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A 3-CYCLE CONSTRUCTION OF COMPLETE ARCS SHARING $(q+3)/2$ POINTS WITH A CONIC

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Abstract. In the projective plane $PG(2,q)$, $q \equiv 2 \pmod 3$ odd prime power, $q \geq 11$, an explicit construction of $\frac{1}{2}(q+7)$-arcs sharing $\frac{1}{2}(q+3)$ points with an irreducible conic is considered. The construction is based on 3-orbits of some projectivity, called 3-cycles. For every q, variants of the construction give non-equivalent arcs. It allows us to obtain complete $\frac{1}{2}(q+7)$-arcs for $q \leq 4523$. Moreover, for $q = 17, 59$ there exist variants that are incomplete arcs. Completing these variants we obtained complete $(\frac{1}{2}(q+3)+\delta)$-arcs with $\delta = 4$, $q = 17$, and $\delta = 3$, $q = 59$; a description of them as union of some symmetrical objects is given.

1. Introduction

Let $PG(2,q)$ be the projective plane over the Galois field F_q of q elements. A k-arc in $PG(2,q)$ is a set of k points, no three of which are collinear. A k-arc in $PG(2,q)$ is complete if it is not contained in a $(k+1)$-arc of $PG(2,q)$. Surveys of results on the sizes of plane complete arcs, methods of their construction and connected problems can be found in [1–12, 15–19, 23, 24, 27], see also the references therein.

The points of a complete k-arc in $PG(2,q)$ can be treated as columns of a parity check matrix of a $[k,k-3,4]_q2$ linear code of length k, codimension 3, minimum distance 4, covering radius 2, over the field F_q [18, 22]. It is a maximum distance separable (MDS) code.

We consider planes with q odd. It is well known that in $PG(2,q)$, $q \geq 5$ odd, an arc, not contained in a conic, can have at most $\frac{1}{2}(q+3)$ points in common with an irreducible conic. In $PG(2,q)$, we denote by $K_q(\delta)$ a *complete* $(\frac{1}{2}(q+3)+\delta)$-arc other than a conic, sharing $\frac{1}{2}(q+3)$ points with an irreducible conic. Denote by Δ_q the maximal possible value of δ.

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Finding the value of Δ_q and obtaining arcs $K_q(\delta)$ with $\delta \geq 2$ are *open problems.* They are connected with the *open problem* on constructions of *complete* $\frac{1}{2}(q+7)$-arcs in $PG(2,q)$, q odd. These problems are considered in [1,7–10,13,15,16,20,21,25,26], see also the references therein.

In [1], the unique arc $K_{13}(4)$ is studied in detail, see also [21]. The arc $K_{17}(4)$ is given in [9, Sec. 2], [10, Sec. 2], [13, Sec. 4, Rem. 3], [16, Rem. 4.2], [21, Sec. 7]. The arcs $K_{19}(3)$ and $K_{43}(3)$ are obtained by Pellegrino [26]. A computer-aided description is given for $K_{19}(3)$ in [9,10,21] and for $K_{43}(3)$ in [21]. The arc $K_{27}(3)$ was first given in [9,10], see also [21]. Finally, the arc $K_{59}(3)$ was first described in [21], see also [10]. In [9, Th. 3], [10, Th. 4] it is shown that the arcs $K_{17}(4), K_{19}(3)$, and $K_{27}(3)$ are unique up to collineations. In [21] this fact is confirmed and also it is proven that the arcs $K_{43}(3)$ and $K_{59}(3)$ are unique.

By [1], $\Delta_{13} = 4$. By [21, Cor. 4.17], $\Delta_{25} = 2$. In [9, Th. 3], [10, Th. 4], it is shown that $\Delta_{17} = 4$, $\Delta_{19} = \Delta_{27} = 3$, $\Delta_q = 2$ for $q = 11, 23, 29, 31$, and there is no arc $K_{17}(3)$. These results are confirmed in [21, Sec. 7], where the following is also shown: $\Delta_q = 2$ if $q \leq 89$, $q \neq 17, 19, 27, 43, 59$; $\Delta_{17} = 4$; $\Delta_q = 3$ for $q = 19, 27, 43, 59$. In [21, Cors 4.16, 4.17, Ths 5.1, 6.1] infinite sequences of q values for which $\Delta_q = 2$ or $\Delta_q \geq 3$ are obtained. In [7], the arcs which consist of $(q+3)/2$ points of a conic and two points external to the conic are classified. Moreover the above mentioned results about Δ_q are confirmed for $q \leq 89$ and it is proven by computer search that $\Delta_q = 2$ for $97 \leq q \leq 223$ and $q = 229, 241, 243$.

In [16], $\frac{1}{2}(q+7)$-arcs in $PG(2,q)$, $q \equiv 1 \pmod 4$, are constructed and it is shown that for $q \leq 337$, $q \neq 17$, these arcs are complete. In [21], constructions of $\frac{1}{2}(q+7)$ -arcs in $PG(2,q)$ are considered and the known conditions of the existence of complete $\frac{1}{2}(q+7)$-arcs in $PG(2,q)$ are summarized.

In [10, Sec. 2], for $PG(2,q)$, $q \equiv 2 \pmod 3$ odd, two explicit Constructions S and C of $\frac{1}{2}(q+7)$-arcs in $PG(2,q)$ are briefly described. In the present work first we present Construction C in detail, proving that 3-*cycles*, introduced in [10], are 3-orbits of some projectivity, see Section 2. Also we show that Construction C gives complete $\frac{1}{2}(q+7)$-arcs in $PG(2,q)$, $q \leq 4523$. In Section 3, we introduce *quadruples* of 3-cycles. It allowed us to partition all arcs of Construction C into three classes and obtain the exact number of nonequivalent $\frac{1}{2}(q+7)$-arcs of Construction C. For $q = 17, 59$ Construction C forms incomplete arcs, which, as it is shown in Section 4, can be completed to arcs $K_{17}(4)$ and $K_{59}(3)$. At the end we give a geometrical description of them as union of some symmetric objects called *flowers*. Finally, in Section 5, for $q \leq 59$ we showed that all $\frac{1}{2}(q+7)$-arcs of Construction S are equivalent arcs of one class of Construction C. This equivalence is unexpected enough as ideas of Construction C and S are quite different. Construction C uses 3-orbits of some projectivity whereas Construction S is based on plane sections of a cap in $PG(3,q)$. And a design of this cap (considered in [13]) is not of geometric nature.

Some of the results from this work were briefly presented without proofs in [14].

2. A 3-cycle construction of $\frac{1}{2}(q+7)$-arcs in $PG(2,q)$

2.1. A conic $\mathcal{C}$. In the following, throughout the paper, $q \equiv 2 \pmod 3$ is an odd prime power, $q \geq 11$. Thus, -3 is a non square [17].

In the homogenous coordinates of a point (x_0, x_1, x_2) w.l.g. $x_0 \in \{0, 1\}$.

We use the conic $\mathcal{C}$ of equation

$$x_0^2 + x_1^2 = 2x_0x_2. \tag{2.1}$$

It can be represented as

$$\mathcal{C} = \{(1, i, \frac{i^2+1}{2})|i \in F_q\} \cup \{(0,0,1)\} \subset PG(2,q). \tag{2.2}$$

Denote the points of the conic $\mathcal{C}$:

$$A_i = (1, i, \frac{i^2+1}{2}),\ i \in F_q; \quad A_\infty = (0,0,1). \tag{2.3}$$

Let

$$F_q^+ = F_q \cup \{\infty\}.$$

2.2. A PROJECTIVITY ψ AND ITS 3-ORBITS (3-CYCLES). We introduce the following projectivity ψ:

$$\psi(x_0, x_1, x_2) = (x_0, x_1, x_2)\mathbf{G}, \quad \mathbf{G} = \begin{bmatrix} 0 & -2 & 2 \\ 1 & -1 & -1 \\ 1 & 1 & 1 \end{bmatrix}. \tag{2.4}$$

Lemma 2.1. *The projectivity ψ has the order three, i.e.*

$$\psi^3(x_0, x_1, x_2) = (x_0, x_1, x_2).$$

Proof. By (2.4), $\mathbf{G}^3 = 8\mathbf{I}_3$ where $\mathbf{I}_3$ is the identity matrix of the order three. □

Definition 2.2. A 3-orbit of the projectivity ψ is called a 3-*cycle*. A 3-cycle containing a point $B \in PG(2,q)$ is denoted by $C(B)$.

We consider three points of $PG(2,q)\backslash\mathcal{C}$:

$$P = (0,1,0),\ Q = (1,-1,-1),\ Q_+ = (1,1,-1).$$

We denote by $\mathcal{L}$ the line of equation $x_0 + x_2 = 0$.

Lemma 2.3. **(i):** *The points P, Q, and Q_+ form a 3-cycle so that*

$$\begin{aligned} C(P) &= C(Q) = C(Q_+) = \{P, Q, Q_+\} \subset PG(2,q)\backslash\mathcal{C}; \\ \psi(P) &= Q,\ \psi(Q) = Q_+,\ \psi(Q_+) = P. \end{aligned} \tag{2.5}$$

(ii): *The points P, Q, and Q_+ are collinear and they lie on the line $\mathcal{L}$.*

Proof. It is easy to check the assertions directly using (2.4) and Lemma 2.1. □

Lemma 2.4. **(i):** *The only point of $PG(2,q)$ fixed by the projectivity ψ is* $(1,0,2)$.

(ii): *The only line fixed by ψ is $\mathcal{L}$.*

Proof. The assertion (i) can be shown by elementary calculations. The line $\mathcal{L}$ is fixed by (2.5). Now recall that for any collineation in $PG(2,q)$, the number of fixed points is equal to the number of fixed lines [27, Th. 2.65]. □

Corollary 2.5. *The projectivity ψ partitions the plane $PG(2,q)$ into $\frac{1}{3}(q^2+q)$ 3-orbits and one 1-orbit $\{(1,0,2)\}$. Moreover, $\frac{1}{3}(q+1)$ 3-orbits lie on the line $\mathcal{L}$.*

Proof. The assertion follows from Lemmas 2.1 and 2.4. □

We denote

$$m(i) = \frac{i-3}{1+i},\ p(i) = \frac{i+3}{1-i}. \tag{2.6}$$

Clearly, $m(1) = -1$, $p(-1) = 1$. It is natural to put

$$m(-1) = p(1) = \infty,\ m(\infty) = 1,\ p(\infty) = -1,\ A_{-\infty} = A_\infty. \tag{2.7}$$

Lemma 2.6. *It holds that*

$$\psi(A_i) = A_{m(i)},\ \psi(A_{m(i)}) = A_{p(i)},\ \psi(A_{p(i)}) = A_i,\ i \in F_q^+. \tag{2.8}$$

$$\psi(A_{-i}) = A_{-p(i)},\ \psi(A_{-p(i)}) = A_{-m(i)},\ \psi(A_{-m(i)}) = A_{-i},\ i \in F_q^+. \tag{2.9}$$

Proof. To save the space, we omit intermediate technical transformations.

Let $i \notin \{\pm 1, \infty\}$. By (2.3), (2.4), (2.6),

$$(1, i, \frac{i^2+1}{2})\mathbf{G} = \frac{(1+i)^2}{2}(1, m(i), \frac{i^2-2i+5}{(1+i)^2}) =$$
$$\frac{(1+i)^2}{2}(1, m(i), \frac{(m(i))^2+1}{2}) = A_{m(i)}.$$

By (2.6), $m(m(i)) = p(i)$. So, $\psi(A_{m(i)}) = A_{m(m(i))} = A_{p(i)}$. Finally, $\psi(A_{p(i)}) = \psi^3(A_i) = A_i$. As $m(-i) = -p(i)$ and $p(-i) = -m(i)$, the relation (2.9) follows from (2.8).

Now let $i \in \{\pm 1, \infty\}$.

Directly by (2.4), $A_\infty \mathbf{G} = A_1$, $A_1\mathbf{G} = A_{-1}$, $A_{-1}\mathbf{G} = A_\infty$. So, taking into account (2.6), (2.7), the relations (2.8), (2.9) are proven also for these i. □

Theorem 2.7. *The projectivity ψ partitions the conic $\mathcal{C}$ of (2.2) into $\frac{1}{3}(q+1)$ 3-orbits of the form*

$$\{A_i, A_{m(i)}, A_{p(i)}\},\ i \in F_q^+.$$

Proof. We use Lemmas 2.4, 2.6 and Corollary 2.5. Note that $(1, 0, 2) \notin \mathcal{C}$. □

Definition 2.8. The 3-cycle containing the point A_i of the conic $\mathcal{C}$ is denoted by C_i, i.e. $C(A_i) = C_i$. The 3-cycle C_{-i} is *opposite* to C_i.

Lemma 2.9. **(i):** *It holds that*

$$C_i = C_{m(i)} = C_{p(i)} = \{A_i, A_{m(i)}, A_{p(i)}\},\ i \in F_q^+.$$
$$C_{-i} = C_{-p(i)} = C_{-m(i)} = \{A_{-i}, A_{-p(i)}, A_{-m(i)}\},\ i \in F_q^+.$$

(ii): *The relation $A_{-i} \in C_i$ holds if and only if $i \in \{0, \pm 1, \pm 3, \infty\}$, i.e. the cycles $C_0 = C_{-3} = C_3$ and $C_\infty = C_1 = C_{-1}$ (and only these cycles) are opposite to themselves.*

Proof. The assertions follow from (2.6), (2.7) and Lemma 2.6. □

For the 3-cycles C_∞ and C_0, which play an important role in the construction to be proposed, we have by Lemmas 2.6 and 2.9:

$$\begin{array}{c} \psi(A_\infty) = A_1,\ \psi(A_1) = A_{-1},\ \psi(A_{-1}) = A_\infty. \\ C_\infty = C_1 = C_{-1} = \{A_\infty, A_1, A_{-1}\}. \\ \psi(A_0) = A_{-3},\ \psi(A_{-3}) = A_3,\ \psi(A_3) = A_0. \\ C_0 = C_{-3} = C_3 = \{A_0, A_{-3}, A_3\}. \end{array} \tag{2.10}$$

Let $\mathcal{C}$ be the conic of (2.2). We denote the $(q-5)$-set

$$\mathcal{C}^* = \mathcal{C} \setminus (C_0 \cup C_1).$$

Theorem 2.10. *The $(q-5)$-set $\mathcal{C}^*$ is partitioned into $\frac{1}{6}(q-5)$ pairs of opposite 3-cycles $\{C_i, C_{-i}\}$ with $i \notin \{0, \pm 3, \infty, \pm 1\}$.*

Proof. We use Theorem 2.7 and Lemma 2.9. □

2.3. THE EXTERNAL POINTS $P = (0, 1, 0)$, $Q = (1, -1, -1)$, $Q_+ = (1, 1, -1)$.

Lemma 2.11. *The points P, Q, and Q_+ are external points to the conic $\mathcal{C}$ of (2.2). The pairs of tangent points of P, Q, and Q_+ on $\mathcal{C}$ are, respectively, A_∞, A_0; A_1, A_{-3}; and A_{-1}, A_3.*

Proof. Let ℓ_P be the polar line in P with respect to $\mathcal{C}$. Then

$$\ell_P \cap \mathcal{C} : x_1 = 0,\ x_0^2 - 2x_0x_2 = 0,$$

whence either $x_0 = 0$ (then $x_2 = 1$) or $x_0 \neq 0$ (then $x_2 = \frac{1}{2}x_0$). As $x_0 \in \{0, 1\}$, the assertion for the point P follows. For the assertions on points Q and Q_+ we use (2.5), (2.10). □

Lemma 2.12. *For any $i \neq j$, the points A_i and A_j are collinear with the point $P = (0, 1, 0)$ if and only if $i + j = 0$.*

Proof. By direct calculation we obtain $(i+j)(i-j) = 0$ and since $i \neq j$, $i+j = 0$. □

Corollary 2.13. *Let $\mathcal{C}$ be the conic of (2.2). Every bisecant of $\mathcal{C}$ through P contains a pair of points of the form $\{A_i, A_{-i}\}$, $i \notin \{0, \infty\}$. The points A_i and A_{-i} belong to distinct opposite cycles C_i and C_{-i}, respectively, except the cases $\{A_1, A_{-1}\} \subset C_1 = C_{-1}$ and $\{A_3, A_{-3}\} \subset C_3 = C_{-3}$.*

Proof. By Lemma 2.12, there are $\frac{q-1}{2}$ needed bisecants through P, A_i, A_{-i}. They are all bisecants through P as it is an external point to $\mathcal{C}$ by Lemma 2.11. We also use (2.10). □

Corollary 2.14. *Let $\mathcal{C}$ be the conic of (2.2). Every bisecant of $\mathcal{C}$ through Q contains a pair of points of the form $\{A_{m(i)}, A_{-p(i)}\}$, $i \notin \{0, -1\}$. The points $A_{m(i)}$ and $A_{-p(i)}$ belong to distinct opposite cycles $C_i = C_{m(i)}$ and $C_{-i} = C_{-p(i)}$, respectively, except the cases $i \in \{1, \infty\}$, $\{A_{m(i)}, A_{-p(i)}\} = \{A_{-1}, A_\infty\} \subset C_{-1} = C_\infty$, and $i \in \{\pm 3\}$, $\{A_{m(i)}, A_{-p(i)}\} = \{A_0, A_3\} \subset C_0 = C_3$.*

Proof. We use (2.5), (2.8)–(2.10), and Lemmas 2.9, 2.11. Taking into account that a projectivity preserves lines, the assertion follows from Corollary 2.13. □

2.4. A 3-CYCLE CONSTRUCTION C. We consider the following construction.
Construction C (3-*cycle construction*). Let $q \equiv 2 \pmod 3$ be a power of an odd prime, $q \geq 11$. Let $i \in F_q^+ \setminus \{0, \pm 1, \pm 3, \infty\}$. We denote by K_1 a $\frac{1}{2}(q-5)$-set consisting of $\frac{1}{6}(q-5)$ 3-cycles C_i. From every pair $\{C_i, C_{-i}\}$ of opposite 3-cycles, exactly one 3-cycle is included in K_1. For this purpose, any of the two 3-cycles can be taken. So, there are $2^{(q-5)/6}$ formally distinct variants of the set K_1. Let

$$S_6 = \{P, Q, A_\infty, A_1, A_0, A_{-3}\} \tag{2.11}$$

be a point 6-set. We form a point $\frac{1}{2}(q+7)$-set

$$K = K_1 \cup S_6. \tag{2.12}$$

By above, the $\frac{1}{2}(q+7)$-set K of Construction C contains $\frac{1}{6}(q-5)$ complete 3-cycles belonging to the conic $\mathcal{C}$ and six points from 3-cycles $C(P)$, $C(A_\infty)$, $C(A_0)$. From each of these three 3-cycles $C(B)$ we take two points: B and $\psi(B)$.

Remark 2.15. If q is not a prime, the points of S_6 have coordinates belonging to the base field.

Theorem 2.16. *Let $q \equiv 2 \pmod 3$ be a power of an odd prime, $q \geq 11$. Then in $PG(2,q)$, for the all $2^{(q-5)/6}$ variants of the set K_1, the set $K = K_1 \cup S_6$ of Construction C is a $\frac{1}{2}(q+7)$-arc other than a conic that shares $\frac{1}{2}(q+3)$ points with an irreducible conic.*

Proof. We should show that the external points P and Q do not lie on any bisecants of K. By Construction C, the bisecants of the conic $\mathcal{C}$ of (2.2) through distinct opposite 3-cycles C_i and C_{-i}, where $i \notin \{0, \pm 1, \pm 3, \infty\}$, are not bisecants of K. So, by Corollaries 2.13, 2.14, we should only consider the bisecants A_1A_{-1}, A_3A_{-3}, $A_{-1}A_\infty$, and A_0A_3. By (2.11), (2.12), these bisecants are not bisecants of K. □

Remark 2.17. In [7, Th. 4] a necessary and sufficient condition on the existence of arcs with $\frac{1}{2}(q+3)$ points on a conic $\mathcal{C}$ and two points external to $\mathcal{C}$ is given for odd q. This condition furnishes a framework for a computer search for arcs of this type. Theorem 4 of [7], given a choice of the external points of $\mathcal{C}$ which determines a parameter n, defines a certain collection T of sets S_i of points of $\mathcal{C}$ with $|\bigcup_{S_i \in T} S_i| = \frac{1}{2}(q+3)$. The theorem states that the points of T together with the two external points form an arc if and only if n is odd and the sets of the collection are disjoint. Theorem 2.16 of this paper gives, with a more restrictive condition on q, an explicit construction of arcs with $\frac{1}{2}(q+3)$ points on a conic and two points external to the conic. In the language of [7, Th. 4], the construction of Theorem 2.16 corresponds to a choice of the external points that implies $n = 3$. Moreover, Theorem 2.16 states that this choice always produces a collection T of disjoint subsets of the conic.

Theorem 2.18. *Let $K = K_1 \cup S_6$ be an arc of Construction C. For $q \leq 4523$, there is a variant of K_1 such that the $\frac{1}{2}(q+7)$-arc K is complete. Also, for $q \leq 149$, $q \neq 17, 59$, all the variants of K_1 give complete arcs K. For $q = 17$ and $q = 59$ there are n_q equivalent variants of K_1 providing incomplete $\frac{1}{2}(q+7)$-arcs K embedded in complete $(\frac{1}{2}(q+3)+\delta)$-arcs with $n_{17} = 2$, $\delta = 4$ for $q = 17$, and $n_{59} = 4$, $\delta = 3$ for $q = 59$.*

Proof. The assertions are proven by computer search. □

Conjecture 2.19. *Let $q \equiv 2 \pmod 3$ be an odd power prime, $q \geq 11$. Then, in Construction C, there is at least one variant of K_1 providing a complete $\frac{1}{2}(q+7)$-arc K.*

Remark 2.20. We denote $\psi(S_6) = S_6^{(1)}$, $\psi^2(S_6) = S_6^{(2)}$. By (2.5), (2.10),

$$S_6^{(1)} = \{Q, Q_+, A_1, A_{-1}, A_{-3}, A_3\}, \ S_6^{(2)} = \{Q_+, P, A_{-1}, A_\infty, A_3, A_0\}.$$

Let $K = K_1 \cup S_6$ be an arc of Construction C. Since $\psi(K_1) = K_1$, we have the following $\frac{1}{2}(q+7)$-arcs $K^{(1)}$ and $K^{(2)}$ as *variants of Construction C*:

$$K^{(1)} = \psi(K) = K_1 \cup S_6^{(1)}, \quad K^{(2)} = \psi^2(K) = K_1 \cup S_6^{(2)}.$$

3. Quadruples of 3-cycles. Equivalent $\frac{1}{2}(q+7)$-arcs

3.1. 3-cycle quadruples and three types of $\frac{1}{2}(q+7)$-arcs.

As $q \equiv 2 \pmod 3$ odd, we have either $q \equiv 5 \pmod{12}$ or $q \equiv 11 \pmod{12}$. Only in the 2-nd case, $\sqrt{3}$ exists.

Lemma 3.1. *Let $q \equiv 11 \pmod{12}$. Then*

$$C_{\sqrt{3}} = \{A_{\sqrt{3}}, A_{3-2\sqrt{3}}, A_{-3-2\sqrt{3}}\}, \quad C_{-\sqrt{3}} = \{A_{-\sqrt{3}}, A_{3+2\sqrt{3}}, A_{-3+2\sqrt{3}}\}.$$

Proof. We use (2.6), Lemmas 2.6 and 2.9, and the following relations:
$(3-2\sqrt{3})(1+\sqrt{3}) = \sqrt{3}-3$, $(-3-2\sqrt{3})(1-\sqrt{3}) = \sqrt{3}+3$. □

Definition 3.2. We call a *quadruple* the set of 3-cycles
(3.1)
$$\mathcal{Q}_i = \{C_i, C_{-3/i}, C_{-i}, C_{3/i}\},\ i \notin \{0, \pm 3, \infty, \pm 1, \pm\sqrt{3}, \pm(3-2\sqrt{3}), \pm(3+2\sqrt{3})\}.$$

By (3.1),

$$\mathcal{Q}_i = \mathcal{Q}_{3/i} = \mathcal{Q}_{-i} = \mathcal{Q}_{-3/i}. \tag{3.2}$$

By (2.6),

$$m(3/i) = \frac{3(1-i)}{i+3},\ p(3/i) = \frac{3(1+i)}{i-3},\ i \notin \{0, \pm 3, \infty, \pm 1\}. \tag{3.3}$$

Lemma 3.3. *Let $i \notin \{0, \pm 3, \infty, \pm 1\}$. Then the cycles C_i and $C_{-3/i}$ are distinct. Also, $C_i = C_{3/i}$ if and only if $q \equiv 11 \pmod{12}$ and $i \in \{\pm\sqrt{3}, \pm(3-2\sqrt{3}), \pm(3+2\sqrt{3})\}$.*

Proof. We use (3.3) and Lemmas 2.6, 2.9, and 3.1. Assume that either $i = -3/i$ or $i = -p(3/i)$, or finally $i = -m(3/i)$. In the all three cases we obtain $i^2 = -3$ that is impossible. Now, assume that $i = 3/i$. Then $i^2 = 3$ and $i = \pm\sqrt{3}$. □

Corollary 3.4. *Let $i \notin \{0, \pm 3, \infty, \pm 1, \pm\sqrt{3}, \pm(3-2\sqrt{3}), \pm(3+2\sqrt{3})\}$. Then the quadruple $\Psi_i = \{C_i, C_{-3/i}, C_{-i}, C_{3/i}\}$ consists of four distinct 3-cycles.*

Theorem 3.5. *Let $\mathcal{C}$ be the conic of (2.2). Let $\mathcal{C}^* = \mathcal{C}\setminus(C_0 \cup C_1)$. If $q \equiv 5 \pmod{12}$ then $\mathcal{C}^*$ is partitioned into $\frac{1}{12}(q-5)$ quadruples $\mathcal{Q}_i$. If $q \equiv 11 \pmod{12}$ then $\mathcal{C}^*$ is partitioned into $\frac{1}{12}(q-11)$ quadruples $\mathcal{Q}_i$ and the pair of 3-cycles $\{C_{\sqrt{3}}, C_{-\sqrt{3}}\}$.*

Proof. We use Lemmas 3.1, 3.3 and Corollary 3.4. □

By Construction C, we include in the set K_1 exactly *two 3-cycles from every quadruple* $\mathcal{Q}_i$. As $\mathcal{Q}_i = \mathcal{Q}_{3/i} = \mathcal{Q}_{-i} = \mathcal{Q}_{-3/i}$, there are exactly two non equivalent variants of such inclusion: $C_i, C_{-3/i}$ and $C_i, C_{3/i}$. We denote by V_- and V_+ the subset of K_1 consisting of the pairs of the form $C_i, C_{-3/i}$ and $C_i, C_{3/i}$, respectively. It is possible that $V_- = \emptyset$ or $V_+ = \emptyset$. Also, if $q \equiv 11 \pmod{12}$, one and only one cycle of $\{C_{\sqrt{3}}, C_{-\sqrt{3}}\}$ is used in K_1 and it is denoted by $V_{\sqrt{3}}$. If $q \equiv 5 \pmod{12}$, then $\sqrt{3}$ does not exist and $V_{\sqrt{3}} = \emptyset$. So, the point $\frac{1}{2}(q+7)$-set of Construction C has the form

$$K = K_1 \cup S_6 = V_- \cup V_+ \cup V_{\sqrt{3}} \cup S_6. \tag{3.4}$$

We partition the set $\mathcal{M}$ of all arcs K of Construction C into three classes $\mathcal{M}_-$, $\mathcal{M}_+$, and $\mathcal{M}_\pm$ consisting, respectively, from arcs of the types K_-, K_+, and $K_\pm$ such that
(3.5)
$$\begin{array}{llllll}
q \equiv 5 \pmod{12}, & K_- = V_- \cup S_6, & V_- \neq \emptyset, & V_+ = \emptyset, & V_{\sqrt{3}} = \emptyset. \\
q \equiv 5 \pmod{12}, & K_+ = V_+ \cup S_6, & V_- = \emptyset, & V_+ \neq \emptyset, & V_{\sqrt{3}} = \emptyset. \\
q \equiv 11 \pmod{12}, & K_+ = V_+ \cup V_{\sqrt{3}} \cup S_6, & V_- = \emptyset, & V_+ \neq \emptyset, & V_{\sqrt{3}} \neq \emptyset. \\
q \equiv 5 \pmod{12}, & K_\pm = V_- \cup V_+ \cup S_6, & V_- \neq \emptyset, & V_+ \neq \emptyset, & V_{\sqrt{3}} = \emptyset. \\
q \equiv 11 \pmod{12}, & K_\pm = V_- \cup V_+ \cup V_{\sqrt{3}} \cup S_6, & V_- \neq \emptyset, & V_+ \neq \emptyset, & V_{\sqrt{3}} \neq \emptyset.
\end{array}$$

By above,

$$\mathcal{M} = \mathcal{M}_- \cup \mathcal{M}_+ \cup \mathcal{M}_\pm. \quad |\mathcal{M}| = |\mathcal{M}_-| + |\mathcal{M}_+| + |\mathcal{M}_\pm|. \tag{3.6}$$

Lemma 3.6. *It holds that* $|\mathcal{M}| = 2^{(q-5)/6}$,

$$
\begin{aligned}
|\mathcal{M}_-| &= \begin{cases} 2^{(q-5)/12} & \text{if } q \equiv 5 \pmod{12} \\ 0 & \text{if } q \equiv 11 \pmod{12} \end{cases}, \\
|\mathcal{M}_+| &= 2^{\lfloor (q+1)/12 \rfloor} = \begin{cases} 2^{(q-5)/12} & \text{if } q \equiv 5 \pmod{12} \\ 2^{(q+1)/12} & \text{if } q \equiv 11 \pmod{12} \end{cases}, \\
|\mathcal{M}_\pm| &= \begin{cases} 2^{(q-5)/6} - 2^{(q+7)/12} & \text{if } q \equiv 5 \pmod{12} \\ 2^{(q-5)/6} - 2^{(q+1)/12} & \text{if } q \equiv 11 \pmod{12} \end{cases}.
\end{aligned} \tag{3.7}
$$

Proof. The value of $|\mathcal{M}|$ follows from Construction C. For $|\mathcal{M}_-|$ and $|\mathcal{M}_+|$ we use Theorem 3.5. Now, by (3.6), $|\mathcal{M}_\pm| = |\mathcal{M}| - |\mathcal{M}_-| - |\mathcal{M}_+|$. □

3.2. PROJECTIVITIES ϕ_1, ϕ_2, ϕ_3 AND THEIR ACTION ON $\frac{1}{2}(q+7)$-ARCS. We use the following three projectivities ϕ_i of order two:

$$\phi_i(x_0, x_1, x_2) = (x_0, x_1, x_2)\mathbf{G}_i, \ i = 1, 2, 3;$$

$$\mathbf{G}_1 = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 3 & 0 \\ -2 & 0 & -1 \end{bmatrix}, \ \mathbf{G}_3 = \begin{bmatrix} 0 & -2 & 2 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \tag{3.8}$$

$$\mathbf{G}_2 = \begin{bmatrix} 4 & 6 & 2 \\ 3 & -3 & -3 \\ 1 & -3 & 5 \end{bmatrix} \text{ if } q \neq 5^{2t+1}, \ \mathbf{G}_2 = \begin{bmatrix} 4 & 1 & 2 \\ 3 & -3 & -3 \\ 1 & -3 & 0 \end{bmatrix} \text{ if } q = 5^{2t+1}.$$

Let ε be the identity transformation of points of $PG(2, q)$, i.e.

$$\varepsilon(x_0, x_1, x_2) = (x_0, x_1, x_2).$$

Lemma 3.7. *It holds that*

$$\phi_1^2 = \phi_2^2 = \phi_3^2 = \phi_1\phi_2\phi_3 = \varepsilon.$$

Proof. It is easy to calculate that $\mathbf{G}_1^2 = 9\mathbf{I}_3$, $\mathbf{G}_2^2 = 36\mathbf{I}_3$, $\mathbf{G}_3^2 = 4\mathbf{I}_3, \mathbf{G}_1\mathbf{G}_2\mathbf{G}_3 = -36\mathbf{I}_3$, where $\mathbf{I}_3$ is the identity matrix of order three. □

Lemma 3.8. *The projectivities ϕ_1, ϕ_2, ϕ_3 preserve the set S_6 of* (2.11), *i.e.*

$$\phi_i(S_6) = S_6, \ i = 1, 2, 3.$$

Proof. The assertions can be checked directly by (3.8). □

Lemma 3.9. *The projectivities ϕ_1, ϕ_2, ϕ_3 transform the following 3-cycles to each other:*

$$\phi_1(C_i) = C_{-3/i}, \ \phi_2(C_i) = C_{3/i}, \ \phi_3(C_i) = C_{-i}, \ i \notin \{0, \pm 3, \infty, \pm 1\}. \tag{3.9}$$

Proof. The assertions can be checked directly using (2.6), (3.3) and Lemmas 2.6 and 2.9. □

Definition 3.10. **(i):** Let W be a set of 3-cycles. Then $\overline{W}$ is the set obtained from W so that every 3-cycle of W is changed by its opposite cycle. If $W = \emptyset$ then $\overline{W}$ is empty too.

(ii): Let $K = K_1 \cup S_6$. Then $\widehat{K} = \overline{K_1} \cup S_6$ is called an *opposite arc* to K.

By (3.9),

$$\begin{aligned} &\phi_1(V_-) = V_-,\ \phi_1(V_+) = \overline{V_+},\ \phi_1(V_{\sqrt{3}}) = \overline{V_{\sqrt{3}}}\ ; \\ &\phi_2(V_-) = \overline{V_-},\ \phi_2(V_+) = V_+,\ \phi_2(V_{\sqrt{3}}) = V_{\sqrt{3}}\ ; \\ &\phi_3(K_1) = \overline{K_1}. \end{aligned} \tag{3.10}$$

Lemma 3.11. *Let $K = K_1 \cup S_6 = V_- \cup V_+ \cup V_{\sqrt{3}} \cup S_6$ be an arc of Construction C. Let the types K_-, K_+, and $K_\pm$ of the arcs K be defined as in (3.5). Then*

$$\phi_1(K) = V_- \cup \overline{V_+} \cup \overline{V_{\sqrt{3}}} \cup S_6, \qquad \phi_1(K_-) = K_-, \qquad \phi_1(K_+) = \phi_3(K_+) = \widehat{K_+};$$
$$\phi_2(K) = \overline{V_-} \cup V_+ \cup V_{\sqrt{3}} \cup S_6,\ \phi_2(K_-) = \phi_3(K_-) = \widehat{K_-}, \qquad \phi_2(K_+) = K_+;$$
$$\phi_3(K) = \widehat{K}.$$

Proof. The assertions follow from Lemma 3.8 and (3.10). □

3.3. STABILIZERS OF $\frac{1}{2}(q+7)$-ARCS OF CONSTRUCTION C. Let T be a point set of $PG(2,q)$. We denote by $W(T)$ and $W^*(T)$ the stabilizer group of T up to projectivities and collineations, respectively, i.e. $W(T) \subseteq PGL(3,q)$ and $W^*(T) \subseteq P\Gamma L(3,q)$. Let

$$\Pi = \{\varepsilon, \phi_1, \phi_2, \phi_3\},\ \Pi_1 = \{\varepsilon, \phi_1\},\ \Pi_2 = \{\varepsilon, \phi_2\}$$

be sets of projectivities. By Lemma 3.7, these sets are groups. Moreover,

$$\Pi \cong \mathbf{Z}_2 \times \mathbf{Z}_2.$$

Lemma 3.12. *Let $q = p^h$, $p \equiv 2 \pmod 3$, p prime, $h = 2t+1 \geq 1$. The following holds for the stabilizer of S_6:*

(i) *$\Pi \subseteq W(S_6)$ for all q;*
(ii) *$W(S_6) = \Pi$, $|W(S_6)| = 4$, if $q \geq 11$;*
(iii) *$W(S_6) = PGL(2,q)$, $|W(S_6)| = q(q^2-1) = 120$, if $q = 5$;*
(iv) *$|W^*(S_6)| = h \cdot |W(S_6)| = \begin{cases} 4h & \text{if } p \geq 11 \\ 120h & \text{if } p = 5 \end{cases}$.*

Proof. (i) By Lemma 3.8, we have $\Pi \subseteq W(S_6)$.

(ii) It should be noted, that for $q \geq 11$ we have $A_{-3} = (1,-3,5)$ with nonzero last coordinate whereas $A_{-3} = (1,-3,0)$ if $q = 5$. For $q \geq 11$, we verified by direct computations that only the projectivities in Π preserve S_6.

(iii) It can be checked directly that S_6 is contained in the conic of equation $2x_0x_1 - 2x_0x_2 + 2x_0^2 - x_1x_2 = 0$. For $q = 5$, S_6 coincides with the conic. Now the assertion follows from [17, Cors 7.13, 7.14].

(iv) By Remark 1, the points of S_6 are fixed by the field automorphisms. □

Lemma 3.13. *Let K be an arc of Construction C. For $q \geq 11$, the stabilizer up to $P\Gamma L(3,q)$ of K is contained in that of S_6, i.e.*

$$W^*(K) \subseteq W^*(S_6) \text{ if } q \geq 11.$$

Proof. Let $\sigma \subset W^*(K)$ be a collineation.

It is known that a collineation sends a conic to a conic. Moreover, we show that $\sigma(\mathcal{C}) = \mathcal{C}$ where $\mathcal{C}$ is the conic of (2.2). Consider a set S of 9 points of K. Let $T = \sigma(S)$. As $T \subseteq K$, there are at least 7 points of T contained in $\mathcal{C}$. Let T_1 be a subset of T of size 7 contained in $\mathcal{C}$. Let $S_1 \subset S$ so that $\sigma(S_1) = T_1$. As $S_1 \subset K$ and $|S_1| = 7$, there are at least 5 points of S_1 contained in $\mathcal{C}$. Let S_2 be a subset of 5 points of S_1 contained in $\mathcal{C}$. Let $T_2 = \sigma(S_2)$. As $T_2 \subset T_1$, we have $T_2 \subset \mathcal{C}$. Now let $\mathcal{C}_1 = \sigma(\mathcal{C})$. The conic $\mathcal{C}_1$ contains T_2, the image of S_2. On the other side, T_2 is contained also in $\mathcal{C}$. As through 5 non collinear points a unique conic passes, we have $\sigma(\mathcal{C}) = \mathcal{C}$.

As $\sigma(K) = K$ and $\sigma(\mathcal{C}) = \mathcal{C}$, it holds that $\sigma(\{P, Q\}) = \{P, Q\}$. Also, $\sigma(\{A_0, A_1, A_\infty, A_{-3}\}) = \{A_0, A_1, A_\infty, A_{-3}\}$. In fact let r be a tangent line to $\mathcal{C}$ through P or Q. Then $\sigma(r)$ is a line through P or Q and is tangent to $\mathcal{C}$. (The collineation σ fixes $\mathcal{C}$, so secants go to secants, tangents go to tangents and external lines go to external lines.) Recall that, by Lemma 2.11, the pairs of tangent points of P and Q on $\mathcal{C}$ are, respectively, A_∞,A_0, and A_1, A_{-3}. So, $\sigma(S_6) = S_6$. □

Remark 3.14. Similarly to Lemma 16 it can be proven that the stabilizer up to $PGL(3, q)$ of K is contained in that of S_6, i.e.

$$W(K) \subseteq W(S_6) \text{ if } q \geq 11.$$

Theorem 3.15. *Let K_-, K_+, and $K_\pm$ be arcs of Construction C defined in* (3.5). *For the stabilizers of these arcs the following holds.*

$$\begin{aligned} \Pi_1 &\subseteq W(K_-), \quad \Pi_2 \subseteq W(K_+). \\ W(K_-) &= \Pi_1,\, W(K_+) = \Pi_2,\, W(K_\pm) = \{\varepsilon\} \text{ if } q \geq 11. \end{aligned}$$

Proof. The assertion follows from Lemmas 3.11, 3.12 and Remark 3.14. □

3.4. Equivalent arcs of Construction C. In this subsection we consider the equivalence of arcs and their stabilizers up to $PGL(3, q)$.

Lemma 3.16. *Let K and K' be equivalent $\frac{1}{2}(q+7)$-arcs of Construction C and let $K' = \varkappa(K)$ where $\varkappa$ is a projectivity. Then $\varkappa \subset \Pi$.*

Proof. By hypothesis, $K = K_1 \cup S_6$, $K' = K'_1 \cup S_6$, $K_1 \subset \mathcal{C}$, $K'_1 \subset \mathcal{C}$, where $\mathcal{C}$ is the conic of (2.2). As in the proof of Lemma 3.13, it can be shown that $\varkappa(\mathcal{C}) = \mathcal{C}$ and $\varkappa(S_6) = S_6$. □

Here and further, the term "*formally distinct sets*" means that the sets are distinct by construction, but they could actually coincide as projective objects.

By Definition 3.10, if $W \neq \emptyset$ then the sets W and $\overline{W}$ are formally distinct. In particular, it means that the opposite arcs K and $\widehat{K}$ are formally distinct. Moreover, they are equivalent as $\phi_3(K) = \widehat{K}$.

Theorem 3.17. *Let K_-, K_+, and $K_\pm$ be arcs of Construction C defined in* (3.5).

- **(i):** *If an $\frac{1}{2}(q+7)$-arc K of Construction C is of type K_- or K_+ then the opposite arc $\phi_3(K) = \widehat{K}$ is the unique formally distinct arc of Construction C equivalent to K.*
- **(ii):** *For every arc $K_\pm = V_- \cup V_+ \cup V_{\sqrt{3}} \cup S_6$ of Construction C with $V_- \neq \emptyset$ and $V_+ \neq \emptyset$ there are exactly three formally distinct arcs of Construction C equivalent to it. These equivalent arcs are*

$$\phi_1(K_\pm) = V_- \cup \overline{V_+} \cup \overline{V_{\sqrt{3}}} \cup S_6,\ \phi_2(K_\pm) = \overline{V_-} \cup V_+ \cup V_{\sqrt{3}} \cup S_6,\ \phi_3(K_\pm) = \widehat{K_\pm}.$$

Proof. The assertion follows from Lemmas 3.11 and 3.16. □

Remark 3.18. For $q = 17$, an incomplete $\frac{1}{2}(q+7)$-arc of Construction C is of type K_+. Therefore here we have two incomplete arcs of Construction C equivalent to each other. For $q = 59$, an incomplete $\frac{1}{2}(q+7)$-arc of Construction C is of type $K_\pm$. Therefore, in this case, we have four incomplete arcs of Construction C equivalent to each other.

We denote by $\mathcal{N}^{\rm neq}$, $\mathcal{N}_-^{\rm neq}$, $\mathcal{N}_+^{\rm neq}$, and $\mathcal{N}_\pm^{\rm neq}$ the number of nonequivalent arcs in the sets $\mathcal{M}$, $\mathcal{M}_-$, $\mathcal{M}_+$, and $\mathcal{M}_\pm$ respectively. By (3.6),

$$\mathcal{N}^{\rm neq} = \mathcal{N}_-^{\rm neq} + \mathcal{N}_+{}^{\rm neq} + \mathcal{N}_\pm^{\rm neq}. \tag{3.11}$$

Theorem 3.19. *The number of nonequivalent $\frac{1}{2}(q+7)$-arcs of Construction C is as follows.*

$$\mathcal{N}_-^{\rm neq} = \frac{1}{2}|\mathcal{M}_-|,\ \mathcal{N}_+^{\rm neq} = \frac{1}{2}|\mathcal{M}_+|,\ \mathcal{N}_\pm^{\rm neq} = \frac{1}{4}|\mathcal{M}_\pm|. \tag{3.12}$$

$$\mathcal{N}^{\rm neq} = \begin{cases} 2^{(q-17)/6} + 2^{(q-17)/12} & \text{if } q \equiv 5 \pmod{12} \\ \lfloor 2^{(q-17)/6} \rfloor + 2^{(q-11)/12} - \lfloor 2^{(q-23)/12} \rfloor & \text{if } q \equiv 11 \pmod{12} \end{cases}. \tag{3.13}$$

Proof. The assertion (3.12) follows from Theorem 3.17. For (3.13) we use (3.7), (3.11), and (3.12). □

Remark 3.20. In [7, Tab. 5], the total number of nonequivalent $\frac{1}{2}(q+7)$-arcs sharing $\frac{1}{2}(q+3)$ points with a conic is given for $q \le 251$. Comparing the number of classes of Theorem 3.19 with the values of [7, Tab. 5], we see that for $q = 11, 23, 47, 191$ Construction C produces all the $\frac{1}{2}(q+7)$-arcs sharing $\frac{1}{2}(q+3)$ points with a conic, while for the other values of q, $q \le 251$, other $\frac{1}{2}(q+7)$-arcs sharing $\frac{1}{2}(q+3)$ points with a conic can be obtained with a different choice of the external point Q, as described in [7, Th. 4].

4. Structures of the arcs $K_{17}(4)$ and $K_{59}(3)$

4.1. On completing points for incomplete K_- and K_+ arcs of Construction C.

Recall, see e.g. [27, Sec. 2.7], that a collineation σ of $PG(2,q)$ is *central* if there is a point C (the center) and a line L (the axis) such that σ fixes any line on C and any point on L. Moreover, a collineation of order 2 is either a central collineation or a Baer collineation. As we consider non square q, all the projectivities ϕ_i of (3.8) are central.

By an elementary calculation we have the following lemma.

Lemma 4.1. **(i):** *For the central projectivities ϕ_1, ϕ_2, and ϕ_3, of (3.8), the axis is the line of equation $x_0 + x_2 = 0$, $2x_0 - 3x_1 - x_2 = 0$, and $2x_0 + x_1 - x_2$, respectively.*

(ii): *The projectivities ϕ_1 and ϕ_2 are homologies with the center $(1, 0, 2)$ and $(1, -3, -1)$, respectively.*

The following lemma is obvious.

Lemma 4.2. *Let the stabilizer group $W(K)$ of a $\frac{q+7}{2}$-arc K of Construction C have order two and let the matrix form of the generator of $W(K)$ be $\mathbf{G}$. Then a point $X \notin K$ lies on a bisecant of K if and only if the point $X\mathbf{G}$ lies on a bisecant of K.*

Lemma 4.3. *Let K be an incomplete arc of Construction C. Then the points, completing K to a complete arc, are external to the conic $\mathcal{C}$ of (2.2).*

Proof. As $|K \cap \mathcal{C}| = \frac{q+3}{2}$, it holds that all points internal to $\mathcal{C}$ are covered by bisecants of K. Also, by Construction C all points of $\mathcal{C}$ not belonging to K are covered too. □

Recall that $\mathcal{L}$ is the line with equation $x_0 + x_2 = 0$.

Theorem 4.4. *Let q be prime.*

(i): *Let K_- be an incomplete arc of Construction C. Let B_- be the set of points completing K_- to a complete arc. Then $|B_-| = 2r$ with r integer and B_- is partitioned into r 2-cycles of the projectivity ϕ_1 so that every 2-cycle is collinear with the point $(1, 0, 2)$. Moreover, $\phi_2(B_-) = \phi_3(B_-)$.*

(ii): *Let K_+ be an incomplete arc of Construction C and let all points on the line of equation $2x_0 - 3x_1 - x_2 = 0$ be covered by bisecants of K_+. Let B_+ be the set of points completing K_- to a complete arc. Then $|B_+| = 2d$ with d integer and B_+ is partitioned into d 2-cycles of the projectivity ϕ_2 so that every 2-cycle is collinear with the point $(1, -3, -1)$. Moreover, $\phi_1(B_+) = \phi_3(B_+)$.*

Proof. We use Theorem 3.15 and Lemmas 3.7, 3.11, 4.1, and 4.2. Note also that the point $(1, 0, 2)$ and the line $\mathcal{L}$ are always covered by bisecants of any arc of Construction C. □

Example 4.5. For $q = 17$ there is an incomplete arc K_+ of Construction C. It is described in detail in Section 4.3 where it is shown that in this case we have $B_+ = \{R = (1, 3, 6), S = (1, 15, 3)\}$, i.e. $d = 1$, see Theorem 9(ii). It is easy to check directly that $\phi_2(R) = S$ and that R, S and $(1, -3, -1)$ are collinear. Moreover, $\phi_1(\{R, S\}) = \{(1, 10, 4), (1, 8, 15)\} = \phi_3(\{R, S\})$.

4.2. DE-CYCLES. Let B be an external point to the conic $\mathcal{C}$ of (2.2). As with the involution σ_Q of [7, Sec. 2] we introduce a function $f_B(i)$ over F_q^+ such that the points $A_i, A_{f_B(i)}$ are collinear with B when $i \neq f_B(i)$. If $i = f_B(i)$ then the line BA_i is tangent to $\mathcal{C}$ at A_i.

Lemma 4.6. *It holds that*

$$f_B(i) = \frac{bi + 1 - 2c}{i - b} \text{ if } B = (1, b, c);\; f_B(i) = 2c - i \text{ if } B = (0, 1, c). \tag{4.1}$$

Proof. The assertion follows from the following two determinants:

$$\begin{vmatrix} 1 & b & c \\ 1 & i & \frac{i^2+1}{2} \\ 1 & j & \frac{j^2+1}{2} \end{vmatrix} = 0, \qquad \begin{vmatrix} 0 & 1 & c \\ 1 & i & \frac{i^2+1}{2} \\ 1 & j & \frac{j^2+1}{2} \end{vmatrix} = 0.$$ □

According to Lemma 2.12 and Corollary 2.14, by using (4.1),

$$f_P(i) = -i,\; f_Q(i) = -m(i). \tag{4.2}$$

Definition 4.7. Let D, E be external points of the conic $\mathcal{C}$ of (2.2). If there exists $H_{DE} = \{d_1, e_1, d_2, e_2, d_3, e_3\} \subset F_q, |H_{DE}| = 6$, such that:

$$e_u = f_E(d_u),\; u = 1, 2, 3,\; d_1 = f_D(e_3),\; d_j = f_D(e_{j-1}),\; j = 2, 3$$

then the 3-sets $C_{d_1}^{DE} = \{A_{d_1}, A_{d_2}, A_{d_3}\}$ and $C_{e_1}^{DE} = \{A_{e_1}, A_{e_2}, A_{e_3}\}$ are called *opposite DE-cycles.*

If we form an arc $\mathcal{K}$ containing D, E and some points of the conic $\mathcal{C}$ then, by the definition of the function $f_B(i)$, only one of two opposite DE-cycles may belong to $\mathcal{K}$.

Lemma 4.8. *The "standard" opposite* 3*-cycles of Lemma* 2.9 *are PQ-cycles, i.e.*

$$C_i = C_i^{PQ},\; C_{-i} = C_{-i}^{PQ},\; i \notin \{0, \pm 3, \infty, \pm 1\}.$$

Proof. The assertion can be checked directly using (2.6), (2.8), (2.9), and (4.2). □

For the 3-cycles $C_i = \{A_i, A_j, A_k\}$ and $C_u^{DE} = \{A_u, A_b, A_c\}$ we denote the corresponding sets of subscripts $\widetilde{C}_i = \{i, j, k\}$ and $\widetilde{C}_u^{DE} = \{u, b, c\}$.

Lemma 4.9. *Let K be a $\frac{1}{2}(q+7)$-arc of Construction C. Let $U \notin K$ be an external point of the conic $\mathcal{C}$. If U does not lie on a bisecant of K, both of the tangent points from U on $\mathcal{C}$ belong to K.*

Proof. An external point U of the conic lies on 2 tangents and $\frac{q-1}{2}$ bisecants. The arc K contains $\frac{q-1}{2}+2$ points of the conic. If at least one tangent point does not belong to K then exactly $\frac{q-1}{2}+1$ points of the conic simultaneously lie on bisecants through U and belong to K. So, at least one bisecant through U contains two points belonging K, that is U is covered. □

4.3. STRUCTURE OF $K_{17}(4)$. Let $q = 17$. So, $q \equiv 5 \pmod{12}$ and $V_{\sqrt{3}} = \emptyset$. Thus,

$$\begin{aligned}\widetilde{C}_2 &= \{2, 11, 12\}, -\frac{3}{2} = 7,\ \widetilde{C}_7 = \{7, 9, 4\},\ \widetilde{C}_{-2} = \{15, 5, 6\},\ \frac{3}{2} = -7,\\ \widetilde{C}_{-7} &= \{10, 13, 8\},\ \mathcal{C}^* = \mathcal{Q}_2 = \{C_2, C_7, C_{-2}, C_{-7}\}.\end{aligned}$$

We put $V_- = \emptyset$, $K_1 = V_+ = C_2 \cup C_{-7}$. The $\frac{1}{2}(q+7)$-arc

$$K = K_+ = V_+ \cup S_6 = C_2 \cup C_{-7} \cup S_6$$

is incomplete. It is equivalent to the opposite arc $\widehat{K} = \overline{V_+} \cup S_6 = C_{-2} \cup C_7 \cup S_6$.

The arc K is embedded in the complete $\frac{1}{2}(q+11)$-arc $K_{17}(4) = K \cup \{R, S\}$ where $R = (1, 3, 6)$, $S = (1, 15, 3)$ are points external to the conic $\mathcal{C}$. The tangent points of R and S on $\mathcal{C}$ are, respectively, A_{13}, A_{10} and A_2, A_{11}. By (4.1), $f_R(i) = (3i+6)/(i-3)$ and $f_S(i) = -(2i+5)/(2+i)$.

By Theorem 4.4(ii), R and S form a 2-cycle of the projectivity ϕ_2, i.e. $R \times \mathbf{G_2} = S$. Moreover, R and S are collinear with the point $(1, -3, -1)$ and $\phi_1(R, S) = \phi_3(R, S)$.

The following 3-cycles of Definition 4.7 can be formed by using the points P, Q, R, and S :

$$\begin{aligned}\widetilde{C}_i^{PQ} &= \{i, \frac{i-3}{1+i}, \frac{i+3}{1-i}\},\ \widetilde{C}_i^{PS} = \{i, \frac{2i+5}{2+i}, \frac{9i+3}{4i+9}\},\\ \widetilde{C}_i^{RQ} &= \{i, \frac{1+7i}{2i}, \frac{7}{2-3i}\},\ \widetilde{C}_i^{RS} = \{i, \frac{3}{5i+11}, \frac{15i-1}{4i}\}.\end{aligned} \tag{4.3}$$

The set

$$\Phi_i = C_i^{PQ} \cup C_i^{PS} \cup C_i^{RQ} \cup C_i^{RS}$$

is called *flower* with *center* A_i. Every 3-cycle is called *petal.*

We denote $\widetilde{\Phi}_i = \widetilde{C}_i^{PQ} \cup \widetilde{C}_i^{PS} \cup \widetilde{C}_i^{RQ} \cup \widetilde{C}_i^{RS}$.

By computer we showed the following.

Proposition 4.10. *The flower $\Phi_i = C_i^{PQ} \cup C_i^{PS} \cup C_i^{RQ} \cup C_i^{RS}$, given by (4.3), is a 9-arc in $PG(2, 17)$.*

Fixing the external points P, Q, R, S and their tangent points $A_\infty, A_0, A_1, A_{-3}, A_{13}, A_{10}, A_2, A_{11}$, we eliminate the points of conic $\mathcal{C}$ collinear with the points chosen. The only points which remain to add are $A_8 = (1, 8, 7)$ and $A_{12} = (1, 12, 13)$. We note that the points of the flower Φ_8 with center A_8 are points already chosen and so not in contrast to each other. The same happens for Φ_{12}. By (4.3),

$$\begin{aligned}\widetilde{\Phi}_8 &= \{8\} \cup \{10, 13\} \cup \{-3, 1\} \cup \{11, 2\} \cup \{\infty, 0\},\\ \widetilde{\Phi}_{12} &= \{12\} \cup \{2, 11\} \cup \{13, 10\} \cup \{0, \infty\} \cup \{1, -3\}.\end{aligned}$$

Finally,

$$K_{17}(4) = \Phi_8 \cup \Phi_{12} \cup \{P, Q, R, S\}.$$

4.4. STRUCTURE OF $K_{59}(3)$. Let $q = 59$. So, $q \equiv 11 \pmod{12}$ and $V_{\sqrt{3}} \neq \emptyset$. It holds that $\sqrt{3} = 11$,

$$\begin{aligned} -\frac{3}{55} &= 45,\ \mathcal{Q}_{55} = \{C_{55}, C_{45}, C_4, C_{14},\},\ -\frac{3}{23} = 5,\ \mathcal{Q}_{23} = \{C_{23}, C_5, C_{36}, C_{54}\},\\ -\frac{3}{7} &= 8,\ \mathcal{Q}_8 = \{C_7, C_8, C_{52}, C_{51}\},\ -\frac{3}{33} = 16,\ \mathcal{Q}_{33} = \{C_{33}, C_{16}, C_{26}, C_{43}\},\\ \mathcal{C}^* &= \mathcal{Q}_{55} \cup \mathcal{Q}_{23} \cup \mathcal{Q}_7 \cup \mathcal{Q}_{33} \cup \{C_{11}, C_{-11}\}. \end{aligned}$$

We put

$$V_- = C_{55} \cup C_{45},\ V_+ = (C_{23} \cup C_{54}) \cup (C_7 \cup C_{51}) \cup (C_{33} \cup C_{43}),\ V_{\sqrt{3}} = C_{11}.$$

The $\frac{1}{2}(q+7)$-arc

$$K = K_\pm = V_- \cup V_+ \cup V_{\sqrt{3}} \cup S_6 = \{C_{55}, C_{45}, C_{23}, C_{54}, C_7, C_{51}, C_{33}, C_{43}, C_{11}\} \cup S_6$$

is incomplete. It is equivalent to the three formally distinct arcs $\phi_1(K), \phi_2(K)$, $\phi_3(K)$ of Construction C.

The arc K is embedded in the complete $\frac{1}{2}(q+9)$-arc $K_{59}(3) = K \cup H$ where $H = (1, 8, 7)$ is an external point to the conic $\mathcal{C}$. The tangent points of H on $\mathcal{C}$ are A_{-5}, A_{21}. By (4.1), $f_H(i) = (8i - 13)/(i - 8)$.

The following 3-cycles of Definition 4.7 can be formed by using P, Q, H :

$$\widetilde{C}_i^{PQ} = \{i, \frac{i-3}{1+i}, \frac{i+3}{1-i}\},\ \widetilde{C}_i^{QH} = \{i, \frac{21i-11}{5+9i}, \frac{12i-9}{2i+15}\}. \tag{4.4}$$

The two 3-cycles of (4.4) form a *flower* $\Phi_i = C_i^{PQ} \cup C_i^{QH}$ with *center* A_i. Every 3-cycle is a *petal*. We denote $\widetilde{\Phi}_i = \widetilde{C}_i^{PQ} \cup \widetilde{C}_i^{QH}$.

By computer we showed the following.

Proposition 4.11. *The flower* $\Phi_i = C_i^{PQ} \cup C_i^{QH}$, *given by* (4.4), *is a* 5*-arc in* $PG(2, 59)$.

Fixing the external points P, Q, H and their tangent points $A_\infty, A_0, A_1, A_{-3}, A_{-5}$, A_{21}, we eliminate the points $A_3, A_5, A_8, A_9, A_{26}, A_{38}, A_{57}, A_{58}$, collinear with the points chosen. By (4.4),

$$\begin{aligned} \widetilde{\Phi}_2 &= \{2\} \cup \{54, 39\} \cup \{27, 7\}, & \widetilde{\Phi}_6 = \{6\} \cup \{10, 51\} \cup \{\infty, 22\},\\ \widetilde{\Phi}_{11} &= \{11\} \cup \{34, 40\} \cup \{18, 40\}, & \widetilde{\Phi}_{15} = \{15\} \cup \{24, 45\} \cup \{46, 51\},\\ \widetilde{\Phi}_{17} &= \{17\} \cup \{43, 27\} \cup \{47, 10\}, & \widetilde{\Phi}_{23} = \{23\} \cup \{31, 50\} \cup \{0, 45\},\\ \widetilde{\Phi}_{30} &= \{30\} \cup \{7, 18\} \cup \{31, 33\}, & \widetilde{\Phi}_{55} = \{55\} \cup \{47, 22\} \cup \{24, 34\}. \end{aligned}$$

Finally,

$$K_{59}(3) = \{\Phi_2, \Phi_6, \Phi_{11}, \Phi_{15}, \Phi_{17}, \Phi_{23}, \Phi_{30}, \Phi_{55}\} \cup \{P, Q, H\} \cup \{A_\infty, A_0, A_1, A_{-3}, A_{-5}, A_{21}\}.$$

5. On relation between Constructions C and S

In [10, Sec. 2], an explicit Construction S of $\frac{1}{2}(q+7)$-arcs in $PG(2,q)$, $q \equiv 2 \pmod 3$ odd prime, is given. The construction is based on plane sections of the complete $\frac{1}{2}(q^2+q+8)$-cap in $PG(3,q)$ of [13] sharing $\frac{1}{2}(q^2+q+2)$ points with an elliptic quadric. It is proven in [10] that every $\frac{1}{2}(q+7)$-set of Construction S is an arc sharing $\frac{1}{2}(q+3)$ points with an irreducible conic. Also, for $q \le 3701$, $q \ne 17$, there is at least one variant of an arc of Construction S that is a complete arc.

By computer, we showed the following interesting fact.

Proposition 5.1. *Let $q = 11, 17, 23, 29, 41, 47, 53, 59$. Then Construction S does not give any arcs nonequivalent to arcs of Construction C. Every $\frac{1}{2}(q+7)$-arc of Construction S is equivalent up to projectivities to an arc K_+ of Construction C. And vice versa, for every arc K_+ of Construction C there exists an arc of Construction S equivalent to it. All arcs K_- and $K_\pm$ of Construction C have no equivalent arcs by Construction S.*

Conjecture 5.2. *The assertion of Proposition 5.1 is true for all prime $q \equiv 2 \pmod 3$.*

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