar_\pm$ and only consider the "+"-case.

Proposition 3.1. *If T_λ is as in* (2.1), *with the phase function* (2.4), *and if the boundary B of an obstacle is convex, then there is the following estimate*♠ *for the operator $T^\hbar$ defined in* (3.2)*:*

$$\|T^\hbar\| \le const\, \lambda^{-\frac{n}{2}}\, \hbar^{-\frac{1}{2}}. \tag{3.5}$$

If, further, the highest order of contact of tangent lines with the boundary B is not greater than k, then the operator $\bar T^\hbar$ in (3.3) *is bounded by*

$$\|\bar T^\hbar\| \le const\, \lambda^{-\frac{n-1}{2}}\, \hbar^{\frac{1}{2}+\frac{1}{2k}}. \tag{3.6}$$

The estimates stated in the Proposition coincide at

$$\hbar = \hbar_o \equiv \lambda^{-\frac{k}{2k+1}}. \tag{3.7}$$

♠we do not consider unnecessarily small values of $\hbar$; see (3.7) and thereafter.

Substituting this value of $\hbar_o$ into (3.4), we obtain the estimate (1.6) for the operator T_λ. (Precisely, we should have taken for $\hbar_o$ a negative power of 2 nearest to $\lambda^{-\frac{k}{2k+1}}$.) This would prove the statement (1.3) of the Main Theorem in the Introduction. In the argument to follow, we will not be considering the estimates on $T^\hbar$ and $\bar{T}^\hbar$ corresponding to the values of $\hbar$ smaller than as specified by (3.7).

To prove Proposition 3.1, we will introduce the almost orthogonal decomposition (with respect to ϑ). In this section, we will prove the individual estimates of the form (3.5) for the parts of $T^\hbar$ (Lemma 3.2), and all relevant almost orthogonality relations for the parts of $T^\hbar$ and $\bar{T}^\hbar$ (Lemma 3.3). We elaborate the proof used by S. Cuccagna [Cu]. The proof of the individual estimates of the form (3.6) for the parts of $\bar{T}^\hbar$ is contained in Section 4 (Lemma 4.1). Only these estimates depend on the highest order of contact of tangent lines with B. Then, the statement of Proposition 3.1 follows from the Cotlar-Stein Lemma [St].

We assume that we have chosen some local coordinates $x = (x_1 \dots, x_n)$ and $\vartheta = (\vartheta_1, \dots, \vartheta_n)$. We take sets of integers, $\Theta \in \mathbb{Z}^n$, and localize the integral kernels of $\bar{T}^\hbar$ and $T^\hbar$ with the aid of the functions

$$\chi(\hbar^{-1}\vartheta - \Theta) = \prod_{i=1}^{n} \chi_1(\hbar^{-1}\vartheta_i - \Theta_i),$$

where $\chi_1 \in C^\infty_{\mathrm{comp}}([-1,1])$ is such that $\sum_{n\in\mathbb{Z}} \chi_1(t-n) \equiv 1$. We decompose $T^\hbar$ into $T^\hbar = \sum\limits_{\Theta\in\mathbb{Z}^n} T^\hbar_\Theta$, where $T^\hbar_\Theta = T^\hbar \circ \chi(\hbar^{-1}\vartheta - \Theta)$. We use the same decomposition for $\bar{T}^\hbar$, representing it by the sum of operators $\bar{T}^\hbar_\Theta = \bar{T}^\hbar \circ \chi(\hbar^{-1}\vartheta - \Theta)$.

Lemma 3.2. *The estimate* (3.5) *holds for each* $T^\hbar_\Theta$*:*

$$\|T^\hbar_\Theta\|^2 \le const\, \lambda^{-n}\hbar^{-1}. \tag{3.8}$$

Proof. We consider the integral kernel of $T^\hbar_\Theta T^{\hbar *}_\Theta$:

$$K(T^\hbar_\Theta T^{\hbar *}_\Theta)(x,y) = \int d^n\vartheta\, e^{i\lambda(S(x,\vartheta)-S(y,\vartheta))} \times \dots . \tag{3.9}$$

We integrate by parts in (3.9), using the operator

$$L_\vartheta = \frac{1}{i\lambda} \cdot \frac{(S_\vartheta(x,\vartheta) - S_\vartheta(y,\vartheta)) \cdot \nabla_\vartheta}{|S_\vartheta(x,\vartheta) - S_\vartheta(y,\vartheta)|^2} \tag{3.10}$$

When acting on cut-offs, ∇_ϑ contributes at most $\hbar^{-1}$. The derivative ∇_ϑ can act on the denominator of L_ϑ itself, also giving a factor bounded by $const\,\hbar^{-1}$. For this, we need to require that the map $\pi_R|_\vartheta : x \mapsto S_\vartheta(x,\vartheta)$ in (2.2) be "pseudoconvex": $|S_\vartheta(x,\vartheta) - S_\vartheta(y,\vartheta)| \ge const\,\hbar\, |x-y|$ on each connected set where $|\det S_{x\vartheta}| \ge \hbar/2$ (in particular, on the support of $T^\hbar_\Theta$; if it is not connected, we might consider the connected components separately). The pseudoconvexity requirement is satisfied for the Radon transform of Melrose and Taylor, for the scattering on a convex obstacle, and can be verified geometrically [Co].

Integration by parts $n+1$ times is thus equivalent to adding the following factor to the integral kernel (3.9) of $T^\hbar_\Theta T^{\hbar *}_\Theta$:

$$const\, \frac{1}{(1 + \lambda\hbar\, |S_\vartheta(x,\vartheta) - S_\vartheta(y,\vartheta)|)^{n+1}}. \tag{3.11}$$

To apply Young's inequality, we need to integrate the absolute value of (3.9) with respect to x. We postpone the integration with respect to ϑ and change the variables $x \mapsto \eta \equiv S_\vartheta(x,\vartheta)$:

$$\int dx = \int \frac{d\eta}{|\det(\frac{\partial\eta}{\partial x})|} = \int \frac{d\eta}{|\det(S_{x\vartheta})|}.$$

Therefore, the integration with respect to x of the expression (3.9), with the extra factor (3.11), gives $const\,(\lambda\hbar)^{-n}\hbar^{-1}$. (We recall that $|\det(S_{x\vartheta})| \ge \hbar/2$ on the support of (3.9).)

The integration $\int d\vartheta$ contributes $\hbar^n$, due to the size of ϑ-support of the integral kernel of $T^\hbar_\Theta$. Therefore, L^1-norm of $K(T^\hbar_\Theta T^{\hbar*}_\Theta)(x,y)$ with respect to x (or y) is bounded by $const\,\lambda^{-n}\hbar^{-1}$, and by Young's inequality $\|T^\hbar_\Theta T^{\hbar*}_\Theta\| \le const\,\lambda^{-n}\hbar^{-1}$. This concludes the proof of Lemma 3.2. $\square$

Lemma 3.3. *Let* $|\partial_{\vartheta_n} h| \ge const > 0$, *and let*

$$\sup_{\operatorname{supp}\psi}\left(\|S^{-1}_{x'\vartheta'}\||S_{x'\vartheta_n}|\right)\sup_{\operatorname{supp}\psi}\frac{|h_{\vartheta'}|}{|h_{\vartheta_n}|} \le \frac{1}{2}. \tag{3.12}$$

Then, as long as $\hbar \ge \hbar_o = \lambda^{-\frac{k}{2k+1}}$, *for some* $k \ge 1$, *the operators* $T^\hbar_\Theta$ *(and* $\bar T^\hbar_\Theta$*) are almost orthogonal:*

$$\left\|T^\hbar_\Theta\, T^{\hbar*}_W\right\|,\ \left\|T^{\hbar*}_\Theta\, T^\hbar_W\right\| \le const\,\lambda^{-n}\,\hbar^{-1} a(\Theta - W), \tag{3.13}$$

$$\left\|\bar T^\hbar_\Theta\, \bar T^{\hbar*}_W\right\|,\ \left\|\bar T^{\hbar*}_\Theta\, \bar T^\hbar_W\right\| \le const\,\lambda^{-n+1}\,\hbar^{\frac{k+1}{k}}\, a(\Theta - W). \tag{3.14}$$

Here $a \in C(\mathbb{Z}^n, \mathbb{R}_+)$ *is a function such that* $\sum_{\Theta\in\mathbb{Z}^n} a^{\frac{1}{2}}(\Theta) \le const$.

Remark. Before turning to the proof, we mention that the conditions of Lemma 3.3 are satisfied if π_L is a Whitney fold (if the direction ∂_{ϑ_n} is chosen to be in a small conical neighborhood of $\operatorname{Ker} d\pi_L$). In particular, they are satisfied for the localization of the operator (2.1) near $(\mathbf{r}_o, \omega_o)$, as long as $|x|$ and $|\vartheta|$ are small (so that $|b_x|$ and $|g_\vartheta|$ are not greater than 1/4).

Proof of Lemma 3.3. The proof is the same for the operators $T^\hbar_\Theta$ and $\bar T^\hbar_\Theta$, and we use a generic notation $T^\hbar_\Theta$ for both of them.

If $|W - \Theta| \ge 2\sqrt{n}$, then $\chi(\hbar^{-1}\vartheta - \Theta)\,\chi(\hbar^{-1}\vartheta - W) = 0$, and hence $T^\hbar_\Theta T^{\hbar*}_W = 0$.

Now we consider the composition $T^{\hbar*}_\Theta T^\hbar_W$. Its integral kernel is given by

$$\begin{aligned} K(T^{\hbar*}_\Theta T^\hbar_W)(\vartheta, w) = \int d^n x\, e^{-i\lambda(S(x,\vartheta)-S(x,w))}\psi(x,\vartheta)\psi(x,w) \\ \times \beta(\hbar^{-1}h(x,\vartheta))\,\beta(\hbar^{-1}h(x,w))\,\chi(\hbar^{-1}\vartheta - \Theta)\,\chi(\hbar^{-1}w - W). \end{aligned} \tag{3.15}$$

If the value of $|W - \Theta|$ is sufficiently large, then ϑ and w on the support of (3.15) satisfy the following relation:

$$|w_n - \vartheta_n| \le \sqrt{2}\,|w' - \vartheta'| \sup_{\operatorname{supp}\psi}\frac{|h_{\vartheta'}|}{|h_{\vartheta_n}|}. \tag{3.16}$$

Indeed, if the contrary is true, $|w_n-\vartheta_n|>\sqrt{2}\,|w'-\vartheta'|\sup\limits_{\operatorname{supp}\psi}\dfrac{|h_{\vartheta'}|}{|h_{\vartheta_n}|}$, then, writing $h(t)=h(x,\vartheta+t(w-\vartheta))$, we obtain:

$$
\begin{aligned}
|h(x,w)-h(x,\vartheta)| &\geq \left|\int_0^1 dt\,\frac{d}{dt}h(t)\right|\\
&\geq \int_0^1 dt\,\left(|w_n-\vartheta_n||h_{\vartheta_n}(t)|-|w'-\vartheta'|\,|h_{\vartheta'}(t)|\right)\\
&> (1-2^{-1/2})|w_n-\vartheta_n|\inf_{\operatorname{supp}\psi}|h_{\vartheta_n}|.
\end{aligned}
$$

The right-hand side is greater than $4\hbar$ if $|W-\Theta|$ is sufficiently large; then the functions $\beta(\hbar^{-1}h(x,\vartheta))$ and $\beta(\hbar^{-1}h(x,w))$ have no common support in x.

Therefore, we may assume that the inequality (3.16) is valid. We employ the operator L_x, given by

$$
L_x=\frac{1}{i\lambda}\cdot\frac{(S_x(x,w)-S_x(x,\vartheta))\cdot\nabla_x}{\left|S_x(x,w)-S_x(x,\vartheta)\right|^2}. \tag{3.17}
$$

We have:

$$
|S_{x'}(x,\vartheta)-S_{x'}(x,w)|\geq|w'-\vartheta'|-|w_n-\vartheta_n|\sup_{\operatorname{supp}\psi}|S_{x'\vartheta_n}|. \tag{3.18}
$$

(Recall that $S_{x'\vartheta'}$ is a unit matrix.) According to (3.12) and (3.16), we have the following bound from below for the right-hand side of (3.18):

$$
|S_{x'}(x,\vartheta)-S_{x'}(x,w)|\geq(1-2^{-1/2})\,|w'-\vartheta'|\geq \mathit{const}\,\hbar\,|W-\Theta|\,. \tag{3.19}
$$

We insert L_x^N into the integral kernel of $T_\Theta^{\hbar *}T_W^\hbar$ given by (3.15) and integrate by parts; this multiplies the integral kernel of $T_\Theta^{\hbar *}T_W^\hbar$ by $\mathit{const}\,(\lambda\cdot\hbar\cdot\hbar|W-\Theta|)^{-N}$. Hence, the L^1-norm of (3.15) with respect to ϑ (or w) is bounded by

$$
\int d\vartheta\,|K(T_\Theta^{\hbar *}T_W^\hbar)(\vartheta,w)|\leq \mathit{const}\,\hbar^n\left(\lambda\hbar^2\;|W-\Theta|\right)^{-N}. \tag{3.20}
$$

According to (3.7), we are only interested in the values of $\hbar$ not less than $\hbar_o\equiv\lambda^{-\frac{k}{2k+1}}$, hence $\lambda\hbar^2\geq\lambda^{\frac{1}{2k+1}}$. We conclude that the left-hand side of (3.20) would be smaller than any negative power of λ, if N were taken large enough. This proves the almost orthogonality relations stated in Lemma 3.3. □

4. Individual estimates near the critical variety

Now we proceed to deriving the most delicate estimate – the one on $\bar{T}_\Theta^\hbar$:

$$
\begin{gathered}
\bar{T}_\Theta^\hbar u(x)=\int e^{i\lambda S(x,\vartheta)}\,\psi(x,\vartheta)\,\bar\beta(\hbar^{-1}h(x,\vartheta))\,\chi(\hbar^{-1}\vartheta-\Theta)\,u(\vartheta)\,d\vartheta,\\
\bar\beta\in C^\infty_{\mathrm{comp}}([-2,2]),\quad \chi\in C^\infty_{\mathrm{comp}}(\mathbb{B}^n_1).
\end{gathered} \tag{4.1}
$$

Lemma 4.1. *If the boundary B admits tangent lines with contact of order not greater than k, with some constant $\varkappa > 0$ in* (1.4), *then*

$$\|\bar{T}^{\hbar}_{\Theta}\| \le const\,\lambda^{-\frac{n-1}{2}}\hbar^{\frac{1}{2}+\frac{1}{2k}}. \tag{4.2}$$

Proof. We fix $\Theta = (0,\dots,0) \in \mathbb{Z}^n$ and introduce the notation $\bar{\tau} \equiv \bar{T}^{\hbar}_{\Theta}$ (to consider other values of Θ, we may change local coordinates). We introduce the fine localization with respect to x, with the step $\delta = \hbar^{1/k}$:

$$\bar{\tau} = \sum_{X\in\mathbb{Z}^n} \bar{\tau}_X, \qquad \bar{\tau}_X = \chi(\delta^{-1}x - X)\circ\bar{\tau}, \tag{4.3}$$

with χ as above. The mixed Hessian of the phase function S is of rank at least $n-1$, while the x_n-support of $\bar{\tau}_X$ is of size $\hbar^{1/k}$, and ϑ_n-support is of size $\hbar$. We apply Hörmander's estimate on non-singular oscillatory integral operators in $n-1$ dimensions and then Young's inequality in x_n and ϑ_n, obtaining

$$\|\bar{\tau}_X\| \le const\,\lambda^{-\frac{n-1}{2}}(\hbar^{\frac{1}{k}}\hbar)^{\frac{1}{2}} \le const\,\lambda^{-\frac{n-1}{2}}\hbar^{\frac{k+1}{2k}}, \tag{4.4}$$

in an agreement with (4.2). Further, we claim that $\bar{\tau}_X$ are almost orthogonal for different $X \in \mathbb{Z}^n$, satisfying the inequalities

$$\|\bar{\tau}^*_X\bar{\tau}_Y\| \le const\,\lambda^{-n+1}\hbar^{1+\frac{1}{k}}a(X-Y), \tag{4.5}$$

$$\|\bar{\tau}_X\bar{\tau}^*_Y\| \le const\,\lambda^{-n+1}\hbar^{1+\frac{1}{k}}a(X-Y), \tag{4.6}$$

where $a \in C(\mathbb{Z}^n, \mathbb{R}_+)$ is a function satisfying $\sum_{X\in\mathbb{Z}^n} a^{\frac{1}{2}}(X) \le const$. By Cotlar-Stein lemma, the inequalities (4.4)-(4.6) would prove Lemma 4.1.

The inequality (4.5) is trivially satisfied, since if $|X-Y|$ is large enough (greater than $2\sqrt{n}$), then $\bar{\tau}^*_X \circ \bar{\tau}_Y = \bar{\tau}^* \circ \chi(\delta^{-1}x - X)\,\chi(\delta^{-1}x - Y)\circ\bar{\tau} = 0$.

Let us turn to the proof of the almost orthogonality relations (4.6). We consider the integral kernel of $\bar{\tau}_X\bar{\tau}^*_Y$:

$$K(\bar{\tau}_X\bar{\tau}^*_Y)(x,y) = \int d\vartheta\, e^{i\lambda(S(x,\vartheta)-S(y,\vartheta))} \times \dots. \tag{4.7}$$

For our convenience, we assume that $v_n = y_n - x_n \ge 0$ (or we interchange the points x and y in the argument below). We will consider two cases: when the vector $v = y - x \in \mathbb{R}^n_L$ is inside the cone of magnitude $2\hbar\sqrt{n}$ around the direction ∂_{x_n} (*the vertical case*), and when it is outside this cone (*the horizontal case*).

Vertical case. We recall that the determinant of the mixed Hessian of the phase function (2.4) can be written as

$$h(x,\vartheta) = \mathbf{k}_R b(x) - \partial_{\vartheta_n} g(\vartheta), \tag{4.8}$$

where the vector field $\mathbf{k}_R$ is defined by $\mathbf{k}_R = g_{\vartheta'}\cdot\nabla_{x'} + \partial_{x_n} \in T_x\mathbb{R}^n_L \times T_\vartheta\mathbb{R}^n_R$, $\mathbf{k}_R|_\Sigma \in \operatorname{Ker} d\pi_R$. Note that $\mathbf{k}_R$ does not depend on x and that $|g_{\vartheta'}(\vartheta)| \approx |\vartheta'| < \hbar\sqrt{n}$. This implies that on the support of $\bar{\tau}$, the vector field $\mathbf{k}_R$ is always inside the cone of magnitude $\hbar\sqrt{n}$ around the vector ∂_{x_n}.

Let $\mathbf{v} \in T\mathbb{R}^n_L \times T\mathbb{R}^n_R$ be the vector field generated by the vector $v = y - x$. Since $\mathbf{v}$ is within $2\hbar\sqrt{n}$-cone of ∂_{x_n}, while $\mathbf{k}_R$ is within $\hbar\sqrt{n}$-cone of ∂_{x_n}, the angle between $\mathbf{k}_R$ and $\mathbf{v}$ is at most $3\hbar\sqrt{n}$, and hence

$$\left|\mathbf{k}_R - \frac{|\mathbf{k}_R|}{|\mathbf{v}|}\mathbf{v}\right| \le 3\hbar\sqrt{n}|\mathbf{k}_R|. \tag{4.9}$$

Using (4.8) and (4.9), we derive that

$$\begin{aligned} |h(y,\vartheta) - h(x,\vartheta)| &= |\mathbf{k}_R b(x+v) - \mathbf{k}_R b(x)| \\ &\ge \frac{|\mathbf{k}_R|}{|\mathbf{v}|}\,|\mathbf{v}b(x+v) - \mathbf{v}b(x)| - 3\hbar\sqrt{n}|\mathbf{k}_R| \cdot 2\sup|\nabla_x b|. \end{aligned} \tag{4.10}$$

Technical Lemma. *If the graph of a convex function $b \in C^1(\mathbb{R}^n)$ admits tangent lines with contact of order not greater than k in a small convex open neighborhood $U \subset \mathbb{R}^n$, with some constant $\varkappa > 0$ in (1.4), then for any $x \in U$ and any vector v such that $|v|$ is sufficiently small and $x + v \in U$, there is the inequality*

$$|\mathbf{v}|^{-1}|\mathbf{v}b(x+v) - \mathbf{v}b(x)| \ge 2\varkappa|v|^k, \tag{4.11}$$

where $\mathbf{v} \in T\mathbb{R}^n$ is the vector field generated by the vector v.

Proof. It suffices to illustrate this statement in the one-dimensional case. Let I be an interval in $\mathbb{R}$. If the graph of a concave (*convex down*) function $b \in C^1(I)$ admits tangent lines with contact of order not greater than $k \in \mathbb{N}$, with some constant $\varkappa > 0$ in (1.4), then for any x and y in I one has

$$b(y) - b(x) - (y-x)b'(x) \ge \varkappa|y-x|^{k+1}. \tag{4.12}$$

The inequality (4.12) follows immediately from Definition 1.1 of contact of order not greater than k (at the point x). Interchanging x and y in (4.12), we get

$$b(x) - b(y) - (x-y)b'(y) \ge \varkappa|y-x|^{k+1}. \tag{4.13}$$

The summation of (4.12) and (4.13) yields $|b'(y) - b'(x)| \ge 2\varkappa|y-x|^k$. □

According to the assumption that B admits tangent lines with contact of order not greater than k and to Technical Lemma above (see (4.11)), the expression (4.10) is greater than $2\varkappa\,|v|^k$. Thus, (4.10) is bounded from below by

$$|h(y,\vartheta) - h(x,\vartheta)| \ge 2\varkappa\,|y-x|^k - 6\hbar\sqrt{n}\,|y-x| \cdot \sup|\nabla_x b|. \tag{4.14}$$

Since $|y-x| \approx \hbar^{\frac{1}{k}}\,|Y-X|$, we conclude that $|h(x,\vartheta) - h(y,\vartheta)| \ge 4\hbar$ if $|Y-X|$ is large, hence either x or y could not be on the support of (4.7).

Horizontal case. Thus, we are left to consider the case when the vector $v = y - x$ is outside the $2\hbar\sqrt{n}$-cone of ∂_{x_n}:

$$|v'| \ge 2\hbar\sqrt{n}|v_n|. \tag{4.15}$$

We are employing the operator L_ϑ as in (3.10). We need to find the bound from below for the difference $S_\vartheta(y,\vartheta)-S_\vartheta(x,\vartheta)$, which is in the denominator of L_ϑ. We have:

$$|S_\vartheta(y,\vartheta)-S_\vartheta(x,\vartheta)| \ge |v'| - \hbar\sqrt{n}|v_n|,$$

since $S_{x'\vartheta'}=\mathbb{I}_{n-1}$ and $|S_{x_n\vartheta'}|=|g_{\vartheta'}(\vartheta)|\approx|\vartheta'|<\hbar\sqrt{n}$. Then we use (4.15), getting

$$|S_\vartheta(y,\vartheta)-S_\vartheta(x,\vartheta)| \ge |y'-x'|/2. \tag{4.17}$$

Now we integrate in (4.7) by parts, with the aid of the operator L_ϑ. Each derivative ∇_ϑ contributes $\hbar^{-1}$ (including the case when the derivative acts on the denominator of L_ϑ itself). Hence, integration by parts adds the factor

$$const\,(1+\lambda\hbar\,|y'-x'|)^{-N}. \tag{4.18}$$

To apply Young's inequality, we integrate the integral kernel (4.7), with the extra factor (4.18), with respect to x (or y). This yields $(\lambda\hbar)^{-(n-1)}\hbar^{\frac{1}{k}}$ (recall that the support in x_n is $\delta=\hbar^{1/k}$). The integration in (4.7) with respect to ϑ yields $\hbar^n$. To control the almost orthogonality, we appeal to the factor (4.18) once again. Due to (4.15),

$$\lambda\hbar\,|y'-x'| \ge const\,\lambda\hbar^2|y-x| \approx const\,\lambda\hbar^{2+\frac{1}{k}}|Y-X|.$$

As long as $\hbar\ge\hbar_o=\lambda^{-\frac{k}{2k+1}}$, the factor $\lambda\hbar^{2+\frac{1}{k}}$ is not less than 1. We may assume we have spared $2n+1$ powers of $(1+\lambda\hbar|y'-x'|)^{-1}$ from (4.18) and substitute them by $(1+|Y-X|)^{-(2n+1)}$. We collect all the factors we mentioned:

$$\|\bar\tau_X\bar\tau_Y^*\| \le const\,\lambda^{-(n-1)}\hbar^{1+\frac{1}{k}}\,(1+|Y-X|)^{-(2n+1)}.$$

This proves the required almost orthogonality relations (4.6) for the operators $\bar\tau_X$ with different indices $X\in\mathbb{Z}^n$, and concludes the proof of Lemma 4.1. □

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