Constructions and Families of Nonbinary Linear Codes with Covering Radius 2

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***Abstract*—New constructions of linear nonbinary codes with covering radius $R = 2$ are proposed. They are in part modifications of earlier constructions by the author and in part are new. Using a starting code with $R = 2$ as a "seed" these constructions yield an infinite family of codes with the same covering radius. New infinite families of codes with $R = 2$ are obtained for all alphabets of size $q \geq 4$ and all codimensions $r \geq 3$ with the help of the constructions described. The parameters obtained are better than those of known codes. New estimates for some partition parameters in earlier known constructions are used to design new code families. Complete caps and other saturated sets of points in projective geometry are applied as starting codes. A table of new upper bounds on the length function for $q = 4, 5, 7, R = 2,$ and $r \leq 24$ is included.**

***Index Terms*—Complete caps, covering codes, covering radius, linear codes, nonbinary codes.**

I. INTRODUCTION

Constructions of linear covering codes with covering radius $R = 2$ and close problems are considered, e.g., in [1]–[16], and the references therein. An introduction to linear covering codes and a general survey is given in [3]. Linear codes with $R = 2$, even basis $q \geq 4$, and codimension $r = 4$ are constructed in [1, p. 104]. In [5] and [6] the construction of [1] is generalized to odd basis $q \geq 5$. In [2] a construction of codes with $R = 2, q = 3, r = 4t + 1$, is given. Using these codes a "doubling" construction of [8, Theorem 3a], [14, Sec. 4] produces codes with $R = 2, q = 3, r = 4t + 2$. In a geometric perspective, linear covering codes with $R = 2$ are studied in [10], [11, Ch. 9], [12, Sec. 2], [15], and [16]. For arbitrary q and r, some code constructions were proposed in [4] and developed in [5], [6], and [9]. They rely on a linear code of covering radius R as a starting code and result in infinite families of linear codes with the same covering radius. These constructions can be called "q^m-concatenating constructions" since a parity-check matrix of a starting code is repeated q^m times.

In this correspondence we use, modify, and develop the approach and constructions considered in a recent paper [6]. In Section II, we survey constructions for $R = 2$ from [6] and propose new constructions. For all the constructions described we obtain estimates of partition parameters useful for constructing new codes by an iterative process when codes obtained in a certain step are starting codes in the next steps. The obtained estimates are better than those in [6] and help us to obtain a number of good code families. In Section III, with the help of the described constructions and the iterative process, we construct new infinite families of covering codes with $R = 2$ and any $q \geq 4, r \geq 3$. Parameters of the new codes are better than those of known codes. The obtained codes give new upper bounds on the length functions. We have included a table of the new upper bounds for codes with $q = 4, 5, 7,\ R = 2,$ and $r \leq 24$. Some results of this work were announced in [7] and [8].

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Let $\boldsymbol{F}_q$ be the Galois field of q elements and $\boldsymbol{F}_q^* = \boldsymbol{F}_q \backslash \{0\}$. Denote by an $[n, n-r]_q R$ code a q-ary linear code of length n, codimension r, and covering radius R. This code is an $(n-r)$-dimensional subspace in the space of n-dimensional row vectors over $\boldsymbol{F}_q$. In the notation $[n, n-r]_q R$ we may omit R and change $n-r$ by its value. The length function $l(r, R; q)$ [1] is the smallest length of a q-ary linear code with codimension r and covering radius R. Let $\mathcal{G}_q^r$ be the space of r-dimensional q-ary column vectors.

All matrices and column vectors below are q-ary. An element of $\boldsymbol{F}_{q^m}$ appearing in a q-ary matrix denotes a column vector of $\mathcal{G}_q^m$ that is a q-ary representation of this element, and vice versa, we can treat a column vector of $\mathcal{G}_q^m$ as an element of $\boldsymbol{F}_{q^m}$.

Let 0^k be the all-zero matrix with k rows. Denote by $\mathbf{0}$ a zero column vector. We specify the dimensions of vectors and matrices unless they are unambiguously defined by the context. In this case we specify the number of q-ary rows in a matrix or positions in a column.

Denote by $\{H\}$ the set of columns of a matrix H. Let $H_{\mathcal{C}}$ be a parity-check matrix of a code $\mathcal{C}$ and let T be the symbol of transposition.

We consider linear combinations of q-ary columns only with *nonzero* q-ary coefficients.

Fact 1: An $[n, n-r]_q R$ code with a parity-check matrix $H = [\boldsymbol{f}_1 \cdots \boldsymbol{f}_n]$, where $\boldsymbol{f}_j \in \mathcal{G}_q^r, j = \overline{1, n}$, has covering radius $R = 2$ if and only if each nonzero column $\boldsymbol{p}$ of $\mathcal{G}_q^r$ can be represented by one of following linear combinations with columns of H:

$$\boldsymbol{p} = \gamma_1 \boldsymbol{f}_{j_1}, \quad \boldsymbol{p} \in \mathcal{G}_q^r, \quad \boldsymbol{p} \neq \mathbf{0}, \quad \boldsymbol{f}_{j_1} \in \{H\}, \quad \gamma_1 \in \boldsymbol{F}_q^* \tag{1}$$

$$\boldsymbol{p} = \gamma_1 \boldsymbol{f}_{j_1} + \gamma_2 \boldsymbol{f}_{j_2}, \quad \boldsymbol{p} \in \mathcal{G}_q^r, \quad \boldsymbol{p} \neq \mathbf{0}, \quad j_1 \neq j_2, \quad \boldsymbol{f}_{j_1}, \boldsymbol{f}_{j_2} \in \{H\}, \quad \gamma_1, \gamma_2 \in \boldsymbol{F}_q^* \tag{2}$$

and, besides, there is a nonzero column $\boldsymbol{p}'$ of $\mathcal{G}_q^r$ that cannot be written in the form of (1).

Note that if $R = 1$, each nonzero column of $\mathcal{G}_q^r$ can be written in the form of (1).

We can treat a nonzero column of $\mathcal{G}_q^r$ as a point of the projective geometry PG$(r-1, q)$. Then a parity-check matrix of an $[n, n-r]_q 2$ code $\mathcal{C}$ is a set $\mathcal{H}$ of n points. A point set of a projective geometry is 1-saturated if any point of the geometry lies on a line through at least two points of this set. By Fact 1, $\mathcal{H}$ is a 1-saturated set. A complete cap is a 1-saturated set such that no three of its points are collinear. If $\mathcal{C}$ has minimum distance at least 4 then $\mathcal{H}$ is a complete cap. For details, see [6, pp. 2072, 2077], [11, Ch. 9], [12, Sec. 1.3], [13, Sec. 12], [15], and [16]. In Section III, we use complete caps and other 1-saturated sets as starting codes.

Definition 1: Let H be a parity-check matrix of an $[n, n-r]_q 2$ code and let $H = [\boldsymbol{f}_1 \cdots \boldsymbol{f}_n]$, where $\boldsymbol{f}_j \in \mathcal{G}_q^r, j = \overline{1, n}$.

a) A partition of the column set $\{H\}$ into nonempty subsets is called a 2-*partition* if each nonzero column $\boldsymbol{p}$ of $\mathcal{G}_q^r$ can be represented in the form of either (1) or (2) where in (2) columns $\boldsymbol{f}_{j_1}$ and $\boldsymbol{f}_{j_2}$ belong to *distinct subsets*.

b) A subset $\mathcal{F}$ of a 2-partition is called a $\mathcal{Q}^+$-*subset* if each column of $\mathcal{F}$ can be written in the form of (2), where columns $\boldsymbol{f}_{j_1}$ and $\boldsymbol{f}_{j_2}$ belong to *distinct subsets* $\mathcal{Y}$ and $\mathcal{Z}$ of this 2-partition and it is possible that $\mathcal{Y} = \mathcal{F}$. A 2-partition is called a 2^+-*partition* if it has a $\mathcal{Q}^+$-subset.

c) A 2-partition is called a 2^E-*partition* if each nonzero column $\boldsymbol{p}$ of $\mathcal{G}_q^r$ can be represented in the form of (2) with columns

$\boldsymbol{f}_{j_1}$ and $\boldsymbol{f}_{j_2}$ belonging to *distinct subsets*. A code is called an *E-code* if its parity-check matrix admits a 2^E-partition.

Remark 1: Definition 1 uses approaches of [6, Definitions 2.4, 6.1, 6.2]. The term "AL2" of [6] is changed to "E" (for "exactly"). Each nonzero column $\boldsymbol{p}$ of $\mathcal{G}_q^r$ is equal to a linear combination of *exactly two columns* from a parity-check matrix of an $[n, n-r]_q 2$ E-code. By Fact 1 and Definition 1, a 2-partition of a parity-check matrix H is a 2^E-partition if and only if each column of H can be written in the form of (2) with columns $\boldsymbol{f}_{j_1}$ and $\boldsymbol{f}_{j_2}$ belonging to distinct subsets. So, each subset of a 2^E-partition is a $\mathcal{Q}^+$-subset.

Denote by $h(H;\mathcal{K})$ (respectively, $h^+(H;\mathcal{K})$ or $h^E(H;\mathcal{K})$) the number of subsets in a 2-partition $\mathcal{K}$ (respectively, 2^+-partition $\mathcal{K}$ or 2^E-partition $\mathcal{K}$) for a matrix H. A partition into one-element subsets is called *trivial*. By Fact 1, for a parity-check matrix of a code with covering radius 2 the trivial partition always is a 2-partition. The minimal number of subsets in any 2-partition for a matrix H is denoted by $h(H)$. So, $h(H) = \min_{\mathcal{K}} h(H;\mathcal{K})$. Similarly, $h^+(H) = \min_{\mathcal{K}} h^+(H;\mathcal{K}), h^E(H) = \min_{\mathcal{K}} h^E(H;\mathcal{K})$.

Example 1: We consider $q = 4, r = 3$. Let α be a primitive element of $\boldsymbol{F}_4$. The construction of [6, Theorem 5.2] gives the $[5,2]_4 2$ code $\mathcal{D}$ with parity-check matrix

$$H_{\mathcal{D}} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & \alpha & 0 & 1 \end{bmatrix} = [\boldsymbol{f}_1 \quad \boldsymbol{f}_2 \quad \boldsymbol{f}_3 \quad \boldsymbol{f}_4 \quad \boldsymbol{f}_5]. \qquad (3)$$

Consider the partition

$$\mathcal{K} = \{\boldsymbol{f}_1, \boldsymbol{f}_2\}, \{\boldsymbol{f}_3\}, \{\boldsymbol{f}_4\}, \{\boldsymbol{f}_5\}$$

(we will write partitions in this form). Assume $\beta, \gamma, \delta \in \boldsymbol{F}_4^*$. We have

$$\begin{aligned}
(\beta,0,0)^T &= \beta \boldsymbol{f}_1, (0,\gamma,0)^T \\
&= \gamma \boldsymbol{f}_4, (0,0,\delta)^T \\
&= \delta \boldsymbol{f}_4 + \delta \boldsymbol{f}_5, (0,\delta,\delta)^T \\
&= \delta \boldsymbol{f}_5, (0,\gamma \neq \delta,\delta)^T \\
&= (\gamma+\delta)\boldsymbol{f}_4 + \delta \boldsymbol{f}_5, (\beta,0,\alpha\beta)^T \\
&= \beta \boldsymbol{f}_3, (\beta,0,\delta \neq \alpha\beta)^T \\
&= (\beta+\delta\alpha^{-1})\boldsymbol{f}_1 + \delta\alpha^{-1}\boldsymbol{f}_3, (\beta,\gamma,0)^T \\
&= \beta \boldsymbol{f}_1 + \gamma \boldsymbol{f}_4, (\beta,\gamma,\alpha^j\beta)^T \\
&= \beta \boldsymbol{f}_{2+j} + \gamma \boldsymbol{f}_4, \qquad j = 0,1, (\beta, \alpha^i\beta, \alpha^2\beta)^T \\
&= \beta \boldsymbol{f}_{3-i} + \alpha^i \beta \boldsymbol{f}_5, \qquad i = \overline{0,2}.
\end{aligned}$$

By Definition 1a), $\mathcal{K}$ is a 2-partition. Since $\boldsymbol{f}_1 = \alpha^2 \boldsymbol{f}_2 + \alpha \boldsymbol{f}_3$, the subset $\{\boldsymbol{f}_1, \boldsymbol{f}_2\}$ is a $\mathcal{Q}^+$-subset. Hence $\mathcal{K}$ is a 2^+-partition and $h^+(H_{\mathcal{D}}) \le 4$.

Example 2: We consider $q = 5, r = 3$. Let $\mathcal{U}$ be a $[6,3]_5$ code with parity-check matrix

$$H_{\mathcal{U}} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & 0 & 1 & 2 \end{bmatrix} = [\boldsymbol{f}_1 \cdots \boldsymbol{f}_6]. \qquad (4)$$

Using Fact 1, we verified directly that $\mathcal{U}$ has covering radius 2. Hence the trivial partition of $H_{\mathcal{U}}$ is a 2-partition. Since

$$\boldsymbol{f}_{1+j} = 2\boldsymbol{f}_{2+j} + 4\boldsymbol{f}_{3+j}, j = 0,3$$

this partition is a 2^E-partition (see Remark 1), $h^E(H_{\mathcal{U}}) \le 6$, and $\mathcal{U}$ is an E-code.

II. Constructions of Codes

Constructions considered in this section are based on the following approach to lengthening nonbinary codes with covering radius $R = 2$ [6]. We use an $[n_0, n_0 - r_0]_q 2$ *starting code* $\mathcal{V}_0$ with a parity-check matrix

$$H_0 = [\boldsymbol{f}_1 \cdots \boldsymbol{f}_{n_0}], \boldsymbol{f}_j \in \mathcal{G}_q^{r_0}, \qquad j = \overline{1, n_0}.$$

Let m be an integer. We put $q^m + 1 \ge n_0$. Denote by W_m a parity-check matrix of the $[w_{m,q}, w_{m,q} - m]_q 1$ Hamming code with $w_{m,q} = (q^m - 1)/(q-1)$. We form an $[n, n-r]_q R_{\mathcal{V}}$ *new code* $\mathcal{V}$ with $n = 2w_{m,q} + n_0 q^m, r = r_0 + 2m$, and parity-check matrix (5) (see at the bottom of this page), where

$$\begin{aligned}
&\boldsymbol{f}_j \in \{H_0\}, \qquad j = \overline{1, n_0} \\
&\{\xi_1, \xi_2, \cdots, \xi_{q^m}\} = \boldsymbol{F}_{q^m} \\
&\beta_j \in \boldsymbol{F}_{q^m}, \qquad j = \overline{1, n_0 - 1}, \ \beta_i \neq \beta_j
\end{aligned}$$

if $i \neq j$, $\boldsymbol{0}$ is the zero column with m positions, 0^k is the zero $k \times w_{m,q}$ matrix for $k = r_0, m$.

From now on $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})^T$ is a column of $\mathcal{G}_q^r$ with $\boldsymbol{a} \in \mathcal{G}_q^{r_0}, \boldsymbol{b}, \boldsymbol{c} \in \mathcal{G}_q^m$. In order to show $R_{\mathcal{V}} = 2$ we represent an arbitrary nonzero column $(\boldsymbol{p}, \boldsymbol{u}_1, \boldsymbol{u}_2)^T \in \mathcal{G}_q^r$ by a linear combination of at most two columns of $H'_{\mathcal{V}}$, see Fact 1. For a nonzero column $\boldsymbol{d}$ of $\mathcal{G}_q^m$ we denote by $\boldsymbol{w}(\boldsymbol{d})$ *the only* column of the matrix W_m and by $\pi(\boldsymbol{d})$ *the only* element of $\boldsymbol{F}_q^*$ with $\boldsymbol{d} = \pi(\boldsymbol{d})\boldsymbol{w}(\boldsymbol{d})$.

1) Assume $\boldsymbol{p} \neq \boldsymbol{0}$. Since the starting code $\mathcal{V}_0$ has $R = 2$ the column $\boldsymbol{p}$ can be written in the form of either (1) or (2) with $\boldsymbol{f}_{j_v} \in \{H_0\}, v = 1, 2$. We denote $\boldsymbol{t} = \boldsymbol{u}_2 - \beta_{j_1}\boldsymbol{u}_1$.

   a) Assume $\boldsymbol{p} = \gamma_1 \boldsymbol{f}_{j_1}, \boldsymbol{f}_{j_1} \in \{H_0\}$, see (1). For $j_1 \neq n_0$ and $j_1 = n_0$ we have, respectively,

$$(\boldsymbol{p}, \boldsymbol{u}_1, \boldsymbol{u}_2)^T = \pi(\boldsymbol{t})(\boldsymbol{0}, \boldsymbol{0}, \boldsymbol{w}(\boldsymbol{t}))^T + \gamma_1(\boldsymbol{f}_{j_1}, \gamma_1^{-1}\boldsymbol{u}_1, \beta_{j_1}\gamma_1^{-1}\boldsymbol{u}_1)^T, \quad \boldsymbol{t} \neq \boldsymbol{0}, \quad \gamma_1 \in \boldsymbol{F}_q^* \qquad (6)$$

$$(\boldsymbol{p}, \boldsymbol{u}_1, \boldsymbol{u}_2)^T = \pi(\boldsymbol{u}_1)(\boldsymbol{0}, \boldsymbol{w}(\boldsymbol{u}_1), \boldsymbol{0})^T + \gamma_1(\boldsymbol{f}_{j_1}, \boldsymbol{0}, \gamma_1^{-1}\boldsymbol{u}_2)^T, \quad \boldsymbol{u}_1 \neq \boldsymbol{0}, \quad \gamma_1 \in \boldsymbol{F}_q^*. \qquad (7)$$

   b) Assume $\boldsymbol{p} = \gamma_1 \boldsymbol{f}_{j_1} + \gamma_2 \boldsymbol{f}_{j_2}$ with $j_1 \neq j_2, \boldsymbol{f}_{j_1}, \boldsymbol{f}_{j_2} \in \{H_0\}$; see (2). For $j_1, j_2 \neq n_0$ we have

$$(\boldsymbol{p}, \boldsymbol{u}_1, \boldsymbol{u}_2)^T = \gamma_1(\boldsymbol{f}_{j_1}, \xi_x, \beta_{j_1}\xi_x)^T + \gamma_2(\boldsymbol{f}_{j_2}, \xi_y, \beta_{j_2}\xi_y)^T, \quad \gamma_1, \gamma_2 \in \boldsymbol{F}_q^*. \qquad (8)$$

   Values of ξ_x, ξ_y can be found from the system

$$\begin{aligned}
\gamma_1 \xi_x + \gamma_2 \xi_y &= \boldsymbol{u}_1 \\
\gamma_1 \beta_{j_1} \xi_x + \gamma_2 \beta_{j_2} \xi_y &= \boldsymbol{u}_2
\end{aligned}$$

   that has determinant $\gamma_1\gamma_2(\beta_{j_2} - \beta_{j_1}) \neq 0$. Here $\beta_{j_1} \neq \beta_{j_2}$ since $j_1 \neq j_2$. For $j_2 = n_0$ we have

$$(\boldsymbol{p}, \boldsymbol{u}_1, \boldsymbol{u}_2)^T = \gamma_1(\boldsymbol{f}_{j_1}, \gamma_1^{-1}\boldsymbol{u}_1, \beta_{j_1}\gamma_1^{-1}\boldsymbol{u}_1)^T + \gamma_2(\boldsymbol{f}_{j_2}, \boldsymbol{0}, \gamma_2^{-1}\boldsymbol{t})^T, \quad \gamma_1, \gamma_2 \in \boldsymbol{F}_q^*. \qquad (9)$$

$$H'_{\mathcal{V}} = \left[\begin{array}{c|c|cccc|c|cccc|cccc}
0^{r_0} & 0^{r_0} & \boldsymbol{f}_1 & \boldsymbol{f}_1 & \cdots & \boldsymbol{f}_1 & \cdots & \boldsymbol{f}_{n_0-1} & \boldsymbol{f}_{n_0-1} & \cdots & \boldsymbol{f}_{n_0-1} & \boldsymbol{f}_{n_0} & \boldsymbol{f}_{n_0} & \cdots & \boldsymbol{f}_{n_0} \\
W_m & 0^m & \xi_1 & \xi_2 & \cdots & \xi_{q^m} & \cdots & \xi_1 & \xi_2 & \cdots & \xi_{q^m} & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\
0^m & W_m & \beta_1\xi_1 & \beta_1\xi_2 & \cdots & \beta_1\xi_{q^m} & \cdots & \beta_{n_0-1}\xi_1 & \beta_{n_0-1}\xi_2 & \cdots & \beta_{n_0-1}\xi_{q^m} & \xi_1 & \xi_2 & \cdots & \xi_{q^m}
\end{array}\right] \qquad (5)$$

2) Assume $\boldsymbol{p} = \boldsymbol{0}$. We have

$$(\boldsymbol{0}, \boldsymbol{u}_1, \boldsymbol{u}_2)^T = \pi(\boldsymbol{u}_1)(\boldsymbol{0}, \boldsymbol{w}(\boldsymbol{u}_1), \boldsymbol{0})^T + \pi(\boldsymbol{u}_2)(\boldsymbol{0}, \boldsymbol{0}, \boldsymbol{w}(\boldsymbol{u}_2))^T, \quad \boldsymbol{u}_1, \boldsymbol{u}_2 \neq \boldsymbol{0}. \tag{10}$$

In (6), (7), and (10) one summand can be absent if $\boldsymbol{t} = \boldsymbol{0}$ or $\boldsymbol{u}_i = \boldsymbol{0}$. Columns $(\boldsymbol{0}, \boldsymbol{w}(\boldsymbol{u}_1), \boldsymbol{0})^T$ and $(\boldsymbol{0}, \boldsymbol{0}, \boldsymbol{w}(\boldsymbol{u}_2))^T$ are taken from two left submatrices of $H'_{\mathcal{V}}$, see (5).

Finally, there is a nonzero column $\boldsymbol{p}' \in \mathcal{G}_q^{r_0}$ that cannot be represented in the form of (1), see Fact 1. A column $(\boldsymbol{p}', \boldsymbol{u}_1, \boldsymbol{u}_2)^T$ cannot be written in the form of (1) as well. So, $R_{\mathcal{V}} = 2$.

The parameter m is bounded only from below, see the inequality $q^m + 1 \geq n_0$. Hence we can obtain an infinite family of the new codes $\mathcal{V}$.

If

$$\bigcup_{i=1}^{n_0-1} \{\beta_i\} = \boldsymbol{F}_{q^m}$$

we can eliminate one or two left submatrices of $H'_{\mathcal{V}}$ and reduce n. For example, assume that $\mathcal{V}_0$ is an E-code and use a parity-check matrix $H'_{\mathcal{V}}$ of the form of (5) without two left submatrices. Since $\mathcal{V}_0$ is an E-code, for $\boldsymbol{p} \neq \boldsymbol{0}$ we always can apply the case (2) and the relations of (8) and (9). If $\boldsymbol{p} = \boldsymbol{0}$ we cannot use the relation of (10). But for $\boldsymbol{u}_1 \neq \boldsymbol{0}$ the situation $\bigcup_i \{\beta_i\} = \boldsymbol{F}_{q^m}$ always permits us to find b with $\boldsymbol{u}_2 = \beta_b \boldsymbol{u}_1$. We have

$$(\boldsymbol{0}, \boldsymbol{u}_1, \boldsymbol{u}_2)^T = (\boldsymbol{f}_b, \boldsymbol{u}_1, \beta_b \boldsymbol{u}_1)^T - (\boldsymbol{f}_b, \boldsymbol{0}, \boldsymbol{0})^T, \quad \boldsymbol{u}_1 \neq \boldsymbol{0}, \quad \boldsymbol{u}_2 = \beta_b \boldsymbol{u}_1 \tag{11}$$

$$(\boldsymbol{0}, \boldsymbol{0}, \boldsymbol{u}_2)^T = (\boldsymbol{f}_{n_0}, \boldsymbol{0}, \boldsymbol{u}_2)^T - (\boldsymbol{f}_{n_0}, \boldsymbol{0}, \boldsymbol{0})^T, \qquad \boldsymbol{u}_1 = \boldsymbol{0}. \tag{12}$$

Constructions described below develop the considered approach to improve parameters of new codes. The restriction $q^m + 1 \geq n_0$ is connected with the condition: $\beta_i \neq \beta_j$ if $i \neq j$. But if columns $\boldsymbol{f}_i, \boldsymbol{f}_j$ are not used together in (8) or (9) then we can put $\beta_i = \beta_j$. Using a 2-partition $\mathcal{K}$ we can assign the same value of β_i to all elements of each subset. Then the bound on m takes another form, e.g.,

$$q^m + 1 \geq h_0 \in \{h(H_0; \mathcal{K}), h^+(H_0; \mathcal{K}), h^E(H_0; \mathcal{K})\}.$$

Since $h_0 \leq n_0$ we can reduce m, obtain $\bigcup_i \{\beta_i\} = \boldsymbol{F}_{q^m}$ and eliminate the left submatrices of $H'_{\mathcal{V}}$.

We define a matrix $B_m(\beta)$ where $\beta \in \boldsymbol{F}_{q^m} \cup \{*\}$ is called an *indicator* of the matrix.

$$B_m(\beta) = \begin{bmatrix} \xi_1 & \xi_2 & \cdots & \xi_{q^m} \\ \beta\xi_1 & \beta\xi_2 & \cdots & \beta\xi_{q^m} \end{bmatrix}, \qquad \text{if } \beta \in \boldsymbol{F}_{q^m}$$

$$B_m(*) = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \xi_1 & \xi_2 & \cdots & \xi_{q^m} \end{bmatrix} \tag{13}$$

where $\{\xi_1, \xi_2, \cdots, \xi_{q^m}\} = \boldsymbol{F}_{q^m}$, $\xi_1 = 0$, $\xi_2 = 1$, and $\boldsymbol{0}$ is the zero column of $\mathcal{G}_q^m$, cf. (5).

Construction $M^{(i)}$*:* The index i denotes a variant of a construction. Let $q \geq 3$. A *starting code* $\mathcal{V}_0$ is an $[n_0, n_0 - r_0]_q 2$ code with a parity-check matrix $H_0 = [\boldsymbol{f}_1 \cdots \boldsymbol{f}_{n_0}]$, $\boldsymbol{f}_j \in \mathcal{G}_q^{r_0}$, $j = \overline{1, n_0}$. Let $\mathcal{K}_0$ be a 2-partition of $\{H_0\}$. Let m be an integer. We form an $[n, n-r]_q R_{\mathcal{V}}$ *new code* $\mathcal{V}$ with $n = n^{(i)} + n_0 q^m$, $r = r_0 + 2m$, and parity-check matrix $H_{\mathcal{V}}$, cf. (5)

$$H_{\mathcal{V}} = [D^{(i)} \quad S], \quad S = [S_1 \cdots S_{n_0}], \quad S_j = \begin{bmatrix} P(\boldsymbol{f}_j) \\ B_m(\beta_j) \end{bmatrix}$$

$$P(\boldsymbol{f}_j) = [\boldsymbol{f}_j \cdots \boldsymbol{f}_j], \qquad j = \overline{1, n_0} \tag{14}$$

where $D^{(i)}$ is an $r \times n^{(i)}$ matrix, the value of $n^{(i)}$ depends on the form of the matrix $D^{(i)}$, the matrix $P(\boldsymbol{f}_j)$ is an $r_0 \times q^m$ matrix of q^m equal columns $\boldsymbol{f}_j \in \{H_0\}$, $j = \overline{1, n_0}$, the assignment of indicators β_j depends on the partition $\mathcal{K}_0$ as follows: if columns $\boldsymbol{f}_i, \boldsymbol{f}_j$ belong to distinct subsets of $\mathcal{K}_0$ then the inequality $\beta_i \neq \beta_j$ must be true, if columns $\boldsymbol{f}_i, \boldsymbol{f}_j$ belong to the same subset of $\mathcal{K}_0$ then we are free to assign the equality $\beta_i = \beta_j$ or the inequality $\beta_i \neq \beta_j$.

We introduce notations for Construction $M^{(i)}$. Let

$$\mathcal{B} = \bigcup_{i=1}^{n_0} \{\beta_i\}.$$

Denote by $\mathcal{L}(\lambda)$ the union of columns of all submatrices S_j having the same indicator $\beta_j = \lambda$, $j = \overline{1, n_0}$. Let $\boldsymbol{d}_k$ be the kth column of the matrix $D^{(i)}$, $k = \overline{1, n^{(i)}}$. Denote by $\boldsymbol{t}_{jc}$ the cth column of the submatrix $B_m(\beta_j)$, $j = \overline{1, n_0}$, $c = \overline{1, q^m}$. Let $\boldsymbol{h}_{jc} = (\boldsymbol{f}_j, \boldsymbol{t}_{jc})^T$ be the cth column of the submatrix S_j. By (13), (14), $\boldsymbol{h}_{jc} = (\boldsymbol{f}_j, \xi_c, \beta_j \xi_c)^T$ if $\beta_j \neq *$, $\boldsymbol{h}_{jc} = (\boldsymbol{f}_j, \boldsymbol{0}, \xi_c)^T$ if $\beta_j = *$. Let $h_0 = h(H_0; \mathcal{K}_0)$. If $\mathcal{K}_0$ is a 2^+-partition or a 2^E-partition we denote $h_0^+ = h^+(H_0; \mathcal{K}_0)$ or $h_0^E = h^E(H_0; \mathcal{K}_0)$. Let $h_0^\square$ be one of the values of either h_0, or h_0^+, or h_0^E.

For the set $\{S\}$ we define a partition $\mathcal{K}_S$ into $h_0^\square$ subsets corresponding to the partition $\mathcal{K}_0$. For every j, all columns of $\{S_j\}$ belong to the same subset of $\mathcal{K}_S$. Columns of sets $\{S_i\}$ and $\{S_j\}$ belong to the same subset of $\mathcal{K}_S$ if and only if columns $\boldsymbol{f}_i$ and $\boldsymbol{f}_j$ belong to the same subset of $\mathcal{K}_0$. The situation when the equality $\beta_i = \beta_j$ always holds if columns $\boldsymbol{f}_i$ and $\boldsymbol{f}_j$ belong to the same subset of $\mathcal{K}_0$ is called an "$h_0^\square$-assignment." In this case, each union $\mathcal{L}(\lambda)$ is a subset of $\mathcal{K}_S$ and $|\mathcal{B}| = h_0^\square$ where $|\mathcal{F}|$ is the cardinality of a set $\mathcal{F}$. Now we define a partition $\overline{\mathcal{K}}_S$ of the set $\{S\}$ into $2h_0^\square$ subsets partitioning every subset $\mathcal{T}$ of the partition $\mathcal{K}_S$ into two subsets so that the first one consists of columns $\boldsymbol{h}_{j1}, \boldsymbol{h}_{j2}$ of all the submatrices S_j which belong to $\mathcal{T}$ and the second one consists of columns $\{\boldsymbol{h}_{jk}: k = \overline{3, q^m}\}$ of these submatrices. By (13), (14), $\boldsymbol{h}_{j1} = (\boldsymbol{f}_j, \boldsymbol{0}, \boldsymbol{0})^T$, $\boldsymbol{h}_{j2} = (\boldsymbol{f}_j, 1, \beta_j)^T$ if $\beta_j \neq *$, $\boldsymbol{h}_{j2} = (\boldsymbol{f}_j, \boldsymbol{0}, 1)^T$ if $\beta_j = *$.

Comment 1:

i) In Construction $M^{(i)}$ we want to obtain $R_{\mathcal{V}} = 2$ where $R_{\mathcal{V}}$ is the covering radius of the new code $\mathcal{V}$. As before, $\boldsymbol{u} = (\boldsymbol{p}, \boldsymbol{u}_1, \boldsymbol{u}_2)^T$ is an arbitrary nonzero column of $\mathcal{G}_q^r$ with $\boldsymbol{p} \in \mathcal{G}_q^{r_0}$, $\boldsymbol{u}_1, \boldsymbol{u}_2 \in \mathcal{G}_q^m$. In order to prove $R_{\mathcal{V}} = 2$ we show that *the column* $\boldsymbol{u}$ *is equal to a linear combination of at most two columns of* $H_{\mathcal{V}}$, see Fact 1. We represent $\boldsymbol{u}$ by a linear combination

$$(\boldsymbol{p}, \boldsymbol{u}_1, \boldsymbol{u}_2)^T = \sum_{p=1}^{y} \nu_p \boldsymbol{d}_{i_p} + \sum_{k=1}^{z} \gamma_k (\boldsymbol{f}_{j_k}, \boldsymbol{t}_{j_k c_k})^T, \quad y + z \leq 2,\ y, z \geq 0,\ \nu_p, \gamma_k \in \boldsymbol{F}_q^*. \tag{15}$$

Since the starting code $\mathcal{V}_0$ has $R = 2$, a nonzero column $\boldsymbol{p}$ can be written in the form of either (1) or (2) with $\boldsymbol{f}_{j_1}, \boldsymbol{f}_{j_2} \in \{H_0\}$. We consider the following situation in (15); see (2)

$$\boldsymbol{p} = \gamma_1 \boldsymbol{f}_{j_1} + \gamma_2 \boldsymbol{f}_{j_2}, \quad \boldsymbol{f}_{j_1}, \boldsymbol{f}_{j_2} \in \{H_0\}, \quad j_1 \neq j_2, \quad \beta_{j_1} \neq \beta_{j_2}, \quad \gamma_1, \gamma_2 \in \boldsymbol{F}_q^*, \quad y = 0, \quad z = 2 \tag{16}$$

where columns $\boldsymbol{f}_{j_1}$ and $\boldsymbol{f}_{j_2}$ belong to *distinct subsets* of $\mathcal{K}_0$ (namely, this fact forces the inequality $\beta_{j_1} \neq \beta_{j_2}$). In the case (16), the combination (15) has the form of either (8), when $\beta_{j_1}, \beta_{j_2} \neq *$, or (9), when $\beta_{j_2} = *$. So, in the case (16) the form of the matrix S in (14) always permits us to find needed columns $t_{j_k c_k}$. Other cases depend on the matrix $D^{(i)}$ and the set $\mathcal{B}$ and are considered in the proofs of Theorems 1–5. In these proofs we represent the column $(\boldsymbol{p}, \boldsymbol{u}_1, \boldsymbol{u}_2)^T$ in the form (15). From Construction $M^{(i)}$ and this comment it follows that for a proof of the relation $R_V = 2$ we should consider in theorem proofs only the cases $\boldsymbol{p} = \gamma_1 \boldsymbol{f}_{j_1} \neq \boldsymbol{0}$, see (1), and $\boldsymbol{p} = \boldsymbol{0}$.

ii) The approach to obtain estimates of $h(H_{\mathcal{V}}), h^E(H_{\mathcal{V}})$ is as follows: we consider all cases of the representation of (15) for different $\boldsymbol{p}, \boldsymbol{u}_1$, and $\boldsymbol{u}_2$ and then construct a 2-partition, or a 2^+-partition, or a 2^E-partition of the column set $\{H_\mathcal{V}\}$ in (14) so that *columns that are used together in considered representations belong to distinct subsets*. Note that for the representations of (8) and (9) we use at most one column from each submatrix S_j.

iii) Let α be a primitive element of $\boldsymbol{F}_q$. We define a function

$$\psi(\nu)=1 \text{ if } \nu\neq 1, \qquad \psi(\nu)=\alpha \text{ if } \nu=1,$$
$$\nu\in\boldsymbol{F}^*_{q^m}, \quad \psi(\nu)\in\boldsymbol{F}^*_q. \tag{17}$$

Assume $q\geq 4$. Then $\alpha\neq\alpha^2\neq 1$ and each column of S is a linear combination (with coefficients from $\boldsymbol{F}^*_q$) of exactly two columns from distinct subsets of the partition $\overline{\mathcal{K}}_S$, e.g.,

$$\boldsymbol{h}_{jp}=(1-\delta)\boldsymbol{h}_{j1}+\delta\boldsymbol{h}_{jv}, p\in\{\overline{2,q^m}\}, \quad \xi_p\in\boldsymbol{F}^*_{q^m},$$
$$\delta=\alpha\psi(\xi_p\alpha^{-1}), \quad \xi_v=\frac{\xi_p}{\delta}\neq 0,1,\xi_p. \tag{18}$$

Theorem 1: We give *Construction* $M^{(1)}$. Let $q\geq 3, q^m+1\geq h_0$, $m\geq 1, h_0\geq 3$

$$D^{(1)}=[A_mQ_m], A_m=\begin{bmatrix}0^{r_0}\\ W_m\\ 0^m\end{bmatrix}, \quad Q_m=\begin{bmatrix}0^{r_0}\\ 0^m\\ W_m\end{bmatrix},$$
$$n^{(1)}=2w_{m,q}, \quad \{0,*\}\subset\mathcal{B}\subseteq\boldsymbol{F}_{q^m}\cup\{*\}.$$

Then the new code $\mathcal{V}$ is an $[n,n-r]_q2$ code with

$$n=2w_{m,q}+n_0q^m, \quad r=r_0+2m, \quad h(H_\mathcal{V})\leq h_0.$$

Besides, for $q\geq 4$ the new code $\mathcal{V}$ is an E-code with

$$h^+(H_\mathcal{V})\leq h_0+1, \quad h^E(H_\mathcal{V})\leq 2h_0.$$

Proof: The conditions $\mathcal{B}\subseteq\boldsymbol{F}_{q^m}\cup\{*\}$ and $q^m+1\geq h_0$ allow us to assign distinct indicators $\beta_i\neq\beta_j$ if columns $\boldsymbol{f}_i, \boldsymbol{f}_j$ belong to distinct subsets of $\mathcal{K}_0$. We use the h_0-assignment.

1) Assume that in (15) $\boldsymbol{p}=\gamma_1\boldsymbol{f}_{j_1}\neq\boldsymbol{0}$, see (1). For $\beta_{j_1}\in\boldsymbol{F}_{q^m}$ we denote $\boldsymbol{t}=\boldsymbol{u}_2-\beta_{j_1}\boldsymbol{u}_1$.

a) Assume $\beta_{j_1}\neq *$. If $\boldsymbol{t}\neq\boldsymbol{0}$ then in (15) $y=z=1$, $i_1>w_{m,q}, \boldsymbol{d}_{i_1}=(\boldsymbol{0},\boldsymbol{0},\boldsymbol{w}(\boldsymbol{t}))^T\in\{Q_m\}$, and we use (6). If $\boldsymbol{t}=\boldsymbol{0}, \boldsymbol{u}_1\neq\boldsymbol{0}$, we can apply for (15) the following two variants:

$$(\boldsymbol{p},\boldsymbol{u}_1,\boldsymbol{u}_2)^T=\gamma_1(\boldsymbol{f}_{j_1},\gamma_1^{-1}\boldsymbol{u}_1,\beta_{j_1}\gamma_1^{-1}\boldsymbol{u}_1)^T$$
$$=\gamma_1((1-\delta)\boldsymbol{h}_{j_11}+\delta\boldsymbol{h}_{j_1v})$$
$$\xi_v=\frac{\gamma_1^{-1}\boldsymbol{u}_1}{\delta}\neq 0,1. \tag{19}$$

In the first variant it holds that $y=0, z=1$, cf. (6). For the second one we put $q\geq 4, \delta=\alpha\psi((\gamma_1^{-1}\boldsymbol{u}_1)\alpha^{-1})$, see (18) with $\xi_p=\gamma_1^{-1}\boldsymbol{u}_1$, and have in (15) $y=0$, $z=2, j_1=j_2$. Finally, if $\boldsymbol{t}=\boldsymbol{0}, \boldsymbol{u}_1=\boldsymbol{0}$, then $\boldsymbol{u}_2=\boldsymbol{0}$ and we apply for (15) the variants

$$(\boldsymbol{p},\boldsymbol{0},\boldsymbol{0})^T=\gamma_1(\boldsymbol{f}_{j_1},\boldsymbol{0},\boldsymbol{0})^T$$
$$=\gamma_1((1-\alpha)\boldsymbol{h}_{j_12}+\alpha\boldsymbol{h}_{j_1v})$$
$$\xi_v=\frac{\alpha-1}{\alpha}\neq 0,1. \tag{20}$$

b) Assume $\beta_{j_1}=*$. If $\boldsymbol{u}_1\neq\boldsymbol{0}$ then in (15) $y=z=1$, $i_1\leq w_{m,q}, \boldsymbol{d}_{i_1}=(\boldsymbol{0},\boldsymbol{w}(\boldsymbol{u}_1),\boldsymbol{0})^T\in\{A_m\}$, and we use (7). If $\boldsymbol{u}_1=\boldsymbol{0}, \boldsymbol{u}_2\neq\boldsymbol{0}$, we can apply two variants, cf. (7), (19)

$$(\boldsymbol{p},\boldsymbol{0},\boldsymbol{u}_2)^T=\gamma_1(\boldsymbol{f}_{j_1},\boldsymbol{0},\gamma_1^{-1}\boldsymbol{u}_2)^T$$
$$=\gamma_1((1-\delta)\boldsymbol{h}_{j_11}+\delta\boldsymbol{h}_{j_1v})$$
$$\xi_v=\frac{\gamma_1^{-1}\boldsymbol{u}_2}{\delta}\neq 0,1. \tag{21}$$

In the second variant we put

$$q\geq 4, \qquad \delta=\alpha\psi((\gamma_1^{-1}\boldsymbol{u}_2)\alpha^{-1})$$

see (18). If $\boldsymbol{u}_1=\boldsymbol{u}_2=\boldsymbol{0}$ we use (20).

2) Assume that in (15) $\boldsymbol{p}=\boldsymbol{0}$.

a) Assume $\boldsymbol{u}_1,\boldsymbol{u}_2\neq\boldsymbol{0}$. Then in (15) $y=2, \boldsymbol{d}_{i_1}\in\{A_m\}$, $\boldsymbol{d}_{i_2}\in\{Q_m\}$, and we use (10).

b) Assume $\boldsymbol{u}_1\neq\boldsymbol{0}, \boldsymbol{u}_2=\boldsymbol{0}$, or $\boldsymbol{u}_1=\boldsymbol{0}, \boldsymbol{u}_2\neq\boldsymbol{0}$. Since $\{0,*\}\subset\mathcal{B}$ we can put $\beta_k=0, \beta_d=*$, for some k,d, and apply for (15) the following variants, cf. (10):

$$(\boldsymbol{0},\boldsymbol{u}_1,\boldsymbol{0})^T=\pi(\boldsymbol{u}_1)(\boldsymbol{0},\boldsymbol{w}(\boldsymbol{u}_1),\boldsymbol{0})^T$$
$$=\psi(\boldsymbol{u}_1)\boldsymbol{h}_{kx}-\psi(\boldsymbol{u}_1)\boldsymbol{h}_{k1}$$
$$\xi_x=\frac{\boldsymbol{u}_1}{\psi(\boldsymbol{u}_1)}\neq 0,1, \qquad \beta_k=0. \tag{22}$$
$$(\boldsymbol{0},\boldsymbol{0},\boldsymbol{u}_2)^T=\pi(\boldsymbol{u}_2)(\boldsymbol{0},\boldsymbol{0},\boldsymbol{w}(\boldsymbol{u}_2))^T$$
$$=\psi(\boldsymbol{u}_2)\boldsymbol{h}_{dy}-\psi(\boldsymbol{u}_2)\boldsymbol{h}_{d1}$$
$$\xi_y=\frac{\boldsymbol{u}_2}{\psi(\boldsymbol{u}_2)}\neq 0,1,\beta_d=*. \tag{23}$$

Columns $\boldsymbol{h}_{kx}$ and $\boldsymbol{h}_{k1}$ (respectively, $\boldsymbol{h}_{dy}$ and $\boldsymbol{h}_{d1}$) are from distinct subsets of $\overline{\mathcal{K}}_S$ for $q\geq 3$.

So, we have considered *all cases* of the representation of (15) and have shown that $R_\mathcal{V}=2$.

The estimate of $h(H_\mathcal{V})$ follows from (6)–(10) and (19)–(23). In (19)–(23) we take the first variant with $y+z=1$. We use the partition $\mathcal{K}_S$ into h_0 subsets. Each union $\mathcal{L}(\lambda)$ is a subset of $\mathcal{K}_S$. By (6) and (7), columns of $\mathcal{L}(*)$ and columns of Q_m are not used together. Hence we can inscribe the columns of Q_m to the subset $\mathcal{L}(*)$. Similarly, we inscribe the columns of A_m to some subset $\mathcal{L}(\lambda_1)$ with $\lambda_1\neq *$. The approach of Comment 1 ii) is implemented. We formed a 2-partition $\mathcal{K}_1$ of the set $\{H_\mathcal{V}\}$ into h_0 subsets.

Assume $q\geq 4$. We use the partition $\mathcal{K}_1$ and partition some subset $\mathcal{L}(\lambda_2)$ with $\lambda_2\notin\{\lambda_1,*\}$ into two subsets so that the first one consists of columns $\boldsymbol{h}_{j1}$ and $\boldsymbol{h}_{j2}$ of all the submatrices S_j which belong to $\mathcal{L}(\lambda_2)$. By (18), each of these two new subsets is a $\mathcal{Q}^+$-subset. So, $h^+(H_\mathcal{V})\leq h_0+1$. An indicator $\lambda_2\notin\{\lambda_1,*\}$ exists since $h_0\geq 3$.

The estimate of $h^E(H_\mathcal{V})$ follows from (6)–(10), (19)–(23), and Comment 1 iii). In (19)–(23) we consider the second variant with $z=2$. These relations show that the column $\boldsymbol{u}$ can be represented by a linear combination of *exactly two* columns of $H_\mathcal{V}$. Hence the new code $\mathcal{V}$ is an E-code. To provide the mentioned second variant we use the partition $\overline{\mathcal{K}}_S$ with $2h_0$ subsets. Then, similarly to the partition $\mathcal{K}_1$, we inscribe the columns of A_m and Q_m to convenient subsets of $\overline{\mathcal{K}}_S$. Recall that in the second variant of (19) and (21) we put $q\geq 4$. □

In Theorems 2, 3, and 5 we will give conditions $\mathcal{B}=\boldsymbol{F}_{q^m}$ or $\mathcal{B}=\boldsymbol{F}_{q^m}\cup\{*\}$. They mean that we *must* use *all* elements of $\boldsymbol{F}_{q^m}$ or $\boldsymbol{F}_{q^m}\cup\{*\}$ as indicators β_j in (14). In this case we have "*a complete set of indicators*" (CSI). For columns $\boldsymbol{u}_1,\boldsymbol{u}_2\in\mathcal{G}_q^m$ with $\boldsymbol{u}_1\neq\boldsymbol{0}$ the property CSI always permits us to find an index b with $\boldsymbol{u}_2=\beta_b\boldsymbol{u}_1$, see, e.g., (11).

Theorem 2: We give *Construction* $M^{(2)}$. Let

$$q\geq 3, \quad n_0\geq q^m\geq h_0, \quad m\geq 1,$$
$$D^{(2)}=Q_m, \quad n^{(2)}=w_{m,q}, \quad \mathcal{B}=\boldsymbol{F}_{q^m}.$$

Then the new code $\mathcal{V}$ is an $[n,n-r]_q2$ code with $n=w_{m,q}+n_0q^m$, $r=r_0+2m$. Besides, for $q\geq 4, m\geq 2$, the new code $\mathcal{V}$ is an E-code with $h(H_\mathcal{V})\leq h^E(H_\mathcal{V})\leq 2h_0+2$.

Proof: The relation $n_0 \geq q^m$ is a necessary condition to have CSI with $\mathcal{B} = \boldsymbol{F}_{q^m}$. If $q^m > h_0$ we cannot use the h_0-assignment. To obtain $\mathcal{B} = \boldsymbol{F}_{q^m}$ we assign $q^m - h_0$ "auxiliary" indicators β_j so that $\beta_i \neq \beta_j$ when columns $\boldsymbol{f}_i, \boldsymbol{f}_j$ belong to the same subset of the partition $\mathcal{K}_0$. In this case, a subset of the partition $\mathcal{K}_S$ can consist of more than one union $\mathcal{L}(\lambda)$. We call such process a "q^m-assignment." The conditions $\mathcal{B} = \boldsymbol{F}_{q^m}$ and $q^m \geq h_0$ allow us to assign distinct indicators $\beta_i \neq \beta_j$ if columns $\boldsymbol{f}_i, \boldsymbol{f}_j$ belong to distinct subsets of $\mathcal{K}_0$.

1) Assume that in (15) $\boldsymbol{p} = \gamma_1 \boldsymbol{f}_{j_1} \neq \boldsymbol{0}$, see (1). Since $\{*\} \notin \mathcal{B}$ we use (6), (19), and (20).
2) Assume that in (15) $\boldsymbol{p} = \boldsymbol{0}$.
   a) Assume $\boldsymbol{u}_1 \neq \boldsymbol{0}$. We find b with $\boldsymbol{u}_2 = \beta_b \boldsymbol{u}_1$ and put in (15) $y = 0, z = 2, j_1 = j_2 = b$.

$$\begin{aligned}(\boldsymbol{0}, \boldsymbol{u}_1, \boldsymbol{u}_2)^T &= \psi(\boldsymbol{u}_1)\left(\boldsymbol{f}_b, \frac{\boldsymbol{u}_1}{\psi(\boldsymbol{u}_1)}, \beta_b \frac{\boldsymbol{u}_1}{\psi(\boldsymbol{u}_1)}\right)^T \\ &\quad - \psi(\boldsymbol{u}_1)(\boldsymbol{f}_b, \boldsymbol{0}, \boldsymbol{0})^T \\ \boldsymbol{u}_2 &= \beta_b \boldsymbol{u}_1, \quad \boldsymbol{u}_1 \neq \boldsymbol{0}. \end{aligned} \tag{24}$$

   By (17), for $q \geq 3$ columns of (24) are from distinct subsets of the partition $\overline{\mathcal{K}}_S$, cf. (11).

   b) Assume $\boldsymbol{u}_1 = \boldsymbol{0}$. Then $\boldsymbol{u}_2 \neq \boldsymbol{0}$. For $m \geq 2$ we treat columns of the matrix W_m as points of the projective geometry PG $(m-1, q)$, see, e.g., [6, Secs. I, IV, V], [11], and [13]. We use the same notation for a column and the corresponding point. We pass $w_{m-1,q}$ lines from some point $\boldsymbol{k}$. Let $\boldsymbol{k}_d^i$ be the dth point of the ith line with $\boldsymbol{k} = \boldsymbol{k}_0^i$. We form a partition $\overline{\mathcal{K}}_W$ of the point set into two subsets

$$\{\boldsymbol{k}, \boldsymbol{k}_1^i\colon i = \overline{1, w_{m-1,q}}\}$$

   and

$$\{\boldsymbol{k}_d^i\colon i = \overline{1, w_{m-1,q}}, d = \overline{2, q}\}.$$

   Every point of a line is a linear combination (with coefficients from $\boldsymbol{F}_q^*$) of two *arbitrary* points of the same line. So we can obtain two variants for (15) where in the second one points are from distinct subsets of $\overline{\mathcal{K}}_W$. Let, e.g., $\boldsymbol{w}(\boldsymbol{u}_2) = \boldsymbol{k}_2^i$. Then

$$\begin{aligned}(\boldsymbol{0}, \boldsymbol{0}, \boldsymbol{u}_2)^T &= \pi(\boldsymbol{u}_2)(\boldsymbol{0}, \boldsymbol{0}, \boldsymbol{w}(\boldsymbol{u}_2))^T \\ &= \sigma_1(\boldsymbol{0}, \boldsymbol{0}, \boldsymbol{k}_1^i)^T \\ &\quad + \sigma_2(\boldsymbol{0}, \boldsymbol{0}, \boldsymbol{k}_3^i)^T, \qquad \sigma_1, \sigma_2 \in \boldsymbol{F}_q^*. \end{aligned} \tag{25}$$

   If $m = 1$ then $|\{W_m\}| = 1$, in (25) only the first variant is possible, and $\mathcal{V}$ is not an E-code.

To estimate $h^E(H_\mathcal{V})$ we use (6), (8), (9), (24), the second variant of (19), (20), and (25), and Comment 1 iii). To provide (24) and the second variant of (19) and (20) we use the partition $\overline{\mathcal{K}}_S$ with $2h_0$ subsets and put $q \geq 4$ for (19). To provide the second variant of (25) we use the partition $\overline{\mathcal{K}}_W$ with two subsets. The inequality $h(H_\mathcal{V}) \leq h^E(H_\mathcal{V})$ follows from Definition 1. □

Theorem 3: We give *Construction* $M^{(3)}$. Let $\mathcal{K}_0$ be a 2^+-partition. Let

$$q \geq 3, \quad n_0 \geq q^m + 1 \geq h_0^+, \quad m \geq 1,$$
$$D^{(3)} = Q_m, \quad n^{(3)} = w_{m,q}, \quad \mathcal{B} = \boldsymbol{F}_{q^m} \cup \{*\}.$$

We assume that if in (14) an indicator $\beta_j = *$ then the column $\boldsymbol{f}_j$ belongs to a $\mathcal{Q}^+$-subset of $\mathcal{K}_0$. Under these conditions the new code $\mathcal{V}$ is an $[n, n-r]_q 2$ code with $n = w_{m,q} + n_0 q^m, r = r_0 + 2m$. Besides, for $q \geq 4$ the new code $\mathcal{V}$ is an E-code with $h(H_\mathcal{V}) \leq h^E(H_\mathcal{V}) \leq 2h_0^+$.

Proof: The condition $n_0 \geq q^m + 1$ is a necessary condition to have CSI with $\mathcal{B} = \boldsymbol{F}_{q^m} \cup \{*\}$. If $q^m + 1 > h_0^+$ we cannot use the h_0^+-assignment. We use the q^m-assignment as in Theorem 2 and besides we assign the indicator $\beta_j = *$ in accordance with the request of Theorem 3.

1) Assume that in (15) $\boldsymbol{p} = \gamma_1 \boldsymbol{f}_{j_1} \neq \boldsymbol{0}$, see (1).
   a) Assume $\beta_{j_1} \neq *$. We can use the relations (6), (19), and (20).
   b) Assume $\beta_{j_1} = *$. Then $\boldsymbol{f}_{j_1}$ belongs to a $\mathcal{Q}^+$-subset of $\mathcal{K}_0$. Hence, by Definition 1, $\boldsymbol{f}_{j_1} = \sigma \boldsymbol{f}_i + \eta \boldsymbol{f}_t$ where $\sigma, \eta \in \boldsymbol{F}_q^*$ and $\boldsymbol{f}_i, \boldsymbol{f}_t$ belong to distinct subsets of $\mathcal{K}_0$. So, we can reduce this case to the situation of (16) and use the relations (8) and (9).
2) Assume that in (15) $\boldsymbol{p} = \boldsymbol{0}$. We can use the relations (23) and (24).

For estimate of $h^E(H_\mathcal{V})$ we use the second variant in (19), (20), and (23) with $q \geq 4$ for (19). To provide it and the relation (24) we use the partition $\overline{\mathcal{K}}_S$ with $2h_0^+$ subsets. Taking into account (6) and the case $\beta_{j_1} = *$ considered in this proof, we see that columns of the union $\mathcal{L}(*)$ and columns of $D^{(3)} = Q_m$ are not used together. Hence, similarly to Theorem 1, we can inscribe the columns of Q_m to a subset of $\overline{\mathcal{K}}_S$ obtained from the union $\mathcal{L}(*)$. □

Remark 2: Under conditions of Theorems 1–3 (with $m \geq 2$ for Theorem 2) the new code $\mathcal{V}$ is an E-code for $q = 3$ as well as for $q \geq 4$. Moreover, for $q = 3, m \geq 2$, the same estimates of $h^E(H_\mathcal{V})$ as for $q \geq 4$ hold. We did not prove these facts for $q = 3$ to save space.

Theorem 4: We give *Construction* $M^{(4)}$. Let $\mathcal{K}_0$ be a 2^E-partition. Let

$$q \geq 3, \quad q^m + 1 \geq h_0^E, \quad m \geq 1,$$
$$D^{(4)} = \begin{bmatrix} P_*(\boldsymbol{f}_1) \\ 0^m \\ W_m \end{bmatrix}, \quad n^{(4)} = w_{m,q},$$
$$\beta_1 \neq *, \quad \mathcal{B} \subseteq \boldsymbol{F}_{q^m} \cup \{*\}$$

where $P_*(\boldsymbol{f}_1) = [\boldsymbol{f}_1 \cdots \boldsymbol{f}_1]$ is an $r_0 \times w_{m,q}$ matrix of $w_{m,q}$ equal columns $\boldsymbol{f}_1 \in \{H_0\}$. Then the new code $\mathcal{V}$ is an $[n, n-r]_q 2$ E-code with $n = w_{m,q} + n_0 q^m,\ r = r_0 + 2m,\ h(H_\mathcal{V}) \leq h^E(H_\mathcal{V}) \leq h_0^E + 1$.

Proof: We use the h_0^E-assignment with $|\mathcal{B}| = h_0^E$. Assume $\boldsymbol{p} \neq \boldsymbol{0}$. Since $\mathcal{K}_0$ is a 2^E-partition we always can use (16) that leads to (8) and (9). Now assume $\boldsymbol{p} = \boldsymbol{0}$. We have

$$\begin{aligned}(\boldsymbol{0}, \boldsymbol{u}_1, \boldsymbol{u}_2)^T &= \pi(\boldsymbol{t})(\boldsymbol{f}_1, \boldsymbol{0}, \boldsymbol{w}(\boldsymbol{t}))^T \\ &\quad - \pi(\boldsymbol{t})\left(\boldsymbol{f}_1, \frac{-\boldsymbol{u}_1}{\pi(\boldsymbol{t})}, \frac{-\beta_1 \boldsymbol{u}_1}{\pi(\boldsymbol{t})}\right)^T, \\ &\qquad \boldsymbol{t} = \boldsymbol{u}_2 - \beta_1 \boldsymbol{u}_1 \neq \boldsymbol{0}. \end{aligned} \tag{26}$$

$$\begin{aligned}(\boldsymbol{0}, \boldsymbol{u}_1, \boldsymbol{u}_2)^T &= (\boldsymbol{f}_1, \boldsymbol{u}_1, \beta_1 \boldsymbol{u}_1)^T - (\boldsymbol{f}_1, \boldsymbol{0}, \boldsymbol{0})^T, \\ &\qquad \boldsymbol{u}_2 = \beta_1 \boldsymbol{u}_1, \ \ \boldsymbol{t} = \boldsymbol{0}. \end{aligned} \tag{27}$$

In (15) it holds that $y = z = 1$ for (26) and $y = 0, z = 2, j_1 = j_2 = 1$, for (27).

To estimate $h^E(H_\mathcal{V})$ we use the relations (8), (9), (26), and (27). First, we use the partition $\mathcal{K}_S$ with h_0^E subsets. Then, to provide the relation (27), we partition the union $\mathcal{L}(\beta_1)$ (that is a subset of $\mathcal{K}_S$) into two subsets so that the first is the column $\boldsymbol{h}_{11}$. To provide the relations (26) we inscribe the columns of $D^{(4)}$ to any union $\mathcal{L}(\lambda)$ with $\lambda \neq \beta_1$. □

Remark 3: Constructions $M^{(1)}, M^{(2)}$, and $M^{(4)}$ of Theorems 1, 2, and 4 are Constructions $A32_2, C12_1$, and $AL2$ of [6, Secs. III, VI, Notation 6.1] except for some details. In Theorems 1, 2, and 4 of this work the upper estimates of $h(H_{\mathcal{V}}), h^E(H_{\mathcal{V}})$ are less (i.e., better) than ones in [6, Secs. IV, VI]. In [6, Sec. VI], e.g., we have $h^E(H_{\mathcal{V}}) \leq 3h_0 + 2(w_{m-1,q}+2)$ for Construction $A32_2$ and $h^E(H_{\mathcal{V}}) \leq 3q^m + w_{m-1,q} + 2$ for Construction $C12_1$, but in Theorems 1, 2 of this work we have $h^E(H_{\mathcal{V}}) \leq 2h_0$ and $h^E(H_{\mathcal{V}}) \leq 2h_0 + 2$. For Construction $AL2$ no estimate of $h^E(H_{\mathcal{V}})$ is given in [6] but one is obtained in Theorem 4 of this work.

These estimates of $h^E(H_{\mathcal{V}})$ are important for constructing new codes by iterative process when newly obtained codes are starting codes for next steps, see Section III. The improved estimates helped us to obtain a number of good code families.

Theorem 5: We give *Construction* $M^{(5)}$. Let $\mathcal{K}_0$ be a 2^E-partition. Let the matrix $D^{(5)}$ be absent, i.e., $n^{(5)} = 0, H_{\mathcal{V}} = S$, and let $q \geq 3$, $n_0 \geq q^m + 1 \geq h_0^E$, $m \geq 1$, and $\mathcal{B} = \boldsymbol{F}_{q^m} \cup \{*\}$. Then the new code $\mathcal{V}$ is an $[n, n-r]_q 2$ E-code with $n = n_0 q^m$, $r = r_0 + 2m, h(H_{\mathcal{V}}) \leq h^E(H_{\mathcal{V}}) \leq 2h_0^E$.

Proof: As before, the condition $n_0 \geq q^m + 1$ is a necessary condition for $\mathcal{B} = \boldsymbol{F}_{q^m} \cup \{*\}$. If $q^m + 1 > h_0^E$ we cannot use the h_0^E-assignment. We use the q^m-assignment as in Theorem 2 and additionally we assign the indicator $\beta_j = *$. Similarly to Theorem 4 for $\boldsymbol{p} \neq \boldsymbol{0}$ we can always use the relations (8) and (9). If $\boldsymbol{p} = \boldsymbol{0}, \boldsymbol{u}_1 \neq \boldsymbol{0}$, we use (24). If $\boldsymbol{p} = \boldsymbol{0}, \boldsymbol{u}_1 = \boldsymbol{0}$, we can apply the second variant of (23). To provide (23), (24) we use the partition $\overline{\mathcal{K}}_S$ with $2h_0^E$ subsets. □

Remark 4: In (13) we can put $\{\xi_2, \cdots, \xi_k\} = \{W_m\}, k = w_{m,q}+1$. Let $\mathcal{K}_u'$ be a partition of $\{S_u\}$ into three subsets $\{\boldsymbol{h}_{u1}\}, \{\boldsymbol{h}_{up}: p = \overline{2,k}\}, \{\boldsymbol{h}_{up}: p = \overline{k+1, q^m}\}$. This partition has the following property: each column of S_u is equal to a linear combination (with coefficients from $\boldsymbol{F}_q^*$) of exactly two columns from distinct subsets of $\mathcal{K}_u'$, e.g.,

$$\boldsymbol{h}_{up} = (1 - \pi(\xi_p))\boldsymbol{h}_{u1} + \pi(\xi_p)\boldsymbol{h}_{ug(p)}$$

where $p = \overline{k+1, q^m}, \xi_{g(p)} = \boldsymbol{w}(\xi_p), g(p) \in \{\overline{2,k}\}$. The matrix Ω_u^r of [6, eqs. (14), (15)] and the matrix S_u in this work are the same. Hence $\mathcal{K}_u'$ is also a partition of $\{\Omega_u^r\}$. Estimates of $h(H_{\mathcal{V}}), h^E(H_{\mathcal{V}})$ in [6, Secs. IV, VI] are right but with the help of the partition $\mathcal{K}_u'$ the estimates of [6, eqs. (20)–(22), (25)] can be explained better than it has been done in [6]. Moreover, using the partition $\overline{\mathcal{K}}_S$ and approaches of Theorems 1–5 one can improve estimates of [6, Sec. IV] changing terms $3h_0, 2|\mathcal{B}|$ and $3|\mathcal{B}|$ by the term $2h_0$.

III. Families of Codes with Covering Radius $R = 2$

Remark 5: With the help of Construction $M^{(5)}$ we design infinite *iterative chains* of new codes. A starting $[n_0, n_0 - r_0]_q 2$ E-code $\mathcal{V}_0$, satisfying the conditions of Theorem 5, is called an *initial* code. The initial code has $n_0 \geq q^{m_i} + 1 \geq h_0^E$ for some values m_i. Using Construction $M^{(5)}$ for $m = m_i$ we obtain new $[n_i, n_i - r_i]_q 2$ E-codes $\mathcal{V}_i$ of the *first level* with $n_i = n_0 q^{m_i}, r_i = r_0 + 2m_i$, $h^E(H_{\mathcal{V}_i}) \leq h_1^E = 2h_0^E$. Then for values of $m_{i,j}$ satisfying the condition $n_i \geq q^{m_{i,j}} + 1 \geq h_1^E$ we obtain $[n_{i,j}, n_{i,j} - r_{i,j}]_q 2$ E-codes $\mathcal{V}_{i,j}$ of the *second level* with $n_{i,j} = n_i q^{m_{i,j}} = n_0 q^{M_{i,j}}$, $r_{i,j} = r_i + 2m_{i,j} = r_0 + 2M_{i,j}, M_{i,j} = m_i + m_{i,j}$. Similarly, we obtain codes of the $(v+1)$th level from codes of the vth level, $v \geq 2$. (Note that distinct codes of the vth level can have the same parameters.) Such iterative process produces an infinite family of $[n, n-r]_q 2$ E-codes $\mathcal{V}(x)$ with $n = n_0 q^x, r = r_0 + 2x, x \rightarrow \infty$. The values of x form an infinite sequence with some gaps in its beginning. There is $x^{\odot}$ such that for every $x \geq x^{\odot}$ we can design at least one code $\mathcal{V}(x)$. We consider gaps and the values of $x^{\odot}$ for concrete code families. We use a development of the notation, e.g., $n_{i,j} \geq q^{m_{i,j,k}} + 1 \geq h_2^E = 2h_1^E, M_{i,j,k} = M_{i,j} + m_{i,j,k}$. We fill gaps by other constructions.

Let

$$\mu_q(n, R, \mathcal{C}) = q^{-r(\mathcal{C})} \sum_{i=0}^{R} (q-1)^i \binom{n}{i}$$

be the covering density of an $[n, n - r(\mathcal{C})]_q R$ code $\mathcal{C}$. For an infinite family $\mathcal{A}$ consisting of $[n, n - r(\mathcal{A}_n)]_q R$ codes $\mathcal{A}_n$ we consider the value

$$\overline{\mu}_q(R, \mathcal{A}) = \liminf_{n \rightarrow \infty} \mu_q(n, R, \mathcal{A}_n).$$

Denote by $n_q^{\bullet}[r, R]$ the least *known* length of a q-ary linear code of codimension r, covering radius R. Let $\mu_q^{\bullet}[n, R]$ be the least *known* covering density and let $r_q^{\bullet}[n, R]$ be the largest *known* codimension of a q-ary *linear* code of length n, covering radius R. If we obtain an $[n, n-r]_q R$ new code $\mathcal{V}$ with $n < n_q^{\bullet}[r, R]$ then $r \geq r_q^{\bullet}[n, R] + 1$ and $\mu_q(n, R, \mathcal{V}) \leq (1/q)\mu_q^{\bullet}[n, R]$. In examples we give $n_q^{\bullet}[r, 2]$ for comparison.

Example 3: We consider $q = 4, r = 2t - 1$, and use the $[5, 2]_4 2$ code $\mathcal{D}$ of Example 1, see (3). Construction $M^{(3)}$ for $\mathcal{V}_0 = \mathcal{D}, m = 1$, gives a $[21, 16]_4 2$ E-code $\mathcal{S}$ with $h^E(H_{\mathcal{S}}) \leq 8$. We take $\mathcal{S}$ as the initial code of a chain with $n_0 = 21, r_0 = 5, h_0^E \leq 8, m_1 = 2$.

Construction $M^{(5)}$ for $m = 2$ forms an $[n_1, n_1 - r_1]_4 2$ E-code $\mathcal{V}_1$ of the first level with $n_1 = 21 \cdot 4^2 = 336, r_1 = 5 + 2 \cdot 2 = 9, h_1^E \leq 2 \cdot 8 = 16$.

The condition $n_1 = 21 \cdot 4^2 \geq 4^{m_{1,j}} + 1 \geq 16$ holds for $m_{1,1} = 2$, $m_{1,2} = 3, m_{1,3} = 4$. So $M_{1,j} = 4, 5, 6$ for $j = 1, 2, 3$. We obtain three E-codes $\mathcal{V}_{1,j}$ of the second level with $n_{1,j} = 21 \cdot 4^{M_{1,j}} = 5376, 21504, 86016, r_{1,j} = 5 + 2M_{1,j} = 13, 15, 17$, and $h_2^E \leq 2 \cdot 16 = 32$.

The condition $n_{1,1} = 21 \cdot 4^4 \geq 4^{m_{1,1,k}} + 1 \geq 32$ holds for $m_{1,1,k} = \overline{3,6}, k = \overline{1,4}$. We obtain four E-codes $\mathcal{V}_{1,1,k}$ of the third level with $k = \overline{1,4}, n_{1,1,k} = 21 \cdot 4^{M_{1,1,k}}, M_{1,1,k} = 4 + m_{1,1,k} = \overline{7,10}, r_{1,1,k} = 5 + 2M_{1,1,k} = 19, 21, 23, 25$. Similarly, the condition $n_{1,2} = 21 \cdot 4^5 \geq 4^{m_{1,2,k}} + 1 \geq 32$ holds for $m_{1,2,k} = \overline{3,7}, k = \overline{1,5}$, and we obtain five E-codes $\mathcal{V}_{1,2,k}$ of the third level with $k = \overline{1,5}, n_{1,2,k} = 21 \cdot 4^{M_{1,2,k}}$, $M_{1,2,k} = \overline{8,12}, r_{1,2,k} = 5 + 2M_{1,2,k} = 21$, 23, 25, 27, 29. Finally we have six E-codes $\mathcal{V}_{1,3,k}$ with $m_{1,3,k} = \overline{3,8}, k = \overline{1,6}, n_{1,3,k} = 21 \cdot 4^{M_{1,3,k}}, M_{1,3,k} = \overline{9,14}, r_{1,3,k} = 23, 25, 27, 29, 31, 33$. So we obtained a number of codes of the third level with equal parameters. Further codimensions $r_{i,j,k,\cdots}$ of the new codes again can be equal to each other but due to it gaps in the sequences of x and $r = 5 + 2x$ are absent if $x \geq x^{\odot} = 4$. Gaps appear for $x = 1, 3, r = 7, 11$. We obtained an infinite family $\mathcal{A}_1$ of $[n, n-r]_4 2$ E-codes with

$$\mathcal{A}_1: q = 4, \quad r = 2t - 1, \quad n = 21 \times 4^{t-3}, \quad t = 3, 5, \quad \text{and} \quad t \geq 7, \quad \overline{\mu}_4(2, \mathcal{A}_1) \approx 1.938. \tag{28}$$

Now we fill the gaps. Construction $M^{(1)}$ with $\mathcal{V}_0 = \mathcal{D}, m = 2$, forms a $[90, 83]_4 2$ E-code, Construction $M^{(4)}$ with $\mathcal{V}_0 = \mathcal{S}, m = 3$, gives a $[1365, 1354]_4 2$ E-code.

We have

$$n_4^{\bullet}[2t - 1, 2] = 25\tfrac{1}{3} \times 4^{t-3} - \tfrac{1}{3}$$

[5, eq. (4.1)]. So, $n < n_4^{\bullet}[r, 2]$ for the family $\mathcal{A}_1$.

Example 4: We consider $q = 5, r = 2t - 1$, and take the E-code $\mathcal{U}$ of Example 2 as the initial code with $n_0 = 6, r_0 = 3, h_0^E \leq 6, m_1 = 1$. We construct a code chain using Construction $M^{(5)}$ similarly to Remark 5 and Example 3. We obtain an E-code $\mathcal{V}_1$ with

$n_1 = 6 \cdot 5^1 = 30, r_1 = 5, h_1^E \le 12$. Since $30 \ge 5^2 + 1 \ge 12$ one has $m_{1,1} = 2, M_{1,1} = 3$. We get a $[750, 741]_5 2$ E-code $\mathcal{V}_{1,1}$ of the second level with $h_2^E \le 24$. The condition $750 \ge 5^{m_{1,1,k}} + 1 \ge 24$ holds for $m_{1,1,k} = 2, 3, 4, k = 1, 2, 3$. We have $M_{1,1,k} = 5, 6, 7$, and obtain three E-codes $\mathcal{V}_{1,1,k}$ of the third level with

$$n_{1,1,k} = 6 \cdot 5^{M_{1,1,k}} = 18750, 93750, 468750,$$
$$r_{1,1,k} = 13, 15, 17, \qquad h_3^E \le 48.$$

Further we get E-codes $\mathcal{V}_{1,1,k,u}$ with

$$r_{1,1,1,u} = 19, 21, 23, 25$$
$$r_{1,1,2,u} = 21, 23, 25, 27, 29$$
$$r_{1,1,3,u} = 23, 25, \cdots, 33.$$

For $x \ge x^{\odot} = 5$ gaps in the sequence $r = 3 + 2x$ are absent. Gaps exist for $x = 2, 4, r = 7, 11$. We obtained a family $\mathcal{A}_2$ of $[n, n-r]_5 2$ E-codes with parameters

$$\mathcal{A}_2\colon q = 5, \quad r = 2t - 1, \quad n = 6 \times 5^{t-2}, \quad t = 2, 3, 5, \text{ and } t \ge 7, \quad \overline{\mu}_5(2, \mathcal{A}_2) \approx 2.304. \quad (29)$$

Construction $M^{(4)}$ with $\mathcal{V}_0 = \mathcal{U}, m = 2$, and $\mathcal{V}_0 = \mathcal{V}_1, m = 3$, gives a $[156, 149]_5 2$ E-code and a $[3781, 3770]_5 2$ E-code and fills the gaps. Note that for $q = 5$ the code family of [6, Example 6.3] cannot be obtained since a needed $[4, 1]_5 2$ code $\mathcal{V}_0$ does not exist [3, Table 6.4].

Example 5: We consider $q \ge 4, r = 2t$. The construction of [1, p. 104] generalized in [5, Theorem 3.1] and [6] gives the $[2q+1, 2q-3]_q 2$ code $\mathcal{M}$ with parity-check matrix

$$\begin{aligned} H_{\mathcal{M}} &= \begin{bmatrix} 1 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\ \xi_1 & \cdots & \xi_q & 1 & 0 & 0 & \cdots & 0 \\ \xi_1^2 & \cdots & \xi_q^2 & 0 & 0 & 1 & \cdots & 1 \\ 0 & \cdots & 0 & 0 & 1 & \xi_2 & \cdots & \xi_q \end{bmatrix} \\ &= [\boldsymbol{f}_1 \cdots \boldsymbol{f}_{2q+1}], \{\xi_1, \cdots, \xi_q\} \\ &= \boldsymbol{F}_q, \xi_1 = 0. \end{aligned} \quad (30)$$

In the proof of [6, Theorem 5.1] it is shown that the partitions

$$\mathcal{K}_q = \{\boldsymbol{f}_1, \cdots, \boldsymbol{f}_{q-1}\}, \{\boldsymbol{f}_q\}, \{\boldsymbol{f}_{q+1}\}, \{\boldsymbol{f}_{q+2}\}, \{\boldsymbol{f}_{q+3}, \cdots, \boldsymbol{f}_{2q+1}\}$$

for even q and

$$\begin{aligned} \mathcal{K}_q = &\{\boldsymbol{f}_1, \cdots, \boldsymbol{f}_{q-4}\}, \{\boldsymbol{f}_{q-3}\}, \{\boldsymbol{f}_{q-2}\}, \{\boldsymbol{f}_{q-1}\}, \\ &\{\boldsymbol{f}_q, \boldsymbol{f}_{q+1}\}, \{\boldsymbol{f}_{q+2}\}, \{\boldsymbol{f}_{q+3}, \cdots, \boldsymbol{f}_{2q+1}\} \end{aligned}$$

for odd q are 2-partitions. It is easy to see that in both cases each column $\boldsymbol{f}_i$ of the subset $\mathcal{J} = \{\boldsymbol{f}_{q+3}, \cdots, \boldsymbol{f}_{2q+1}\}$ is equal to a linear combination of the columns $\boldsymbol{f}_{q+2} \notin \mathcal{J}$ and $\boldsymbol{f}_j \in \mathcal{J}$. So $\mathcal{J}$ is a $\mathcal{Q}^+$-subset and $\mathcal{K}_q$ are 2^+-partitions. Let $q = 5, \xi_{i+1} = i, i = \overline{1, 4}$. We verified by computer that

$$\mathcal{K}_5 = \{\boldsymbol{f}_1, \boldsymbol{f}_6\}, \{\boldsymbol{f}_2, \boldsymbol{f}_5\}, \{\boldsymbol{f}_3, \boldsymbol{f}_4\}, \{\boldsymbol{f}_7\}, \{\boldsymbol{f}_8, \cdots, \boldsymbol{f}_{11}\}$$

is a 2^+-partition. So for $q \ge 4$ we have $h^+(H_{\mathcal{M}}) \le q + 1$. Construction $M^{(3)}$ with $\mathcal{V}_0 = \mathcal{M}$, $\mathcal{K}_0 = \mathcal{K}_q$, $m = 1$, $h_0^+ \le q + 1$, gives an E-code $\mathcal{P}$. We take $\mathcal{P}$ as the initial code with $n_0 = f_q(r_0), r_0 = 6, h_0^E \le 2q + 2, m_1 = 2$, where $f_q(2t) = 2q^{t-1} + q^{t-2} + q^{t-3}$.

We design a code chain by Construction $M^{(5)}$, see Remark 5 and Examples 3, 4. We obtain an E-code $\mathcal{V}_1$ with $n_1 = f_q(r_1)$, $r_1 = 10, h_1^E \le 4q + 4$. Let $q \ge 5$. Then $q^2 + 1 \ge 4q + 4$, $q^3 + 1 \ge 8q + 8$. For $j = \overline{1, 3}$ we get E-codes $\mathcal{V}_{1,j}$ with $n_{1,j} = f_q(r_{1,j}), r_{1,j} = 14, 16, 18, h_2^E \le 8q + 8$, and E-codes $\mathcal{V}_{1,j,k}$ with $n_{1,j,k} = f_q(r_{1,j,k}), r_{1,1,k} = 20, 22, 24, 26$, $r_{1,2,k} = 22, 24, 26, 28, 30, r_{1,3,k} = 24, 26, \cdots, 34$. We have $x^{\odot} = 4$. Gaps appear for $x = 1, 3, r = 8, 12$. If $q = 4$ we have gaps for $r = 8, 12, 14, 20$ since $4^2 + 1 < 4q + 4$. We got a family $\mathcal{A}_3$ of $[n, n-r]_q 2$ E-codes with

$$\begin{aligned} \mathcal{A}_3\colon \quad q \ge 4, \quad r = 2t, \quad n = 2q^{t-1} + q^{t-2} + q^{t-3}, \\ t = 3, 5, \text{ and } t \ge 7 \text{ if } q \ge 5, \\ t = 3, 5, 8, 9, \text{ and } t \ge 11 \text{ if } q = 4, \\ \overline{\mu}_q(2, \mathcal{A}_3) \approx 2 - 2q^{-1} + 0.5q^{-2} - 2q^{-3} + q^{-4}. \end{aligned} \quad (31)$$

By (31)

$$\overline{\mu}_4(2, \mathcal{A}_3) \approx 1.504$$
$$\overline{\mu}_5(2, \mathcal{A}_3) \approx 1.6056$$
$$\overline{\mu}_7(2, \mathcal{A}_3) \approx 1.7191$$
$$\overline{\mu}_8(2, \mathcal{A}_3) \approx 1.7542$$
$$\overline{\mu}_9(2, \mathcal{A}_3) \approx 1.7814.$$

To fill the gaps for $q \ge 4$ Construction $M^{(1)}$ with $\mathcal{V}_0 = \mathcal{M}, m = 2$, and Construction $M^{(4)}$ with $\mathcal{V}_0 = \mathcal{P}, m = 3$, form $[n, n-r]_q 2$ E-codes with $n = f_q(8) + q + 2, r = 8$, and $n = f_q(12) + q^2 + q + 1, r = 12$. For $q = 4$ Construction $M^{(4)}$ with $\mathcal{V}_0 = \mathcal{P}, m = 2$, gives an auxiliary $[597, 587]_4 2$ E-code $\mathcal{L}$ with $h^E(H_{\mathcal{L}}) \le 11$. Construction $M^{(5)}$ with $\mathcal{V}_0 = \mathcal{L}, m = 2$, forms an E-code $\mathcal{T}$ with $n = 9552, r = 14$, $h^E(H_{\mathcal{T}}) \le 22$. Construction $M^{(5)}$ with $\mathcal{V}_0 = \mathcal{T}, m = 3$, gives an E-code with $n = 611328, r = 20$.

Due to [5, eq. (1.5)], [6, Example 6.1] we have

$$n_q^{\bullet}[2t, 2] = f_q(2t) + q^{t-4} + w_{t-2,q}$$

for $q = 4, 5$ and

$$n_q^{\bullet}[2t, 2] = f_q(2t) + w_{t-3,q}$$

for $q \ge 7$. So, $n = f_q(2t) < n_q^{\bullet}[r, 2]$ for the family $\mathcal{A}_3$.

Example 6: We consider $q \ge 7, r = 2t - 1$. For $q \ge 7$ there exists always an $[n_q, n_q - 3]_q 2$ code $\mathcal{W}$ with $n_q < q$. For example, we can treat points of a small complete cap as columns of a parity-check matrix of a code $\mathcal{W}$ [6, pp. 2072, 2077], [12, Sec. 1.3]. Then by [10, p. 59], [11, Table 9.3], and [15, Table 1], we have $n_q \le \lfloor q/2 \rfloor + 2$ for $q \ge 8$ and $n_q \le 6, 6, 6, 7, 8, 9, 10, 10, 10, 12, 12, 13, 14, 14$, for $q = 7, 8, 9, 11, 13, 16, 17, 19, 23, 25, 27, 29, 31, 32$, respectively. Using the trivial partition Construction $M^{(1)}$ with $\mathcal{V}_0 = \mathcal{W}, m = 1$, $h_0 = n_q$, forms an $[n, n-5]_q 2$ E-code $\mathcal{E}$ with $n = n_q q + 2$, $h^+(H_{\mathcal{E}}) \le n_q + 1, h^E(H_{\mathcal{E}}) \le 2n_q$. If $2n_q \le q + 1$ we take $\mathcal{E}$ as the initial code. If $2n_q > q + 1$ Construction $M^{(3)}$ with $\mathcal{V}_0 = \mathcal{E}, m = 1$, gives an $[n, n-7]_q 2$ initial E-code $\mathcal{N}$ with $n = n_q q^2 + 2q + 1, h^E(H_{\mathcal{N}}) \le 2n_q + 2 < q^2$. Similarly to Examples 3–5, Construction $M^{(5)}$ designs a family $\mathcal{A}_4$ of $[n, n-r]_q 2$ E-codes with properties

$$\begin{aligned} &\mathcal{A}_4\colon q \ge 7, \quad r = 2t - 1, \quad t = 4, 6, \text{ and } t \ge 8, \\ &n_q < q, \text{ there exists an } [n_q, n_q - 3]_q 2 \text{ code}, \\ &n = n_q q^{t-2} + 2q^{t-3} \text{ for } q + 1 \ge 2n_q, \\ &n = n_q q^{t-2} + 2q^{t-3} + q^{t-4} \text{ for } q + 1 < 2n_q, \\ &\overline{\mu}_q(2, \mathcal{A}_4) \approx \frac{1}{8}(q + 4 + 6q^{-1} - 11q^{-3}) \\ &\qquad \text{for odd } q \ge 9 \text{ with } n_q = \frac{q-1}{2} + 2, \\ &\overline{\mu}_q(2, \mathcal{A}_4) \approx \frac{1}{8}(q + 6 + 9q^{-1} - 4q^{-2} - 16q^{-3}) \\ &\qquad \text{for even } q \ge 8 \text{ with } n_q = \frac{q}{2} + 2. \end{aligned} \quad (32)$$

TABLE I
UPPER BOUNDS ON THE LENGTH FUNCTION $l(r, 2; q)$ FOR $q = 4, 5, 7$

r	$l(r,2;4) \le$	r_0	i	$l(r,2;5) \le$	r_0	i	$l(r,2;7) \le$	r_0	i
3	5^{c}			6			6^{d}		
4	9^{a}			11^{bc}			15^{bc}		
5	21^{*}	3	3	30^{*}	3	5	44^{*}	3	1
6	37^{*}	4	3	$56^{\bullet}$	4	3	106^{c}	4	3
7	90^{*}	3	1	156^{*}	3	4	309^{*}	5	3
8	154^{b}	4	1	287^{*}	4	1	751^{*}	4	1
9	336^{*}	5	5	750^{*}	5	5	2164^{*}	5	4
10	592^{*}	6	5	$1400^{\bullet}$	6	5	5194^{*}	6	5
11	1365^{*}	5	4	3781^{*}	5	4	15141^{*}	7	5
12	2389^{*}	6	4	$7031^{\bullet}$	6	4	36415^{*}	6	4
13	$5376^{\bullet}$	9	5	$18750^{\bullet}$	9	5	106036^{*}	9	5
14	$9552^{\bullet}$	10	5	$35000^{\bullet}$	10	5	$254506^{\bullet}$	10	5
15	21504^{*}	9	5	93750^{*}	9	5	$741909^{\bullet}$	11	5
16	37888^{*}	10	5	$175000^{\bullet}$	10	5	1781542^{*}	10	5
17	86016^{*}	9	5	468750^{*}	9	5	5193363^{*}	11	5
18	151552^{*}	10	5	$875000^{\bullet}$	10	5	12470794^{*}	10	5
19	$344064^{\bullet}$	13	5	$2343750^{\bullet}$	13	5	36353541^{*}	11	5
20	$611328^{\bullet}$	14	5	$4375000^{\bullet}$	14	5	$87295558^{\bullet}$	14	5
21	$1376256^{\bullet}$	13	5	$11718750^{\bullet}$	13	5	$254474787^{\bullet}$	15	5
22	$2424832^{\bullet}$	16	5	$21875000^{\bullet}$	14	5	$611068906^{\bullet}$	14	5
23	5505024^{*}	13	5	58593750^{*}	13	5	$1781323509^{\bullet}$	15	5
24	9699328^{*}	16	5	$109375000^{\bullet}$	14	5	4277482342^{*}	14	5

Key to Table I: ^{a}[1] ^{b}[5] ^{c}[6] ^{d}[11]

$^{\bullet}$ bounds obtained in this work

* bounds described briefly without proofs in [7],[8] and considered in details in this work

r_0 – codimension of a starting code, i – number i of construction $M^{(i)}$

starting codes are codes from this table excepting $r = 14$, $q = 4$.

By (32)

$$\overline{\mu}_8(2, \mathcal{A}_4) \approx 1.88$$
$$\overline{\mu}_9(2, \mathcal{A}_4) \approx 1.707$$
$$\overline{\mu}_{11}(2, \mathcal{A}_4) \approx 1.943.$$

Besides, since $n_7 = 6$, we have

$$\overline{\mu}_7(2, \mathcal{A}_4) \approx 2.09.$$

To fill the gaps Construction $M^{(4)}$ with $\mathcal{V}_0 = \mathcal{E}, m = 2$, forms an E-code $\mathcal{C}$ with $n = n_q q^3 + 2q^2 + q + 1, r = 9, h^E(H_{\mathcal{C}}) \le 2n_q + 1 < q^2$. Construction $M^{(5)}$ with $\mathcal{V}_0 = \mathcal{C}, m = 2$, gives an E-code with $n = n_q q^5 + 2q^4 + q^3 + q^2, r = 13$.

For $q = p^2 \ge 16$ we take as $\mathcal{W}$ the code of [6, Theorem 5.2] with $n_q = 3p - 1$. By [6, eq. (30)], it can be shown that $h^E(H_{\mathcal{W}}) \le 4$. Construction $M^{(4)}$ with $\mathcal{V}_0 = \mathcal{W}, m = 1$, forms an initial code. Construction $M^{(5)}$ gives a family $\mathcal{A}_5$ of $[n, n-r]_q 2$ E-codes with

$$\begin{aligned} \mathcal{A}_5 \colon q = p^2 \ge 16, \quad & r = 2t - 1 \ge 5, \\ n = (3p-1)q^{t-2} &+ q^{t-3}, \\ \overline{\mu}_q(2, \mathcal{A}_5) \approx 4.5 - \frac{3}{p} &- \frac{17}{2q} + \frac{9}{pq}. \end{aligned} \tag{33}$$

For growing $q = p^2$ codes of (33) have a covering density 4.5 but if we do not consider the q value as square we obtain a density of $q/8$, see (32).

By [6, Example 6.3]

$$n_q^{\bullet}[2t-1, 2] = n_q q^{t-2} + 2q^{t-3} + q^{t-4} + w_{t-4,q}$$

if an $[n_q, n_q - 3]_q 2$ code with $n_q < q$ exists and

$$n_q^{\bullet}[2t-1, 2] = (3p-1)q^{t-2} + q^{t-3} + w_{t-3,q}$$

if $q = p^2 \ge 9$. So, $n < n_q^{\bullet}[r, 2]$ for the families $\mathcal{A}_4$ and $\mathcal{A}_5$.

Results of Examples 3–6 form Table I. For $r = 3, 4$ we use (3), (4), and (30).

Remark 6: In the binary case, constructions of [9, Theorem 3.1] and their modifications [9, Sec. 3] are effective. We can treat Constructions $M^{(1)}, M^{(2)}$ as nonbinary generalizations of these constructions. Ideas of Constructions $M^{(3)}$–$M^{(5)}$ mainly work in the binary case. But we have no improvements of binary results from [9] with the help of $M^{(3)}$–$M^{(5)}$.

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REFERENCES

[1] R. A. Brualdi, V. S. Pless, and R. M. Wilson, "Short codes with a given covering radius," *IEEE Trans. Inform. Theory*, vol. 35, pp. 99–109, Jan. 1989.
[2] J. C. Cock and P. R. J. Östergård, "Ternary covering codes derived from BCH codes," *J. Combin. Theory, Ser. A*, vol. 80, pp. 283–289, 1997.
[3] G. D. Cohen, I. S. Honkala, S. N. Litsyn, and A. C. Lobstein, *Covering Codes*, (Elsevier, Math. Library, vol. 54). Amsterdam, The Netherlands: North-Holland, 1997.
[4] A. A. Davydov, "Construction of linear covering codes," *Probl. Pered. Inform.* (in Russian), vol. 26, no. 4, pp. 38–55, Oct.–Dec. 1990. English transl. in *Probl. Inform. Transm.*, vol. 26, no. 4, pp. 317–331, Oct.–Dec. 1990.
[5] ——, "Construction of codes with covering radius 2," *Lecture Notes in Computer Science, Algebraic Coding*, no. 573. New York: Springer-Verlag, 1992, pp. 23–31.
[6] ——, "Constructions and families of covering codes and saturated sets of points in projective geometry," *IEEE Trans. Inform. Theory*, vol. 41, pp. 2071–2080, Nov. 1995.
[7] ——, "On linear nonbinary codes with covering radius two," preprint, Dec. 1994.
[8] ——, "On nonbinary linear codes with covering radius two," in *Proc. V Int. Workshop Algebr. Combin. Coding Theory* (Sozopol, Bulgaria, June 1996), pp. 105–110.
[9] A. A. Davydov and A. Yu. Drozhzhina-Labinskaya, "Constructions, families and tables of binary linear covering codes," *IEEE Trans. Inform. Theory*, vol. 40, pp. 1270–1279, July 1994.
[10] J. W. P. Hirschfeld, "Maximum sets in finite projective spaces," in *Surveys in Combinatorics*, E. K. Lloyd, Ed. (London Math. Soc. Lecture Note Ser., vol. 82). Cambridge, U.K.: Cambridge Univ. Press, 1983, pp. 55–76.
[11] ——, *Projective Geometries over Finite Fields*, 2nd ed. Oxford, U.K.: Clarendon, 1998.
[12] J. W. P. Hirschfeld and L. Storme, "The packing problem in statistics, coding theory and finite projective spaces," *J. Statist. Planning Infer.*, vol. 72, pp. 355–380, 1998.
[13] H. Janwa, "Some optimal codes from algebraic geometry and their covering radii," *Europ. J. Combin.*, vol. 11, no. 3, pp. 249–266, 1990.
[14] P. R. J. Östergård, "New constructions for q-ary covering codes," *Ars Combin.*, to be published.
[15] F. Pambianco and L. Storme, "Small complete caps in spaces of even characteristic," *J. Combin. Theory, Ser. A*, vol. 75, pp. 70–84, 1996.
[16] E. Ughi, "Saturated configurations of points in projective Galois spaces," *Europ. J. Combin.*, vol. 8, pp. 325–334, 1987.