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New upper bounds on the smallest size of a complete arc in a finite Desarguesian projective plane

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Abstract. In the projective planes $\mathrm{PG}(2,q)$, more than 1230 new small complete arcs are obtained for $q \le 13627$ and $q \in G$ where G is a set of 38 values in the range $13687,\dots,45893$; also, $2^{18} \in G$. This implies new upper bounds on the smallest size $t_2(2,q)$ of a complete arc in $\mathrm{PG}(2,q)$. From the new bounds it follows that

$$t_2(2,q) < 4.5\sqrt{q} \text{ for } q \le 2647$$

and $q = 2659, 2663, 2683, 2693, 2753, 2801$. Also,

$$t_2(2,q) < 4.8\sqrt{q} \text{ for } q \le 5419$$

and $q = 5441, 5443, 5449, 5471, 5477, 5479, 5483, 5501, 5521$. Moreover,

$$t_2(2,q) < 5\sqrt{q} \text{ for } q \le 9497$$

and $q = 9539, 9587, 9613, 9623, 9649, 9689, 9923, 9973$. Finally,

$$t_2(2,q) < 5.15\sqrt{q} \text{ for } q \le 13627$$

and $q = 13687, 13697, 13711, 14009$. Using the new arcs it is shown that

$$t_2(2,q) < \sqrt{q}\ln^{0.73} q \text{ for } 109 \le q \le 13627 \text{ and } q \in G.$$

Also, as q grows, the positive difference $\sqrt{q}\ln^{0.73} q - \bar{t}_2(2,q)$ has a tendency to increase whereas the ratio $\bar{t}_2(2,q)/(\sqrt{q}\ln^{0.73} q)$ tends to decrease. Here $\bar{t}_2(2,q)$ is the smallest known size of a complete arc in $\mathrm{PG}(2,q)$. These properties allow us to conjecture that the estimate $t_2(2,q) < \sqrt{q}\ln^{0.73} q$ holds for all $q \ge 109$. The new upper bounds are obtained by finding new small complete arcs in $\mathrm{PG}(2,q)$ with the help of a computer search using randomized greedy algorithms. Finally, new forms of the upper bound on $t_2(2,q)$ are proposed.

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1. Introduction

Let $\mathrm{PG}(2,q)$ be the projective plane over the Galois field F_q. An n-arc is a set of n points no three of which are collinear. An n-arc is called complete if it is not contained in an $(n+1)$-arc of $\mathrm{PG}(2,q)$. For an introduction to projective geometries over finite fields, see [26,41,43].

In [27,28] the relationship among the theory of n-arcs, coding theory, and mathematical statistics is presented. In particular, a complete arc in a plane $\mathrm{PG}(2,q)$, the points of which are treated as 3-dimensional q-ary columns, defines a parity check matrix of a q-ary linear code with codimension 3, Hamming distance 4, and covering radius 2. Arcs can be interpreted as linear maximum distance separable (MDS) codes [47, Sec. 7], [49] and they are related to optimal coverings arrays [24] and to superregular matrices [29].

One of the main problems in the study of projective planes, which is also of interest in coding theory, is finding of the spectrum of possible sizes of complete arcs. In particular, the value of $t_2(2,q)$, the smallest size of a complete arc in $\mathrm{PG}(2,q)$, is interesting. Finding estimates of the minimum size $t_2(2,q)$ is a hard open problem.

This paper is devoted to *upper bounds* for $t_2(2,q)$.

Surveys of results on the sizes of plane complete arcs, methods of their construction and comprehension of the relating properties can be found in [3,8,9, 15,25–28,30,32,37–39,41–48]. Problems connected with small complete plane arcs are considered in [1–12,14–23,25–28,30–48,50–52]; see also the references in these articles.

The exact values of $t_2(2,q)$ are known only for $q \le 32$; see [34] and recent work [35] where the equalities $t_2(2,31) = t_2(2,32) = 14$ are established.

There are the following lower bounds (see [2,40,41]):

$$t_2(2,q) > \begin{cases} \sqrt{2q}+1 & \text{for any } q, \\ \sqrt{3q}+\frac{1}{2} & \text{for } q = p^h,\ p \text{ prime},\ h = 1,2,3. \end{cases}$$

Let $t(\mathcal{P}_q)$ be the size of the smallest complete arc in any (not necessarily Galois) projective plane $\mathcal{P}_q$ of order q. In [30], for sufficiently large q, the following result is proved by probabilistic methods (we give it in the form of [28, Tab. 2.6]):

$$t(\mathcal{P}_q) \le D\sqrt{q}\log^C q, C \le 300, \tag{1.1}$$

where C and D are constants independent of q. The logarithm basis is not noted as the estimate is asymptotic. The authors of [30] conjecture that the constant can be reduced to $C = 10$ but this conjecture is not proved. A survey and analysis of random constructions for geometrical objects can be found in [17]; see also the references therein.

Regarding complete arcs of sizes smaller than $\frac{1}{2}q$ obtained by algebraic constructions, we note (following [28, p. 209]) that complete arcs in $\mathrm{PG}(2,q)$ have

been constructed with sizes approximately $\frac{1}{3}q$ (see [1,3,31,44,45,52]), $\frac{1}{4}q$ (see [3,31,46]), $2q^{0.9}$ (see [44] where such arcs are constructed for $q > 7^{10}$). It is noted in [17, Sec. 8], that the smallest size of a complete arc in $\mathrm{PG}(2,q)$ obtained via algebraic constructions is $cq^{3/4}$ where c is a universal constant; see [46, Sec. 3] and [47, Th. 6.8].

Denote by $\bar{t}_2(2,q)$ the smallest *known* size of a complete arc in $\mathrm{PG}(2,q)$. Clearly,

$$t_2(2,q) \le \bar{t}_2(2,q). \tag{1.2}$$

For even $q = 2^h$, $11 \le h \le 15$, the smallest known sizes of complete n-arcs in $\mathrm{PG}(2,q)$ are obtained in [3,11]; see also [9, p. 35]. They are as follows:

$$\bar{t}_2(2,2^{11}) = 199, \quad \bar{t}_2(2,2^{12}) = 301, \quad \bar{t}_2(2,2^{13}) = 450 \qquad [3];$$
$$\bar{t}_2(2,2^{14}) = 665, \quad \bar{t}_2(2,2^{15}) = 993 \qquad [11].$$

Also, $(6\sqrt{q}-6)$-arcs in $\mathrm{PG}(2,q)$, $q = 4^{2h+1}$, are constructed in [10]; for $h \le 4$ it is proved that they are complete. It gives a complete 3066-arc in $\mathrm{PG}(2,2^{18})$. Note that we obtain $\bar{t}_2(2,2^{13}) = 449$ and $\bar{t}_2(2,2^{15}) = 987$ that improves the results of [3,11].

We denote the following sets of q values:

$$\begin{aligned}
T_2 &= \{5119, 5147, 5153, 5209, 5231, 5237, 5261, 5279, 5281, 5303, 5347,\\
&\quad 5641, 5843, 6011, 2^{13}, 2^{14}, 2^{15}, 2^{18}\};\\
Q &= \{961, 1024, 1369, 1681, 2401\} = \{31^2, 2^{10}, 37^2, 41^2, 7^4\};\\
G &= \{13687, 13697, 13711, 14009, 15013, 16001, 2^{14}, 17011, 18013, 19001,\\
&\quad 20011, 21001, 22003, 23011, 24001, 25013, 26003, 27011, 28001, 29009,\\
&\quad 30011, 31013, 32003, 2^{15}, 33013, 34019, 35023, 36007, 37003, 38011,\\
&\quad 39019, 40009, 41011, 42013, 43003, 44017, 45007, 45893, 2^{18}\}.
\end{aligned}$$

For $q \le 841$ the values of $\bar{t}_2(2,q)$ (up to August 2011) are collected in [3, Table 1] whence it follows that $\bar{t}_2(2,q) < 4\sqrt{q}$ for $q \le 841$. In [18], see also [22], complete $(4\sqrt{q}-4)$-arcs are obtained for $q = p^2$ odd, $q \le 1681$ or $q = 2401$. In [11,14] complete $(4\sqrt{q}-4)$-arcs are obtained for even $q = 64, 256, 1024$.

In this paper it is shown that $\bar{t}_2(2,857) = 117 < 4\sqrt{857}$. So,

$$t_2(2,q) < 4\sqrt{q} \quad \text{for } 2 \le q \le 841,\ q = 857,\ q \in Q. \tag{1.3}$$

For $q \le 5107$ and $q \in T_2$ the values of $\bar{t}_2(2,q)$ (up to August 2011) are collected in [3, Tables 1, 2, 3, 4], see also [3, Theorems 1.1, 1.2, 4.1], where the following results are obtained:

$$
\begin{aligned}
&t_2(2,q) < 4.5\sqrt{q} \quad \text{for } q \le 2593,\ q = 2693, 2753;\\
&t_2(2,q) < 4.79\sqrt{q} \quad \text{for } q \le 5107;\\
&t_2(2,q) < 4.98\sqrt{q} \quad \text{for } q \in T_2 \setminus \{2^{14}, 2^{15}, 2^{18}\};\\
&t_2(2,q) < 0.9987\sqrt{q}\ln^{0.75} q \quad \text{for } 23 \le q \le 5107, q \in T_2.
\end{aligned} \tag{1.4}
$$

Also, in [3, Conject. 1.3], it was conjectured that $t_2(2,q) < \sqrt{q}\ln^{0.75} q$ for all $q \ge 23$.

Here, we obtain more than 1230 new small complete arcs in $\mathrm{PG}(2,q)$ and extended and improved results of [3], including the ones of (1.4). In particular, we obtained complete arcs with sizes smaller than in [3]; that is, we improved the upper bounds for $t_2(2,q)$ of [3] for more than 230 values of q. These improvements are in the bold font in Tables 1, 2, 3, 4, 5, 6 of Sect. 3.

The following results have been shown.

Theorem 1.1. *In* $\mathrm{PG}(2,q)$, *the following hold:*

$t_2(2,q) < 3.8\sqrt{q}$	*for* $q \le 541$,	$q = 601, 661.$
$t_2(2,q) < 4\sqrt{q}$	*for* $q \le 841$,	$q = 857, 31^2, 2^{10}, 37^2, 41^2, 7^4.$
$t_2(2,q) < 4.3\sqrt{q}$	*for* $q \le 1627$,	$q = 1657, 1663, 41^2, 1697, 7^4.$
$t_2(2,q) < 4.5\sqrt{q}$	*for* $q \le 2647$,	$q = 2659, 2663, 2683, 2693, 2753, 2801.$
$t_2(2,q) < 4.8\sqrt{q}$	*for* $q \le 5419$,	$q = 5441, 5443, 5449, 5471, 5477, 5479,$ $5483, 5501, 5521.$
$t_2(2,q) < 5\sqrt{q}$	*for* $q \le 9497$,	$q = 9539, 9587, 9613, 9623, 9649, 9689,$ $9923, 9973.$
$t_2(2,q) < 5.15\sqrt{q}$	*for* $q \le 13627$,	$q = 13687, 13697, 13711, 14009.$
$t_2(2,q) < 5.5\sqrt{q}$	*for* $q \le 13627$,	$q \in G$ *with* $q \le 40009.$

Theorem 1.2. *In* $\mathrm{PG}(2,q)$, *the following hold:*

(i)

$$
t_2(2,q) < \sqrt{q}\ln^{0.73} q \quad \text{for } 109 \le q \le 13627, q \in G. \tag{1.5}
$$

(ii) *In particular,*

$$
\begin{aligned}
&t_2(2,q) < \sqrt{q}\ln^{0.72983} q \quad \text{for } 109 \le q \le 13627,\ q \in G;\\
&t_2(2,q) < \sqrt{q}\ln^{0.72969} q \quad \text{for } 3797 \le q \le 13627,\ q \in G;\\
&t_2(2,q) < \sqrt{q}\ln^{0.72953} q \quad \text{for } 5557 \le q \le 13627,\ q \in G;\\
&t_2(2,q) < \sqrt{q}\ln^{0.72904} q \quad \text{for } 6983 \le q \le 13627,\ q \in G;\\
&t_2(2,q) < \sqrt{q}\ln^{0.72891} q \quad \text{for } 7639 \le q \le 13627,\ q \in G;\\
&t_2(2,q) < \sqrt{q}\ln^{0.72766} q \quad \text{for } 8929 \le q \le 13627,\ q \in G;\\
&t_2(2,q) < \sqrt{q}\ln^{0.72703} q \quad \text{for } 9817 \le q \le 13627,\ q \in G;\\
&t_2(2,q) < \sqrt{q}\ln^{0.72694} q \quad \text{for } 10627 \le q \le 13627,\ q \in G;
\end{aligned}
$$

TABLE 1 The smallest known sizes $\bar{t}_2 = \bar{t}_2(2,q) < 4\sqrt{q}$ of complete arcs in planes PG$(2,q)$, $q \leq 841$. $A_q = \left\lfloor 4\sqrt{q} - \bar{t}_2(2,q) \right\rfloor$, $B_q \geq \bar{t}_2(2,q)/\sqrt{q}$

q	$\bar{t}_2$	A_q	B_q	q	$\bar{t}_2$	A_q	B_q	q	$\bar{t}_2$	A_q	B_q
2	4	1	2.83	193	46	9	3.32	503	85	4	3.79
3	4	2	2.31	197	47	9	3.35	509	85	5	3.77
4	6	2	3.00	199	47	9	3.34	**512**	**85**	**5**	**3.76**
5	6	2	2.69	211	49	9	3.38	521	86	5	3.77
7	6	4	2.27	223	51	8	3.42	523	86	5	3.77
8	6	5	2.13	227	51	9	3.39	529	87	5	3.79
9	6	6	2.00	229	51	9	3.38	**541**	**88**	**5**	**3.79**
11	7	6	2.12	233	52	9	3.41	547	89	4	3.81
13	8	6	2.22	239	53	8	3.43	557	90	4	3.82
16	9	7	2.25	241	53	9	3.42	563	91	3	3.84
17	10	6	2.43	243	53	9	3.40	569	91	4	3.82
19	10	7	2.30	251	55	8	3.48	**571**	**91**	**4**	**3.81**
23	10	9	2.09	256	55	9	3.44	577	92	4	3.84
25	12	8	2.40	257	55	9	3.44	587	93	3	3.84
27	12	8	2.31	263	56	8	3.46	593	94	3	3.87
29	13	8	2.42	269	57	8	3.48	599	94	3	3.85
31	14	8	2.52	271	57	8	3.47	601	90	8	3.68
32	14	8	2.48	277	58	8	3.49	607	95	3	3.86
37	15	9	2.47	281	59	8	3.52	613	96	3	3.88
41	16	9	2.50	283	59	8	3.51	617	96	3	3.87
43	16	10	2.45	289	60	8	3.53	619	96	3	3.86
47	18	9	2.63	293	60	8	3.51	625	96	4	3.84
49	18	10	2.58	307	62	8	3.54	631	97	3	3.87
53	18	11	2.48	311	63	7	3.58	641	98	3	3.88
59	20	10	2.61	313	63	7	3.57	**643**	**98**	**3**	**3.87**
61	20	11	2.57	317	63	8	3.54	647	99	2	3.90
64	22	10	2.75	331	65	7	3.58	**653**	**99**	**3**	**3.88**
67	23	9	2.81	337	66	7	3.60	659	100	2	3.90
71	22	11	2.62	343	66	8	3.57	661	90	12	3.51
73	24	10	2.81	347	67	7	3.60	673	101	2	3.90
79	26	9	2.93	349	67	7	3.59	677	102	2	3.93
81	26	10	2.89	353	68	7	3.62	683	102	2	3.91
83	27	9	2.97	**359**	**68**	**7**	**3.59**	691	103	2	3.92
89	28	9	2.97	361	69	7	3.64	701	104	1	3.93
97	30	9	3.05	**367**	**69**	**7**	**3.61**	709	104	2	3.91
101	30	10	2.99	373	70	7	3.63	**719**	**105**	**2**	**3.92**
103	31	9	3.06	379	71	6	3.65	727	106	1	3.94
107	32	9	3.10	383	71	7	3.63	729	104	4	3.86
109	32	9	3.07	389	72	6	3.66	733	107	1	3.96
113	33	9	3.11	397	73	6	3.67	739	107	1	3.94

TABLE 1 continued

q	$\bar{t}_2$	A_q	B_q	q	$\bar{t}_2$	A_q	B_q	q	$\bar{t}_2$	A_q	B_q
121	34	10	3.10	**401**	**73**	**7**	**3.65**	743	108	1	3.97
125	35	9	3.14	409	74	6	3.66	751	108	1	3.95
127	35	10	3.11	**419**	**75**	**6**	**3.67**	757	109	1	3.97
128	34	11	3.01	421	76	6	3.71	761	109	1	3.96
131	36	9	3.15	431	77	6	3.71	769	110	0	3.97
137	37	9	3.17	433	77	6	3.71	**773**	**110**	**1**	**3.96**
139	37	10	3.14	439	78	5	3.73	**787**	**111**	**1**	**3.96**
149	39	9	3.20	443	78	6	3.71	797	112	0	3.97
151	39	10	3.18	449	79	5	3.73	809	113	0	3.98
157	40	10	3.20	457	80	5	3.75	811	113	0	3.97
163	41	10	3.22	461	80	5	3.73	821	114	0	3.98
167	42	9	3.26	463	80	6	3.72	823	114	0	3.98
169	42	10	3.24	467	81	5	3.75	827	115	0	4.00
173	43	9	3.27	479	82	5	3.75	829	115	0	4.00
179	44	9	3.29	487	83	5	3.77	839	115	0	3.98
181	44	9	3.28	491	83	5	3.75	841	112	4	3.87
191	46	9	3.33	499	84	5	3.77				

$$\begin{aligned} t_2(2,q) &< \sqrt{q}\ln^{0.72687} q \quad \textit{for } 11927 \le q \le 13627,\ q \in G;\\ t_2(2,q) &< \sqrt{q}\ln^{0.72508} q \quad \textit{for } q \in G \textit{ with } 16001 \le q. \end{aligned} \tag{1.6}$$

(iii) *Alternatively,*

$$\begin{aligned} t_2(2,q) &< 0.99963\sqrt{q}\ln^{0.73} q \quad \textit{for } \ 109 \le q \le 13627,\ q \in G;\\ t_2(2,q) &< 0.99934\sqrt{q}\ln^{0.73} q \quad \textit{for } 3797 \le q \le 13627,\ q \in G;\\ t_2(2,q) &< 0.99897\sqrt{q}\ln^{0.73} q \quad \textit{for } 5557 \le q \le 13627,\ q \in G;\\ t_2(2,q) &< 0.99790\sqrt{q}\ln^{0.73} q \quad \textit{for } 6983 \le q \le 13627,\ q \in G;\\ t_2(2,q) &< 0.99760\sqrt{q}\ln^{0.73} q \quad \textit{for } 7639 \le q \le 13627,\ q \in G;\\ t_2(2,q) &< 0.99482\sqrt{q}\ln^{0.73} q \quad \textit{for } 8929 \le q \le 13627,\ q \in G;\\ t_2(2,q) &< 0.99340\sqrt{q}\ln^{0.73} q \quad \textit{for } 9817 \le q \le 13627,\ q \in G;\\ t_2(2,q) &< 0.99319\sqrt{q}\ln^{0.73} q \quad \textit{for } 10627 \le q \le 13627,\ q \in G;\\ t_2(2,q) &< 0.99299\sqrt{q}\ln^{0.73} q \quad \textit{for } 11927 \le q \le 13627,\ q \in G;\\ t_2(2,q) &< 0.98887\sqrt{q}\ln^{0.73} q \quad \textit{for } q \in G \textit{ with } 16001 \le q. \end{aligned} \tag{1.7}$$

Let $\bar{c}(q)$ be given by relation

$$\bar{t}_2(2,q) = \sqrt{q}\ln^{\bar{c}(q)} q. \tag{1.8}$$

We make the following observation on the values of $\bar{t}_2(2,q)$; see Figs. 1, 2, 3, 4 and Tables 1, 2, 3, 4, 5, 6.

TABLE 2 The smallest known sizes $\bar{t}_2 = \bar{t}_2(2,q) < 4.5\sqrt{q}$ of complete arcs in planes PG$(2,q)$, $853 \le q \le 2647$, $A_q = \lfloor a_q\sqrt{q} - \bar{t}_2(2,q) \rfloor$, $B_q \ge \bar{t}_2(2,q)/\sqrt{q}$

q	$\bar{t}_2$	A_q	B_q	q	$\bar{t}_2$	A_q	B_q	q	$\bar{t}_2$	A_q	B_q
853	117	14	4.01	1429	160	10	4.24	**2027**	**197**	**5**	**4.38**
857	*117*	*0*	*4.00*	**1433**	**160**	**10**	**4.23**	2029	198	4	4.40
859	118	13	4.03	1439	161	9	4.25	**2039**	**198**	**5**	**4.39**
863	118	14	4.02	**1447**	**161**	**10**	**4.24**	2048	199	4	4.40
877	119	14	4.02	1451	162	9	4.26	2053	199	4	4.40
881	**119**	**14**	**4.01**	1453	162	9	4.25	**2063**	**199**	**5**	**4.39**
883	120	13	4.04	1459	162	9	4.25	2069	200	4	4.40
887	120	14	4.03	1471	163	9	4.25	2081	201	4	4.41
907	122	13	4.06	1481	164	9	4.27	2083	201	4	4.41
911	122	13	4.05	1483	164	9	4.26	2087	201	4	4.40
919	**122**	**14**	**4.03**	1487	164	9	4.26	2089	201	4	4.40
929	**123**	**14**	**4.04**	1489	164	9	4.26	2099	202	4	4.41
937	124	13	4.06	**1493**	**164**	**9**	**4.25**	**2111**	**202**	**4**	**4.40**
941	**124**	**14**	**4.05**	1499	165	9	4.27	**2113**	**202**	**4**	**4.40**
947	125	13	4.07	**1511**	**165**	**9**	**4.25**	**2129**	**203**	**4**	**4.40**
953	**125**	**13**	**4.05**	**1523**	**166**	**9**	**4.26**	**2131**	**203**	**4**	**4.40**
961	*120*	*4*	*3.88*	1531	167	9	4.27	2137	204	4	4.42
967	**126**	**13**	**4.06**	1543	167	9	4.26	**2141**	**204**	**4**	**4.41**
971	127	13	4.08	1549	168	9	4.27	**2143**	**204**	**4**	**4.41**
977	127	13	4.07	**1553**	**168**	**9**	**4.27**	2153	205	3	4.42
983	128	13	4.09	1559	169	8	4.29	2161	205	4	4.41
991	127	14	4.04	**1567**	**169**	**9**	**4.27**	**2179**	**206**	**4**	**4.42**
997	129	13	4.09	**1571**	**169**	**9**	**4.27**	2187	207	3	4.43
1009	130	12	4.10	1579	170	8	4.28	**2197**	**207**	**3**	**4.42**
1013	130	13	4.09	**1583**	**170**	**9**	**4.28**	2203	208	3	4.44
1019	**130**	**13**	**4.08**	**1597**	**171**	**8**	**4.28**	2207	208	3	4.43
1021	131	12	4.10	**1601**	**171**	**9**	**4.28**	2209	208	3	4.43
1024	*124*	*4*	*3.88*	1607	172	8	4.30	**2213**	**208**	**3**	**4.43**
1031	**131**	**13**	**4.08**	1609	172	8	4.29	**2221**	**208**	**4**	**4.42**
1033	132	12	4.11	**1613**	**172**	**8**	**4.29**	**2237**	**209**	**3**	**4.42**
1039	132	13	4.10	1619	173	8	4.30	2239	210	2	4.44
1049	133	12	4.11	1621	173	8	4.30	2243	210	3	4.44
1051	133	12	4.11	**1627**	**173**	**8**	**4.29**	**2251**	**210**	**3**	**4.43**
1061	134	12	4.12	1637	174	8	4.31	2267	211	3	4.44
1063	134	12	4.11	1657	175	8	4.30	**2269**	**211**	**3**	**4.43**
1069	**134**	**13**	**4.10**	**1663**	**175**	**8**	**4.30**	**2273**	**211**	**3**	**4.43**
1087	136	12	4.13	1667	176	7	4.32	2281	212	2	4.44
1091	136	12	4.12	1669	176	7	4.31	**2287**	**212**	**3**	**4.44**
1093	136	12	4.12	*1681*	*160*	*4*	*3.91*	2293	213	2	4.45
1097	**136**	**13**	**4.11**	**1693**	**177**	**8**	**4.31**	2297	213	2	4.45

TABLE 2 continued

q	$\bar{t}_2$	A_q	B_q	q	$\bar{t}_2$	A_q	B_q	q	$\bar{t}_2$	A_q	B_q
1103	137	12	4.13	**1697**	**177**	**8**	**4.30**	**2309**	**213**	**3**	**4.44**
1109	**137**	**12**	**4.12**	1699	178	7	4.32	2311	214	2	4.46
1117	138	12	4.13	1709	178	8	4.31	2333	215	2	4.46
1123	**138**	**12**	**4.12**	1721	179	7	4.32	**2339**	**215**	**2**	**4.45**
1129	139	12	4.14	**1723**	**179**	**7**	**4.32**	**2341**	**214**	**3**	**4.43**
1151	**140**	**12**	**4.13**	1733	180	7	4.33	**2347**	**215**	**3**	**4.44**
1153	141	11	4.16	**1741**	**180**	**7**	**4.32**	2351	216	2	4.46
1163	**141**	**12**	**4.14**	1747	181	7	4.34	**2357**	**216**	**2**	**4.45**
1171	142	11	4.15	1753	181	7	4.33	2371	217	2	4.46
1181	143	11	4.17	**1759**	**181**	**7**	**4.32**	2377	216	3	4.44
1187	**143**	**12**	**4.16**	**1777**	**182**	**7**	**4.32**	2381	217	2	4.45
1193	144	11	4.17	1783	183	7	4.34	2383	218	1	4.47
1201	**144**	**11**	**4.16**	1787	183	7	4.33	2389	218	1	4.47
1213	145	11	4.17	**1789**	**183**	**7**	**4.33**	2393	218	2	4.46
1217	**145**	**11**	**4.16**	1801	184	6	4.34	**2399**	**218**	**2**	**4.46**
1223	146	11	4.18	1811	184	7	4.33	*2401*	*192*	*4*	*3.92*
1229	146	11	4.17	**1823**	**185**	**7**	**4.34**	**2411**	**219**	**1**	**4.47**
1231	**146**	**11**	**4.17**	1831	186	6	4.35	**2417**	**219**	**2**	**4.46**
1237	147	11	4.18	1847	187	6	4.36	2423	220	1	4.47
1249	148	11	4.19	1849	187	6	4.35	**2437**	**220**	**2**	**4.46**
1259	**148**	**11**	**4.18**	1861	188	6	4.36	2441	221	1	4.48
1277	150	10	4.20	1867	188	6	4.36	2447	221	1	4.47
1279	150	10	4.20	**1871**	**188**	**6**	**4.35**	2459	222	1	4.48
1283	150	11	4.19	**1873**	**188**	**6**	**4.35**	**2467**	**222**	**1**	**4.47**
1289	**150**	**11**	**4.18**	1877	189	5	4.37	**2473**	**222**	**1**	**4.47**
1291	151	10	4.21	1879	189	6	4.37	2477	223	0	4.49
1297	151	11	4.20	**1889**	**189**	**6**	**4.35**	2503	224	1	4.48
1301	**151**	**11**	**4.19**	1901	190	6	4.36	2521	225	0	4.49
1303	151	11	4.19	**1907**	**190**	**6**	**4.36**	**2531**	**225**	**1**	**4.48**
1307	152	10	4.21	1913	191	5	4.37	2539	226	0	4.49
1319	**152**	**11**	**4.19**	1931	192	5	4.37	2543	226	0	4.49
1321	153	10	4.21	1933	192	5	4.37	2549	226	1	4.48
1327	153	10	4.21	1949	193	5	4.38	2551	227	0	4.50
1331	**153**	**11**	**4.20**	1951	193	5	4.37	2557	227	0	4.49
1361	**155**	**11**	**4.21**	**1973**	**194**	**5**	**4.37**	2579	228	0	4.49
1367	156	10	4.22	1979	195	5	4.39	2591	229	0	4.50
1369	*144*	*4*	*3.90*	**1987**	**195**	**5**	**4.38**	2593	229	0	4.50
1373	**156**	**10**	**4.22**	**1993**	**195**	**5**	**4.37**	**2609**	**229**	**0**	**4.49**
1381	157	10	4.23	1997	196	5	4.39	**2617**	**230**	**0**	**4.50**
1399	158	10	4.23	1999	196	5	4.39	**2621**	**230**	**0**	**4.50**
1409	159	9	4.24	**2003**	**196**	**5**	**4.38**	**2633**	**230**	**0**	**4.49**
1423	160	9	4.25	2011	197	4	4.40	**2647**	**231**	**0**	**4.49**
1427	160	9	4.24	2017	197	5	4.39				

TABLE 3 The smallest known sizes $\bar{t}_2 = \bar{t}_2(2,q) < 4.8\sqrt{q}$ of complete arcs in planes PG$(2,q)$, $2657 \le q \le 5419$, $A_q = \left\lfloor a_q\sqrt{q} - \bar{t}_2(2,q) \right\rfloor$, $B_q \ge \bar{t}_2(2,q)/\sqrt{q}$

q	$\bar{t}_2$	A_q	B_q	q	$\bar{t}_2$	A_q	B_q	q	$\bar{t}_2$	A_q	B_q
2657	**232**	**25**	**4.51**	3539	275	22	4.63	4463	315	19	4.72
2659	*232*	*0*	*4.50*	**3541**	**275**	**22**	**4.63**	4481	315	19	4.71
2663	*232*	*0*	*4.50*	**3547**	**275**	**22**	**4.62**	**4483**	**316**	**18**	**4.72**
2671	233	25	4.51	**3557**	**276**	**22**	**4.63**	4489	316	19	4.72
2677	**233**	**25**	**4.51**	3559	276	22	4.63	4493	317	18	4.73
2683	*233*	*0*	*4.50*	3571	277	21	4.64	**4507**	**317**	**18**	**4.73**
2687	234	25	4.52	**3581**	**277**	**22**	**4.63**	**4513**	**317**	**18**	**4.72**
2689	234	25	4.52	3583	277	22	4.63	4517	317	19	4.72
2693	*233*	*0*	*4.49*	3593	278	21	4.64	4519	318	18	4.74
2699	**234**	**25**	**4.51**	3607	278	22	4.63	4523	318	18	4.73
2707	235	25	4.52	**3613**	**278**	**22**	**4.63**	4547	319	18	4.74
2711	235	25	4.52	3617	278	22	4.63	4549	319	18	4.73
2713	235	25	4.52	3623	279	21	4.64	4561	319	18	4.73
2719	**235**	**25**	**4.51**	**3631**	**279**	**22**	**4.64**	4567	320	17	4.74
2729	236	25	4.52	3637	280	21	4.65	4583	320	18	4.73
2731	236	25	4.52	3643	278	23	4.61	4591	321	17	4.74
2741	**236**	**25**	**4.51**	3659	281	21	4.65	4597	321	18	4.74
2749	237	25	4.53	**3671**	**281**	**21**	**4.64**	**4603**	**321**	**18**	**4.74**
2753	*236*	*0*	*4.50*	**3673**	**280**	**23**	**4.63**	**4621**	**321**	**18**	**4.73**
2767	238	25	4.53	**3677**	**281**	**22**	**4.64**	4637	322	18	4.73
2777	238	25	4.52	**3691**	**282**	**21**	**4.65**	4639	323	17	4.75
2789	239	25	4.53	**3697**	**282**	**22**	**4.64**	4643	323	17	4.75
2791	239	25	4.53	**3701**	**282**	**22**	**4.64**	4649	323	17	4.74
2797	**239**	**25**	**4.52**	3709	283	21	4.65	4651	323	17	4.74
2801	*238*	*0*	*4.50*	**3719**	**283**	**21**	**4.65**	4657	323	18	4.74
2803	240	24	4.54	**3721**	**283**	**22**	**4.64**	4663	323	18	4.74
2809	240	25	4.53	3727	284	21	4.66	**4673**	**323**	**18**	**4.73**
2819	**240**	**25**	**4.53**	3733	284	21	4.65	4679	324	18	4.74
2833	**240**	**26**	**4.51**	3739	283	22	4.63	4691	324	18	4.74
2837	**241**	**25**	**4.53**	**3761**	**284**	**22**	**4.64**	4703	325	17	4.74
2843	242	24	4.54	**3767**	**285**	**21**	**4.65**	4721	326	17	4.75
2851	**242**	**24**	**4.54**	3769	286	20	4.66	4723	326	17	4.75
2857	**242**	**25**	**4.53**	3779	286	21	4.66	**4729**	**325**	**18**	**4.73**
2861	243	24	4.55	3793	287	20	4.67	4733	326	17	4.74
2879	**243**	**25**	**4.53**	3797	287	21	4.66	**4751**	**327**	**17**	**4.75**
2887	243	25	4.53	3803	287	21	4.66	4759	327	17	4.75
2897	**244**	**25**	**4.54**	3821	288	21	4.66	4783	328	17	4.75
2903	245	24	4.55	**3823**	**288**	**21**	**4.66**	4787	329	16	4.76
2909	245	24	4.55	**3833**	**288**	**21**	**4.66**	4789	329	17	4.76
2917	**245**	**25**	**4.54**	**3847**	**288**	**22**	**4.65**	**4793**	**329**	**17**	**4.76**

TABLE 3 continued

q	$\bar{t}_2$	A_q	B_q	q	$\bar{t}_2$	A_q	B_q	q	$\bar{t}_2$	A_q	B_q
2927	245	25	4.53	**3851**	**288**	**22**	**4.65**	**4799**	**329**	**17**	**4.75**
2939	246	25	4.54	3853	289	21	4.66	4801	329	17	4.75
2953	**246**	**25**	**4.53**	3863	290	20	4.67	4813	330	16	4.76
2957	**247**	**24**	**4.55**	**3877**	**290**	**21**	**4.66**	4817	329	18	4.75
2963	248	24	4.56	**3881**	**290**	**21**	**4.66**	4831	329	18	4.74
2969	248	24	4.56	3889	291	20	4.67	4861	331	17	4.75
2971	**247**	**25**	**4.54**	3907	292	20	4.68	4871	332	16	4.76
2999	**249**	**24**	**4.55**	3911	292	20	4.67	4877	332	17	4.76
3001	**249**	**24**	**4.55**	**3917**	**291**	**21**	**4.65**	4889	333	16	4.77
3011	**250**	**24**	**4.56**	3919	292	21	4.67	**4903**	**333**	**17**	**4.76**
3019	**250**	**24**	**4.55**	**3923**	**292**	**21**	**4.67**	**4909**	**332**	**18**	**4.74**
3023	**249**	**25**	**4.53**	3929	293	20	4.68	4913	334	16	4.77
3037	**251**	**24**	**4.56**	3931	293	20	4.68	4919	334	16	4.77
3041	**251**	**24**	**4.56**	**3943**	**293**	**20**	**4.67**	**4931**	**334**	**17**	**4.76**
3049	252	24	4.57	**3947**	**293**	**21**	**4.67**	**4933**	**334**	**17**	**4.76**
3061	**252**	**24**	**4.56**	3967	294	20	4.67	4937	335	16	4.77
3067	253	23	4.57	**3989**	**295**	**20**	**4.68**	4943	334	17	4.76
3079	253	24	4.56	4001	296	20	4.68	4951	335	16	4.77
3083	253	24	4.56	4003	296	20	4.68	4957	335	17	4.76
3089	254	23	4.58	**4007**	**296**	**20**	**4.68**	4967	336	16	4.77
3109	255	23	4.58	4013	296	20	4.68	4969	336	16	4.77
3119	255	24	4.57	4019	296	20	4.67	**4973**	**336**	**16**	**4.77**
3121	255	24	4.57	4021	296	21	4.67	4987	336	17	4.76
3125	256	23	4.58	4027	296	21	4.67	4993	337	16	4.77
3137	**256**	**24**	**4.58**	4049	298	20	4.69	**4999**	**337**	**16**	**4.77**
3163	257	24	4.57	**4051**	**298**	**20**	**4.69**	5003	337	16	4.77
3167	258	23	4.59	4057	298	20	4.68	5009	337	16	4.77
3169	258	23	4.59	4073	299	20	4.69	5011	337	16	4.77
3181	**258**	**24**	**4.58**	**4079**	**299**	**20**	**4.69**	5021	338	16	4.78
3187	258	24	4.58	4091	300	19	4.70	**5023**	**338**	**16**	**4.77**
3191	259	23	4.59	4093	300	19	4.69	**5039**	**338**	**16**	**4.77**
3203	259	23	4.58	**4096**	**300**	**20**	**4.69**	5041	339	16	4.78
3209	260	23	4.59	4099	300	20	4.69	**5051**	**339**	**16**	**4.77**
3217	**260**	**23**	**4.59**	**4111**	**300**	**20**	**4.68**	5059	339	16	4.77
3221	**260**	**23**	**4.59**	**4127**	**301**	**20**	**4.69**	**5077**	**340**	**16**	**4.78**
3229	260	24	4.58	**4129**	**301**	**20**	**4.69**	**5081**	**340**	**16**	**4.77**
3251	262	23	4.60	**4133**	**301**	**20**	**4.69**	**5087**	**340**	**16**	**4.77**
3253	261	24	4.58	4139	302	19	4.70	**5099**	**341**	**16**	**4.78**
3257	262	23	4.60	4153	301	21	4.68	**5101**	**341**	**16**	**4.78**
3259	**262**	**23**	**4.59**	**4157**	**302**	**20**	**4.69**	5107	341	16	4.78
3271	**262**	**23**	**4.59**	4159	302	20	4.69	**5113**	**341**	**16**	**4.77**
3299	264	23	4.60	4177	303	20	4.69	5119	341	16	4.77
3301	264	23	4.60	**4201**	**304**	**20**	**4.70**	5147	343	15	4.79

TABLE 3 continued

q	$\bar{t}_2$	A_q	B_q	q	$\bar{t}_2$	A_q	B_q	q	$\bar{t}_2$	A_q	B_q
3307	**264**	**23**	**4.60**	4211	305	19	4.71	**5153**	**342**	**16**	**4.77**
3313	265	22	4.61	4217	305	19	4.70	**5167**	**344**	**15**	**4.79**
3319	265	23	4.60	4219	305	19	4.70	**5171**	**344**	**15**	**4.79**
3323	265	23	4.60	4229	305	20	4.70	**5179**	**344**	**15**	**4.79**
3329	**265**	**23**	**4.60**	4231	306	19	4.71	**5189**	**344**	**16**	**4.78**
3331	**265**	**23**	**4.60**	4241	306	19	4.70	**5197**	**345**	**15**	**4.79**
3343	265	24	4.59	4243	306	19	4.70	**5209**	**345**	**15**	**4.79**
3347	266	23	4.60	4253	306	20	4.70	**5227**	**346**	**15**	**4.79**
3359	267	22	4.61	4259	307	19	4.71	**5231**	**346**	**15**	**4.79**
3361	267	22	4.61	4261	307	19	4.71	**5233**	**346**	**15**	**4.79**
3371	**267**	**23**	**4.60**	4271	307	19	4.70	**5237**	**346**	**15**	**4.79**
3373	**267**	**23**	**4.60**	4273	306	20	4.69	**5261**	**347**	**15**	**4.79**
3389	268	23	4.61	4283	308	19	4.71	**5273**	**348**	**15**	**4.80**
3391	**267**	**24**	**4.59**	4289	308	19	4.71	**5279**	**348**	**15**	**4.79**
3407	**269**	**22**	**4.61**	4297	308	19	4.70	**5281**	**348**	**15**	**4.79**
3413	269	23	4.61	4327	310	18	4.72	**5297**	**349**	**14**	**4.80**
3433	270	22	4.61	**4337**	**310**	**19**	**4.71**	5303	349	15	4.80
3449	**271**	**22**	**4.62**	**4339**	**310**	**19**	**4.71**	**5309**	**349**	**15**	**4.79**
3457	271	22	4.61	4349	311	18	4.72	**5323**	**349**	**15**	**4.79**
3461	**271**	**23**	**4.61**	4357	311	19	4.72	**5329**	**350**	**15**	**4.80**
3463	272	22	4.63	4363	310	20	4.70	**5333**	**349**	**16**	**4.78**
3467	272	22	4.62	4373	312	18	4.72	**5347**	**350**	**15**	**4.79**
3469	272	22	4.62	**4391**	**312**	**19**	**4.71**	**5351**	**350**	**15**	**4.79**
3481	272	23	4.62	4397	313	18	4.73	**5381**	**352**	**14**	**4.80**
3491	273	22	4.63	**4409**	**313**	**19**	**4.72**	**5387**	**352**	**14**	**4.80**
3499	273	22	4.62	4421	314	18	4.73	**5393**	**352**	**15**	**4.80**
3511	274	22	4.63	4423	312	20	4.70	**5399**	**352**	**15**	**4.80**
3517	274	22	4.63	**4441**	**314**	**19**	**4.72**	**5407**	**352**	**15**	**4.79**
3527	**274**	**22**	**4.62**	4447	314	19	4.71	**5413**	**353**	**14**	**4.80**
3529	275	22	4.63	**4451**	**315**	**18**	**4.73**	**5417**	**353**	**15**	**4.80**
3533	275	22	4.63	4457	315	18	4.72	**5419**	**353**	**15**	**4.80**

Observation 1.3. (i) *The following inequalities hold:*

$$\sqrt{q}\ln^{0.73} q - \bar{t}_2(2,q) > 0,\ \frac{\bar{t}_2(2,q)}{\sqrt{q}\ln^{0.73} q} < 1,\ \bar{c}(q) < 0.73,$$
$$\text{for } 109 \le q \le 13627,\ q \in G.$$

(ii) *In the interval* $3797 \le q \le 13627$ *and* $q \in G$, *when* q *increases, the positive difference* $\sqrt{q}\ln^{0.73} q - \bar{t}_2(2,q)$ *tends to increase, whereas the ratio* $\bar{t}_2(2,q)/(\sqrt{q}\ln^{0.73} q)$ *and the value of* $\bar{c}(q)$ *tend to decrease.*

In the region $q \le 13627$ and $q \in G$, see Theorem 1.2, we confirm the conjecture of [3, Conject. 1.3] written after relations (1.4). Moreover, on the base

TABLE 4 The smallest known sizes $\bar{t}_2 = \bar{t}_2(2,q) < 5\sqrt{q}$ of complete arcs in planes PG$(2,q)$, $5431 \le q \le 9497$, $A_q = \left\lfloor 5\sqrt{q} - \bar{t}_2(2,q) \right\rfloor$, $B_q \ge \bar{t}_2(2,q)/\sqrt{q}$

q	$\bar{t}_2$	A_q	B_q	q	$\bar{t}_2$	A_q	B_q	q	$\bar{t}_2$	A_q	B_q
5431	354	14	4.81	6763	402	9	4.89	8161	448	3	4.96
5437	354	14	4.81	6779	402	9	4.89	8167	448	3	4.96
5441	*354*	*14*	*4.80*	6781	402	9	4.89	8171	448	3	4.96
5443	*354*	*14*	*4.80*	6791	403	9	4.90	8179	448	4	4.96
5449	*354*	*15*	*4.80*	6793	403	9	4.89	8191	446	6	4.93
5471	*355*	*14*	*4.80*	6803	402	10	4.88	**8192**	**449**	**3**	**4.97**
5477	*355*	*15*	*4.80*	6823	404	9	4.90	8209	448	5	4.95
5479	*355*	*15*	*4.80*	6827	404	9	4.89	8219	450	3	4.97
5483	*355*	*15*	*4.80*	6829	404	9	4.89	8221	449	4	4.96
5501	*356*	*14*	*4.80*	6833	404	9	4.89	8231	450	3	4.97
5503	357	13	4.82	6841	404	9	4.89	8233	449	4	4.95
5507	357	14	4.82	6857	405	9	4.90	8237	450	3	4.96
5519	357	14	4.81	6859	405	9	4.90	8243	449	4	4.95
5521	*356*	*15*	*4.80*	6863	405	9	4.89	8263	451	3	4.97
5527	357	14	4.81	6869	405	9	4.89	8269	449	5	4.94
5531	358	13	4.82	6871	405	9	4.89	8273	451	3	4.96
5557	358	14	4.81	6883	406	8	4.90	8287	450	5	4.95
5563	359	13	4.82	6889	406	9	4.90	8291	452	3	4.97
5569	359	14	4.82	6899	406	9	4.89	8293	452	3	4.97
5573	359	14	4.81	6907	407	8	4.90	8297	452	3	4.97
5581	359	14	4.81	6911	407	8	4.90	8311	452	3	4.96
5591	359	14	4.81	6917	406	9	4.89	8317	453	2	4.97
5623	361	13	4.82	6947	408	8	4.90	8329	453	3	4.97
5639	362	13	4.83	6949	408	8	4.90	8353	454	2	4.97
5641	**362**	**13**	**4.82**	6959	408	9	4.90	8363	454	3	4.97
5647	362	13	4.82	6961	408	9	4.90	8369	454	3	4.97
5651	362	13	4.82	6967	409	8	4.91	8377	454	3	4.97
5653	362	13	4.82	6971	409	8	4.90	8387	455	2	4.97
5657	362	14	4.82	6977	409	8	4.90	8389	454	3	4.96
5659	362	14	4.82	6983	408	9	4.89	8419	456	2	4.97
5669	363	13	4.83	6991	409	9	4.90	8423	456	2	4.97
5683	363	13	4.82	6997	409	9	4.89	8429	455	4	4.96
5689	363	14	4.82	7001	409	9	4.89	8431	456	3	4.97
5693	363	14	4.82	7013	410	8	4.90	8443	457	2	4.98
5701	364	13	4.83	7019	410	8	4.90	8447	456	3	4.97
5711	363	14	4.81	7027	410	9	4.90	8461	456	3	4.96
5717	364	14	4.82	7039	410	9	4.89	8467	457	3	4.97
5737	365	13	4.82	7043	411	8	4.90	8501	457	4	4.96
5741	365	13	4.82	7057	410	10	4.89	8513	457	4	4.96
5743	365	13	4.82	7069	412	8	4.91	8521	459	2	4.98

Table 4 continued

q	$\bar{t}_2$	A_q	B_q	q	$\bar{t}_2$	A_q	B_q	q	$\bar{t}_2$	A_q	B_q
5749	365	14	4.82	7079	412	8	4.90	8527	458	3	4.96
5779	367	13	4.83	7103	412	9	4.89	8537	460	1	4.98
5783	366	14	4.82	7109	413	8	4.90	8539	458	4	4.96
5791	367	13	4.83	7121	414	7	4.91	8543	460	2	4.98
5801	367	13	4.82	7127	414	8	4.91	8563	460	2	4.98
5807	368	13	4.83	7129	414	8	4.91	8573	460	2	4.97
5813	366	15	4.81	7151	415	7	4.91	8581	460	3	4.97
5821	368	13	4.83	7159	415	8	4.91	8597	461	2	4.98
5827	369	12	4.84	7177	416	7	4.92	8599	461	2	4.98
5839	369	13	4.83	7187	415	8	4.90	8609	461	2	4.97
5843	**368**	**14**	**4.82**	7193	415	9	4.90	8623	462	2	4.98
5849	369	13	4.83	7207	417	7	4.92	8627	462	2	4.98
5851	369	13	4.83	7211	417	7	4.92	8629	462	2	4.98
5857	370	12	4.84	7213	417	7	4.91	8641	462	2	4.98
5861	370	12	4.84	7219	417	7	4.91	8647	462	2	4.97
5867	370	12	4.84	7229	418	7	4.92	8663	463	2	4.98
5869	370	13	4.83	7237	418	7	4.92	8669	463	2	4.98
5879	370	13	4.83	7243	418	7	4.92	8677	463	2	4.98
5881	371	12	4.84	7247	416	9	4.89	8681	464	1	4.99
5897	371	12	4.84	7253	418	7	4.91	8689	464	2	4.98
5903	371	13	4.83	7283	418	8	4.90	8693	464	2	4.98
5923	372	12	4.84	7297	420	7	4.92	8699	464	2	4.98
5927	372	12	4.84	7307	418	9	4.89	8707	464	2	4.98
5939	373	12	4.85	7309	420	7	4.92	8713	464	2	4.98
5953	372	13	4.83	7321	421	6	4.93	8719	464	2	4.97
5981	374	12	4.84	7331	421	7	4.92	8731	464	3	4.97
5987	374	12	4.84	7333	421	7	4.92	8737	466	1	4.99
6007	375	12	4.84	7349	422	6	4.93	8741	466	1	4.99
6011	**373**	**14**	**4.82**	7351	422	6	4.93	8747	466	1	4.99
6029	375	13	4.83	7369	422	7	4.92	8753	466	1	4.99
6037	376	12	4.84	7393	423	6	4.92	8761	465	3	4.97
6043	376	12	4.84	7411	423	7	4.92	8779	467	1	4.99
6047	377	11	4.85	7417	423	7	4.92	8783	466	2	4.98
6053	377	12	4.85	7433	424	7	4.92	8803	467	2	4.98
6067	376	13	4.83	7451	422	9	4.89	8807	466	3	4.97
6073	377	12	4.84	7457	424	7	4.92	8819	468	1	4.99
6079	378	11	4.85	7459	425	6	4.93	8821	468	1	4.99
6089	378	12	4.85	7477	426	6	4.93	8831	469	0	5.00
6091	378	12	4.85	7481	426	6	4.93	8837	468	2	4.98
6101	379	11	4.86	7487	426	6	4.93	8839	468	2	4.98
6113	379	11	4.85	7489	426	6	4.93	8849	468	2	4.98
6121	379	12	4.85	7499	426	6	4.92	8861	470	0	5.00
6131	380	11	4.86	7507	427	6	4.93	8863	468	2	4.98

TABLE 4 continued

q	$\bar{t}_2$	A_q	B_q	q	$\bar{t}_2$	A_q	B_q	q	$\bar{t}_2$	A_q	B_q
6133	380	11	4.86	7517	427	6	4.93	8867	470	0	5.00
6143	379	12	4.84	7523	426	7	4.92	8887	469	2	4.98
6151	380	12	4.85	7529	426	7	4.91	8893	470	1	4.99
6163	380	12	4.85	7537	428	6	4.93	8923	471	1	4.99
6173	381	11	4.85	7541	428	6	4.93	8929	471	1	4.99
6197	382	11	4.86	7547	428	6	4.93	8933	471	1	4.99
6199	381	12	4.84	7549	428	6	4.93	8941	471	1	4.99
6203	381	12	4.84	7559	428	6	4.93	8951	471	2	4.98
6211	383	11	4.86	7561	429	5	4.94	8963	472	1	4.99
6217	383	11	4.86	7573	429	6	4.93	8969	472	1	4.99
6221	383	11	4.86	7577	428	7	4.92	8971	471	2	4.98
6229	383	11	4.86	7583	429	6	4.93	8999	471	3	4.97
6241	384	11	4.87	7589	428	7	4.92	9001	472	2	4.98
6247	384	11	4.86	7591	430	5	4.94	9007	473	1	4.99
6257	384	11	4.86	7603	430	5	4.94	9011	472	2	4.98
6263	384	11	4.86	7607	430	6	4.94	9013	473	1	4.99
6269	384	11	4.85	7621	431	5	4.94	9029	473	2	4.98
6271	384	11	4.85	7639	431	6	4.94	9041	474	1	4.99
6277	385	11	4.86	7643	431	6	4.93	9043	474	1	4.99
6287	385	11	4.86	7649	431	6	4.93	9049	474	1	4.99
6299	385	11	4.86	7669	431	6	4.93	9059	475	0	5.00
6301	386	10	4.87	7673	432	5	4.94	9067	475	1	4.99
6311	386	11	4.86	7681	432	6	4.93	9091	475	1	4.99
6317	385	12	4.85	7687	433	5	4.94	9103	475	2	4.98
6323	387	10	4.87	7691	433	5	4.94	9109	476	1	4.99
6329	387	10	4.87	7699	433	5	4.94	9127	477	0	5.00
6337	386	12	4.85	7703	433	5	4.94	9133	477	0	5.00
6343	387	11	4.86	7717	434	5	4.95	9137	477	0	5.00
6353	388	10	4.87	7723	433	6	4.93	9151	476	2	4.98
6359	387	11	4.86	7727	434	5	4.94	9157	477	1	4.99
6361	388	10	4.87	7741	434	5	4.94	9161	478	0	5.00
6367	388	10	4.87	7753	435	5	4.95	9173	478	0	5.00
6373	388	11	4.87	7757	433	7	4.92	9181	477	2	4.98
6379	389	10	4.88	7759	435	5	4.94	9187	478	1	4.99
6389	388	11	4.86	7789	435	6	4.93	9199	477	2	4.98
6397	389	10	4.87	7793	436	5	4.94	9203	479	0	5.00
6421	390	10	4.87	7817	437	5	4.95	9209	479	0	5.00
6427	390	10	4.87	7823	436	6	4.93	9221	480	0	5.00
6449	391	10	4.87	7829	437	5	4.94	9227	480	0	5.00
6451	390	11	4.86	7841	438	4	4.95	9239	480	0	5.00
6469	391	11	4.87	7853	438	5	4.95	9241	480	0	5.00
6473	392	10	4.88	7867	438	5	4.94	9257	481	0	5.00
6481	392	10	4.87	7873	437	6	4.93	9277	481	0	5.00

Table 4 continued

q	$\bar{t}_2$	A_q	B_q	q	$\bar{t}_2$	A_q	B_q	q	$\bar{t}_2$	A_q	B_q
6491	393	9	4.88	7877	438	5	4.94	9281	480	1	4.99
6521	392	11	4.86	7879	438	5	4.94	9283	481	0	5.00
6529	393	11	4.87	7883	438	5	4.94	9293	482	0	5.00
6547	394	10	4.87	7901	440	4	4.96	9311	481	1	4.99
6551	395	9	4.89	7907	440	4	4.95	9319	482	0	5.00
6553	395	9	4.88	7919	439	5	4.94	9323	482	0	5.00
6561	395	10	4.88	7921	440	5	4.95	9337	483	0	5.00
6563	395	10	4.88	7927	440	5	4.95	9341	483	0	5.00
6569	395	10	4.88	7933	440	5	4.95	9343	483	0	5.00
6571	394	11	4.87	7937	440	5	4.94	9349	483	0	5.00
6577	396	9	4.89	7949	441	4	4.95	9371	483	1	4.99
6581	396	9	4.89	7951	441	4	4.95	9377	484	0	5.00
6599	396	10	4.88	7963	441	5	4.95	9391	484	0	5.00
6607	397	9	4.89	7993	441	6	4.94	9397	484	0	5.00
6619	396	10	4.87	8009	443	4	4.96	9403	484	0	5.00
6637	398	9	4.89	8011	443	4	4.95	9409	484	1	4.99
6653	398	9	4.88	8017	443	4	4.95	9413	485	0	5.00
6659	398	10	4.88	8039	444	4	4.96	9419	484	1	4.99
6661	398	10	4.88	8053	444	4	4.95	9421	485	0	5.00
6673	399	9	4.89	8059	445	3	4.96	9431	485	0	5.00
6679	399	9	4.89	8069	445	4	4.96	9433	484	1	4.99
6689	399	9	4.88	8081	445	4	4.96	9437	484	1	4.99
6691	399	9	4.88	8087	444	5	4.94	9439	484	1	4.99
6701	400	9	4.89	8089	445	4	4.95	9461	486	0	5.00
6703	400	9	4.89	8093	446	3	4.96	9463	486	0	5.00
6709	400	9	4.89	8101	446	4	4.96	9467	486	0	5.00
6719	400	9	4.88	8111	446	4	4.96	9473	486	0	5.00
6733	401	9	4.89	8117	447	3	4.97	9479	486	0	5.00
6737	401	9	4.89	8123	446	4	4.95	9491	487	0	5.00
6761	402	9	4.89	8147	447	4	4.96	9497	486	1	4.99

of our results and observations, we slightly strengthen this conjecture of [3, Conject. 1.3] suggesting Conjecture 1.4.

Conjecture 1.4. *In* PG$(2, q)$,

$$t_2(2, q) < \sqrt{q}\ln^{0.73} q \quad \textit{for all } q \geq 109.$$

Also, we conjecture the following.

Conjecture 1.5. *In* PG$(2, q)$,

$$t_2(2, q) < 5.5\sqrt{q} \quad \textit{for } q \leq 40009. \tag{1.9}$$

TABLE 5 The smallest known sizes $\bar{t}_2 = \bar{t}_2(2,q)$ of complete arcs in planes PG$(2,q)$, $9511 \le q \le 13627$, $B_q \ge \bar{t}_2(2,q)/\sqrt{q}$

q	$\bar{t}_2$	B_q	q	$\bar{t}_2$	B_q	q	$\bar{t}_2$	B_q	q	$\bar{t}_2$	B_q
9511	488	5.01	9521	489	5.02	9533	489	5.01	*9539*	*487*	*4.99*
9547	489	5.01	9551	489	5.01	*9587*	*489*	*5.00*	9601	490	5.01
9613	*489*	*4.99*	9619	491	5.01	*9623*	*490*	*5.00*	9629	492	5.02
9631	492	5.02	9643	491	5.01	*9649*	*490*	*4.99*	9661	492	5.01
9677	492	5.01	9679	493	5.02	*9689*	*492*	*5.00*	9697	493	5.01
9719	494	5.02	9721	494	5.02	9733	494	5.01	9739	494	5.01
9743	495	5.02	9749	495	5.02	9767	496	5.02	9769	496	5.02
9781	495	5.01	9787	495	5.01	9791	495	5.01	9803	496	5.01
9811	497	5.02	9817	497	5.02	9829	497	5.02	9833	497	5.02
9839	497	5.02	9851	498	5.02	9857	498	5.02	9859	498	5.02
9871	498	5.02	9883	498	5.01	9887	499	5.02	9901	498	5.01
9907	499	5.02	*9923*	*498*	*5.00*	9929	499	5.01	9931	499	5.01
9941	500	5.02	9949	500	5.02	9967	501	5.02	*9973*	*499*	*5.00*
10007	502	5.02	10009	502	5.02	10037	503	5.03	10039	503	5.03
10061	504	5.03	10067	504	5.03	10069	504	5.03	10079	504	5.03
10091	505	5.03	10093	504	5.02	10099	505	5.03	10103	504	5.02
10111	505	5.03	10133	506	5.03	10139	506	5.03	10141	506	5.03
10151	504	5.01	10159	505	5.02	10163	506	5.02	10169	507	5.03
10177	507	5.03	10181	507	5.03	10193	507	5.03	10201	508	5.03
10211	507	5.02	10223	508	5.03	10243	508	5.02	10247	509	5.03
10253	509	5.03	10259	509	5.03	10267	510	5.04	10271	510	5.04
10273	510	5.04	10289	509	5.02	10301	510	5.03	10303	510	5.03
10313	511	5.04	10321	511	5.03	10331	511	5.03	10333	511	5.03
10337	512	5.04	10343	511	5.03	10357	512	5.04	10369	513	5.04
10391	512	5.03	10399	512	5.03	10427	514	5.04	10429	513	5.03
10433	514	5.04	10453	515	5.04	10457	514	5.03	10459	515	5.04
10463	515	5.04	10477	515	5.04	10487	514	5.02	10499	516	5.04
10501	516	5.04	10513	517	5.05	10529	516	5.03	10531	516	5.03
10559	518	5.05	10567	518	5.04	10589	518	5.04	10597	519	5.05
10601	519	5.05	10607	519	5.04	10609	519	5.04	10613	520	5.05
10627	519	5.04	10631	520	5.05	10639	519	5.04	10651	520	5.04
10657	519	5.03	10663	521	5.05	10667	520	5.04	10687	521	5.04
10691	522	5.05	10709	521	5.04	10711	522	5.05	10723	522	5.05
10729	522	5.04	10733	523	5.05	10739	523	5.05	10753	523	5.05
10771	524	5.05	10781	524	5.05	10789	524	5.05	10799	523	5.04
10831	525	5.05	10837	525	5.05	10847	526	5.06	10853	526	5.05
10859	526	5.05	10861	526	5.05	10867	527	5.06	10883	527	5.06
10889	527	5.06	10891	527	5.05	10903	527	5.05	10909	527	5.05
10937	529	5.06	10939	528	5.05	10949	528	5.05	10957	529	5.06
10973	528	5.05	10979	529	5.05	10987	530	5.06	10993	530	5.06
11003	530	5.06	11027	531	5.06	11047	530	5.05	11057	531	5.05

TABLE 5 continued

q	$\bar{t}_2$	B_q	q	$\bar{t}_2$	B_q	q	$\bar{t}_2$	B_q	q	$\bar{t}_2$	B_q
11059	532	5.06	11069	532	5.06	11071	531	5.05	11083	533	5.07
11087	533	5.07	11093	533	5.07	11113	533	5.06	11117	533	5.06
11119	533	5.06	11131	534	5.07	11149	534	5.06	11159	535	5.07
11161	534	5.06	11171	535	5.07	11173	535	5.07	11177	535	5.07
11197	535	5.06	11213	535	5.06	11239	536	5.06	11243	537	5.07
11251	537	5.07	11257	537	5.07	11261	537	5.07	11273	537	5.06
11279	538	5.07	11287	537	5.06	11299	539	5.08	11311	539	5.07
11317	538	5.06	11321	539	5.07	11329	539	5.07	11351	540	5.07
11353	539	5.06	11369	541	5.08	11383	541	5.08	11393	539	5.05
11399	541	5.07	11411	540	5.06	11423	542	5.08	11437	542	5.07
11443	542	5.07	11447	543	5.08	11449	543	5.08	11467	543	5.08
11471	543	5.07	11483	544	5.08	11489	543	5.07	11491	544	5.08
11497	544	5.08	11503	544	5.08	11519	544	5.07	11527	544	5.07
11549	545	5.08	11551	545	5.08	11579	546	5.08	11587	546	5.08
11593	546	5.08	11597	546	5.08	11617	547	5.08	11621	547	5.08
11633	547	5.08	11657	548	5.08	11677	549	5.09	11681	549	5.08
11689	548	5.07	11699	550	5.09	11701	549	5.08	11717	550	5.09
11719	548	5.07	11731	550	5.08	11743	551	5.09	11777	552	5.09
11779	551	5.08	11783	551	5.08	11789	552	5.09	11801	552	5.09
11807	552	5.09	11813	551	5.07	11821	551	5.07	11827	551	5.07
11831	551	5.07	11833	552	5.08	11839	553	5.09	11863	553	5.08
11867	554	5.09	11881	554	5.09	11887	554	5.09	11897	555	5.09
11903	555	5.09	11909	555	5.09	11923	556	5.10	11927	555	5.09
11933	554	5.08	11939	555	5.08	11941	555	5.08	11953	556	5.09
11959	556	5.09	11969	556	5.09	11971	556	5.09	11981	556	5.08
11987	556	5.08	12007	557	5.09	12011	557	5.09	12037	557	5.08
12041	556	5.07	12043	558	5.09	12049	556	5.07	12071	558	5.08
12073	559	5.09	12097	559	5.09	12101	560	5.10	12107	560	5.09
12109	559	5.08	12113	560	5.09	12119	559	5.08	12143	561	5.10
12149	561	5.09	12157	562	5.10	12161	561	5.09	12163	560	5.08
12167	561	5.09	12197	562	5.09	12203	562	5.09	12211	562	5.09
12227	563	5.10	12239	564	5.10	12241	563	5.09	12251	561	5.07
12253	563	5.09	12263	564	5.10	12269	563	5.09	12277	565	5.10
12281	564	5.09	12289	565	5.10	12301	564	5.09	12323	563	5.08
12329	566	5.10	12343	564	5.08	12347	566	5.10	12373	568	5.11
12377	567	5.10	12379	567	5.10	12391	568	5.11	12401	567	5.10
12409	567	5.09	12413	566	5.09	12421	567	5.09	12433	568	5.10
12437	569	5.11	12451	568	5.10	12457	569	5.10	12473	568	5.09
12479	566	5.07	12487	570	5.11	12491	571	5.11	12497	571	5.11
12503	571	5.11	12511	571	5.11	12517	571	5.11	12527	572	5.12
12539	572	5.11	12541	572	5.11	12547	571	5.10	12553	572	5.11
12569	570	5.09	12577	573	5.11	12583	571	5.10	12589	573	5.11
12601	573	5.11	12611	572	5.10	12613	573	5.11	12619	574	5.11

TABLE 5 continued

q	$\bar{t}_2$	B_q	q	$\bar{t}_2$	B_q	q	$\bar{t}_2$	B_q	q	$\bar{t}_2$	B_q
12637	571	5.08	12641	573	5.10	12647	574	5.11	12653	575	5.12
12659	573	5.10	12671	575	5.11	12689	575	5.11	12697	576	5.12
12703	576	5.12	12713	576	5.11	12721	576	5.11	12739	577	5.12
12743	577	5.12	12757	578	5.12	12763	576	5.10	12769	577	5.11
12781	577	5.11	12791	577	5.11	12799	577	5.11	12809	578	5.11
12821	578	5.11	12823	578	5.11	12829	578	5.11	12841	579	5.11
12853	578	5.10	12889	581	5.12	12893	580	5.11	12899	581	5.12
12907	581	5.12	12911	580	5.11	12917	580	5.11	12919	580	5.11
12923	581	5.12	12941	582	5.12	12953	582	5.12	12959	581	5.11
12967	582	5.12	12973	582	5.11	12979	582	5.11	12983	583	5.12
13001	582	5.11	13003	581	5.10	13007	583	5.12	13009	583	5.12
13033	584	5.12	13037	585	5.13	13043	583	5.11	13049	585	5.13
13063	585	5.12	13093	586	5.13	13099	586	5.13	13103	585	5.12
13109	585	5.11	13121	586	5.12	13127	587	5.13	13147	587	5.12
13151	587	5.12	13159	585	5.10	13163	587	5.12	13171	587	5.12
13177	588	5.13	13183	587	5.12	13187	588	5.13	13217	589	5.13
13219	590	5.14	13229	588	5.12	13241	590	5.13	13249	590	5.13
13259	590	5.13	13267	590	5.13	13291	590	5.12	13297	590	5.12
13309	590	5.12	13313	592	5.14	13327	591	5.12	13331	592	5.13
13337	592	5.13	13339	592	5.13	13367	592	5.13	13381	593	5.13
13397	593	5.13	13399	594	5.14	13411	594	5.13	13417	594	5.13
13421	593	5.12	13441	593	5.12	13451	593	5.12	13457	595	5.13
13463	594	5.12	13469	595	5.13	13477	596	5.14	13487	594	5.12
13499	595	5.13	13513	596	5.13	13523	596	5.13	13537	597	5.14
13553	597	5.13	13567	598	5.14	13577	598	5.14	13591	598	5.13
13597	598	5.13	13613	598	5.13	13619	600	5.15	13627	600	5.14

Finally, we suppose the following new forms for the upper bound on $t_2(2,q)$:

$$t_2(2,q) < \sqrt{q}\ln^{c(q)} q, \tag{1.10}$$

$$t_2(2,q) < D\sqrt{q}\ln^{c(q;D)} q,\ D<1, \tag{1.11}$$

$$t_2(2,q) < d(q)\sqrt{q}\ln q, \tag{1.12}$$

where D is an universal constant independent of q whereas $c(q)$, $c(q;D)$, and $d(q)$ are *decreasing functions of* q.

We introduce the values $\overline{c}(q;D)$ and $\overline{d}(q)$.

$$\bar{t}_2(2,q) = D\sqrt{q}\ln^{\overline{c}(q;D)} q, \quad \bar{t}_2(2,q) = \overline{d}(q)\sqrt{q}\ln q. \tag{1.13}$$

By (1.2), (1.8), and (1.10)–(1.13), it holds that

$$c(q) \le \overline{c}(q),\ c(q;D) \le \overline{c}(q;D),\ d(q) \le \overline{d}(q). \tag{1.14}$$

Some results of this paper were presented in [4,5].

TABLE 6 The smallest known sizes $\bar{t}_2 = \bar{t}_2(2,q)$ of complete arcs in planes $PG(2,q)$, $q \in G$, $B_q \geq \bar{t}_2(2,q)/\sqrt{q}$, $\overline{\Phi}_q = \sqrt{q}\ln^{0.73} q - \bar{t}_2(2,q)$, $\bar{c}(q)$ see (1.8), $\overline{\Psi}_q = 0.7\sqrt{q}\ln^{\ln^{-1} q+0.7805} q - \bar{t}_2(2,q)$, $\overline{d}_*(q) = \bar{t}_2(2,q)/(\sqrt{q}\ln^{0.73} q)$

q	$\bar{t}_2$	B_q	$\overline{\Phi}_q$	$\overline{\Psi}_q$	$\overline{d}_*(q)$	$\bar{c}(q)$
13687	601	5.14	5.3	1.56	0.9912	0.7261
13697	602	5.15	4.6	0.81	0.9925	0.7266
13711	602	5.15	4.9	1.16	0.9919	0.7264
14009	608	5.14	6.5	2.57	0.9894	0.7253
15013	633	5.17	6.5	2.04	0.9898	0.7255
16001	656	5.19	7.4	2.42	0.9888	0.7251
2^{14}	665	5.20	7.5	2.31	0.9889	0.7251
17011	679	5.21	8.2	2.67	0.9881	0.7247
18013	702	5.24	8.2	2.13	0.9885	0.7249
19001	723	5.25	9.3	2.75	0.9873	0.7244
20011	745	5.27	9.4	2.33	0.9876	0.7246
21001	765	5.28	10.5	3.01	0.9864	0.7240
22003	785	5.30	11.5	3.51	0.9855	0.7237
23011	805	5.31	12.2	3.72	0.9850	0.7235
24001	826	5.34	11.2	2.18	0.9866	0.7242
25013	843	5.34	14.2	4.72	0.9834	0.7228
26003	864	5.36	12.5	2.48	0.9858	0.7238
27011	881	5.37	14.7	4.27	0.9836	0.7229
28001	900	5.38	14.3	3.41	0.9843	0.7232
29009	918	5.39	15.0	3.60	0.9839	0.7230
30011	934	5.40	17.3	5.39	0.9819	0.7222
31013	954	5.42	15.3	2.93	0.9843	0.7232
32003	969	5.42	17.8	5.00	0.9820	0.7222
2^{15}	**987**	**5.46**	**13.2**	0.04	**0.9868**	**0.7243**
33013	988	5.44	16.4	3.18	0.9836	0.7230
34019	1004	5.45	17.8	4.06	0.9826	0.7225
35023	1019	5.45	19.8	5.69	0.9809	0.7218
36007	1039	5.48	16.4	1.77	0.9845	0.7234
37003	1054	5.48	17.9	2.86	0.9833	0.7228
38011	1070	5.49	18.4	2.94	0.9831	0.7228
39019	1085	5.50	19.8	3.83	0.9821	0.7223
40009	1098	5.49	22.6	6.26	0.9798	0.7214
41011	1115	5.51	21.5	4.70	0.9811	0.7219
42013	1131	5.52	21.2	3.98	0.9816	0.7221
43003	1145	5.53	22.6	4.91	0.9807	0.7218
44017	1162	5.54	21.1	3.05	0.9821	0.7224
45007	1175	5.54	23.2	4.67	0.9807	0.7218
45893	1192	5.57	19.5	0.64	0.9839	0.7232
2^{18}	3066	5.99	165.6	79.75	0.9487	0.7092

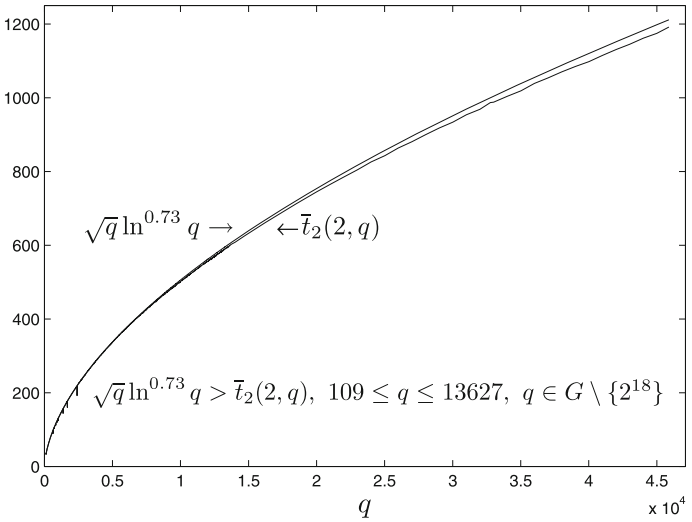

FIGURE 1 The values of $\sqrt{q}\ln^{0.73} q$ (*the top curve*) and $\overline{t}_2(2,q)$ (*the bottom curve*)

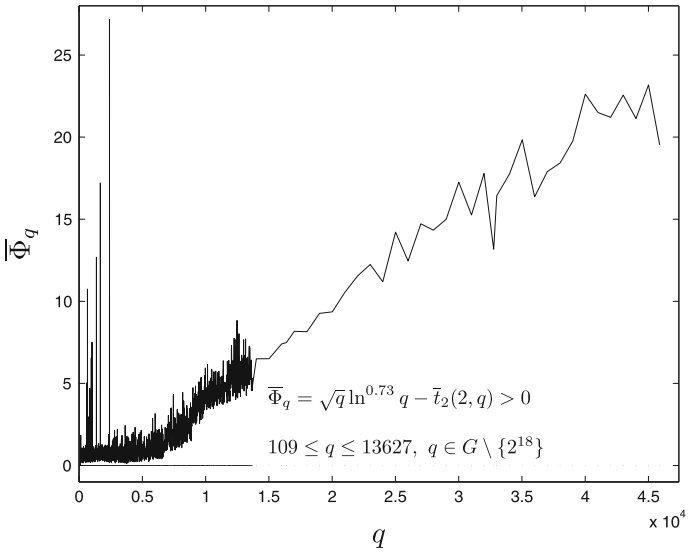

FIGURE 2 The difference $\overline{\Phi}_q = \sqrt{q}\ln^{0.73} q - \overline{t}_2(2,q) > 0$

In Sect. 2, the randomized greedy algorithms for obtaining small complete arcs are described. In Sect. 3, for $q \le 13627$ and $q \in G$, tables of $\overline{t}_2(2,q)$ collecting our new results and the known ones are given. The corresponding theorems describing the results in detail are also given. In all tables we denote $\overline{t}_2 = \overline{t}_2(2,q)$. In Sect. 4 some observations and analysis of the values $\overline{t}_2(2,q)$ are presented; the corresponding plots are given.

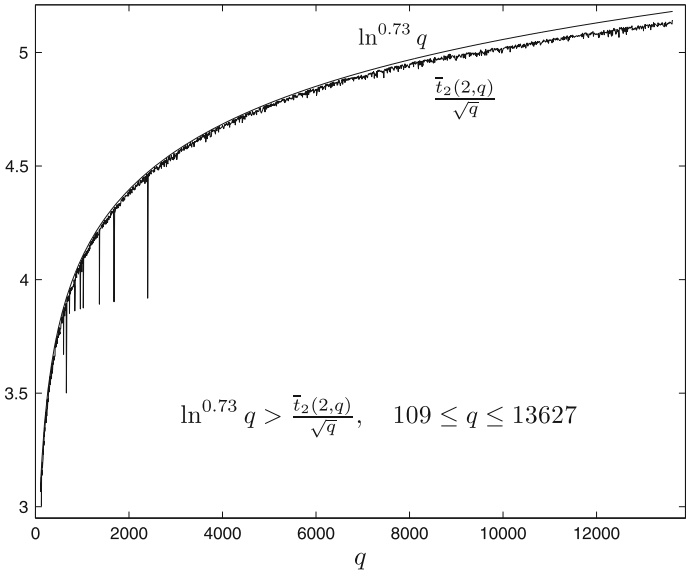

FIGURE 3 The values of $\ln^{0.73} q$ (*the top curve*) and $\bar{t}_2(2,q)/\sqrt{q}$ (*the bottom curve*)

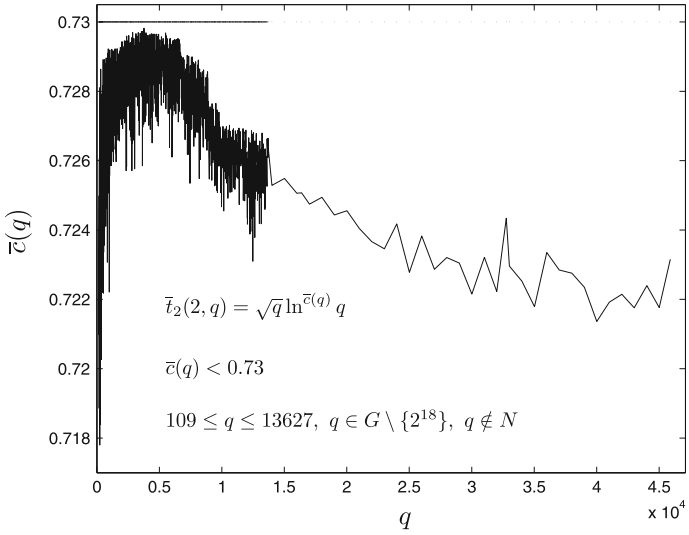

FIGURE 4 The values of $\overline{c}(q)$

2. Greedy algorithms for small complete arcs in PG$(2, q)$

In this paper we use for computer search randomized greedy algorithms [3, Sec. 2], [8, Sec. 2], [13, Sec. 2]. At every step an algorithm maximizes an objective function f but some steps are executed in a random manner. The number

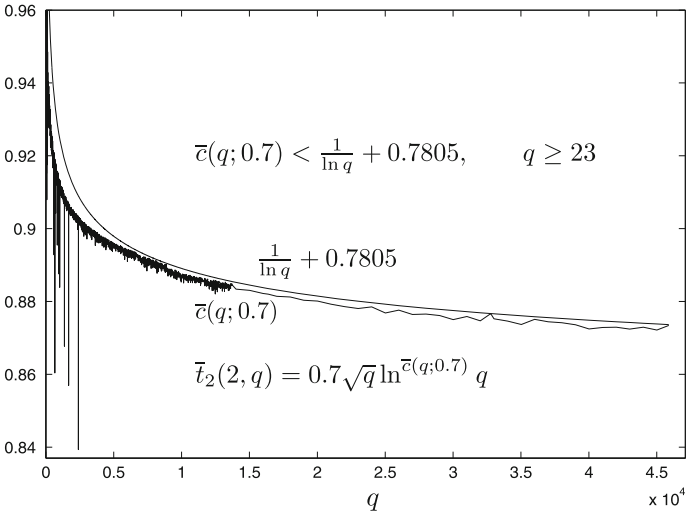

FIGURE 5 The values of $\frac{1}{\ln q} + 0.7805$ (*the top curve*) and $\overline{c}(q;0.7)$ (*the bottom curve*), $23 \leq q \leq 13627$, $q \in G \setminus \{2^{18}\}$

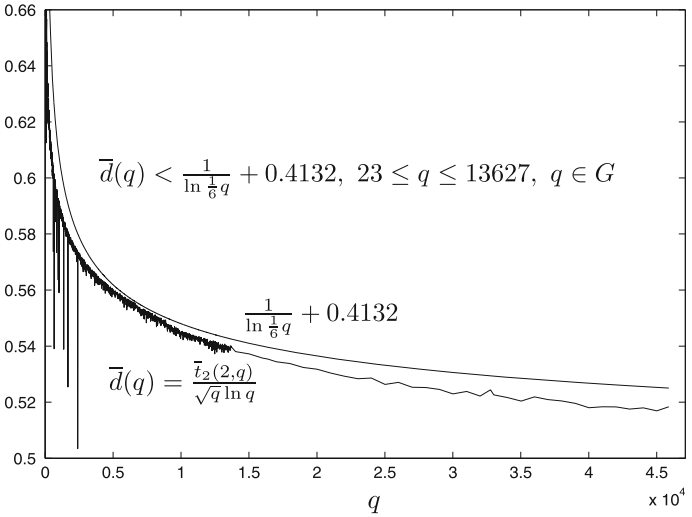

FIGURE 6 The values of $1/\ln \frac{1}{6}q + 0.4132$ (*the top curve*) and $\overline{d}(q) = \overline{t}_2(2,q)/(\sqrt{q}\ln q)$ (*the bottom curve*)

of these steps, their ordinal numbers and some other parameters of the algorithm have been taken intuitively. Also, if the same maximum of f can be obtained in distinct ways, one way is chosen randomly.

We begin to construct a complete arc by using a starting set S_0 of points. At the i-th step one point is added to the set S_{i-1} and we obtain a point set S_i.

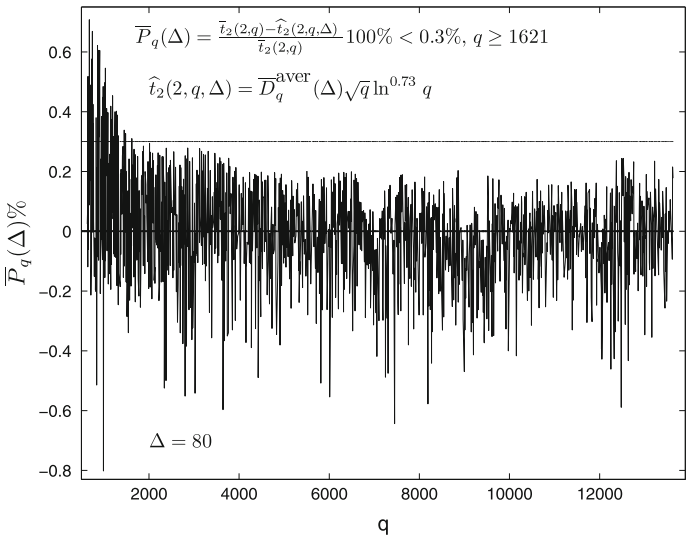

FIGURE 7 The values of $\overline{P}_q(\Delta) = \frac{\overline{t}_2(2,q) - \widehat{t}_2(2,q,\Delta)}{\overline{t}_2(2,q)} 100\ \%$, $631 \leq q \leq 13627$, $q \notin N$, $\Delta = 80$

As the value of the objective function f we consider *the number of covered* points in $\mathrm{PG}(2,q)$, that is, points that lie on bisecants of the obtained set.

On every "random" i-th step we take $d_{q,i}$ *randomly chosen points* of $\mathrm{PG}(2,q)$ *uncovered by* S_{i-1} and compute the objective function f adding each of these $d_{q,i}$ points to S_{i-1}. The point providing the maximum of f is included into S_i.

On every "non-random" j-th step we consider *all points uncovered by* S_{j-1} and add to S_{j-1} the point providing the maximum of f.

As S_0 we can use a subset of points of an arc obtained in previous stages of the search.

A generator of random numbers is used for a random choice. To get arcs with distinct sizes, starting conditions of the generator are changed for the same set S_0. In this way the algorithm works in a convenient limited region of the search space to obtain examples improving the size of the arc from which the fixed points have been taken.

In order to obtain arcs with new sizes sufficiently many attempts should be made with randomized greedy algorithms. The predicted sizes $\widehat{t}_2(2,q,\Delta)$ considered in Sect. 4 are useful for understanding if a good result have been obtained. Here Δ is a parameter used for a prediction. If the result is not close to the predicted size, the attempts are continued.

We obtain small complete arcs in $\mathrm{PG}(2,q)$ in two stages.

At the 1-st stage, we take the frame as S_0 and create a starting complete arc K_0 using in the beginning of the process $\widehat{t}_2(2,q,\Delta) - \delta_q$ "random" steps

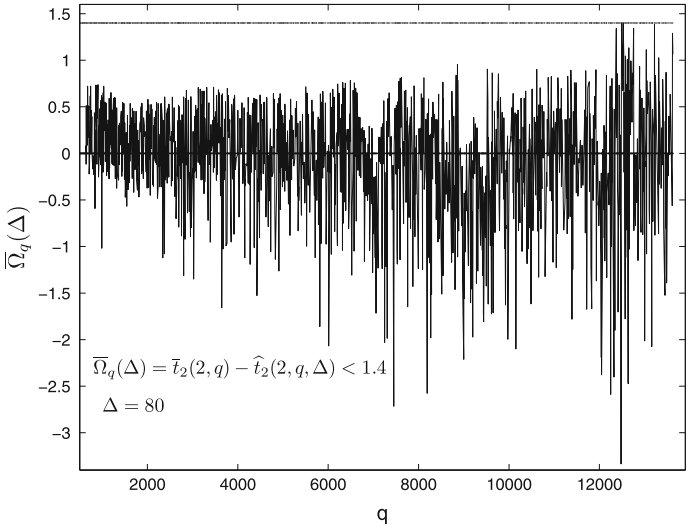

FIGURE 8 The values of $\overline{\Omega}_q(\Delta) = \bar{t}_2(2,q) - \widehat{t}_2(2,q,\Delta)$, $631 \le q \le 13627$, $q \notin N$, $\Delta = 80$

with distinct $d_{q,i}$. All the next steps up to obtaining a complete arc are "non-random".

At the 2-nd stage we execute n attempts to get a complete arc. For every attempt, the starting conditions of the random generator are other than in the previous ones. But the set S_0 is the same; it consists of the first $\widehat{t}_2(2,q,\Delta) - r_q$ points of K_0. Two or three of the first five steps of every attempt are "random", the rest of ones are "non-random".

The values of $d_{q,i}$, δ_q, r_q, n, Δ, are given intuitively depending upon q and (for $d_{q,i}$) upon $|S_{i-1}|$ and the stage of the process. For example, on the 1-st stage one can put $d_{q,i} = 100\beta$ whereas on the 2-nd stage it is useful to take $d_{q,i} = 5\beta$ with $\beta = 1, 2, \ldots$.

3. Small complete arcs in $\mathbf{PG}(2,q)$, $q \le 13627$, $q \in G$

The current values of $\bar{t}_2(2,q)$ for $q \le 13627$, $q \in G$, are represented in Tables 1, 2, 3, 4, 5, 6.

Throughout the paper, in all tables we denote $A_q = \lfloor a_q\sqrt{q} - \bar{t}_2(2,q) \rfloor$, where

$$a_q = \begin{cases} 4 & \text{if } q \le 841,\ q = 857,\ q \in Q, \\ 4.5 & \text{if } 853 \le q \le 2647,\ q = 2659, 2663, 2683, 2693, 2753, 2801, \\ & \quad q \ne 857,\ q \notin Q, \\ 5 & \text{if } 2657 \le q \le 9497,\ q = 9539, 9587, 9613, 9623, 9649, 9689, \\ & \quad 9923, 9973,\ q \notin \{2659, 2663, 2683, 2693, 2753, 2801\}. \end{cases}$$

Also, in all tables, B_q is an upper bound for $\bar{t}_2(2,q)/\sqrt{q}$.

For $q \le 841$, the values of $\bar{t}_2(2,q)$ (up to August 2011) are collected in [3, Tab. 1]. In this work we obtained a number of complete arcs with smaller sizes than in [3]. The current values of $\bar{t}_2(2,q)$ for $q \le 841$ are given in Table 1. The data improving results of [3, Tab. 1] are written in Table 1 in bold font. For $q \le 32$, we have the exact values $\bar{t}_2(2,q) = t_2(2,q)$, see [34,35].

For $853 \le q \le 2647$, the values of $\bar{t}_2(2,q)$ (up to August 2011) are collected in [3, Tabs 2, 3]. Here, we obtain many small complete arcs with new sizes in the region noted. The current values of $\bar{t}_2(2,q) < 4.5\sqrt{q}$ for $853 \le q \le 2647$ are given in Table 2. The data improving results of [3, Tabs 2, 3] are written in Table 2 in bold font. The data for $q = p^2$ with $\bar{t}_2(2,q) = 4\sqrt{q} - 4$ [11,14,18,22] and for $q = 857$ with $\bar{t}_2(2,857) = 117 < 4\sqrt{857}$ are written in italic font.

For $2657 \le q \le 5107$ and $q \in T_2$, the values of $\bar{t}_2(2,q)$ (up to August 2011) are collected in [3, Tabs 3, 4]. Here, in the region noted, we obtain many small complete arcs with sizes smaller than in [3]. The current values of $\bar{t}_2(2,q) < 4.8\sqrt{q}$ for $2657 \le q \le 5419$ are given in Table 3. The data obtained in this work improving results of [3, Tabs 3, 4] are written in Table 3 in bold font. The data for $q = 5113, 5167, 5171, 5179, 5189, 5197, 5227, 5233, 5273, 5297, 5309, 5323, 5329, 5333$, and $q \ge 5351$ are new; they have been obtained in this work and they also are written in bold font. The data for $q = 2659, 2663, 2683, 2693, 2753, 2801$, with $\bar{t}_2(2,q) < 4.5\sqrt{q}$ are written in italic font.

The current values of $\bar{t}_2(2,q) < 5\sqrt{q}$ for $5431 \le q \le 9497$ are given in Table 4. All results in Table 4 are new and they have been obtained by computer search, based on randomized greedy algorithms. Data for $q = 5641, 5843, 6011, 8192$ improving results of [3] are written in bold font. Data for $q = 5441, 5443, 5449, 5471, 5477, 5479, 5483, 5501, 5521$, with $\bar{t}_2(2,q) < 4.8\sqrt{q}$ are written in Table 4 in italic font.

Table 5 contains the current values of $\bar{t}_2(2,q)$ for $9511 \le q \le 13627$. All results in Table 5 are new and they have been obtained in this work by computer search, based on randomized greedy algorithms. Data for $q = 9539, 9587, 9613, 9623, 9649, 9689, 9923, 9973$ with $\bar{t}_2(2,q) < 5\sqrt{q}$ are written in Table 5 in italic font.

Table 6 contains the values of $\bar{t}_2(2,q)$ for $q \in G$. Excepting $q = 2^{14}, 2^{18}$, all $\bar{t}_2(2,q)$ values in Table 6 are new. Data for $q = 2^{15}$ improving results of [3] are written in bold font. Also, in Table 6 the values of $\overline{\Phi}_q = \sqrt{q}\ln^{0.73} q - \bar{t}_2(2,q)$, $\overline{\Psi}_q = 0.7\sqrt{q}\ln^{\ln^{-1} q + 0.7805} q - \bar{t}_2(2,q)$, $\bar{d}_*(q) = \bar{t}_2(2,q)/(\sqrt{q}\ln^{0.73} q)$, and $\bar{c}(q)$, see (1.8), are given.

Theorems 1.1 and 1.2 of Introduction are obtained from Tables 1, 2, 3, 4, 5, 6.

Theorems 3.1 and 3.2, improving and extending the results of [3, Theorems 3.1,3.2], are obtained from Tables 1, 2.

Theorem 3.1. *In* $\mathrm{PG}(2,q)$*, the following hold:*

$$\begin{aligned}
&t_2(2,q) < \ 4\sqrt{q} \ \ \textit{for } 2 \le q \le 841,\ q = 857,\ q \in Q;\\
&t_2(2,q) \le \ 3\sqrt{q} \ \ \textit{for } 2 \le q \le 89,\ \ q = 101;\\
&t_2(2,q) < 3.5\sqrt{q} \ \ \textit{for } 2 \le q \le 277;\\
&t_2(2,q) < 3.6\sqrt{q} \ \ \textit{for } 2 \le q \le 349,\ q = 359, 661;\\
&t_2(2,q) < 3.7\sqrt{q} \ \ \textit{for } 2 \le q \le 419,\ q = 601;\\
&t_2(2,q) < 3.8\sqrt{q} \ \ \textit{for } 2 \le q \le 541;\\
&t_2(2,q) < 3.9\sqrt{q} \ \ \textit{for } 2 \le q \le 673,\ q = 729, 841, 961, 1024, 1369. \quad (3.1)
\end{aligned}$$

Also,

$$\begin{aligned}
&t_2(2,q) \le 4\sqrt{q} - 9 \ \ \textit{for } \ 37 \le q \le 211, \qquad q = 23, 227, 229, 233, 241, 243,\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad 256, 257, 661;\\
&t_2(2,q) \le 4\sqrt{q} - 8 \ \ \textit{for } \ 23 \le q \le 307, \qquad q = 317, 343, 601;\\
&t_2(2,q) \le 4\sqrt{q} - 7 \ \ \textit{for } \ 19 \le q \le 373, \qquad q = 16, 383, 401;\\
&t_2(2,q) \le 4\sqrt{q} - 6 \ \ \textit{for } \ \ 9 \le q \le 433, \qquad q = 443, 463;\\
&t_2(2,q) \le 4\sqrt{q} - 5 \ \ \textit{for } \ \ 8 \le q \le 499, \qquad q = 509, 512, 521, 523, 529, 541;\\
&t_2(2,q) \le 4\sqrt{q} - 4 \ \ \textit{for } \ \ 7 \le q \le 557, \qquad q = 569, 571, 577, 625, 729, 841,\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad q \in Q;\\
&t_2(2,q) < 4\sqrt{q} - 3 \ \ \textit{for } \ \ 7 \le q \le 643, \qquad q = 653;\\
&t_2(2,q) \le 4\sqrt{q} - 2 \ \ \textit{for } \ \ 3 \le q \le 691, \qquad q = 709, 719;\\
&t_2(2,q) < 4\sqrt{q} - 1 \ \ \textit{for } \ \ 2 \le q \le 761, \qquad q = 773, 787.
\end{aligned}$$

Theorem 3.2. *In* $\mathrm{PG}(2,q)$*, the following hold:*

$$\begin{aligned}
&t_2(2,q) < 4.5\sqrt{q} \ \ \textit{for } q \le 2647,\ q = 2659, 2663, 2683, 2693,\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad 2753, 2801;\\
&t_2(2,q) < 4.1\sqrt{q} \ \ \textit{for } q \le 1031,\ q = 1039, 1069, 1369, 1681, 2401;\\
&t_2(2,q) < 4.2\sqrt{q} \ \ \textit{for } q \le 1289,\ q = 1297, 1301, 1303, 1319, 1331;\\
&t_2(2,q) < 4.3\sqrt{q} \ \ \textit{for } q \le 1627,\ q = 1657, 1663, 1697;\\
&t_2(2,q) < 4.4\sqrt{q} \ \ \textit{for } q \le 2069,\ q = 2087, 2089, 2111, 2113, 2129, 2131.
\end{aligned} \quad (3.2)$$

Also,

$$
\begin{aligned}
&t_2(2,q) < 4.5\sqrt{q} - 13 \ \textit{for } q \le\ 997,\ q = 1013, 1019, 1024, 1031, 1039,\\
&\qquad\qquad 1069, 1097, 1369, 1681, 2401;\\
&t_2(2,q) < 4.5\sqrt{q} - 12 \ \textit{for } q \le 1151,\ q = 1163, 1187;\\
&t_2(2,q) < 4.5\sqrt{q} - 11 \ \textit{for } q \le 1259,\ q = 1283, 1289, 1297, 1301, 1303,\\
&\qquad\qquad 1319, 1331, 1361;\\
&t_2(2,q) < 4.5\sqrt{q} - 10 \ \textit{for } q \le 1399,\ q = 1429, 1433, 1447;\\
&t_2(2,q) < 4.5\sqrt{q} - 9 \ \ \textit{for } q \le 1553,\ q = 1567, 1571, 1583, 1601;\\
&t_2(2,q) < 4.5\sqrt{q} - 8 \ \ \textit{for } q \le 1663,\ q = 1693, 1697, 1709;\\
&t_2(2,q) < 4.5\sqrt{q} - 7 \ \ \textit{for } q \le 1789,\ q = 1811, 1823;\\
&t_2(2,q) < 4.5\sqrt{q} - 6 \ \ \textit{for } q \le 1873,\ q = 1879, 1889, 1901, 1907;\\
&t_2(2,q) < 4.5\sqrt{q} - 5 \ \ \textit{for } q \le 2003,\ q = 2017, 2027, 2039, 2063;\\
&t_2(2,q) < 4.5\sqrt{q} - 4 \ \ \textit{for } q \le 2143,\ q = 2161, 2179, 2221;\\
&t_2(2,q) < 4.5\sqrt{q} - 3 \ \ \textit{for } q \le 2237,\ q = 2243, 2251, 2267, 2269, 2273,\\
&\qquad\qquad 2287, 2309, 2341, 2347, 2377;\\
&t_2(2,q) < 4.5\sqrt{q} - 2 \ \ \textit{for } q \le 2381,\ q = 2393, 2399, 2417, 2437;\\
&t_2(2,q) < 4.5\sqrt{q} - 1 \ \ \textit{for } q \le 2473,\ q = 2503, 2531, 2549.
\end{aligned}
$$

Theorem 3.3, improving and extending the results of [3, Th. 3.3], is obtained from Tables 3, 4, 5.

Theorem 3.3. *In* $\mathrm{PG}(2,q)$, *the following hold:*

$$
\begin{aligned}
&t_2(2,q) < 5\sqrt{q} \quad \textit{for } q \le 9497,\ q = 9539, 9587, 9613, 9623, 9649,\\
&\qquad\qquad 9689, 9923, 9973;\\
&t_2(2,q) < 4.6\sqrt{q} \ \textit{for } q \le 3307,\ q = 3319, 3323, 3329, 3331, 3343, 3347,\\
&\qquad\qquad 3371, 3373, 3391;\\
&t_2(2,q) < 4.7\sqrt{q} \ \textit{for } q \le 4201,\ q = 4217, 4219, 4229, 4241, 4243, 4253,\\
&\qquad\qquad 4271, 4273, 4297, 4363, 4423;\\
&t_2(2,q) < 4.8\sqrt{q} \ \textit{for } q \le 5419,\ q = 5441, 5443, 5449, 5471, 5477, 5479,\\
&\qquad\qquad 5483, 5501, 5521;\\
&t_2(2,q) < 4.9\sqrt{q} \ \textit{for } q \le 6961,\ q = 6971, 6977, 6983, 6991, 6997, 7001,\\
&\qquad\qquad 7013, 7019, 7027, 7039, 7043, 7057,\\
&\qquad\qquad 7079, 7103, 7109, 7187, 7193, 7247,\\
&\qquad\qquad 7283, 7307, 7451. \qquad (3.3)
\end{aligned}
$$

Also,

$$t_2(2,q) < 5\sqrt{q} - 22 \text{ for } q \le 3559,\ q = 3581, 3583, 3607, 3613, 3617, 3631, 3643, 3673, 3677, 3697, 3701, 3721, 3739, 3761, 3847, 3851;$$

$$t_2(2,q) < 5\sqrt{q} - 21 \text{ for } q \le 3767,\ q = 3779, 3797, 3803, 3821, 3823, 3833, 3853, 3877, 3881, 3917, 3919, 3923, 3947, 4021, 4027, 4153;$$

$$t_2(2,q) < 5\sqrt{q} - 20 \text{ for } q \le 4079,\ q = 4096, 4099, 4111, 4127, 4129, 4133, 4157, 4159, 4177, 4201, 4229, 4253, 4273, 4363, 4423;$$

$$t_2(2,q) < 5\sqrt{q} - 19 \text{ for } q \le 4297,\ q = 4337, 4339, 4357, 4391, 4409, 4441, 4447, 4463, 4481, 4489, 4517;$$

$$t_2(2,q) < 5\sqrt{q} - 16 \text{ for } q \le 5119,\ q = 5153, 5189, 5333;$$

$$t_2(2,q) < 5\sqrt{q} - 14 \text{ for } q \le 5501,\ q = 5507, 5519, 5521, 5527, 5557, 5569, 5573, 5581, 5591, 5657, 5659, 5689, 5693, 5711, 5717, 5749, 5783, 5813, 5843, 6011;$$

$$t_2(2,q) < 5\sqrt{q} - 12 \text{ for } q \le 6043,\ q = 6053, 6067, 6073, 6089, 6091, 6121, 6143, 6151, 6163, 6199, 6203, 6317, 6337;$$

$$t_2(2,q) < 5\sqrt{q} - 6 \text{ for } q \le 7559,\ q = 7573, 7577, 7583, 7589, 7607, 7639, 7643, 7649, 7669, 7681, 7723, 7757, 7789, 7823, 7873, 7993, 8191.$$

Using Tables 1, 2, 3, 4, 5, 6, we obtain the lists of (1.6), (1.7) in Theorem 1.2, Observation 1.3, and Figs. 1, 2, 3, 4, 5, 6, 7, 8 discussed in Sect. 4.

4. Observations on the values of $\bar{t}_2(2,q)$

We look for upper estimates for the values of $\bar{t}_2(2,q)$ from Tables 1, 2, 3, 4, 5, 6 in the form (1.1), see [30], [28, Tab. 2.6], [3, Sec. 4], and in the forms (1.10)–(1.12). For definiteness, we use natural logarithms. Let

$$N = \{601, 5^4, 661, 3^6, 29^2, 31^2, 2^{10}, 37^2, 41^2, 7^4, 2^{18}\}.$$

For $109 \le q \le 13627$, $q \in G \setminus \{2^{18}\}$, the graphs of values of $\sqrt{q}\ln^{0.73} q$ (*the top curve*) and $\bar{t}_2(2,q)$ (*the bottom curve*) are shown on Fig. 1. One can see on Fig. 1 that for $q < 8000$ the curves practically coalesce with each other, but always $\bar{t}_2(2,q) < \sqrt{q}\ln^{0.73} q$. Also, when $q > 8000$ grows, the graphs $\sqrt{q}\ln^{0.73} q$

and $\bar{t}_2(2,q)$ diverge so that the positive difference $\overline{\Phi}_q = \sqrt{q}\ln^{0.73} q - \bar{t}_2(2,q)$ increases. For $109 \le q \le 13627$, $q \in G \backslash \{2^{18}\}$, this difference is shown on Fig. 2. The relative position of $\ln^{0.73} q$ and $\frac{\bar{t}_2(2,q)}{\sqrt{q}}$ is also interesting, see Fig. 3 where $\ln^{0.73} q$ is the top curve while $\frac{\bar{t}_2(2,q)}{\sqrt{q}}$ is the bottom one. Again, for all q, we have $\bar{t}_2(2,q)/\sqrt{q} < \ln^{0.73} q$. And again, when q grows, the graphs $\ln^{0.73} q$ and $\bar{t}_2(2,q)/\sqrt{q}$ diverge so that the positive difference $\ln^{0.73} q - \bar{t}_2(2,q)/\sqrt{q}$ increases.

Figures 1, 2, 3 are the foundation for Observation 1.3 in Introduction. Data for relatively large $q \in G$ in Table 6 confirm the tendencies indicated in Observation 1.3.

Remark 4.1. On Figs. 1, 2, 3 and in Table 6, for $q > 8000$ and $q \in G$, one can see some decreasing values of $\bar{t}_2(2,q)$ and some increasing difference $\sqrt{q}\ln^{0.73} q - \bar{t}_2(2,q)$. This situation strengthens our confidence in the validity of Conjecture 1.4 on the upper bounds on $t_2(2,q)$. Also, it seems that in fact the upper bound on $t_2(2,q)$ is more complicated than $D\sqrt{q}\ln^C q$ where D and C are universal constants. It is possible that the bound can be improved and be written in another form, e.g. (1.10)–(1.12).

For example, in Figs. 4, 5, 6 the values of $\overline{c}(q)$, $\overline{c}(q;0.7)$, and $\overline{d}(q)$, defined in (1.8), (1.13), are shown. Using Figs. 4, 5, 6 and inequalities (1.14), we can propose the forms of the upper bound written in (1.10)–(1.12). Also, from Figures 5, 6 and Tables 1, 2, 3, 4, 5, 6, we have

Theorem 4.2. *In* $\mathrm{PG}(2,q)$, *the following hold:*

$$\bar{t}_2(2,q) < 0.7\sqrt{q}\ln^{\ln^{-1} q + 0.7805} q,\ \ 23 \le q \le 13627,\ \ q \in G; \tag{4.1}$$

$$\bar{t}_2(2,q) < \left(\frac{1}{\ln \frac{1}{6}q} + 0.4132\right)\sqrt{q}\ln q,\ \ 23 \le q \le 13627,\ \ q \in G. \tag{4.2}$$

By Theorem 4.2, we can conjecture that, for q large enough, it holds that

$$t_2(2,q) < 0.7\sqrt{q}\ln^{0.7805} q, \quad t_2(2,q) < 0.4132\sqrt{q}\ln q.$$

Observation 1.3, Theorem 1.2, Remark 4.1, and Figs. 1, 2, 3, 4 are the foundation for Conjecture 1.4.

Finally, we consider some prediction for the greedy algorithms for computer search; see Sect. 2. Let $q \notin N$. Let $\overline{D}_q^{\mathrm{aver}}(\Delta)$ be the average value of the ratio $\bar{t}_2(2,q)/(\sqrt{q}\ln^{0.73} q)$ calculated from the Δ values of q preceding the given q written as the subscript of $\overline{D}_q^{\mathrm{aver}}(\Delta)$. In other words, let us write the list of the prime powers (not belonging to N) as $q_1 = 2, q_2 = 3, q_3 = 4, q_4 = 5, q_5 = 7, \ldots, q_i = \ldots$ Let $q = q_j$ for some $j > \Delta$. Then

$$\overline{D}_q^{\mathrm{aver}}(\Delta) = \overline{D}_{q_j}^{\mathrm{aver}}(\Delta) = \frac{1}{\Delta}\sum_{i=j-\Delta}^{j-1} \frac{\bar{t}_2(2,q_i)}{\sqrt{q_i}\ln^{0.73} q_i}.$$

We denote

$$\widehat{t}_2(2,q,\Delta) = \overline{D}_q^{\text{aver}}(\Delta)\sqrt{q}\ln^{0.73} q,\ \overline{\Omega}_q(\Delta) = \bar{t}_2(2,q) - \widehat{t}_2(2,q,\Delta),$$
$$\overline{P}_q(\Delta) = \frac{\overline{\Omega}_q(\Delta)}{\bar{t}_2(2,q)}100\ \%.$$

One can treat $\widehat{t}_2(2,q,\Delta)$ as a *predicted* value of $t_2(2,q)$. Then $\overline{\Omega}_q(\Delta) = \bar{t}_2(2,q) - \widehat{t}_2(2,q,\Delta)$ is the difference between the smallest known size $\bar{t}_2(2,q)$ of complete arcs and the predicted value. Finally, $\overline{P}_q(\Delta)$ is this difference in percentage terms of the smallest known size. From Tables 1, 2, 3, 4, 5, we obtain Figures 7, 8 and Observation 4.3.

Observation 4.3. *The following results hold*:

$$-0.79\ \% < \overline{P}_q(80) < 0.71\ \%, \qquad 631 \le q \le 13627,\ q \notin N;$$
$$-0.79\ \% < \overline{P}_q(80) < 0.30\ \%, \qquad 1621 \le q \le 13627,\ q \notin N;$$
$$-3.34 < \overline{\Omega}_q(80) < 1.40, \qquad 631 \le q \le 13627,\ q \notin N.$$

By Observation 4.3 and Figures 7, 8, the upper bounds of $\overline{P}_q(80)$ and $\overline{\Omega}_q(80)$ are relatively small; therefore the predicted values of $\widehat{t}_2(2,q)$ are useful for computer search of small arcs via randomized greedy algorithms.

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