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ASYMPTOTICS OF THE ARNOLD TONGUES IN PROBLEMS AT INFINITY

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Abstract. We consider discrete time systems $x_{k+1} = U(x_k;\lambda)$, $x \in \mathbb{R}^N$, with a complex parameter λ, and study their trajectories of large amplitudes. The expansion of the map $U(\cdot;\lambda)$ at infinity contains a principal linear term, a bounded positively homogeneous nonlinearity, and a smaller vanishing part. We study Arnold tongues: the sets of parameter values for which the large-amplitude periodic trajectories exist. The Arnold tongues in problems at infinity generically are thick triangles [4]; here we obtain asymptotic estimates for the length of the Arnold tongues and for the length of their triangular part. These estimates allow us to study subfurcation at infinity. In the related problems on small-amplitude periodic orbits near an equilibrium, similarly defined Arnold tongues have the form of narrow beaks. For standard pictures associated with the Neimark-Sacker bifurcation of smooth discrete time systems at an equilibrium, the Arnold tongues have asymptotically zero width except for the strong resonance points. The different shape of the tongues in the problem at infinity is due to the non-polynomial form of the principal homogeneous nonlinear term of the map $U(\cdot;\lambda)$: this form implies non-degeneracy of the nonlinear terms in the expansion of the map iterations and non-degeneracy of the corresponding resonance functions.

1. Introduction.

We study bifurcations of periodic trajectories from infinity for the discrete time system

$$x_{k+1} = U(x_k;\lambda), \qquad x \in \mathbb{R}^N,\ N \geq 2, \quad \lambda \in D \subset \mathbb{C}. \tag{1.1}$$

We suppose that for x with sufficiently large norm $|x|$ the map U is continuous with respect to the set of its arguments and has the form

$$U(x;\lambda) = A(\lambda)x + \Phi(x;\lambda) + \xi(x;\lambda); \tag{1.2}$$

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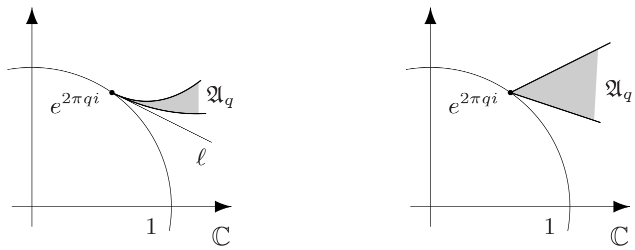

FIGURE 1. Typical shapes of the Arnold tongues at zero (left) and at infinity (right).

here $A(\lambda)x$ is the linear part, Φ is a bounded positively homogeneous nonlinearity, and ξ is a small term (exact assumptions are in Section 2.1).

For any $\lambda \in D$ consider the set $\mathfrak{P}_\lambda \subset \mathbb{R}^N$ of all periodic points of system (1.1). If the matrices $A = A(\lambda)$ have no eigenvalues in a fixed vicinity of the unit circle $S \in \mathbb{C}$, then the union $\mathfrak{P} = \bigcup_{\lambda \in D} \mathfrak{P}_\lambda$ is bounded: this follows from representation (1.2). If the matrix $A = A(\lambda)$ has a pair of simple complex conjugate eigenvalues $\mu(\lambda), \bar{\mu}(\lambda)$ and the range of $\mu(\lambda)$ has a limit point in S, then the set $\mathfrak{P}$ may be unbounded. The two cases are possible:

(i) For some fixed n the subset $\mathfrak{P}^n$ of $\mathfrak{P}$ that contains all periodic points of the period n is unbounded.
(ii) For any n the set $\mathfrak{P}^n$ is bounded, while $\mathfrak{P}$ is unbounded, i.e., there exist sequences n_k and $x_k \in \mathfrak{P}^{n_k}$ such that $n_k, |x_k| \to \infty$.

Cases (i) and (ii) are referred to as subharmonic bifurcation and subfurcation [3]. As we will see below, both of them can take place in a vicinity of the same limit point $\mu_q = e^{2\pi qi}$ of the range of $\mu(\lambda)$ if q is a rational number. We shall present sufficient conditions for these bifurcations.

To stress the difference between bifurcations at infinity and at zero, let us recall some relevant facts about bifurcation of periodic trajectories from the zero equilibrium of a smooth system (1.1) (see, e.g., [3]). Let for small $|x|$

$$U(x;\lambda) = A(\lambda)x + A_2(x;\lambda) + A_3(x;\lambda) + \dots, \tag{1.3}$$

where all N components of each term $A_s : \mathbb{R}^N \times \mathbb{C} \to \mathbb{R}^N$ are polynomials of the degree s with respect to all x-variables. Then, for any rational $q = m/n$ $(n > 4)$ with co-prime positive integer m and n there exists an open set $\mathfrak{A}_q$ called the Arnold tongue such that (see, e.g., [1]): μ_q belongs to the closure $\overline{\mathfrak{A}}_q$ of $\mathfrak{A}_q$; system (1.1) has a periodic point x^n_λ of the period n whenever $A(\lambda)$ has an eigenvalue $\mu(\lambda)$ in $\mathfrak{A}_q$; and $|x^n_\lambda| \to 0$ as $\mu(\lambda) \to \mu_q$ along the set $\mathfrak{A}_q$. Six points on the unit circle that correspond to the values $q = m/n$ with the denominators $n = 1, 2, 3, 4$ ($q \in \{1, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}\}$ modulo 1) are called strong resonances. For any q except for these special values, the Arnold tongue $\mathfrak{A}_q$ has the shape shown on the left picture of Fig. 1. Locally, $\mathfrak{A}_q$ is a cusp between two curves that meet at the point μ_q and have the same tangent straight line ℓ at this point. Moreover, the width w of the cusp at a distance d from the vertex μ_q is of order $d^{n/2-1}$ as $d \to 0$ (see [8]), i.e., the order of tangency of the cusp borders at the vertex μ_q grows as n increases. For the strong resonances the set $\mathfrak{A}_q$ has the shape shown on the right picture of Fig. 1 (at least $\mathfrak{A}_q$ contains such a triangular part).

In [4], it is proved that in the problem on bifurcation of periodic orbits from infinity the Arnold tongues are thick sectors, shown on the right picture of Fig. 1. The angle between the tangent lines to the borders of any tongue $\mathfrak{A}_q$ at its vertex μ_q is positive except for non-generic degenerate situations. This angle tends to zero as the denominator n of q increases.

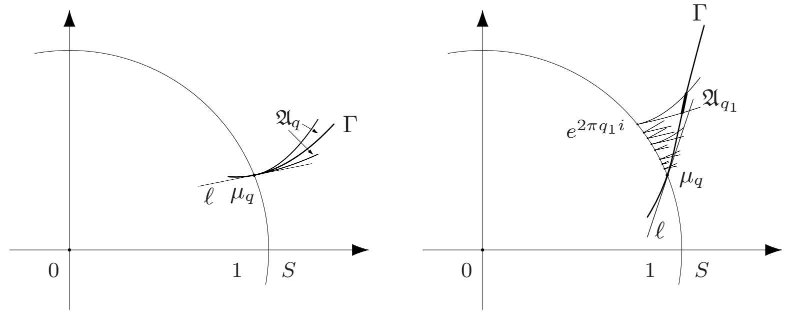

FIGURE 2. Subharmonic bifurcation and subfurcation.

Now, assume that the parameter λ varies so that the eigenvalue $\mu(\lambda)$ of $A(\lambda)$ moves along a smooth curve Γ through the point $\mu_q \in S$ with a rational q, and ℓ is a tangent line of Γ at this point. According to the approach of [1], one observes the subharmonic bifurcation if some arc of the curve Γ with one end at μ_q is contained in the Arnold tongue $\mathfrak{A}_q$ (as on the left picture of Fig. 2), and the subfurcation if the curve Γ crosses infinitely many Arnold tongues in any neighborhood of the point μ_q (as on the right picture of Fig. 2). Figs. 1 and 2 show that the subharmonic bifurcation can occur for a unique direction of ℓ in the local problem at zero of a smooth system (1.1) (and thus the subharmonic bifurcation is a specific non-generic case for this problem, except for points of strong resonances), while there is a bunch of such directions in the problem at infinity. The crucial fact underlying the subfurcation scenario on the right picture of Fig. 2 is that infinitely many Arnold tongues are long enough to intersect the curve Γ. The focus of this paper is estimation of the length of the Arnold tongues from below, justifying this fact.

A simple change of variables of the type $x \to Bx/|x|^2$ transforms the bifurcation problem at infinity to that near the origin. Consequently all our results can be formulated equivalently in terms of the local behavior of the transformed system near its zero equilibrium. However, the evolution operator of the latter system generically does not have the form (1.2). To be more specific, consider two-dimensional systems which provide understanding also of the N-dimensional case in a simpler setting. With complex number notations, the evolution operator of system (1.1) in proper polar coordinates at infinity reads as

$$U(r, \varphi; \lambda) = r\mu(\lambda)e^{i\varphi} + \Phi(\varphi; \lambda) + \xi(r, \varphi; \lambda),$$

where ξ vanishes as $r \to \infty$. In the problem on bifurcation from infinity, we consider positively homogeneous nonlinearities represented by generic complex valued 2π-periodic functions $\Phi(\cdot; \lambda)$ with infinitely many harmonics in their Fourier series. Then the thick Arnold tongues are easily described in terms of some closed curves defined by the Fourier series of Φ. Now, the inversion $re^{i\varphi} \to r^{-1}e^{-i\varphi}$ in $\mathbb{C}$ sends infinity to the origin and transforms the evolution operator to the form

$$\tilde{U}(r, \varphi; \lambda) = \mu(\lambda)^{-1}re^{i\varphi} + r^2\tilde{\Phi}(\varphi; \lambda) + o(r^2), \qquad r \to 0.$$

An important point is that the function $\tilde{\Phi}$ here has infinite Fourier series whenever Φ does, and consequently in this case the map $\tilde{U}$ does not have the form (1.3) at the zero equilibrium whatever smoothness is possessed by the original function Φ. Sufficiently smooth maps admitting the Taylor expansion (1.3) near its zero equilibrium have the form

$$U(r,\varphi;\lambda) = \mu(\lambda) r e^{i\varphi} + r^2 \Phi_2(\varphi;\lambda) + r^3 \Phi_3(\varphi;\lambda) + \dots,$$

where $\Phi_s(\cdot;\lambda)$ is a complex-valued trigonometric polynomial of order at most s. This particular form of the map U defines the shape of the classical thin Arnold tongues. Most results of this paper are inapplicable to such maps, which are a degenerate case in our setting, because the closed curve that determines the shape of a thick Arnold tongue $\mathfrak{A}_q$ shrinks to a point if $\tilde{\Phi} = \Phi_s$ and $n > s+1$. Finally note that we consider the set of all 2π-periodic functions Φ (equivalently, $\tilde{\Phi}$) satisfying some smoothness requirements as a natural class of nonlinearities in a problem at infinity, while all trigonometric polynomials Φ_s form only a narrow special subclass in this generic class.

The paper is organized as follows. Section 2.1 contains notation and main assumptions. In Section 2.2, we present Theorem 2.1 (it is a simpler version of the main result from [4]) on the existence of a thick Arnold tongue $\mathfrak{A}_q$ with the vertex at the point μ_q for a given rational q and discuss its geometric interpretation for two-dimensional systems. Section 3 contains main results: asymptotic estimates of the lengths of the Arnold tongue and its thick part as n increases to infinity. In Section 4, these results are used to analyze subfurcation of discrete time systems with one scalar parameter. In Section 5, a nonlinear change of variables is suggested to reduce the system to the simplest possible form. This form is then used to derive sufficient conditions for the existence of a large invariant annulus. In the rest of the paper we present the proofs. The basis of all the proofs is Theorem 6.1 of Section 6.2, which provides an approximation to system (1.1). Periodic trajectories of system (1.1) (as well as other information about its dynamics) can be recovered from that of the approximation whenever the latter is non-degenerate and thus is the principal part of (1.1).

The shapes of the Arnold tongues at the points of strong resonances in the bifurcation problem at the origin are similar to that of the Arnold tongues $\mathfrak{A}_q$ at infinity. Moreover, there are indications that this analogy can be extended to other aspects of dynamics at infinity (at least for finite ranges of the denominator n of $q = m/n$, where a particular range is defined by the nonlinearity Φ in representation (1.2) and can be arbitrarily large). Our numerical experiments with systems (1.1) near the points $\mu_q = e^{2\pi q i}$ reveal a rich variety of dynamical scenarios at infinity, which are typical for strong resonances of smooth systems at the origin. We briefly discuss them in Section 5. The focus of this paper is periodic trajectories, the shape of the Arnold tongues, and estimates of their length.

2. **Preliminary definitions and results.**

2.1. **Assumptions on the map and basic notation.**

Consider system (1.1) with the map U of the form (1.2). Assume that the function $\Phi = \Phi(x;\lambda)$ is positively homogeneous of order zero in x for each $\lambda \in D$, and the term $\xi = \xi(x;\lambda)$ vanishes at infinity as $O(|x|^s)$, i.e.,

$$\Phi(\theta x;\lambda) = \Phi(x;\lambda) \qquad \text{for all} \qquad |x| = 1,\ \theta > 0,\ \lambda \in D, \tag{2.4}$$

and

$$\limsup_{|x|\to\infty} \sup_{\lambda\in D} |x|^{-s}|\xi(x;\lambda)| < \infty \tag{2.5}$$

for some $s > 0$. Assume additionally that Φ satisfies the Hölder condition

$$|\Phi(x_1;\lambda) - \Phi(x_2;\lambda)| \le K|x_1 - x_2|^\tau, \qquad |x_1| = |x_2| = 1, \quad 0 < \tau \le 1. \tag{2.6}$$

For the sake of simplicity it is assumed that λ and $\bar\lambda$ are the eigenvalues of the matrix $A(\lambda)$, i.e., a simple[1] complex eigenvalue λ of $A = A(\lambda)$ is used as a parameter of the system. Throughout the paper it is supposed that λ ranges over some neighborhood D of a closed subset Λ of the unit circle $S = \{\lambda \in \mathbb{C} : \ |\lambda| = 1\}$ and that all the eigenvalues $\sigma_j = \sigma_j(\lambda)$ of $A(\lambda)$ different from λ and $\bar\lambda$ satisfy

$$\big||\sigma_j(\lambda)| - 1\big| \ge \delta > 0 \qquad \text{for all} \qquad \lambda \in D,\ j = 1, \dots, N-2. \tag{2.7}$$

All three terms in the right-hand side of (1.2) are supposed to be continuous with respect to the set of their arguments x, λ. Non-constant homogeneous functions Φ satisfying (2.4) have discontinuities at the origin; we consider our systems for $x \ne 0$.

Denote by E_λ the proper plane of the matrix $A(\lambda)$ corresponding to the pair of its eigenvalues $\lambda, \bar\lambda$ and by E'_λ the proper $(N-2)$-dimensional subspace of $A(\lambda)$, complementary to E_λ. Denote by P_λ the linear projector onto the plane E_λ along the subspace E'_λ; P_λ commutes with $A(\lambda)$. Choose a basis $\{e_1^\lambda, e_2^\lambda\}$ in E_λ such that the restriction of $A(\lambda)$ to E_λ reads

$$\begin{pmatrix} \Re e\,\lambda & -\Im m\,\lambda \\ \Im m\,\lambda & \Re e\,\lambda \end{pmatrix}.$$

Let Q_λ be the linear invertible map of the complex plane $\mathbb{C}$ onto E_λ defined by $Q_\lambda 1 = e_1^\lambda, Q_\lambda i = e_2^\lambda$. Hence, Q_λ satisfies

$$Q_\lambda^{-1} A(\lambda) Q_\lambda\, z = \lambda z, \qquad z \in \mathbb{C}, \tag{2.8}$$

and Q_λ depends continuously on $\lambda \in D$. Formulations of all the further theorems are independent of the particular choice of the map Q_λ satisfying (2.8) (i.e., of the basis $\{e_1^\lambda, e_2^\lambda\}$) and of the norm $|\cdot|$ in $\mathbb{R}^N$; once chosen, they are fixed up to the end of the paper[2]. The map Q_λ defines a complex structure in E_λ, which simplifies further notation.

Consider the complex-valued 2π-periodic function

$$\Psi_\lambda(\varphi) = Q_\lambda^{-1} P_\lambda \Phi(Q_\lambda e^{i\varphi};\lambda), \qquad \varphi \in \mathbb{R}, \tag{2.9}$$

and its Fourier series

$$\Psi_\lambda(\varphi) = \sum_{k=-\infty}^{\infty} \psi_k^\lambda e^{ik\varphi}, \qquad \psi_k^\lambda \in \mathbb{C}. \tag{2.10}$$

For any positive q denote $\lambda_q = e^{2\pi q i}$; if $q = m/n$ is rational (m and n are coprime) then define the continuous 2π-periodic function $\Psi_q^{res} : \mathbb{R} \to \mathbb{C}$ by

$$\Psi_q^{res}(\varphi) = -\sum_{k=-\infty}^{\infty} \psi_{kn+1}^{\lambda_q} e^{ik\varphi} \equiv -\frac{e^{-i\varphi/n}}{n} \sum_{j=0}^{n-1} \Psi_{\lambda_q}\Big(\frac{\varphi + 2j\pi}{n}\Big) e^{-2j\pi i/n}. \tag{2.11}$$

[1] If either $\lambda = 1$ or $\lambda = -1$, then λ has multiplicity 2: these values are not considered.

[2] It is natural to suppose that $|Q_\lambda z| = |z|$ for $z \in \mathbb{C}$.

2.2. **Local theorem.** Suppose that function (2.11) has no zeros. Then one can fix some continuous branch of the function $\arg\Psi_q^{res}(\varphi)$. Denote by $\mathfrak{B}_q$ the range of this branch of $\arg\Psi_q^{res}$: the set $\mathfrak{B}_q$ is a closed interval.

Our main assumption is that the interval $\mathfrak{B}_q$ has a non-empty interior $\mathrm{Int}\,\mathfrak{B}_q$ and its length is less than 2π. This is a generic case if $\psi_1^{\lambda_q} \neq 0$, the function Ψ_{λ_q} is not a trigonometric polynomial (it has infinitely many non-zero Fourier coefficients), and n is sufficiently large.

Theorem 2.1. *Let $\mathfrak{B} \subset \mathfrak{B}_q$ be a closed non-empty interval. Then there are numbers $\varepsilon = \varepsilon(q, \mathfrak{B})$, $r_j = r_j(q, \mathfrak{B}) > 0$, $j = 1, 2$ such that for every λ from the sector*

$$S_q = S_q(\mathfrak{B}) = \{\lambda \in D \subset \mathbb{C} :\ 0 < |\lambda - \lambda_q| \leq \varepsilon,\ \arg(\lambda - \lambda_q) \in \mathfrak{B}\}$$

system (1.1) has at least one periodic point x_λ with the minimal period n and a norm $|x_\lambda|$ in the interval $r_1|\lambda - \lambda_q|^{-1} < |x_\lambda| < r_2|\lambda - \lambda_q|^{-1}$.

A more general version of Theorem 2.1 was proved in [4].

Assume that we let λ vary along the ray $\arg(\lambda - \lambda_q) = \nu$ for some fixed $\nu \in \mathfrak{B}_q$. Consider the points φ_ν satisfying $\arg\Psi_q^{res}(\varphi_\nu) = \nu$. Let us call φ_λ robust if the function $\arg\Psi_q^{res}$ is strictly monotone in a vicinity of φ_λ. Generically, all φ_λ are robust and there exists a finite number (typically, exactly 2) of such points. Each robust point φ_ν defines an n-periodic orbit $(x_1, \dots, x_n)$, which satisfies asymptotically (as $|\lambda - \lambda_q| \to 0$)

$$x_k = |\lambda - \lambda_q|^{-1}|\Psi_q^{res}(\varphi_\nu)|Q_{\lambda_q}(e^{i(\varphi_\nu^j + 2\pi k)/n}) + o(|\lambda - \lambda_q|^{-1}). \tag{2.12}$$

If an open set $\mathfrak{B}'$ contains the closure $\overline{\mathfrak{B}}_q$ of the set $\mathfrak{B}_q$, then system (1.1) does not have n-periodic points with a sufficiently large norm for $\arg(\lambda - \lambda_q) \notin \mathfrak{B}'$ for sufficiently small $|\lambda - \lambda_q|$. In this sense, Theorem 2.1 describes the Arnold tongue near its vertex λ_q pretty accurately. In the two-dimensional case, system (1.1) can be equivalently rewritten as

$$z_{k+1} = U(z_k; \lambda), \quad z = re^{i\varphi} \in \mathbb{C}, \quad U(z; \lambda) = \lambda z + \Psi_\lambda(\varphi) + o(1), \tag{2.13}$$

where the last term vanishes as $z \to \infty$, the phase plane is complexified, and $\Psi_\lambda(\varphi) = \Phi(e^{i\varphi}; \lambda)$. Neglecting the small terms in the equation $z = U^n(z; \lambda)$, we arrive at its natural approximation

$$(\lambda - \lambda_q)r = \Psi_q^{res}(n\varphi). \tag{2.14}$$

Solutions of the equations $z = U^n(z, \lambda)$ and (2.14) are close for small $|\lambda - \lambda_q|$. The solutions $z = re^{i\varphi}$ of equation (2.14) and the set of parameter values $\lambda - \lambda_q$ for which these solutions exist are described straightforwardly in terms of the curve $\Gamma_q = \{\Psi_q^{res} : \mathbb{R} \to C\}$ (see details in [4]).

A discrete time system (1.1) naturally arises as the Poincare map of a differential equation. Let us consider the simplest two-dimensional example where the Poincare map has the form (2.13). It reads

$$z' = \nu z + a(t)f(z) + b(t), \qquad z \in \mathbb{C}, \tag{2.15}$$

in the complex notation. Here $a = a(t)$, $b = b(t)$ are complex valued 2π-periodic functions and ν is a complex parameter. Assume that the function $f : \mathbb{C} \to \mathbb{C}$ has a radial limit $F : \mathbb{C} \to \mathbb{C}$ at infinity:

$$\lim_{\theta\to\infty} \sup_{|z|=1} |f(\theta z) - F(z)| = 0, \qquad F(z) = F(z/|z|). \tag{2.16}$$

Then the Poincare map of (2.15) is $U(z;\nu) = \lambda z + \Phi(z;\nu) + o(1)$ in a vicinity of infinity, where $\lambda = e^{2\pi\nu}$ is our standard parameter, which lies on the complex unit circle if ν is on the imaginary axis, and Φ is the bounded homogeneous part of U. In terms of the Fourier series of the functions

$$a(t) = \sum_{k=-\infty}^{\infty} \alpha_k e^{ikt}, \qquad F(e^{i\varphi}) = \sum_{k=-\infty}^{\infty} \phi_k e^{ik\varphi},$$

one obtains

$$\Psi_q^{res}(\varphi) = -2\pi e^{2\pi q i} \sum_{k=-\infty}^{\infty} \alpha_{-mk}\phi_{nk+1} e^{ik\varphi}.$$

If $a = const$, then Theorem 2.1 is inapplicable, because $\Psi_q^{res} \equiv const$ too.

3. **Estimates of the length of tongues.** In this section we estimate the length of the Arnold tongues $\mathfrak{A}_q$ for system (1.1) and the length of their thick part asymptotically as $n \to \infty$.

Let $\Lambda \subset \{\lambda \in D \subset \mathbb{C} : |\lambda| = 1\}$ be a closed set. Let

$$\psi(\lambda) := -\frac{1}{2\pi} \int_0^{2\pi} e^{-i\varphi} \Psi_\lambda(\varphi)\, d\varphi \neq 0, \qquad \lambda \in \Lambda, \tag{3.17}$$

where Ψ_λ is function (2.9) (note that $-\psi(\lambda)$ equals the coefficient ψ_1^λ of the first harmonic in (2.10) and $\psi(\lambda)$ is the average value of function (2.11)). Given a rational q and a $\nu > 0$, consider the angle

$$\mathcal{A} = \{\lambda \in \mathbb{C} :\ |\Im m\big((\lambda - \lambda_q)\bar{\psi}(\lambda_q)\big)| \leq \nu\, \Re e\big((\lambda - \lambda_q)\bar{\psi}(\lambda_q)\big\}, \tag{3.18}$$

where $\bar{\psi}(\lambda)$ is the complex conjugate of $\psi(\lambda)$. The constant ν defines the spread of this angle with the vertex λ_q and the bisector $\{\lambda : \lambda - \lambda_q = r\psi(\lambda), r > 0\}$. We say that the length of the Arnold tongue $\mathfrak{A}_q$ in the angle (3.18) is not less then δ, if on any continuous curve

$$L = \big\{\lambda \in \mathbb{C} :\ \lambda = \lambda^*(\rho),\ \rho \in [0,1]\big\} \subset \mathcal{A}\ \cap\ \big\{\lambda :\ 0 < |\lambda - \lambda_q| < \delta\big\} \tag{3.19}$$

that has its ends on the different sides of the angle (3.18) there is at least one $\lambda = \lambda^*(\rho_0) \in L$ such that system (1.1) with this λ has an n-periodic point x_L, and $|x_L| \to \infty$ as $\max\{|\lambda - \lambda_q| : \lambda \in L\} \to 0$ (i.e., as L shrinks to λ_q).

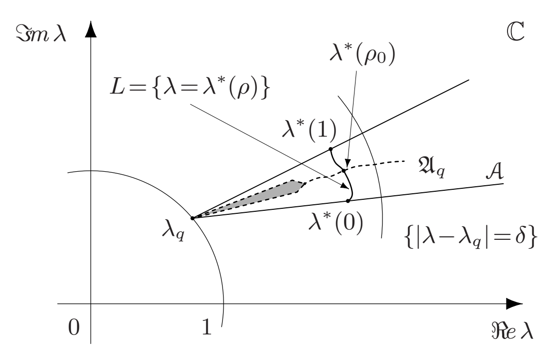

FIGURE 3. The length of the Arnold tongue and of its triangular part.

Theorem 3.1. *Let (3.17) be valid. Then given any $\nu > 0$, there is an $\varepsilon = \varepsilon(\nu, \Lambda) > 0$ such that for every rational positive $q = m/n$ with a sufficiently large n and with $\lambda_q \in \Lambda$, the length of the Arnold tongue $\mathfrak{A}_q$ in the angle (3.18) is not less then ε/n.*

Assumption (3.17) implies that the angular width of the tongues $\mathfrak{A}_q$ goes to zero as the denominator n of q increases. More precisely, for any $\nu > 0$ the intersection of the tongue $\mathfrak{A}_q$ with the circle $|\lambda - \lambda_q| \le \varepsilon(\nu, \Lambda)/n$ lies in the angle (3.18) for any $\lambda_q \in \Lambda$ such that the denominator n of q is sufficiently large (i.e., there are no n-periodic points with a sufficiently large norm if λ lies in the circle $|\lambda - \lambda_q| \le \varepsilon(\nu, \Lambda)/n$ outside this angle).

On Fig. 3 we show schematically two parts of the Arnold tongue $\mathfrak{A}_q$, the existence of which is ensured by Theorems 2.1 and 3.1: the shadowed solid triangular part and the rest, drawn conditionally as a dashed curve. If the curve (3.19) crosses the triangular part of $\mathfrak{A}_q$, then the intersection is an arc with a non-empty interior on this curve L. However, the triangular part is shorter than the length $\delta = \varepsilon(\nu, \Lambda)/n$ of $\mathfrak{A}_q$, ensured by Theorem 3.1. For curves (3.19) lying within the distance δ from λ_q but outside the circle where Theorem 2.1 ensures the triangular shape of $\mathfrak{A}_q$, we can guarantee only one intersection point of L with $\mathfrak{A}_q$ (i.e., with the dashed tail on Fig. 3).

Theorem 2.1 states the existence of a triangular part of the tongue $\mathfrak{A}_q$. In the next Theorem 3.2 we suggest a lower asymptotic estimate for the length of this part as $n \to \infty$ under some additional assumptions: this estimate is uniform for λ. Denote by $\Lambda_\triangle$ the set of all $\lambda_q \in \Lambda$ with rational q such that $\operatorname{Int} \mathfrak{B}_q \ne \emptyset$. Suppose that $\Lambda_\triangle$ is infinite, hence the function Ψ_λ has infinitely many non-zero Fourier coefficients. From (2.11) and (3.17) it follows that if the denominator n of q is sufficiently large, then the function Ψ_q^{res} does not have zeros and $\mathfrak{B}_q$ is a segment, i.e., $\mathfrak{B}_q = [M_q^-, M_q^+]$ with $M_q^+ > M_q^-$ for $\lambda_q \in \Lambda_\triangle$. Denote the positive length of this segment by

$$\Delta_q = M_q^+ - M_q^-.$$

Theorem 2.1 implies that for any $\theta \in (0,1)$ there is a $\delta_q(\theta) > 0$ such that for each λ from the set

$$\{\lambda : \; |\arg(\lambda - \lambda_q) - (M_q^+ + M_q^-)/2| \le \theta\Delta_q/2, \;\; 0 < |\lambda - \lambda_q| \le \delta_q(\theta)\}$$

system (1.1) has at least one periodic point x_λ with the minimal period n and a norm $|x_\lambda|$ in the interval $r_1^q|\lambda - \lambda_q|^{-1} < |x_\lambda| < r_2^q|\lambda - \lambda_q|^{-1}$, where $r_j^q > 0$ do not depend on λ. We say that $\delta_q(\theta)$ is a lower estimate for the length of the triangular part of the Arnold tongue $\mathfrak{A}_q$ for a given θ.

Let $0 < s_1 < s$, $0 < \tau_1 < \tau$, where s and τ are the exponents in (2.5) and (2.6). Let a sequence $\rho_n > 0$ satisfy

$$\sup_{q=m/n, \; \lambda_q \in \Lambda_\Delta} \frac{n^{\tau_1}\rho_n^{\tau_1} + \rho_n^{s_1}}{\Delta_q} \to 0 \quad \text{as} \quad n \to \infty, \tag{3.20}$$

where we take supremum over m for a fixed n. Assume that the

$$|\Psi_{\lambda_1}(\varphi) - \Psi_{\lambda_2}(\varphi)| \le K'|\lambda_1 - \lambda_2|^\tau, \quad \varphi \in \mathbb{R}, \; \lambda_1, \lambda_2 \in D \tag{3.21}$$

for function (2.9). For example, this holds if $A(\lambda)$, $\Phi(x;\lambda)$ are Lipschitz continuous in λ and Q_λ is chosen to be Lipschitz continuous as well.

Theorem 3.2. *Let relations (3.17), (3.20), and (3.21) hold. Then, given a $\theta \in (0,1)$ the value $\delta_q(\theta) = \rho_n$ is a lower estimate for the length of the thick part of* *the Arnold tongue $\mathfrak{A}_q$ for any rational $q = m/n$ with sufficiently large n such that $\lambda_q \in \Lambda_\triangle$ (how large n is depends on θ).*

Condition (3.20) can be specified in terms of the rate at which Δ_q decreases to zero. This rate depends on the smoothness of the function Ψ_λ, which defines how quickly the Fourier coefficients ψ_k^λ of Ψ_λ converge to zero as $k \to \infty$. For example, let $\psi_k^\lambda \sim k^{-2}$ (this corresponds to Lipschitzian but non-differentiable functions Ψ_λ). Generically, in this case $\Delta_q \sim n^{-2}$. If $\tau = 1$ in (2.6), (3.21), i.e., functions Φ, Ψ_λ are Lipschitz in both arguments, and $s \geq 2/3$ in (2.5), then Theorem 3.2 ensures that $n^{-3-\varepsilon}$ is a lower estimate for the length of the triangular part of the tongue $\mathfrak{A}_q$ for any $\varepsilon > 0$.

4. **Systems with a scalar parameter.** Suppose that the complex parameter λ is a function of some real parameter μ: $\lambda = \lambda(\mu)$, $\mu \in (0,1)$, hence the range of λ is a curve on the complex plane. Assume that this curve is smooth and intersects transversally the unit circle at the point $\lambda(\mu_0) = \lambda_q$ with a real (either rational or irrational) q. Then subfurcation can occur for values of μ close to μ_0, like in the case of local dynamics near the origin considered in [3].

Consider system (1.1) with $\lambda = \lambda(\mu) : (0,1) \to D \subset \mathbb{C}$. Let us say that this system undergoes subfurcation at infinity near the point μ_0 if for any $\varepsilon > 0$ there is at least one $\mu_\varepsilon \in (\mu_0 - \varepsilon, \mu_0 + \varepsilon)$ such that system (1.1) with $\lambda = \lambda(\mu_\varepsilon)$ has a periodic point x_ε with the norm $|x_\varepsilon| > 1/\varepsilon$ and a minimal period $n_\varepsilon > 1/\varepsilon$ (i.e., there exist periodic points of arbitrarily large minimal period and norm for parameter values μ from any vicinity of μ_0).

Theorem 4.1. *Suppose that function (3.17) is non-zero on a closed set $\Lambda \subset \{\lambda \in \mathbb{C} : |\lambda| = 1\}$. Suppose that for some non-zero $b \in \mathbb{C}$*

$$\arg\lambda - \arg b \neq \pi/2 + \pi k, \quad \arg\psi(\lambda) - \arg b \neq \pi k \tag{4.22}$$

for all $\lambda \in \Lambda$ and all integer k. Given such a number b, if, additionally, $\lambda'_\mu(\mu_0) = b$ and $\lambda(\mu_0) = \lambda_q$ for either an irrational q or a rational $q = m/n$ with a sufficiently large n, then system (1.1) with $\lambda = \lambda(\mu)$ undergoes subfurcation at infinity near the point μ_0.

This theorem follows from Theorem 3.1, because under the assumptions of Theorem 4.1 the curve $\lambda(\mu)$ intersects infinitely many Arnold tongues $\mathfrak{A}_{q_k}$, where $q_k = m_k/n_k$ are good enough one-sided rational approximations of q. If q is irrational, then this is true for the sequence of convergents q_k of q with either even or odd k. Indeed, the convergents satisfy $|q - q_k| \leq n_k^{-2}$ (see, e.g., [2]) and hence $|\lambda_q - \lambda_{q_k}| = O(n_k^{-2})$, while the length of the Arnold tongue $\mathfrak{A}_{q_k}$ is not less than εn_k^{-1} according to Theorem 3.1. Consequently, condition (4.22) implies that the curve $\lambda(\mu)$ intersects the tongue $\mathfrak{A}_{q_k}$ (by ensuring that the curve and the tongue are not 'parallel') whenever n_k is sufficiently large and the tongue vertex λ_{q_k} lies on the proper side from the point $\lambda_q = \lambda(\mu_0)$ on an arc of the unit circle. The latter condition means equivalently that q_k is either always a lower or always an upper approximation to q, i.e., the index k has the same proper parity for all q_k.

If $q = m/n$ is rational, then a similar argument works for the sequence of rational $q_k = m_k/n_k$ such that $|mn_k - nm_k| = 1$ and the difference $mn_k - nm_k$ has the same proper sign for all k (both infinite sequences $q_k = m_k/n_k$ with $mn_k - nm_k = 1$ and $mn_k - nm_k = -1$ exist, because m and n are coprime). Again, the curve $\lambda(\mu)$ intersects the Arnold tongue $\mathfrak{A}_{q_k}$ for large enough n_k whenever λ_{q_k} lies on the proper side from λ_q and the tongue's length is sufficiently large compared to the difference $|q - q_k|$, which now equals $|m/n - m_k/n_k| = (nn_k)^{-1}$. Consequently, the lower estimate ε/n_k of the tongue's length ensures that the curve $\lambda(\mu)$ passing through the point λ_q intersects infinitely many Arnold tongues if the denominator n of q is sufficiently large. This construction sketches the proof of Theorem 4.1.

Theorem 4.1 does not answer the questions of whether subfurcation at infinity occurs for some finite range of the denominator n of q and how large this range is. Note that smooth systems undergo subfurcation at the origin for all irrational q and all rational $q = m/n$ with $n > 4$.

An interesting problem is to estimate from below the length of the intervals of intersection of the curve $\lambda(\mu)$ with the Arnold tongues. At zero these intervals are very short: if q is irrational, $q_k = m_k/n_k$ is an approximation of q, and $\varepsilon_{n_k} = |q - q_k|$, then the length of the corresponding interval is at most of order $\varepsilon_{n_k}^{n_k/2-1}$, where generically $\varepsilon_{n_k} \sim n_k^{-2}$: on Fig. 2 one of such intervals (in a tongue $\mathfrak{A}_{q_1}$) is shown as a bold segment. For the problem at infinity these intervals may be wider, at least for some values of q. If q is the so-called well approximable irrational number (this means that ε_{n_k} tends to zero quicker than n_k^{-2}), than the curve $\lambda(\mu)$ may intersect triangular parts of the tongues $\mathfrak{A}_{q_k}$. For this case it is possible to obtain polynomial estimates n_k^{-K} from below for the width of the intervals of intersection, where K is independent from n_k and depends on the well approximable irrational q and the smoothness of Ψ_λ. The set of well approximable numbers is dense but has measure zero on the real axis.

5. **Invariant annulus.** Here we consider sufficient conditions for the existence of invariant sets for (1.1) that are situated far from the origin. Let us start with the simple example of a two-dimensional map (2.13) with the term $o(1)$ identically equal to zero. In complex notation, the corresponding system (1.1) has the form

$$z_{k+1} = \lambda z_k + \Psi_\lambda(\varphi_k), \qquad z_k = r_k e^{\varphi_k i}. \tag{5.23}$$

Assume that for every λ from some set $\tilde{\Lambda} \subset \{z \in \mathbb{C}, |z| < 1\}$

$$\Re e\big(\lambda^{-1} e^{-i\varphi} \Psi_\lambda(\varphi)\big) \ge \varepsilon_0 > 0, \qquad \varphi \in \mathbb{R}, \tag{5.24}$$

and consequently the numbers

$$M_\lambda^{\min} = \min_{\varphi\in\mathbb{R}} \Re e\big(\lambda^{-1} e^{-i\varphi} \Psi_\lambda(\varphi)\big), \qquad M_\lambda^{\max} = \max_{\varphi\in\mathbb{R}} \Re e\big(\lambda^{-1} e^{-i\varphi} \Psi_\lambda(\varphi)\big)$$

satisfy $M_\lambda^{\max} \ge M_\lambda^{\min} > \varepsilon_0$. It is straightforward to check that under the assumption (5.24) for any $\varepsilon \in (0, \varepsilon_0)$ the annulus

$$K'_\varepsilon = \{z \in \mathbb{C} : M_\lambda^{\min} - \varepsilon \le |z|(1 - |\lambda|) \le M_\lambda^{\max} + \varepsilon\}$$

is invariant for system (5.23) whenever $1 - |\lambda| > 0$ is sufficiently small. Moreover, every annulus $K_{r_1,r_2} = \{z \in \mathbb{C} : r_1 \le |z| \le r_2\}$ that contains K'_ε and has sufficiently large internal radius r_1 is invariant for system (5.23) and K'_ε is attractive: if an initial point z_0 lies outside K'_ε far enough from the origin, then after a number of iterations $z_k \in K'_\varepsilon$. Similar facts are true for small perturbations (2.13) of system (5.23) and for N-dimensional system (1.1).

Consider the iterated system $z_{k+1} = U^n(z_k; \lambda)$ with the n-th iteration of the map $U(z; \lambda) = \lambda z + \Psi_\lambda(\varphi)$ for λ close enough to the point λ_q with $q = m/n$. The analogue of condition (5.24) for this system is $\Re e\big(\lambda_q^{-1} \Psi_q^{res}(\varphi)\big) < -\varepsilon_0$, which is less restrictive than (5.24). Below we show that under this condition system (2.13) itself also has an invariant annulus after a proper change of variable of the form

$y = x+$'homogeneous term'. We formulate this type of result for more general system (1.1) in $\mathbb{R}^N$.

Assume that all the eigenvalues of the matrix $A(\lambda)$ different from λ and $\bar{\lambda}$ lie in the open unit circle of the complex plane, i.e.,

$$|\sigma_j(\lambda)| < 1 - \delta_*, \qquad \lambda \in D,\ \delta_* > 0,\ j = 1, \ldots, N-2. \tag{5.25}$$

Also, assume that the function $\Phi = \Phi(x;\lambda)$ is Lipschitz continuous in x. For λ sufficiently close to some λ_q (q may be either rational or irrational), define a new variable $y \in \mathbb{R}^N$ by the formula

$$y = B_q(x) := x + Q_{\lambda_q}\Theta_q(Q_{\lambda_q}^{-1}P_{\lambda_q}x) \tag{5.26}$$

with a positively homogeneous function $\Theta_q : \mathbb{C} \to \mathbb{C}$. As we show in the proofs below, such change of variables in the phase space $\mathbb{R}^N$ is continuously invertible in some vicinity of infinity for any positively homogeneous Lipschitz Θ_q. We choose Θ_q in different ways for rational and irrational q.

Let the number q be rational. Set

$$\rho^{\min} = \min_\varphi |\,\Re e\big(\lambda_q^{-1}\,\Psi_q^{res}(\varphi)\big)|, \qquad \rho^{\max} = \max_\varphi |\,\Re e\big(\lambda_q^{-1}\,\Psi_q^{res}(\varphi)\big)|.$$

Theorem 5.1. *Let* (5.25) *hold and*

$$\Re e\big(\lambda_q^{-1}\,\Psi_q^{res}(\varphi)\big) < 0 \qquad \text{for all} \qquad \varphi \in \mathbb{R} \tag{5.27}$$

for some rational $q = m/n$. Set

$$\Theta_q(re^{\varphi i}) = \sum_{k \not\equiv 1\,(\mathrm{mod}\,n)} \psi_k^{\lambda_q} e^{-2\pi qi}(1 - e^{2\pi(k-1)qi})^{-1}e^{k\varphi i}, \tag{5.28}$$

where $\psi_k^{\lambda_q}$ are the Fourier coefficients of (2.10), *and define the change of variable $y = B_q(x)$ by* (5.26). *Then there is $R > 0$ such that for any $\delta > 0$*

$$\Pi_q = \{x = B_q^{-1}(y) :\ \rho^{\min} - \delta \le |\lambda - \lambda_q| \cdot |P_{\lambda_q}y| \le \rho^{\max} + \delta,\ |y - P_{\lambda_q}y| \le R\}$$

is an invariant set of system (1.1) *for any λ from the set $\Lambda_q^\varepsilon = \{|\lambda| < 1, |\lambda_q - \lambda| < \varepsilon\}$ with some $\varepsilon = \varepsilon(\delta) > 0$.*

The constant R in the definition of the set Π_q may be chosen independently from δ, ε. It depends on the value of $1 - \delta_*$, geometric properties of the matrices $A(\lambda)$, and estimates of the bounded terms Φ and ξ of (1.2).

Let us note that the series in (5.28) converges uniformly to the Lipschitzian function Θ_q if Ψ_{λ_q} is Lipschitzian. This follows from

$$\Theta_q(e^{\varphi i}) = \sum_{\ell=0,2,3,\ldots,n-1} e^{-2\pi qi}(1 - e^{2\pi(\ell-1)qi})^{-1} \sum_{k \equiv \ell\,(\mathrm{mod}\,n)} \psi_k^{\lambda_q} e^{k\varphi i},$$

since any function

$$\sum_{k \equiv \ell\,(\mathrm{mod}\,n)} \psi_k^{\lambda_q} e^{k\varphi i} = \frac{1}{n}\sum_{j=0}^{n-1} e^{-2j\ell\pi i/n}\Psi_{\lambda_q}(\varphi + 2j\pi/n)$$

is Lipschitzian together with function (2.9). Also, note that the change of variable (5.26) is the same for all $\lambda \in \Lambda_q^\varepsilon$.

Geometrically, condition (5.27) means that the Arnold tongue $\mathfrak{A}_q$ points inside the unit disc in $\mathbb{C}$.

Suppose that the assumptions of Theorem 5.1 are satisfied for two-dimensional system (2.13), and hence Π_q is an invariant set for this system. Numerical experiments show that if $\lambda \notin \mathfrak{A}_q$, then, generically, the set Π_q contains an attractive invariant curve of the system. For $\lambda \in \mathfrak{A}_q$, we observed different phase portraits: with and without the invariant curve. If it exists, periodic trajectories sometimes belong to this curve and sometimes do not. When λ follows some continuous curve close enough to the point λ_q, the system undergoes various bifurcations. For example, a saddle-node bifurcations takes place on the boundaries of each Arnold tongue $\mathfrak{A}_q$. One scenario is that a saddle-node is born on the invariant curve like in the case of weak resonances for smooth systems near the equilibrium. In another scenario, the saddle-node is born away from the invariant curve: in this case the saddle point can collide with the invariant curve and destroy it as parameter varies inside the Arnold tongue. The study of such bifurcations will be an object of another paper.

The next theorem is an analogue of Theorem 5.1 for an irrational q.

Theorem 5.2. *Let* (5.25) *hold and*

$$\rho := \Re e\big(\lambda_q^{-1}\,\psi(\lambda_q)\big) < 0 \tag{5.29}$$

for an irrational q. Then for any $\delta > 0$ there are an $\varepsilon = \varepsilon(\delta) > 0$ and an integer $K = K(\delta)$ such that

$$\Pi_q = \{x = B_q^{-1}(y) : \ |\rho| - \delta \le |\lambda - \lambda_q| \cdot |P_{\lambda_q} y| \le |\rho| + \delta, |y - P_{\lambda_q} y| \le R\}$$

is an invariant set of system (1.1) *for any $\lambda \in \Lambda_q^\varepsilon$ if $\Theta_q = \Theta_{q,K}$ in the change of variable $y = B_q(x)$ is defined by*

$$\Theta_{q,K}(re^{\varphi i}) = \sum_{|k|\le K,\ k\ne 1} \psi_k^{\lambda_q} e^{-2\pi q i}(1 - e^{2\pi(k-1)qi})^{-1} e^{k\varphi i}. \tag{5.30}$$

Condition (5.29) means that the Arnold tongues $\mathfrak{A}_p$ point inside the unit disc for all rational p close enough to q. The constant R in the definition of Π_q is independent of δ, like in Theorem 5.1.

Now we combine Theorems 5.1 and 5.2 to formulate a non-local result with respect to λ. Let $\mathfrak{L}$ be an arbitrary closed subset of the set $\{\lambda \in \mathbb{C} : |\lambda| = 1, \Re e\big(\bar\lambda\,\psi(\lambda)\big) < 0\}$. Then there exists an integer $n_0 > 0$ (depending on $\mathfrak{L}$) such that if $q = m/n$ is rational with $n > n_0$ and $\lambda_q \in \mathfrak{L}$, then (5.27) holds, i.e., the tongue $\mathfrak{A}_q$ points inside the unit disc. For $\delta, \varepsilon > 0$ denote

$$\tilde O(\delta) = \{\lambda \in \mathbb{C} : \min_{q=m/n,\ n\le n_0} |\lambda - \lambda_q| < \delta\},$$

$$\mathfrak{L}_{\delta,\varepsilon} = \{\lambda \in \mathbb{C} : |\lambda| < 1,\ \inf_{\nu\in\mathfrak{L}} |\lambda - \nu| < \varepsilon\} \setminus \tilde O(\delta).$$

Theorem 5.3. *Given any $\delta > 0$, there are $\varepsilon = \varepsilon(\delta) > 0$, $R > 0$ and a finite number of the maps $y = B_{q_k}(x)$ of the form* (5.26) *such that for any $\lambda \in \mathfrak{L}_{\delta,\varepsilon}$ at least one of the sets*

$$\Pi_k(\lambda) = \{x = B_{q_k}^{-1}(y) : \rho(\lambda) - \delta \le (1 - |\lambda|)|P_\lambda y| \le \rho(\lambda) + \delta,\ |y - P_\lambda y| \le R\}$$

with $\rho(\lambda) = |\,\Re e(\bar\lambda\,\psi(\lambda))|$ is invariant for system (1.1).

6. Proofs.

6.1. **Main theorem.** We denote by E_λ^+ the proper subspace of the matrix $A(\lambda)$ corresponding to all its eigenvalues $\sigma_j(\lambda)$ satisfying $|\sigma_j(\lambda)| > 1$, and by E_λ^- the proper subspace of $A(\lambda)$ corresponding to its eigenvalues $\sigma_j(\lambda)$ satisfying $|\sigma_j(\lambda)| < 1$. Consequently, $E_\lambda^+ \oplus E_\lambda^- = E_\lambda'$ and $E_\lambda^+ \oplus E_\lambda^- \oplus E_\lambda = \mathbb{R}^N$. Let P_λ^+ be the linear projector onto the subspace E_λ^+ along the complementary proper subspace $E_\lambda^- \oplus E_\lambda$ of $A(\lambda)$, and let P_λ^- be the linear projector onto E_λ^- along the complementary proper subspace $E_\lambda^+ \oplus E_\lambda$ of $A(\lambda)$ so that $P_\lambda^+ + P_\lambda^- + P_\lambda = I$. By these definitions, there is a norm $|\cdot|_\lambda$ in $\mathbb{R}^N$ depending continuously on λ and a $\kappa < 1$ such that for all $\lambda \in D$

$$|A(\lambda)x|_\lambda \le \kappa |x|_\lambda, \quad x \in E_\lambda^-; \qquad |A(\lambda)x|_\lambda \ge \kappa^{-1}|x|_\lambda, \quad x \in E_\lambda^+. \tag{6.31}$$

From relations (6.31) and from the boundedness of the nonlinear terms Φ and ξ of representation (1.2), it follows the existence of a $\beta > 0$ such that

$$|P_\lambda^- U(x;\lambda)|_\lambda < \beta \qquad \text{if} \qquad |P_\lambda^- x|_\lambda \le \beta, \tag{6.32}$$

$$|P_\lambda^- U(x;\lambda)|_\lambda < |P_\lambda^- x|_\lambda \qquad \text{if} \qquad |P_\lambda^- x|_\lambda \ge \beta, \tag{6.33}$$

$$|\eta P_\lambda^+ U(x;\lambda) + (1-\eta) P_\lambda^+ A(\lambda) x|_\lambda > |P_\lambda^+ x|_\lambda \qquad \text{if} \qquad |P_\lambda^+ x|_\lambda \ge \beta \tag{6.34}$$

for every $\lambda \in D$ and every $\eta \in [0,1]$. Such a β is fixed up to the end of the proofs. Estimates (6.32), (6.33), and (6.34) (the latter with $\eta = 1$) imply that all the periodic points of the map $U(\cdot;\lambda)$ lie in the interior of the set

$$\Omega_\lambda = \{x \in \mathbb{R}^N : \ |P_\lambda^- x|_\lambda \le \beta, \ |P_\lambda^+ x|_\lambda \le \beta\}. \tag{6.35}$$

Consider the nonlinear projector onto the set $\{x \in \mathbb{R}^N : |P_\lambda^+ x|_\lambda \le \beta\}$ defined by

$$W_\lambda(x) = x + \Big(\frac{\min\{\beta, |P_\lambda^+ x|_\lambda\}}{|P_\lambda^+ x|_\lambda} - 1 \Big) P_\lambda^+ x,$$

and define the map

$$V_\lambda(x) = U(W_\lambda(x);\lambda), \qquad x \in \mathbb{R}^N, \ \lambda \in D.$$

If (5.25) holds, then $V_\lambda(x) = U(x;\lambda)$ for all $x \in \mathbb{R}^N$; if not, then $V_\lambda(x) = U(x;\lambda)$ on the set $\{x \in \mathbb{R}^N : \ |P_\lambda^+ x|_\lambda \le \beta\} \supset \Omega_\lambda$. Therefore in any case each periodic point of the map $U(\cdot;\lambda)$ is a periodic point of the map V_λ too. On the other hand, relations (6.32) and $P_\lambda^- W_\lambda(x) = P_\lambda^- x$ imply

$$|P_\lambda^- V_\lambda(x)|_\lambda < \beta \qquad \text{if} \qquad |P_\lambda^- x|_\lambda \le \beta, \tag{6.36}$$

while relations (6.34) (with $\eta = 1$) and

$$|P_\lambda^+ W_\lambda(x)|_\lambda = \beta \qquad \text{if} \qquad |P_\lambda^+ x|_\lambda \ge \beta$$

imply

$$|P_\lambda^+ V_\lambda(x)|_\lambda > \beta \qquad \text{if} \qquad |P_\lambda^+ x|_\lambda \ge \beta. \tag{6.37}$$

Combining estimates (6.36) and (6.37), one concludes that for any periodic point x of V_λ lying in Ω_λ all the iterations $V_\lambda^k(x)$, $k \in \mathbb{N}$, also belong to the set Ω_λ, where V_λ coincides with $U(\cdot;\lambda)$. Consequently, the periodic points of $U(\cdot;\lambda)$ coincide with the periodic points of V_λ lying in Ω_λ. The map V_λ is defined in such a way that the images of the set Ω_λ under the iterated maps V_λ^k have uniformly bounded projections onto the subspace E'_λ along the subspace E_λ of $\mathbb{R}^N$ for all k, i.e.,

$$|(I - P_\lambda) V_\lambda^k(x)|_\lambda \le c_0, \qquad x \in \Omega_\lambda, \ k \in \mathbb{N}. \tag{6.38}$$

Set

$$\Psi_{\lambda,n}^{res}(\varphi) = -\sum_{k=-\infty}^{\infty} \psi_{kn+1}^\lambda e^{ik\varphi} \equiv -\frac{e^{-i\varphi/n}}{n} \sum_{j=0}^{n-1} \Psi_\lambda \left(\frac{\varphi + 2j\pi}{n} \right) e^{-2j\pi i/n}.$$

By definition, this function equals (2.11) for $\lambda = \lambda_q$ with $q = m/n$. The following statement plays the main role for the further proofs.

Theorem 6.1. *For any $\tau_1 < \tau$, $s_1 < s$, and $0 < \kappa_1 < \kappa_2$ there is a $\delta = \delta(\tau_1, s_1, \kappa_1, \kappa_2)$ such that if $0 < n|\lambda - \lambda_q| < \delta$ for a positive rational $q = m/n$ then the n-th iteration of the operator V_λ satisfies*

$$\big|Q_\lambda^{-1} P_\lambda V_\lambda^n(x) - z - n\lambda_q^{-1}\big((\lambda - \lambda_q)z - e^{i\varphi}\Psi_{\lambda,n}^{res}(n\varphi)\big)\big| \\ \leq 2n\big(n^{\tau_1}|\lambda - \lambda_q|^{\tau_1} + |\lambda - \lambda_q|^{s_1}\big), \quad (6.39)$$

with $z = Q_\lambda^{-1} P_\lambda x$, $\varphi = \arg z$ for all x from the set

$$\Omega_\lambda^{q,\kappa_1,\kappa_2} = \{x \in \Omega_\lambda : \ \kappa_1 \leq |\lambda - \lambda_q||Q_\lambda^{-1} P_\lambda x| \leq \kappa_2\}.$$

6.2. **Proof of Theorem 6.1.** All the constants c, $\bar{c}$, c_1 etc. are absolute in this proof, i.e., they do not depend on λ and q. First note that estimates

$$\left|\frac{x_1}{|x_1|} - \frac{x_2}{|x_2|}\right| \leq \frac{2|x_1 - x_2|}{|x_1|}, \qquad x_1, x_2 \neq 0,$$

and (2.6) imply

$$|\Phi(x_1; \lambda) - \Phi(x_2; \lambda)| \leq \frac{2^\tau K|x_1 - x_2|^\tau}{|x_1|^\tau}. \quad (6.40)$$

Let $x \in \Omega_\lambda^{q,\kappa_1,\kappa_2}$. Define $x_k = V_\lambda^k(x)$, $z_k = Q_\lambda^{-1} P_\lambda x_k$, and $z = Q_\lambda^{-1} P_\lambda x$. Assume that for some $1 \leq k \leq n$

$$\left|z_k - \lambda^k z - \sum_{j=0}^{k-1} \lambda^{k-j-1} Q_\lambda^{-1} P_\lambda \Phi(Q_\lambda(\lambda^j z); \lambda)\right| \leq k\big(k^{\tau_1}|\lambda - \lambda_q|^{\tau_1} + |\lambda - \lambda_q|^{s_1}\big). \quad (6.41)$$

In particular, this is true for $k = 1$ if $|\lambda - \lambda_q|$ is sufficiently small. Indeed, relations

$$z_1 = Q_\lambda^{-1} P_\lambda V_\lambda(x) = \lambda z + Q_\lambda^{-1} P_\lambda \Phi(x; \lambda) + Q_\lambda^{-1} P_\lambda \xi(x; \lambda)$$

imply

$$|z_1 - \lambda z - Q_\lambda^{-1} P_\lambda \Phi(Q_\lambda z; \lambda)| \leq |Q_\lambda^{-1} P_\lambda(\Phi(x; \lambda) - \Phi(Q_\lambda z; \lambda) + \xi(x; \lambda))|,$$

where due to (6.40) and (2.5)

$$\Phi(x; \lambda) - \Phi(Q_\lambda z; \lambda) = O(|z|^{-\tau}), \qquad \xi(x; \lambda) = O(|z|^{-s})$$

(when applying (6.40) we take into account that the value $x - Q_\lambda z = x - P_\lambda x$ is bounded for all $x \in \Omega_\lambda$). Since $|z| \geq \kappa_1/|\lambda - \lambda_q|$, we arrive at

$$|z_1 - \lambda z - Q_\lambda^{-1} P_\lambda \Phi(Q_\lambda z; \lambda)| \leq |\lambda - \lambda_q|^{\tau_1} + |\lambda - \lambda_q|^{s_1}$$

for small $|\lambda - \lambda_q|$, which is (6.41) for $k = 1$.

From the equality

$$z_{k+1} = \lambda z_k + Q_\lambda^{-1} P_\lambda \Phi(W_\lambda(x_k); \lambda) + Q_\lambda^{-1} P_\lambda \xi(W_\lambda(x_k); \lambda)$$

and relations $\xi(W_\lambda(x_k); \lambda) = O(|z_k|^{-s})$ and (6.41), it follows that

$$\begin{aligned} &\left|z_{k+1} - \lambda^{k+1} z - \textstyle\sum_{j=0}^{k} \lambda^{k-j} Q_\lambda^{-1} P_\lambda \Phi(Q_\lambda(\lambda^j z); \lambda)\right| \\ &\quad \leq |\lambda| k\big(k^{\tau_1}|\lambda - \lambda_q|^{\tau_1} + |\lambda - \lambda_q|^{s_1}\big) + \bar{c}|z_k|^{-s} \\ &\qquad + |Q_\lambda^{-1} P_\lambda(\Phi(W_\lambda(x_k); \lambda) - \Phi(Q_\lambda(\lambda^k z); \lambda))|. \end{aligned} \quad (6.42)$$

We assume that the product $n|\lambda - \lambda_q|$ is sufficiently small, which implies $1/2 \leq |\lambda|^k \leq 2$ for all $1 \leq k \leq n$. Therefore (6.41) implies $|z_k - \lambda^k z| \leq kc$. Combining the estimate $|P_\lambda^- x_k|_\lambda \leq \beta$ (which follows from (6.36) for all k) with the relations

$(I-P_\lambda^+)W_\lambda(\tilde x)=(I-P_\lambda^+)\tilde x$, $|P_\lambda^+W_\lambda(\tilde x)|_\lambda\le\beta$, valid for all $\tilde x\in\mathbb{R}^N$, we obtain $|W_\lambda(x_k)-P_\lambda x_k|_\lambda\le 2\beta$ and consequently

$$|W_\lambda(x_k)-Q_\lambda(\lambda^k z)|\le|W_\lambda(x_k)-P_\lambda x_k|+|Q_\lambda(z_k-\lambda^k z)|\le\bar c+k\tilde c.$$

This relation and (6.40) imply the estimates

$$|\Phi(W_\lambda(x_k);\lambda)-\Phi(Q_\lambda(\lambda^k z);\lambda)|\le\tilde c_1k^\tau|z|^{-\tau}\le\tilde c_2k^\tau|\lambda-\lambda_q|^\tau$$

for the last term of (6.42). Taking into account that

$$|z_k|\ge|\lambda^k z|-|z_k-\lambda^k z|\ge|z|/2-kc\ge\kappa_1/(2|\lambda-\lambda_q|)-nc$$

and therefore $|z_k|\ge\tilde c_3|\lambda-\lambda_q|^{-1}$ whenever $n|\lambda-\lambda_q|$ is sufficiently small, we obtain from (6.42) the estimate

$$\begin{aligned}\Big|z_{k+1}-\lambda^{k+1}z-\sum_{j=0}^{k}\lambda^{k-j}Q_\lambda^{-1}P_\lambda\Phi(Q_\lambda(\lambda^j z);\lambda)\Big|&\\ \le|\lambda|k\big(k^{\tau_1}|\lambda-\lambda_q|^{\tau_1}+|\lambda-\lambda_q|^{s_1}\big)+c_1|\lambda-\lambda_q|^s+c_1k^\tau|\lambda-\lambda_q|^\tau.&\quad(6.43)\end{aligned}$$

Note that one obtains the left-hand side of (6.43) from the left-hand side of (6.41) by replacing k with $k+1$. For sufficiently small $n|\lambda-\lambda_q|$ the relations

$$|\lambda|k\le(1+|\lambda-\lambda_q|)k\le k+1/2$$

and

$$c_1|\lambda-\lambda_q|^s+c_1k^\tau|\lambda-\lambda_q|^\tau<\big(|\lambda-\lambda_q|^{s_1}+k^{\tau_1}|\lambda-\lambda_q|^{\tau_1}\big)/2$$

are valid, and consequently the right-hand side of (6.43) is less than

$$(k+1)(k^{\tau_1}|\lambda-\lambda_q|^{\tau_1}+|\lambda-\lambda_q|^{s_1}),$$

which implies that if (6.41) holds for some $k\le n$ then it also holds for $k+1$ in place of k. By induction we conclude that (6.41) is valid for all $1\le k\le n$.

From (6.40) it follows

$$|\Phi(Q_\lambda(\lambda^k z);\lambda)-\Phi(Q_\lambda(\lambda_q^k z);\lambda)|\le\hat c|\lambda^k-\lambda_q^k|^\tau.$$

Since $|\lambda^k-\lambda_q^k|\le 2k|\lambda-\lambda_q|$ for every $1\le k\le n$, this implies

$$\begin{aligned}\Big|\sum_{j=0}^{n-1}\lambda^{n-j-1}Q_\lambda^{-1}P_\lambda\Phi(Q_\lambda(\lambda^j z);\lambda)&\\ -\sum_{j=0}^{n-1}\lambda_q^{n-j-1}Q_\lambda^{-1}P_\lambda\Phi(Q_\lambda(\lambda_q^j z);\lambda)\Big|\le\hat c_1n^{1+\tau}|\lambda-\lambda_q|^\tau.&\quad(6.44)\end{aligned}$$

The relation $\lambda_q^n=1$ implies

$$\lambda^n=\Big(1+\frac{\lambda-\lambda_q}{\lambda_q}\Big)^n=1+n\lambda_q^{-1}(\lambda-\lambda_q)+O(n^2(\lambda-\lambda_q)^2),\qquad n(\lambda-\lambda_q)\to 0,$$

and consequently

$$|\lambda^n z-z-n\lambda_q^{-1}(\lambda-\lambda_q)z|\le\hat c_2n^2|\lambda-\lambda_q|,\qquad(6.45)$$

where the estimate $|z| \le \kappa_2|\lambda-\lambda_q|$ is used. Combining estimates (6.44), (6.45) and (6.41) with $k=n$ and taking into account that $\tau_1<\tau\le 1$, we arrive at

$$\Big|z_n - z - n\lambda_q^{-1}(\lambda-\lambda_q)z - \sum_{j=0}^{n-1}\lambda_q^{n-j-1}Q_\lambda^{-1}P_\lambda\Phi(Q_\lambda(\lambda_q^j z);\lambda)\Big| \\ \le 2n\big(n^{\tau_1}|\lambda-\lambda_q|^{\tau_1}+|\lambda-\lambda_q|^{s_1}\big) \quad (6.46)$$

for all sufficiently small $n|\lambda-\lambda_q|$. Here

$$\sum_{j=0}^{n-1}\lambda_q^{n-j-1}Q_\lambda^{-1}P_\lambda\Phi(Q_\lambda(\lambda_q^j z);\lambda) = -n\lambda_q^{-1}e^{i\varphi}\Psi_{\lambda,n}^{res}(n\varphi)$$

with $\varphi=\arg z$, which follows from the relations

$$\sum_{j=0}^{n-1}\lambda_q^{n-j-1}Q_\lambda^{-1}P_\lambda\Phi(Q_\lambda(\lambda_q^j z);\lambda) = \sum_{j=0}^{n-1}\lambda_q^{n-j-1}\Psi_\lambda(\arg(\lambda_q^j z))$$

$$= \sum_{j=0}^{n-1}e^{2\pi q(n-j-1)i}\Psi_\lambda(2\pi qj+\varphi) = \sum_{j=0}^{n-1}e^{-2\pi q(j+1)i}\sum_{k=-\infty}^{\infty}\psi_k^\lambda e^{ik(2\pi qj+\varphi)} \\ = e^{-2\pi qi}\sum_{k=-\infty}^{\infty}\psi_k^\lambda e^{ik\varphi}\sum_{j=0}^{n-1}e^{2\pi qj(k-1)i}$$

and

$$\sum_{j=0}^{n-1}e^{2\pi qj(k-1)i} = \begin{cases} n & \text{if } k-1 \text{ is a multiple of n;} \\ 0 & \text{otherwise.}\end{cases}$$

Hence (6.46) coincides with (6.39). This completes the proof of Theorem 6.1.

6.3. **Proof of Theorem 3.2.**

6.3.1. We start with sketching the idea of the proof. Fix a $\theta\in(0,1)$ and a $q=m/n$ such that $\lambda_q\in\Lambda_\Delta$. Denote $z=re^{\varphi i}\in\mathbb{C}$, $h\in E'_\lambda$, $x=Q_\lambda z+h\in\mathbb{R}^N$, and put

$$\mathbb{H}_\lambda(x) = h-(I-P_\lambda)V_\lambda^n(x)\in E'_\lambda.$$

Consider for any λ close to λ_q the space $\mathbb{E}_\lambda=\mathbb{C}\times E'_\lambda$ of pairs (z,h). Two continuous vector fields

$$\mathbb{W}_\lambda(z,h) = \big(n^{-1}\lambda_q e^{-i\varphi}Q_\lambda^{-1}P_\lambda(V_\lambda^n(x)-x),\mathbb{H}_\lambda(x)\big),$$
$$\mathbb{V}_\lambda(z,h) = \big((\lambda-\lambda_q)r-\Psi_q^{res}(n\varphi),\mathbb{H}_\lambda(x)\big)$$

are analyzed on the boundary ∂G_λ of a set $G_\lambda\subset\mathbb{E}_\lambda$ defined below.

The equation $x=V_\lambda^n(x)$ is equivalent to $\mathbb{W}_\lambda(z,h)=0$ for any λ. Hence, to prove the existence of a fixed point x_* of V_λ^n we shall show that for proper values of λ the vector field $\mathbb{W}_\lambda$ is non-degenerate on ∂G_λ and its rotation $\gamma(\mathbb{W}_\lambda,\partial G_\lambda)$ is not equal to 0. For this purpose, using Theorem 6.1, we prove that the vector field $\mathbb{V}_\lambda$ is non-degenerate and is the principal part of the field $\mathbb{W}_\lambda$. Due to Rouche's Theorem, this implies that $\mathbb{V}_\lambda$ and $\mathbb{W}_\lambda$ are homotopic on ∂G_λ and have the same rotation. Finally, we show that the rotation of the field $\mathbb{V}_\lambda$ equals ±1, hence $\gamma(\mathbb{V}_\lambda,\partial G_\lambda)=\gamma(\mathbb{W}_\lambda,\partial G_\lambda)\ne 0$, which ensures that $\mathbb{W}_\lambda$ has a zero $(z_*,h_*)\in G_\lambda$.

6.3.2. The first step of the proof is to construct the set G_λ. Fix $\kappa_{1,2}$ independent of q and λ to satisfy

$$\kappa_1 = \min_{\lambda\in\Lambda} |\psi(\lambda)|/2, \qquad \kappa_2 = 2\max_{\lambda\in\Lambda} |\psi(\lambda)|. \tag{6.47}$$

Let $\varphi_q^-, \varphi_q^+ \in [0, 2\pi]$ be any numbers such that

$$\arg \Psi_q^{res}(n\varphi_q^\pm) = M_q^\pm, \qquad \arg \Psi_q^{res}(n\varphi) \in (M_q^-, M_q^+), \quad \varphi \in (\varphi_q^-, \varphi_q^+).$$

Now for any λ from the set

$$0 < |\lambda - \lambda_q| \le \rho_n, \qquad |\arg(\lambda-\lambda_q) - (M_q^+ + M_q^-)/2| \le \theta\Delta_q/2 \tag{6.48}$$

define the intervals

$$J^r = [\kappa_1/|\lambda-\lambda_q|, \kappa_2/|\lambda-\lambda_q|], \qquad J^\varphi = [\varphi_q^-, \varphi_q^+]. \tag{6.49}$$

Finally, consider the set

$$G = G_\lambda = \big\{(z,h) : r \in J^r, \varphi \in J^\varphi, h \in \Omega_\lambda \cap E'_\lambda\big\} \subset \mathbb{E}_\lambda,$$

with Ω_λ defined by (6.35). The boundary of this set is

$$\partial G_\lambda = (G_\lambda^z \times G_\lambda^h) \cup (G_\lambda^z \times \partial G_\lambda^h),$$

where

$$G_\lambda^z = \{z \in \mathbb{C} : |z| \in J^r, \arg z \in J^\varphi\}, \qquad G_\lambda^h = \{h \in E'_\lambda : \Omega_\lambda \cap E'_\lambda\}.$$

6.3.3. The next step is to prove that the vector field $\mathbb{V}_\lambda$ is non-degenerate on ∂G_λ. More precisely, the first component $(\lambda-\lambda_q)r - \Psi_q^{res}(n\varphi)$ of this field (it is independent from h) is non-degenerate on the set ∂G_λ^z and the second component $\mathbb{H}_\lambda$ is non-degenerate on the set ∂G_λ^h for any $z \in G_\lambda^z$. To show this, let us note that the Fourier series of a Hölder function converges uniformly, therefore

$$|\Psi_q^{res}(n\varphi) - \psi(\lambda_q)| \to 0 \tag{6.50}$$

as $n \to \infty$. Hence, (6.47) implies that there exists n_0 such that for $n > n_0$

$$3\kappa_1/2 < \min_{\lambda\in\Lambda} |\Psi_q^{res}(n\varphi)|, \qquad 2\kappa_2/3 > \max_{\lambda\in\Lambda} |\Psi_q^{res}(n\varphi)|. \tag{6.51}$$

The set ∂G_λ^z consists of the points $z = re^\varphi$, where either $r = \kappa_j/|\lambda-\lambda_q|$ or $\varphi = \varphi^\pm$. If $r = \kappa_j/|\lambda-\lambda_q|$, then $|\lambda-\lambda_q|r = |\Psi_q^{res}(n\varphi)|$, which contradicts (6.51). If $\varphi = \varphi^\pm$, then $\arg(\lambda-\lambda_q) = \arg\Psi_q^{res}(n\varphi)$, which contradicts the second inequality in (6.48). On the boundary ∂G_λ^h the equality $h = (I - P_\lambda)V_\lambda^n(x)$ contradicts the definition of the set Ω_λ. This proves non-degeneracy of $\mathbb{V}_\lambda$ on ∂G_λ.

6.3.4. To calculate the rotation $\gamma(\mathbb{V}_\lambda, \partial G_\lambda)$ of the vector field $\mathbb{V}_\lambda$ on the boundary of the set $G_\lambda = G_\lambda^z \times G_\lambda^h$, we use the Rotation Product Formula $\gamma(\mathbb{V}_\lambda, \partial G_\lambda) = \gamma_z\gamma_h$, where $\gamma_z = \gamma\big((\lambda-\lambda_q)r - \Psi_q^{res}(n\varphi), \partial G_\lambda^z\big)$ is the rotation of the first component of the field $\mathbb{V}_\lambda$ on ∂G_λ^z and $\gamma_h = \gamma(\mathbb{H}_\lambda, \partial G_\lambda^h)$ is the rotation of the second component of $\mathbb{V}_\lambda$ on ∂G_λ^h (see, e.g., [5]).

Let us show that $\gamma_z = -1$. First note that γ_z is equal to the rotation of the complex vector field $r - Z(z) = r - (\lambda-\lambda_q)^{-1}\Psi_q^{res}(n\varphi)$, $z = re^{\varphi i}$, on ∂G_λ^z. Relations (6.48) and (6.50) imply $\Re e\, Z > 0$ and $|\Im m\, Z|/\Re e\, Z \to 0$ for sufficiently large n, hence $|Z| < 3\,\Re e\, Z/2$.

If $\varphi = \varphi_q^+$, then $\Im m(r - Z(z)) = -\Im m\, Z(z) = -\Im m\big((\lambda-\lambda_q)^{-1}\Psi_q^{res}(n\varphi_q^+)\big) < 0$, since (6.48) implies $\arg\Psi_q^{res}(n\varphi_q^+) - \arg(\lambda-\lambda_q) = M_q^+ - \arg(\lambda-\lambda_q) > 0$. Similarly, $\Im m(r - Z(z)) > 0$ for $\varphi = \varphi_q^-$.

If $r = \kappa_2|\lambda - \lambda_q|^{-1}$, then the second estimate of (6.51) implies $\Re e(r - Z(z)) > 0$. If $r = \kappa_1|\lambda - \lambda_q|^{-1}$, then $\Re e(r - Z(z)) < 0$ according to the first estimate of (6.51) and $|Z(z)| < 3\,\Re e\, Z(z)/2$.

Now, the relation $\gamma_z = -1$ follows from the definition of the winding number.

Lemma 6.2. *The fields* $h-(I-P_\lambda)V_\lambda^n(x)$ *and* $h-(I-P_\lambda)A^n(\lambda)x = h-A^n(\lambda)h$ *are linearly homotopic on* ∂G_λ^h.

This lemma implies $|\gamma_h| = 1$ and therefore $|\gamma(\mathbb{V}_\lambda, \partial G_\lambda)| = |\gamma_z||\gamma_h| = 1$, which finalizes this step of the proof. Lemma 6.2 is proved in the next section.

6.3.5. Finally, let us show that the field $\mathbb{V}_\lambda$ is the principal part of the field $\mathbb{W}_\lambda$. Since the second components of these fields coincide, it is sufficient to prove the relation

$$|(\lambda-\lambda_q)r-\Psi_q^{res}(n\varphi)| > |(\lambda-\lambda_q)r-\Psi_q^{res}(n\varphi)-n^{-1}\lambda_q e^{-i\varphi}Q_\lambda^{-1}P_\lambda(V_\lambda^n(x)-x)| =: w$$

for all $h \in G_\lambda^h$ and $z \in \partial G_\lambda^z$. From Theorem 6.1, it follows that

$$w \le 2\big(n^{\tau_1}|\lambda-\lambda_q|^{\tau_1} + |\lambda-\lambda_q|^{s_1}\big) + |\Psi_{\lambda,n}^{res}(n\varphi) - \Psi_q^{res}(n\varphi)|.$$

From assumption (3.21) and from the definition of the function $\Psi_{\lambda,n}^{res}$ it follows that it is Hölder in λ with the same constants K' and τ:

$$|\Psi_{\lambda,n}^{res}(n\varphi) - \Psi_q^{res}(n\varphi)| \le K'|\lambda-\lambda_q|^{\tau} = o(|\lambda-\lambda_q|^{\tau_1}).$$

If $\varphi = \varphi^{\pm}$, then $\arg \Psi_q^{res}(n\varphi) = M_q^{\pm}$ and therefore

$$|(\lambda-\lambda_q)r - \Psi_q^{res}(n\varphi)| \ge \kappa_1|\sin(\arg(\lambda-\lambda_q) - M_q^{\pm})| \ge \kappa_1 \sin((1-\theta)\Delta_q/2).$$

If $r = \kappa_j|\lambda-\lambda_q|^{-1}$, then (6.51) implies $|(\lambda-\lambda_q)r - \Psi_q^{res}(n\varphi)| \ge \kappa_1/2$. Now, for sufficiently large n, the inequality $|(\lambda-\lambda_q)r - \Psi_q^{res}(n\varphi)| > w$ follows from assumption (3.20). Theorem 3.2 is completely proved.

6.4. **Proof of Lemma 6.2.** The inclusion $h \in \partial G_\lambda^h$ means either $|P_\lambda^+ h|_\lambda = \beta$ or $|P_\lambda^- h|_\lambda = \beta$. Consider these two cases separately.

Assume $|P_\lambda^+ h|_\lambda = \beta$. Then relation (6.37) implies $|P_\lambda^+ V_\lambda^{k-1}(x)|_\lambda \ge \beta$ for all $k = 1, \dots, n$ (here $V_\lambda^0 = I$). Therefore

$$W_\lambda(V_\lambda^{k-1}(x)) = V_\lambda^{k-1}(x) + \Big(\frac{\beta}{|P_\lambda^+ V_\lambda^{k-1}(x)|_\lambda} - 1\Big)P_\lambda^+ V_\lambda^{k-1}(x),$$

and hence

$$P_\lambda^+ W_\lambda(V_\lambda^{k-1}(x)) = \beta\, P_\lambda^+ V_\lambda^{k-1}(x)/|P_\lambda^+ V_\lambda^{k-1}(x)|_\lambda. \tag{6.52}$$

Since $|P_\lambda^+ W_\lambda(V_\lambda^{k-1}(x))|_\lambda = \beta$, relation (6.34) implies

$$|\eta P_\lambda^+ U(W_\lambda(V_\lambda^{k-1}(x));\lambda) + (1-\eta)P_\lambda^+ A(\lambda)W_\lambda(V_\lambda^{k-1}(x))|_\lambda > \beta,$$

which, due to $U(W_\lambda(x);\lambda) = V_\lambda(x)$, $P_\lambda^+ A(\lambda) = A(\lambda)P_\lambda^+$ and (6.52), is equivalent to

$$|\eta P_\lambda^+ V_\lambda^k(x) + (1-\eta)\alpha\, P_\lambda^+ A(\lambda)V_\lambda^{k-1}(x)|_\lambda > \beta,$$

with $\alpha = \beta/|P_\lambda^+ V_\lambda^{k-1}(x)|_\lambda$. Furthermore, because $\alpha \le 1$ and this estimate holds for all $\eta \in [0,1]$, it implies

$$|\eta P_\lambda^+ V_\lambda^k(x) + (1-\eta)P_\lambda^+ A(\lambda)V_\lambda^{k-1}(x)|_\lambda > \beta$$

for all $\eta \in [0,1]$. Applying to this the second of relations (6.31), we obtain

$$|\eta P_\lambda^+ A^{n-k}(\lambda)V_\lambda^k(x) + (1-\eta)P_\lambda^+ A^{n-k+1}(\lambda)V_\lambda^{k-1}(x)|_\lambda > \beta, \quad \eta \in [0,1].$$

Consequently, the equality $|P_\lambda^+ h|_\lambda = \beta$ ensures that

$$\eta P_\lambda^+ A^{n-k}(\lambda) V_\lambda^k(x) + (1-\eta) P_\lambda^+ A^{n-k+1}(\lambda) V_\lambda^{k-1}(x) \neq P_\lambda^+ h,$$

and hence

$$(I - P_\lambda)\big(\eta A^{n-k}(\lambda) V_\lambda^k(x) + (1-\eta) A^{n-k+1}(\lambda) V_\lambda^{k-1}(x)\big) \neq h.$$

Therefore the vector fields

$$h - (I - P_\lambda) A^{n-k}(\lambda) V_\lambda^k(x) \quad \text{and} \quad h - (I - P_\lambda) A^{n-k+1}(\lambda) V_\lambda^{k-1}(x)$$

are linearly homotopic for each $k = 1, \dots, n$ on the part of ∂G_λ^h where $|P_\lambda^+ h|_\lambda = \beta$, and thus the vector fields

$$h - (I - P_\lambda) V_\lambda^n(x) \quad \text{and} \quad h - (I - P_\lambda) A^n(\lambda) x = h - A^n(\lambda) h \tag{6.53}$$

are also homotopic.

The case $|P_\lambda^- h|_\lambda = \beta$ is simpler. Relation (6.36) implies $|P_\lambda^- V_\lambda^n(x)|_\lambda < \beta$ and the first of relations (6.31) implies $|P_\lambda^- A^n(\lambda) h|_\lambda < \beta$, therefore

$$\eta P_\lambda^- V_\lambda^n(x) + (1-\eta) P_\lambda^- A^n(\lambda) h \neq P_\lambda^- h, \qquad \eta \in [0,1],$$

and the vector fields (6.53) are again linearly homotopic.

6.5. **Proof of Theorem 3.1.**

6.5.1. Sketch of the proof. We use the notation of the previous subsection. We introduce additionally an $(N-2)$-dimensional space Y and the family of linear invertible maps $T_\lambda : Y \to E'_\lambda$ such that Y with some fixed norm $|\cdot|$ is independent of λ, while the maps T_λ depend on λ continuously; the norm in E'_λ is $|\cdot|_\lambda$. The idea of the proof is to exchange the roles of the parameter λ and the unknown φ. We take an arbitrary $\varphi = \varphi_0$, which is now fixed until the end of the proof, and given a curve $L = \{\lambda \in \mathbb{C} : \lambda = \lambda^*(\rho),\ 0 \le \rho \le 1\}$ show that the equation $x = V_\lambda^n(x)$ has a solution $x = Q_\lambda(re^{i\varphi_0}) + T_\lambda y$ $(y \in Y)$ for some $\lambda = \lambda^*(\rho) \in L$. In other words, we need to find a triple $(r, \rho, y) \in \mathbb{R}_+ \times [0,1] \times Y$ such that

$$re^{i\varphi_0} = Q_\lambda^{-1} P_\lambda V_\lambda^n(x), \qquad y = T_\lambda^{-1}(I - P_\lambda) V_\lambda^n(x); \tag{6.54}$$

here and henceforth $\lambda = \lambda^*(\rho)$ and $x = Q_\lambda(re^{i\varphi_0}) + T_\lambda y$. For this purpose, let us rewrite the first equation of (6.54) as

$$(\lambda - \lambda_q)\bar\psi(\lambda_q) r + |\psi(\lambda_q)|^2 - S = 0,$$

where $\bar\psi$ is the complex conjugate of ψ and $S = S(r, \rho, y, q)$ is defined by

$$S = \Big(\big(re^{i\varphi_0} - Q_\lambda^{-1} P_\lambda V_\lambda^n(x)\big) n^{-1} \lambda_q e^{-i\varphi_0} + (\lambda - \lambda_q) r + \psi(\lambda_q)\Big)\bar\psi(\lambda_q).$$

In this notation, solutions of system (6.54) are zeros of the vector field

$$\tilde{\mathbb{V}}_q(r, \rho, y) = \begin{pmatrix} r\,\Re e\big((\lambda - \lambda_q)\bar\psi(\lambda_q)\big) - |\psi(\lambda_q)|^2 - \Re e\, S \\ r\,\Im m\big((\lambda - \lambda_q)\bar\psi(\lambda_q)\big) - \Im m\, S \\ y - T_\lambda^{-1}(I - P_\lambda) V_\lambda^n(x) \end{pmatrix}$$

in the N-dimensional space of triples (r, ρ, y). We shall prove that this vector field has at least one zero in the set

$$G_q = \{(r, \rho, y) :\ r|\lambda^*(\rho) - \lambda_q| \in [\kappa_1, \kappa_2];\ \rho \in [0,1];\ T_{\lambda^*(\rho)} y \in \Omega_{\lambda^*(\rho)}\},$$

where κ_1, κ_2 is a fixed pair of positive numbers such that

$$\kappa_1 < \min_{\lambda \in \Lambda} |\psi(\lambda)|, \quad \kappa_2 > \sqrt{1 + \nu^2} \max_{\lambda \in \Lambda} |\psi(\lambda)|. \tag{6.55}$$

The proof will follow the same line as that of the previous subsection. Thus, we first of all show that for $(r,\rho,y)\in G_q$ the first component $\tilde{\mathbb{V}}_q^r$ of the vector field $\tilde{\mathbb{V}}_q$ is negative if $r=\kappa_1/|\lambda-\lambda_q|$ and positive if $r=\kappa_2/|\lambda-\lambda_q|$; the second component $\tilde{\mathbb{V}}_q^\rho$ has different signs for $\rho=0$ and $\rho=1$; and the third component $\tilde{\mathbb{V}}_q^y$ is nonzero and is homotopic to the vector field $y-T_\lambda^{-1}A^n(\lambda)T_\lambda y$ on the boundary of the domain $T_\lambda^{-1}(E'_\lambda\cap\Omega_\lambda)$. Therefore the rotation $\gamma(\tilde{\mathbb{V}}_q,G_q)$ is defined. However, because the set G_q here is not a direct product of domains in the three subspaces corresponding to the components of $\tilde{\mathbb{V}}_q$, the Rotation Product Formula can not be applied directly for calculation of the rotation $\gamma(\tilde{\mathbb{V}}_q,G_q)$, at least in the same simple form as we used above. Therefore, we shall complete the proof by some additional standard argument enabling the use of this formula.

6.5.2. Case $\rho=0,1$. For these values of ρ,

$$|r\,\Im m((\lambda-\lambda_q)\bar\psi(\lambda_q))|=r\nu\,\Re e\big((\lambda-\lambda_q)\bar\psi(\lambda_q)\big)=r\nu(1+\nu^2)^{-1/2}|\lambda-\lambda_q||\psi(\lambda_q)|$$

(with $\lambda=\lambda^*(\rho)$) and because $r\ge\kappa_1/|\lambda-\lambda_q|$ in G_q,

$$|r\,\Im m((\lambda-\lambda_q)\bar\psi(\lambda_q))|\ge\kappa_1\nu(1+\nu^2)^{-1/2}\min_{\lambda\in\Lambda}|\psi(\lambda)|>0. \tag{6.56}$$

To estimate S, consider that $(r,\rho,y)\in G_q$ implies $x=Q_\lambda(re^{i\varphi_0})+T_\lambda y\in\Omega_\lambda^{q,\kappa_1,\kappa_2}$, which by Theorem 6.1 ensures estimate (6.39). From (6.39) it follows

$$|S|/|\psi(\lambda_q)|\le|\Psi_{\lambda,n}^{res}(n\varphi_0)-\psi(\lambda_q)|+2(n^{\tau_1}|\lambda-\lambda_q|^{\tau_1}+|\lambda-\lambda_q|^{s_1}),$$

provided that $n|\lambda-\lambda_q|<\delta$ for some fixed $\delta>0$. Hence, for any $\varepsilon<\delta$

$$|S|/|\psi(\lambda_q)|\le|\Psi_{\lambda,n}^{res}(n\varphi_0)-\psi(\lambda_q)|+2(\varepsilon^{\tau_1}+\varepsilon^{s_1})\quad\text{for}\quad n|\lambda-\lambda_q|<\varepsilon.$$

Combining this with (6.50), we see that for some fixed function $\tilde\chi=\tilde\chi(n,\varepsilon)$

$$|S|\le\tilde\chi(n,\varepsilon)\quad\text{if}\quad|\lambda-\lambda_q|<\varepsilon/n, \tag{6.57}$$

where $\tilde\chi(n,\varepsilon)\to0$ as $n\to\infty$, $\varepsilon\to0$. This estimate and (6.56) imply that the sign of the second component $\tilde{\mathbb{V}}_q^\rho$ of the vector field $\tilde{\mathbb{V}}_q$ is the same as that of the function $r\,\Im m((\lambda-\lambda_q)\bar\psi(\lambda_q))$ for $\rho=0,1$ provided that ε and n^{-1} are sufficiently small. Consequently this component has different signs for $\rho=0$ and $\rho=1$.

6.5.3. Case $r|\lambda-\lambda_q|=\kappa_1,\kappa_2$. For $r|\lambda-\lambda_q|=\kappa_1$,

$$r\,\Re e((\lambda-\lambda_q)\bar\psi(\lambda_q))-|\psi(\lambda_q)|^2\le(\kappa_1-|\psi(\lambda_q)|)|\psi(\lambda_q)|,$$

while for $r|\lambda-\lambda_q|=\kappa_2$

$$r\,\Re e((\lambda-\lambda_q)\bar\psi(\lambda_q))-|\psi(\lambda_q)|^2\ge(\kappa_2(1+\nu^2)^{-1/2}-|\psi(\lambda_q)|)|\psi(\lambda_q)|.$$

These estimates and relations (6.55) and (6.57) imply that the first component $\tilde{\mathbb{V}}_q^r$ of the vector field $\tilde{\mathbb{V}}_q$ is negative whenever $r|\lambda-\lambda_q|=\kappa_1$ and positive whenever $r|\lambda-\lambda_q|=\kappa_2$ in G_q if ε and n^{-1} are sufficiently small.

6.5.4. Case $T_\lambda y\in\partial(E'_\lambda\cap\Omega_\lambda)$. According to Lemma 6.2, the vector fields $h-(I-P_\lambda)V^n(x)$ and $h-A^n(\lambda)h$ are linearly homotopic on the boundary of the domain $E'_\lambda\cap\Omega_\lambda$ for any fixed q,λ,r,φ. This implies that the third component $\tilde{\mathbb{V}}_q^y(r,\rho,y)=y-T_\lambda^{-1}(I-P_\lambda)V^n(x)$ of $\tilde{\mathbb{V}}_q$ is homotopic to the vector field $y-T_\lambda^{-1}A^n(\lambda)T_\lambda y$ on the boundary of the domain $T_\lambda^{-1}(E'_\lambda\cap\Omega_\lambda)$ for $\varphi=\varphi_0$ and any fixed r and ρ.

6.5.5. Completion of the proof. Consider the vector field

$$\tilde{\mathbb{U}}_q=(\tilde{\mathbb{V}}_q^r,\tilde{\mathbb{V}}_q^\rho,y-T_\lambda^{-1}A^n(\lambda)T_\lambda y).$$

It has the same two first components as the vector field $\tilde{\mathbb{V}}_q = (\tilde{\mathbb{V}}_q^r, \tilde{\mathbb{V}}_q^\rho, \tilde{\mathbb{V}}_q^y)$. The vector fields $\tilde{\mathbb{U}}_q$ and $\tilde{\mathbb{V}}_q$ are homotopic on the boundary ∂G_q of the domain G_q, because their first two components are non-zero on the parts of ∂G_q where $r = \kappa_{1,2}/|\lambda - \lambda_q|$ and $\rho = 0, 1$, and their third components are homotopic on the part of ∂G_q where $T_\lambda y \in \partial(E'_\lambda \cap \Omega_\lambda)$ (∂G_q is the union of all these parts). Therefore $\gamma(\tilde{\mathbb{V}}_q, G_q) = \gamma(\tilde{\mathbb{U}}_q, G_q)$. To calculate this rotation, consider the domain

$$\tilde{G} = \{(r, \rho, y) : \ r|\lambda^*(\rho) - \lambda_q| \in [\kappa_1, \kappa_2]; \ \rho \in [0, 1]; \ |y| \le \eta\},$$

where $\eta > 0$ is sufficiently small such that $\tilde{G} \subset G_q$. Since $y = 0$ for each zero (r, ρ, y) of the vector field $\tilde{\mathbb{U}}_q$, this field does not have zeros in the set $G_q \setminus \tilde{G}$ and consequently $\gamma(\tilde{\mathbb{U}}_q, G_q) = \gamma(\tilde{\mathbb{U}}_q, \tilde{G})$. Now, fix a sufficiently large $R > 0$ such that

$$[\kappa_1/|\lambda^*(\rho) - \lambda_q|, \ \kappa_2/|\lambda^*(\rho) - \lambda_q|] \subset [-R, R] \quad \text{for all} \quad \rho \in [0, 1],$$

and define a continuous extension $\tilde{\mathbb{U}}_q^+$ of the vector field $\tilde{\mathbb{U}}_q$ from the domain $\tilde{G}$ to the cylinder

$$\tilde{G}^+ = \{r : \ r \in [-R, R]\} \times \{\rho : \ \rho \in [0, 1]\} \times \{y \in Y : \ |y| \le \eta\} \supset \tilde{G}$$

by the formula $\tilde{\mathbb{U}}_q^+(r, \rho, y) = \tilde{\mathbb{U}}_q(\Pi(r), \rho, y)$ with the continuous projector

$$\Pi(r) = \begin{cases} \kappa_1/|\lambda - \lambda_q| & \text{if} \quad r \in [-R, \kappa_1/|\lambda - \lambda_q|], \\ r & \text{if} \quad r \in [\kappa_1/|\lambda - \lambda_q|, \kappa_2/|\lambda - \lambda_q|], \\ \kappa_2/|\lambda - \lambda_q| & \text{if} \quad r \in [\kappa_2/|\lambda - \lambda_q|, R]. \end{cases}$$

Because $\Pi(r) = \kappa_{1,2}/|\lambda - \lambda_q|$ for $(r, \rho, y) \in \tilde{G}^+ \setminus \tilde{G}$ and the first component of the vector field $\tilde{U}_q(r, \rho, y)$ is non-zero whenever $r = \kappa_{1,2}/|\lambda - \lambda_q|$, the extension $\tilde{U}_q^+$ does not have zeros in the set $G^+ \setminus \tilde{G}$, hence $\gamma(\tilde{\mathbb{U}}_q, \tilde{G}) = \gamma(\tilde{\mathbb{U}}_q^+, \tilde{G}^+)$ and we conclude that $\gamma(\tilde{\mathbb{V}}_q, G_q) = \gamma(\tilde{\mathbb{U}}_q^+, \tilde{G}^+)$. The latter rotation can now be calculated by the Rotation Product Formula as the domain $\tilde{G}^+$ has the appropriate structure. From the fact that the first component of the vector fields $\tilde{\mathbb{V}}_q, \tilde{\mathbb{U}}_q$ is negative if $r = \kappa_1/|\lambda - \lambda_q|$ and positive if $r = \kappa_2/|\lambda - \lambda_q|$, it follows that the first component of the vector field $\tilde{\mathbb{U}}_q^+$ is negative if $r = -R$ and positive if $r = R$ for every fixed $\rho \in [0, 1]$, $|y| \le \eta$, and therefore the rotation γ^r of the first component of $\tilde{\mathbb{U}}_q^+$ on the boundary of the segment $[-R, R] \ni r$ equals 1. Similarly, since the second component of $\tilde{\mathbb{V}}_q, \tilde{\mathbb{U}}_q$ has different signs at the ends of the segment $[0, 1] \ni \rho$, the same is true for the second component of $\tilde{\mathbb{U}}_q^+$ for every fixed $r \in [-R, R]$, $|y| \le \eta$, hence the rotation γ^ρ of the second component of $\tilde{\mathbb{U}}_q^+$ on the boundary of the segment $[0, 1] \ni \rho$ equals either 1 or -1. Finally, the rotation γ^y of the third component $y - T_\lambda^{-1} A^n(\lambda) T_\lambda y$ of $\tilde{\mathbb{U}}_q^+$ on the sphere $|y| = \eta$ is either 1 or -1, because the matrix $I - T_\lambda^{-1} A^n(\lambda) T_\lambda$ is invertible. The Rotation Product Formula implies $\gamma(\tilde{\mathbb{U}}_q^+, \tilde{G}^+) = \gamma^r \gamma^\rho \gamma^y$, consequently the rotations $\gamma(\tilde{\mathbb{V}}_q, G_q) = \gamma(\tilde{\mathbb{U}}_q^+, \tilde{G}^+)$ equal either 1 or -1. This completes the proof.

6.6. **Sketch of the proof of Theorems 5.1 – 5.3.** To be simple, we sketch the proof for two-dimensional system (2.13) on the complex plane $z \in \mathbb{C}$. We use the notation $\Psi_\lambda(z) = \Psi_\lambda(e^{\varphi i}) = \Psi_\lambda(\varphi)$. Consider the change of variable $\zeta = B(z) = z + \Theta(z)$ in (2.13), where $\Theta(\alpha z) = \Theta(z)$, $\alpha > 0$, is a homogeneous Lipschitz function, unknown *a priori*.

Lemma 6.3. *The map B is one-to-one for sufficiently large $|z|$. The inverse operator has the form $B^{-1}(\zeta) = \zeta - \Theta(\zeta) + o(1)$, $|\zeta| \to \infty$.*

The lemma follows from the Banach Contraction Mapping Principle: the equation $\zeta = z + \Theta(z)$ has a unique solution z for any ζ with a sufficiently large $|\zeta|$, since the function $\zeta - \Theta(z)$ has small Lipschitz constant for large $|z|$ and relations $|z| \to \infty$ and $|\zeta| \to \infty$ are equivalent, because Θ is bounded. The form of the inverse operator follows from direct computations.

After the change of the variable, the evolution map becomes

$$\begin{aligned} BUB^{-1}(\zeta) &= UB^{-1}(\zeta) + \Theta(UB^{-1}(\zeta)) \\ &= \lambda B^{-1}(\zeta) + \Psi_\lambda(B^{-1}(\zeta)) + \Theta\big(\lambda B^{-1}(\zeta) + \Psi_\lambda(B^{-1}(\zeta))\big) + o(1) \\ &= \lambda\zeta - \lambda\Theta(\zeta) + \Psi_\lambda(\zeta) + \Theta(\lambda\zeta) + o(1) = \lambda\zeta + \Delta_\lambda(\zeta) + o(1), \end{aligned}$$

where $\Delta_\lambda(\zeta) = -\lambda\Theta(\zeta) + \Psi_\lambda(\zeta) + \Theta(\lambda\zeta)$ and the terms $o(1)$ vanish as $|z|, |\zeta| \to \infty$. Now we choose the function Θ in such a way that the positively homogeneous function Δ_{λ_q} has the smallest possible number of non-zero terms in its Fourier series

$$\begin{aligned} \Delta_{\lambda_q}(e^{\varphi i}) &= \sum_{k=-\infty}^{\infty} \Big(-e^{2\pi q i}\theta_k e^{k\varphi i} + \psi_k^{\lambda_q} e^{k\varphi i} + \theta_k e^{k\varphi i} e^{2\pi k q i}\Big) \\ &= \sum_{k=-\infty}^{\infty} \Big(\psi_k^{\lambda_q} - \theta_k e^{2\pi q i}(1 - e^{2\pi(k-1)qi})\Big) e^{k\varphi i}, \end{aligned}$$

where $\varphi = \arg\zeta$ and θ_k are Fourier coefficients of the function $\Theta(e^{\varphi i})$. If $q = m/n$ is a rational number, then $1 = e^{2\pi(k-1)qi}$ for $k = 1 \pmod n$. Hence, for such k, the k-th Fourier coefficient of Δ_{λ_q} is independent of θ_k and we set $\theta_k = 0$. For all the other k we put $\theta_k = \psi_k^{\lambda_q} e^{-2\pi q i}(1 - e^{2\pi(k-1)qi})^{-1}$, which implies

$$\Delta_{\lambda_q}(e^{\varphi i}) = \sum_{k=1 \pmod n} \psi_k^{\lambda_q} e^{k\varphi i} = e^{\varphi i} \sum_{s=-\infty}^{\infty} \psi_{ns+1}^{\lambda_q} e^{ns\varphi i} = -e^{\varphi i}\Psi_q^{res}(n\varphi).$$

Thus, for $\lambda = \lambda_q$, the change of the variable $\zeta = B_q(z) = z + \Theta_q(z)$, with Θ_q defined by (5.28), transforms the evolution map of system (2.13) to $BUB^{-1}(\zeta) = \lambda_q\zeta - e^{\varphi i}\Psi_q^{res}(n\varphi) + o(1)$.

If q is irrational, then $e^{2\pi q i}(1 - e^{2\pi(k-1)qi}) \neq 0$ for any $k \neq 1$. In this case, we choose a large K and set $\theta_k = \psi_k^{\lambda_q}(e^{2\pi q i} - e^{2\pi k q i})^{-1}$ for $|k| \le K,\ k \neq 1$ and $\theta_k = 0$ for $|k| > K$, i.e., we define $\Theta = \Theta_{q,K}$ by (5.30). Then

$$\Delta_{\lambda_q}(e^{\varphi i}) = \psi_1^{\lambda_q} e^{\varphi i} + \sigma_K = -\psi(\lambda_q)e^{\varphi i} + \sigma_K, \qquad \sigma_K = \sum_{|k|>K} \psi_k^{\lambda_q} e^{k\varphi i},$$

where the term σ_K can be made arbitrarily small by choosing K large enough. This implies $BUB^{-1}(\zeta) = \lambda_q\zeta - e^{\varphi i}\psi(\lambda_q) + o(1)$ for $\lambda = \lambda_q$.

Using the above formulas for BUB^{-1}, one can finalize the proof of Theorems 5.1 and 5.2 by the type of argument outlined in the beginning of Section 5.

Theorem 5.3 follows from Theorems 5.1 and 5.2, which imply that any $\lambda_q \in \mathfrak{L}$, except for those with rational $q = m/n$ with $n \le n_0$, has a neighborhood O_q such that for all $\lambda \in O_q$ with $|\lambda| < 1$ the set Π_q is invariant for system (1.1). These neighborhoods O_q plus the neighborhood $\tilde O(\delta)$ of the finite set of points λ_q with $q = m/n$, $n \le n_0$, constitute a cover of the closed subset $\mathfrak{L}$ of the unit circle. If we choose a finite subcover O from this cover, then $O \setminus \tilde O(\delta)$ contains the set $\mathfrak{L}_{\delta,\varepsilon}$ for a sufficiently small $\varepsilon > 0$ and the conclusion of the theorem holds for this set $\mathfrak{L}_{\delta,\varepsilon}$.

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