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Constructions and Families of Covering Codes and Saturated Sets of Points in Projective Geometry

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***Abstract*— In a recent paper by this author, constructions of linear binary covering codes are considered. In this work, constructions and techniques of the earlier paper are developed and modified for q-ary linear nonbinary covering codes, $q \geq 3$, and new constructions are proposed. The described constructions design an infinite family of codes with covering radius R based on a starting code of the same covering radius. For arbitrary $R \geq 2$, $q \geq 3$, new infinite families of nonbinary covering codes with "good" parameters are obtained with the help of an iterative process when constructed codes are the starting codes for the following steps. The table of upper bounds on the length function for codes with $q = 3$, $R = 2$, 3, and codimension up to 24 is given. We propose to use saturated sets of points in projective geometries over finite fields as parity check matrices of starting codes. New saturated sets are obtained.**

***Index Terms*—Covering radius, covering codes, nonbinary codes, saturated sets of points in projective geometry, density of a covering.**

I. INTRODUCTION

This work is devoted to design of linear q-ary covering codes and infinite families of the codes. Similar questions are considered, e.g., in [2]–[4], [6]–[16], [20]–[23], [25], [26], [29]–[31], and [33]. In [15], constructions direct sum (DS) and amalgamated DS (ADS) are given. An effective approach to these problems is proposed in the paper [8], where constructions of linear *binary* covering codes are described. Using a starting code, these constructions form an infinite family of new codes with the same covering radius. In [8], graphs $\Gamma(J)$ associated with a parity check matrix and R^*, l-subsets are introduced to improve parameters of new codes. However, nonbinary codes and iterative design of codes, using the graph $\Gamma(J)$, are considered in [8] very briefly.

Here, constructions and techniques of [8] are developed and modified for q-ary codes, $q > 2$. Many approaches and results of this work are essentially *nonbinary* (But some new constructions can also be used for $q = 2$). For ease of representation we consider R, l-objects (instead of R^*, l-subsets) and R, l-partitions of parity check matrices (instead of graphs $\Gamma(J)$), where R is the covering radius of a code. We give new constructions CSI with "a complete set of indicators" for any $R \geq 2$, $q \geq 2$, and their variants for $R = 3, 4$, $q \geq 2$. We propose constructions "almost $l = 2$" (AL2) for $R = 2$, $q \geq 2$. The R, l-partitions allow us to reduce restrictions on a parameter m (see Section III), to construct codes with relatively small codimensions and to extend the useful region for constructions CSI and AL2. We describe methods to increase l for R, l-objects with $q > 2$. Estimates of l and the number of subsets in R, l-partitions are obtained for new codes designed by constructions considered. The estimates are important for an iterative process when constructed codes are starting codes for following steps. For $q \geq 3$ a number of new infinite families of covering codes with "good" parameters are designed. They give new upper bounds on the length functions (Note that R, l-objects, R, l-partitions, and constructions CSI are considered in [11] for $q = 2$, $R \leq 4$).

Saturated sets of points in projective geometries are described in [1] and [32]. A correspondence between these sets and parity check matrices of covering codes is considered in [4], [12], [14], [17], [22], and [25]. In [4] the saturated sets are called "R-spanning sets."

Here we *use saturated sets of points as starting codes* and conversely treat parity check matrices of designed codes as saturated sets. In the last case, parameters obtained are better than those known. In addition, we obtained new constructions of saturated sets.

Some results of this work were presented at conferences [9], [10].

Denote by E_q^n the space of n-dimensional row vectors over the Galois field GF(q), $q \geq 2$. Let an $[n, n-r]_q R$ code be a q-ary *linear* code of length n, codimension r, and covering radius R. The code is a subspace of E_q^n with dimension $k = n - r$. In the notation $[n, n-r]_q R$ we may omit R and change $n - r$ by k. Let an $[n, k, d]_q R$ code be an $[n, k]_q R$ code of minimum distance d. Denote by $d(x, z)$ the Hamming distance between vectors x and z. The sphere of radius R with center z in E_q^n is the set $\{x\colon x \in E_q^n,\ R \geq d(x, z)\}$. Let $V_q(n, R)$ be the cardinality of the sphere.

$$V_q(n, R) = \sum_{i=0}^{R} (q-1)^i \binom{n}{i}. \tag{1}$$

The covering radius of an $[n, n - r(C)]_q$ code C is the least integer R such that E_q^n is covered by spheres of radius R whose centers are codewords of C. The *density of* this *covering* [7] is denoted by $\mu_q(n, R, C)$.

$$\begin{aligned}\mu_q(n, R, C) &\triangleq q^{n-r(C)} V_q(n, R)/q^n \\ &= V_q(n, R)/q^{r(C)} \geq 1.\end{aligned} \tag{2}$$

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(See the *sphere bound* [6]). For an infinite family U consisting of $[n, n-r(U_n)]_q R$ codes U_n we consider the value $\overline{\mu}_q(R, U)$ [8], [15]

$$\overline{\mu}_q(R, U) \triangleq \liminf_{n\to\infty} \mu_q(n, R, U_n), \quad U_n \in U. \quad (3)$$

Further, all columns and matrices are q-ary. An element h of GF(q^m) written as an element of a q-ary matrix denotes a column m-dimensional vector that is a q-ary representation of h. An upper index in a notation of a matrix is the number of q-ary rows in it. Let T be the symbol of transposition of a column. Denote by F_q^r the space of r-dimensional q-ary column vectors. If a column φ belongs to F_q^r then

$$\varphi = (b_1, \cdots, b_r)^T, \quad b_i \in GF(q), \; i = \overline{1, r}.$$

The *length function* $l(r, R; q)$ [4] is the smallest length n of a q-ary $[n, n-r]_q R$ code with codimension r.

Fact 1.1 [6], [16]: The covering radius of a q-ary $[n, n-r]_q$ code with a parity check matrix H^r is the least integer R with the following property: for any q-ary r-dimensional column π there exists a linear combination $L(\pi)$ of at most R columns of H^r such that $\pi = L(\pi)$.

An s-dimensional projective geometry PG(s, q) over the field GF(q) is described in [18], [27]. A *nonzero column* $a = (a_1, \cdots, a_r)^T$ of F_q^r with $a_i \in$ GF(q), $i = \overline{1, r}$, can be treated as a *point* of the geometry PG$(r-1, q)$. A column $\lambda a = (\lambda a_1, \cdots, \lambda a_r)^T$, $\lambda \in$ GF(q), $\lambda \neq 0$, corresponds to the same point.

Definition 1.1 [32]: A set H^r *of points in a projective geometry* PG$(r-1, q)$ is called $(R-1)$-*saturated* if R is the least integer with the following property: for any point x of PG$(r-1, q)$ there exist R independent points of H^r such that x lies in the subspace of PG$(r-1, q)$ generated by these points.

We added to the definition of [32] "R is the *least* integer... ." We use the notation H^r for parity check matrices and saturated sets.

Fact 1.2 [22]:

1) Let H^r be a parity check matrix of an $[n, n-r]_q R$ code of covering radius R. Then in the geometry PG$(r-1, q)$ the set of n points corresponding to columns of H^r is $(R-1)$-saturated.

2) Let H^r be an $(R-1)$-saturated set of n points of PG$(r-1, q)$. The aggregate of r-dimensional columns corresponding to the points of H^r is a parity check matrix of an $[n, n-r]_q R$ code of covering radius R.

From an $[n_1, n_1-r_1]_q R_1$ and an $[n_2, n_2-r_2]_q R_2$ code Construction DS [15] forms an $[n_1+n_2, n_1+n_2-r_1-r_2]_q R$ code, $R = R_1 + R_2$. Let

$$f_{t,q} \triangleq (q^t - 1)/(q-1). \quad (4)$$

From the $[f_{s,q}, f_{s,q}-s]_q 1$ Hamming codes Construction DS produces a code family HM consisting of $[n^*_{r,R}, n^*_{r,R}-r]_q R$ codes with

$$n^*_{r,R} = (R-\Delta) f_{t,q} + \Delta f_{t+1,q}, \quad t = \lfloor r/R \rfloor, \; \Delta = r - tR. \quad (5)$$

We use the family HM for comparisons. We note that $n^*_{tR,R} = Rf_{t,q}$ and we have, e.g., $\overline{\mu}_q(2, \text{HM}) = 2$, $\overline{\mu}_q(3, \text{HM}) = 4.5$. So, it is of interest to obtain families U with $\overline{\mu}_q(2, U) < 2$, $\overline{\mu}_q(3, U) < 4.5$. In the general case, it is of interest to get infinite families consisting of $[n_{r,R}, n_{r,R}-r]_q R$ codes with $n_{r,R} < n^*_{r,R}$. Binary linear codes satisfying the inequalities are obtained, e.g., in [4], [8], [11], [14], [15].

In this work for *arbitrary* $q \geq 3$, $R \geq 2$ we obtain new *infinite* families of codes with $n_{r,R} < n^*_{r,R}$ and with $\overline{\mu}_q(2, U) < 2$, $\overline{\mu}_q(3, U) < 4.5$.

In Section II we introduce R, l-objects and R, l-partitions. In Section III constructions of covering codes are considered. In Section IV we estimate the parameter l and the number of subsets in R, l-partitions for new codes. In Section V we give parameters of the known saturated sets and design new ones. In Section VI for $R = 2$, $q \geq 2$, we proposed constructions AL2 and design new code families. In Sections VII and VIII we obtain new code families with $R \geq 3$, $q \geq 3$.

II. R, l-OBJECTS AND R, l-PARTITIONS

Let GF$^*(q) \triangleq$ GF$(q)\backslash\{0\}$. GF$^*(q)$ is the set of *nonzero* elements of GF(q). We consider *linear combinations* of q-ary *columns* only with *nonzero* q-ary *coefficients*. If formally the number of summands in a linear combination equals zero then this combination is treated as the zero column. So, we consider only linear combinations L of the form

$$L = \sum_{k=1}^{z} a_k \varphi_k, \quad \varphi_k \in F_q^s, \; a_k \in \text{GF}^*(q), \; k = \overline{1, z},$$
$$\text{if } z = 0 \text{ then } L = 0.$$

Here φ_k are q-ary columns, a_k are nonzero q-ary coefficients.

Definition 2.1: Let C be a q-ary linear $[Y, Y-s]_q R$ code of length Y, codimension s, covering radius R, with a parity check matrix Φ^s. Let l be an integer, $R \geq l \geq 0$. The code C is called an R, l-*object* of the space E_q^Y and is denoted by a $[Y, Y-s]_q R, l$ code if for any q-ary s-dimensional column π of F_q^s (including the zero column) there exists a linear combination $L(\pi)$ of at least l and at most R distinct columns of the matrix Φ^s such that $\pi = L(\pi)$. All coefficients in the combination $L(\pi)$ are q-ary and nonzero. For $l = 0$ we can treat the zero column as the linear combination of 0 columns of Φ^s. A $[Y, Y-s]_q R, l$ code with $l > 0$ is also a $[Y, Y-s]_q R, l_1$ code with $l_1 = 0, 1, \cdots, l-1$.

Comment 2.1: Let Φ^s be a parity check matrix of a $[Y, Y-s]_q R, l$ code

$$\Phi^s = [\varphi_1 \cdots \varphi_Y], \quad \varphi_u \in F_q^s, \; u = \overline{1, Y}.$$

By Definition 2.1, for any q-ary s-dimensional column π of F_q^s there exists a representation of the form

$$\pi = L(\pi) = \sum_{k=1}^{z(\pi)} a_k \varphi_{j_k}, \quad R \geq z(\pi) \geq l, \; a_k \in \text{GF}^*(q),$$
$$k = \overline{1, z(\pi)} \quad (6)$$

where $j_k \in \{\overline{1, Y}\}$, $k = \overline{1, z(\pi)}$, $j_p \neq j_t$ if $p \neq t, p, t \in \{\overline{1, z(\pi)}\}$; the integer $z(\pi)$ and the sets $\{a_1, \cdots, a_{z(\pi)}\}$, $\{j_1, \cdots, j_{z(\pi)}\}$ depend on π.

Definition 2.2: A linear or nonlinear code C of length Y and covering radius R is called an R, l-object of the space E_q^Y if for any vector x of E_q^Y there exists a word $w(x)$ of C such that $R \geq d(x, w(x)) \geq l$.

For linear codes Definitions 2.1 and 2.2 are equivalent.

R, l-objects are a subclass of R^*, l-subsets proposed in [8, p. 321].

Remark 2.1: For $R \geq l \geq 0$ a *spherical* R, l-*capsule* with center w in the space E_q^Y is the set $\{x: x \in E_q^Y, \; R \geq d(x, w) \geq l\}$ [8]. Spherical R, l-capsules centered at vectors of an R, l-object cover the space E_q^Y.

Definition 2.3:

1) Let $\theta \geq 3$. Distinct nonzero linearly dependent columns $\varphi_1, \cdots, \varphi_\theta$ of F_q^s are called a θ-*group* if there exist *nonzero* q-ary

coefficients $c_1, \cdots, c_\theta$ such that

$$\sum_{k=1}^{\theta} c_k \varphi_k = 0, \quad \varphi_k \in F_q^s, \ \varphi_k \neq 0, \ c_k \in \mathrm{GF}^*(q), \ k = \overline{1, \theta}$$

where $\varphi_p \neq \varphi_t$ if $p \neq t$, $p, t \in \{\overline{1, \theta}\}$.

2) In the geometry PG $(s-1, q)$ θ distinct points are called a *θ-group* if corresponding columns of F_q^s (see Section I) are a *θ-group.*

We can increase l for fixed R using θ-groups in a parity check matrix

$$\Phi^s = [\varphi_1 \cdots \varphi_Y], \quad \varphi_u \in F_q^s, \ u = \overline{1, Y}.$$

Let

$$A = c_1 \varphi_{u_1} + \cdots + c_\theta \varphi_{u_\theta} = 0$$
$$X \in \mathrm{GF}^*(q)$$
$$\pi = L(\pi) = a_1 \varphi_{j_1} + \cdots + a_z \varphi_{j_z}$$

(see (6)). We add XA to $L(\pi)$.

$$\pi = \sum_{k=1}^{z} a_k \varphi_{j_k} + \sum_{i=1}^{\theta} X c_i \varphi_{u_i} = \sum_{e=1}^{t(X)} w_e \varphi_{p_e}, \ t(X) = z + \Delta(X) \tag{7}$$

where $X, a_k, c_i, w_e \in \mathrm{GF}^*(q)$ for all k, i, e; $\Delta(X) = \theta - \gamma(X) - \Gamma$; $\gamma(X)$ is the number of the cases $a_k = -X c_i$ with $j_k = u_i$, $\gamma(X) \in \{\overline{0, \Gamma}\}$; $\Gamma \triangleq |\{j_1, \cdots, j_z\} \cap \{u_1, \cdots, u_\theta\}|$. The situations $R \geq t(X) > z$ are useful.

Lemma 2.1: In the situation (7) there exist X_1 and X_2 such that

$$\theta - \Gamma \geq \Delta(X_1) \geq \theta - \Gamma - \xi(\Gamma, q)$$
$$\theta - \Gamma - \xi^*(\Gamma, q) \geq \Delta(X_2) \geq \theta - 2\Gamma$$

where

$$\xi(\Gamma, q) \triangleq \lfloor \Gamma/(q-1) \rfloor$$
$$\xi^*(\Gamma, q) \triangleq \lceil \Gamma/(q-1) \rceil.$$

Proof: The sum of $\gamma(X)$ over $q-1$ distinct X is equal to Γ. Hence there exist elements X_1, X_2 such that $\gamma(X_1) \leq \xi(\Gamma, q)$, $\gamma(X_2) \geq \xi^*(\Gamma, q)$.

Example 2.1: By Lemma 2.1, there exist elements $X_1 - X_6$ with the properties: $\Delta(X_1) = \theta - 2$ if $\Gamma = 1$, $q \geq 2$; $\Delta(X_2) = \theta - 1$ if $\Gamma = 1$, $q \geq 3$;

$$\theta - 2 \geq \Delta(X_3) \geq \theta - 3 \geq \Delta(X_4) \geq \theta - 4, \text{ for } \Gamma = 2, \ q = 3;$$

$\Delta(X_5) = \theta - 2$ and $\theta - 3 \geq \Delta(X_6) \geq \theta - 4$ for $\Gamma = 2$, $q \geq 4$. For $\Gamma = 0$ always $\Delta(X) = \theta$.

Example 2.2: Let C be an $[n, k, d]_q R$ code. In (7) we put $\theta = d$. For $z = 0$ we have $t(X) = d$. For $z = 1$ we have $\Gamma \leq 1$, $d + 1 \geq t(X) \geq d - 1$. Hence C is an $[n, k]_q R, l$ code with $l \geq 1$ for $R = d$ and $l \geq 2$ for $R > d$.

Example 2.3: Direct sum of the $[4, 2]_3 1$ Hamming code and the $[4, 1]_3 2$ repetition code gives an $[8, 3]_3 3$ code C with the parity check matrix

$$\Phi^5 = \left[\begin{array}{cccc|cccc} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \end{array}\right] = [\varphi_1 \ldots \varphi_8], \ \varphi_u \in F_3^5, \ u = \overline{1, 8}. \tag{8}$$

Here $\varphi_1, \varphi_2, \varphi_3$ and $\varphi_1, \varphi_2, \varphi_4$ are 3-groups, $\varphi_5, \cdots, \varphi_8$ is a 4-group. For any column $\pi = \varphi_u$, $u = \overline{1, 8}$, we have a θ-group with $\theta \in \{3, 4\}$, $\Gamma = 1$, and we can find an element X with $t(X) = 3$ (see Example 2.1). If $\pi = \varphi_1$ we take $X = 1$, $\pi = \varphi_1 + (\varphi_1 + \varphi_2 + 2\varphi_3) = 2\varphi_1 + \varphi_2 + 2\varphi_3$. But the column $\pi = \varphi_5 + \varphi_6$ is not equal to a linear combination of three columns of Φ^5. So, C is an $[8, 3]_3 3$, 2 code and C is not an $[8, 3]_3 3$, 3 code.

Definition 2.4: Let Φ^s be a parity check matrix of a q-ary linear $[Y, Y-s]_q R$, l code of length Y, codimension s, covering radius R. A partition of the column set of the matrix Φ^s into nonempty subsets is called an *R, l-partition* if for any q-ary s-dimensional column π of F_q^s (including the zero column) there exists a linear combination $L(\pi)$ of at least l and at most R columns from *distinct subsets* such that $\pi = L(\pi)$. No two columns from the same subset should occur in this linear combination. All coefficients in the combination $L(\pi)$ are q-ary and nonzero. For an R, 0-partition we can treat the zero column as the linear combination of 0 columns of Φ^s.

Denote by $h(\Phi^s, l; K)$ the number of subsets in an R, l-partition K for a matrix Φ^s. The value of R is defined by context. Clearly, $R \leq h(\Phi^s, l; K) \leq Y$. A partition K is called *trivial* if $h(\Phi^s, l; K) = Y$. We define the minimum over all R, l-partitions K

$$h(\Phi^s, l) \triangleq \min_K h(\Phi^s, l; K).$$

Remark 2.2: Let K_1 be an R, l_1-partition of the column set of Φ^s that is a parity check matrix of a $[Y, Y-s]_q R$, l_1 code V with $l_1 > 0$. We may treat V as a $[Y, Y-s]_q R$, l_2 code with $l_1 > l_2$. Obviously, there exists an R, l_2-partition K_2 such that $h(\Phi^s, l_2; K_2) \leq h(\Phi^s, l_1; K_1)$: in any case, we can take $K_2 = K_1$. Hence, $h(\Phi^s, l_2) \leq h(\Phi^s, l_1)$ for $l_2 < l_1$.

Example 2.4: The matrix Φ^5 in (8) has a 3, 2-partition K_1 and a 3, 0-partition K_2, where K_1 is $\{\varphi_i\}$, $i = 1, 2, 5, 6, 7, 8$, $\{\varphi_3, \varphi_4\}$ and K_2 is $\{\varphi_i\}$, $i = \overline{5, 8}$, $\{\varphi_1, \varphi_2, \varphi_3, \varphi_4\}$. So

$$h(\Phi^5, 0; K_2) = 5 < h(\Phi^5, 2; K_1) = 7.$$

III. Constructions of Covering Codes

Definition 3.1 [8]: Let ρ be an integer. Let $G(\rho)$ be a class of vectors with integer positive components. A vector g belongs to $G(\rho)$ if

$$g = (\rho_1, \rho_2, \cdots, \rho_\gamma), \ \rho_\lambda \in \{\overline{1, \rho}\}, \ \lambda = \overline{1, \gamma}, \ \sum_{i=1}^{\gamma} \rho_i = \rho \tag{9}$$

and any integer $t \in \{\overline{1, \rho}\}$ can be represented by a sum of the vector components, i.e., there exists a subset $\theta(t)$ of the set $\{\rho_1, \cdots, \rho_\gamma\}$ such that t is equal to the sum of all elements of $\theta(t)$.

Let $G_a(\rho, \nu)$ be a subclass of $G(\rho)$ with $\nu \in \{\overline{1, \rho - 1}\}$, $a \in \{0, 1\}$.

$$G_a(\rho, \nu) \triangleq \{g\colon \ g \in G(\rho); \ \rho_\lambda = 1, \ \lambda = \overline{1, \nu}; \ \rho_f \leq \nu + a, \ f = \overline{\nu + 1, \gamma}\}.$$

For a vector g of $G_a(\rho, \nu)$ we design a subset $\theta_1(t)$ as a variant of $\theta(t)$. The subset $\theta_1(t)$ consists of vector components with *consecutive indices* and at least one of two components ρ_ν, $\rho_{\nu+1}$ belongs to $\theta_1(t)$.

$$\theta_1(t) \triangleq \{\rho_\omega, \rho_{\omega+1}, \cdots, \rho_{\nu+\xi}\}.$$

If $t < \rho_{\nu+1}$ then $\omega = \nu + 1 - t$, $\xi = 0$. If $t \geq \rho_{\nu+1}$ then

$$\omega = \nu + 1 - t + d(\xi), \ d(j) \triangleq \rho_{\nu+1} + \cdots + \rho_{\nu+j}, \ i \leq \xi$$

and $\rho_{\nu+i} \in \theta_1(t)$ if and only if $d(i) \leq t$. By design, $t = \rho_\omega + \cdots + \rho_{\nu+\xi}$. For example, $g = (1, 1, 1, 3, 4, 4) \in G_1(14, 3)$. $\theta_1(9) = \{\rho_2, \rho_3, \rho_4, \rho_5\}$.

From matrices $A_1, \cdots, A_\tau$ we construct the matrix $\mathrm{DS}(A_1, \cdots, A_\tau)$.

$$\mathrm{DS}(A_1, \cdots, A_\tau) \triangleq \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_\tau \end{bmatrix} \tag{10}$$

where 0 is a zero matrix of a suitable size. (DS means "direct sum.")

Denote by W_q^m the parity check matrix of the $[f_{m,q}, f_{m,q} - m]_q 1$ Hamming code. Let 0^s be a zero matrix with s rows. The number of columns in 0^s is defined by context. Denote by $\overline{g}_\rho$ the vector of ρ ones, i.e., $\overline{g}_\rho \triangleq (1, \cdots, 1) \in G(\rho)$. We introduce matrices $D_\rho^{mR}(g)$ and $B^{mR}(b)$.

$$D_\rho^{mR}(g) \triangleq \left[\begin{array}{c|c|c} \mathrm{DS}(A_1, \cdots, A_\varkappa) & & 0^{mL} \\ \hline & 0^{m\Lambda} & \\ \hline 0^{mU} & & \mathrm{DS}(A_{\varkappa+1}, \cdots, A_\gamma) \end{array}\right] \tag{11}$$

where $g = (\rho_1, \cdots, \rho_\gamma) \in G(\rho)$; $\rho_1 + \cdots + \rho_\gamma = \rho$; A_λ is a parity check matrix of an $[N_\lambda, N_\lambda - m\rho_\lambda]_q \rho_\lambda$ code with covering radius ρ_λ, $\lambda = \overline{1, \gamma}$; if $\rho_\lambda = 1$ then $A_\lambda = W_q^m$; $\Lambda = R - \rho \geq 0$; $N(m) = N_1 + \cdots + N_\gamma$; $0^{m\Lambda}$ is the zero $m\Lambda \times N(m)$ matrix; if $g \neq \overline{g}_\rho$ and $g \in G_a(\rho, \nu)$ then $\varkappa = \nu$; if $g = \overline{g}_\rho$ or $g \in \{G(\rho) \backslash G_a(\rho, \nu)\}$ then $\varkappa = 0$ and the submatrices $\mathrm{DS}(A_1, \cdots, A_\varkappa)$, 0^{mL}, 0^{mU} are absent; $L = \rho_1 + \cdots + \rho_\varkappa$; 0^{mL} is the zero $mL \times (N_{\varkappa+1} + \cdots + N_\gamma)$ matrix; $U = \rho_{\varkappa+1} + \cdots + \rho_\gamma$; 0^{mU} is the zero $mU \times (N_1 + \cdots + N_\varkappa)$ matrix.

$$B^{mR}(b) \triangleq \begin{bmatrix} 0^{m(R-1)} \\ e_1 \cdots e_M \end{bmatrix}, \quad \text{for } b = *$$

$$B^{mR}(b) \triangleq \begin{bmatrix} 0^m \\ e_1 \ldots e_M \\ 0^m \end{bmatrix}, \quad \text{for } b = \#,\ R = 3$$

$$B^{mR}(b) \triangleq \begin{bmatrix} e_1 & e_2 & \cdots & e_M \\ e_1 b & e_2 b & \cdots & e_M b \\ e_1 b^2 & e_2 b^2 & \cdots & e_M b^2 \\ \vdots & \vdots & \ddots & \vdots \\ e_1 b^{R-1} & e_2 b^{R-1} & \cdots & e_M b^{R-1} \end{bmatrix}, \quad \text{for } b \in \mathrm{GF}(q^m) \tag{12}$$

where $M = q^m$; b^u is the uth power of b, $u = \overline{1, R-1}$; $e_j \in \mathrm{GF}(q^m)$, $j = \overline{1, M}$; $e_i \neq e_j$ if $i \neq j$, $i, j \in \{\overline{1, M}\}$, i.e., $\{e_1, \cdots, e_M\} = GF(q^m)$; 0^U is the zero $U \times M$ matrix. The element b is called an *indicator* of $B^{mR}(b)$.

A *starting code* V_0 is a linear $[Y, Y - s]_q R$, l_0 code of covering radius R with a parity check matrix Φ^s that is called a *starting matrix.*

$$\Phi^s = [\varphi_1 \cdots \varphi_Y], \quad \varphi_u \in F_q^s,\ u = \overline{1, Y}. \tag{13}$$

Let K_0 be an R, l_0-partition of the column set of the matrix Φ^s. Denote by $P^s(\varphi)$ a matrix of equal s-dimensional columns φ. $P^s(\varphi) = [\varphi \ldots \varphi]$. The number of columns in $P^s(\varphi)$ is defined by context.

The *base construction* forms a *new* $[n, n - r]_q R_V$, l *code* V with a covering radius R_V and with the parity check matrix

$$H^r = \left[\begin{array}{c|ccc} 0^s & P^s(\varphi_1) & \cdots & P^s(\varphi_Y) \\ D_\rho^{mR}(g) & B^{mR}(b_1) & \cdots & B^{mR}(b_Y) \end{array}\right], \quad r = s + mR \tag{14}$$

where $D_\rho^{mR}(g)$ is the $mR \times N(m)$ matrix; 0^s is the zero $s \times N(m)$ matrix; the matrices 0^s, $D_\rho^{mR}(g)$ are absent if $l_0 = R$; $P^s(\varphi_e)$ is an $s \times q^m$ matrix, φ_e is a column of Φ^s, $b_e \in GF(q^m) \cup \{*, \#\}$, $e = \overline{1, Y}$; assignment of indicators b_i depends on the partition K_0 as follows: if columns φ_i, φ_j of Φ^s belong to distinct subsets of K_0 then the inequality $b_i \neq b_j$ should be true, if columns φ_u, φ_t belong to the same subset of K_0 then we are free to assign the equality $b_u = b_t$ or the inequality $b_u \neq b_t$.

If the partition K_0 is trivial, all indicators b_i must be distinct, i.e., $b_i \neq b_j$ if $i \neq j$. We introduce notations for the matrix H^r.

$$\Psi^r \triangleq \begin{bmatrix} 0^s \\ D_\rho^{mR}(g) \end{bmatrix}$$

$$\Omega_u^r \triangleq \begin{bmatrix} P^s(\varphi_u) \\ B^{mR}(b_u) \end{bmatrix}, \quad u = \overline{1, Y}$$

$$\Xi_t^r \triangleq \begin{bmatrix} 0^{s+mt-m} \\ W_q^m \\ 0^{m(R-t)} \end{bmatrix}, \quad t = \overline{1, R}$$

$$H_\Omega^r \triangleq [\Omega_1^r \ldots \Omega_Y^r]$$

$$\beta \triangleq \bigcup_{i=1}^{Y} \{b_i\}, \quad h_0 \triangleq h(\Phi^s, l_0; K_0). \tag{15}$$

β is the set of all indicators of the submatrices $B^{mR}(b_i)$ in H^r.

By the construction in (14), the new code V has covering radius $R_V \geq R$. We give groups *Conditions* A and C *sufficient* for the *equality* $R_V = R$. Each condition includes a subset $\overline{B}$ of the set $\mathrm{GF}(q^m) \cup \{*, \#\}$. In Conditions A we have $\beta \subseteq \overline{B}$, $|\overline{B}| \geq h_0$. It permits us to assign distinct indicators $b_i \neq b_j$ if columns φ_i, φ_j belong to distinct subsets of K_0. For Conditions C we must use *all* elements of $\overline{B}$ as indicators b, i.e., $\beta = \overline{B}$. Here the inequalities $Y \geq |\overline{B}| \geq h_0$ are necessary. Conditions C define *constructions with a complete set of indicators* (CSI).

Sufficient Conditions A and C:

A1 $R \geq 2$, $\beta \subseteq \mathrm{GF}(q^m) \backslash \{0\}$, $q^m - 1 \geq h_0$, $\rho = R - l_0$, $g \in G_1(\rho, \nu)$, $q \geq 2$.

A2 $R \geq 2$, $\beta \subseteq \mathrm{GF}(q^m)$, $q^m \geq h_0$, $\rho = R - l_0$, $g = \overline{g}_\rho$, $q \geq 2$.

A3 $R \geq 2$, $\beta \subseteq \mathrm{GF}(q^m) \cup \{*\}$, $q^m + 1 \geq h_0$, $\rho = R$, $l_0 = 0$, $g = \overline{g}_\rho$, $q \geq 2$.

A4 $R \geq 2$, $\beta \subseteq \mathrm{GF}(q^m) \cup \{*\}$, $q^m + 1 \geq h_0$, $l_0 = R$, $q \geq 2$.

C1 $R \geq 2$, $\beta = \mathrm{GF}(q^m)$, $Y \geq q^m \geq h_0$, $l_0 = 0$, $\rho = R - 1$, $g = \overline{g}_\rho$, $q \geq 2$.

C2 $R \geq 3$, $\beta = \mathrm{GF}(q^m) \cup \{*\}$, $Y \geq q^m + 1 \geq h_0$, $l_0 = 0$, $\rho = R - 1$, $g = \overline{g}_\rho$, $q \geq 2$.

C3 $R = 4$, $\beta = \mathrm{GF}(q^m)$, $Y \geq q^m \geq h_0$, $l_0 = 0$, $\rho = 3$, $g = (2, 1)$, $q \geq 2$, $(q^m - 1)/3$ is not an integer.

C4 $R = 3$, $\beta = \mathrm{GF}(q^m) \cup \{\#\}$, $Y \geq q^m + 1 \geq h_0$, $l_0 = 0$, $\rho = 1$, $g = \overline{g}_1$, $q = 2^i$, $i \geq 1$.

Comment 3.1: In Conditions A, the parameter m is not bounded from above. So, we can get an *infinite family* of the *new* codes V. In Conditions C the parameter m is bounded from above but Conditions C improve parameters of the new codes and are useful in the beginning of an iterative process of code design (see Sections VI and VII). For all Conditions A and C the parameter m is bounded from below. The inequality $q^m + 1 \geq h_0$ is better than $q^m \geq h_0$ or $q^m - 1 \geq h_0$ in respect to restrictions for m. Besides, we can reduce h_0 treating V_0 as a code with $l_0^* < l_0$ (see Remark 2.2). On the other hand, we want to decrease the length n of the new code V by reducing of $N(m)$. This reduction is provided by increasing Λ. Conditions A1–A3, C1–C3, and C4 give $\Lambda = l_0$, $\Lambda = l_0 + 1$, and $\Lambda = l_0 + 2$, respectively. If $l_0 = R$ then $N(m) = 0$ (see A4). For decreasing of $N(m)$ vectors of $G_a(\rho, \nu)$ are preferential to $g = \overline{g}_\rho$.

Theorem 3.1: Let $q \geq 2$. Let a *starting* code V_0 be a $[Y, Y - s]_q R$, l_0 code of length Y, codimension s, and covering radius R

with a parity check matrix Φ^s of the form (13), let K_0 be an R, l_0-partition of the column set of the matrix Φ^s into h_0 subsets, and let a *new code* V be a code with a parity check matrix H^r of the form (14). If any of Sufficient Conditions A and C holds then the new code V is an $[n, n-r]_q R$, l code with the same covering radius R as the starting code and with the following values of length n, codimension r, and the parameter l:

$$n = Yq^m + N(m), \quad r = s + mR, \quad l \geq l_0 \tag{16}$$

where $N(m)$ is the number of columns of the matrix $D_\rho^{mR}(g)$ of (14).

Proof: The proof is based on Fact 1.1. Let λ_μ be the μth column of $D_\rho^{mR}(g)$. Denote by t_{ei} the ith column of $B^{mR}(b_e)$. We represent arbitrary column U of F_q^r as a linear combination of the form (see (6))

$$U \triangleq \begin{bmatrix} \pi \\ u \end{bmatrix} = \sum_{k=1}^{z} a_k \begin{bmatrix} \varphi_{j_k} \\ t_{j_k i_k} \end{bmatrix} + \sum_{t=1}^{Q} f_t \begin{bmatrix} 0^s \\ \lambda_{\mu_t} \end{bmatrix}, \; s + mR = r \tag{17}$$

where $\pi \in F_q^s$, $u \in F_q^{mR}$; $Q + z \leq R$; $R \geq z \geq l_0$; $Q \geq 0$; $a_k, f_t \in \mathrm{GF}^*(q)$, $k = \overline{1, z}$, $t = \overline{1, Q}$; under Conditions A for $p \neq c$ and $p, c \in \{\overline{1, z}\}$ we have $\varphi_{j_p} \neq \varphi_{j_c}$; under Conditions C the equality $\varphi_{j_p} = \varphi_{j_c}$ is possible for $p \neq c$; $u = (u_1, \cdots, u_R)^T$, $u_i \in \mathrm{GF}(q^m)$, $i = \overline{1, R}$.

We partition the rows of matrices $D_\rho^{mR}(g)$ and $B^{mR}(b_e)$ into R equal groups with numbers $1, \cdots, R$. Every group contains m rows. A column of a matrix is treated as an aggregate of R q-ary representations of elements of $\mathrm{GF}(q^m)$. The ith row group corresponds to the element u_i of u.

We consider Condition A1. Let in (17) $\pi \neq 0$ or $l_0 \geq 1$. Then $z \geq 1$. We use an index collection $J = \{j_1, \cdots, j_z\}$ such that $\pi = a_1\varphi_{j_1} + \cdots + a_z\varphi_{j_z}$ and all columns φ_{j_k} belong *distinct subsets* of K_0. Then $b_{j_p} \neq b_{j_c}$ if $p \neq c$ and $p, c \in \{\overline{1, z}\}$ (see the assignment of indicators b_i in (14)). We prove that for the index collection J the representation of (17) is possible, i.e., one can find convenient columns $t_{j_k i_k}$, λ_{μ_t}.

Let $E \triangleq z - \Lambda$. Then $\rho - E = R - z$. Using Definition 3.1 we consider a subset $\theta(E)$ for the vector g of the matrix $D_\rho^{mR}(g)$. Let $\theta(0) \triangleq \varnothing$. Let d^{mR} be the matrix consisting of the columns of $D_\rho^{mR}(g)$ that contain *all* submatrices A_ε such that ρ_ε does *not belong* to $\theta(E)$. $R - z$ rows groups of the matrix d^{mR} that contain the submatrices A_ε form a parity check matrix of a code with covering radius $R - z$. The remaining z row groups of d^{mR} contain zero rows. Denote by $\tau_1, \cdots, \tau_z$ numbers of the zero row groups. These groups correspond to the submatrix $0^{m\Lambda}$ of $D_\rho^{mR}(g)$ and submatrices A_f such that $\rho_f \in \theta(E)$. On the other hand, the zero row groups of d^{mR} correspond to elements $u_{\tau_1}, \cdots, u_{\tau_z}$ of the column u.

Denote by U^* a column of F_q^r coinciding with the column U on the elements π, $u_{\tau_1}, \cdots, u_{\tau_z}$. Let Π^r be the matrix obtained by addition s upper zero rows to the matrix d^{mR}. We design a subset $\theta(E)$ providing existence of indices $i_1, \cdots, i_z$ such that the 1st sum in (17) is equal to U^*. Then we can obtain the column U summing the column U^* with $Q \leq R - z$ columns of Π^r (see the second sum in (17)). "Locations" e_{i_k} of columns $t_{j_k i_k}$ in the 1st sum of (17) are a solution of the system

$$\sum_{k=1}^{z} a_k e_{i_k} b_{j_k}^{\tau_c - 1} = u_{\tau_c}, \quad c = \overline{1, z}, \quad b_{j_p} \neq b_{j_t} \text{ if } p \neq t. \tag{18}$$

Let Δ be the determinant of the system in (18). For the proof it is sufficient that $\Delta \neq 0$. If $\tau_i = \tau_1 + i - 1$, $i = \overline{2, z}$, then $\Delta \neq 0$ since we have the Vandermonde matrix with *consecutive degrees of distinct elements* of $\mathrm{GF}(q^m)$. The subset $\theta_1(E)$ provides the condition $\tau_i = \tau_1 + i - 1$. Zero-row groups of the matrix d^{mR} with numbers $L + i$, $i = \overline{1, \Lambda}$, $L = \rho_1 + \cdots + \rho_\varkappa$, correspond to the submatrix $0^{m\Lambda}$. For Condition A1 we have

$$L = \varkappa = \nu, \; \{\overline{\nu + 1, \nu + \Lambda}\} \subseteq \{\tau_1, \cdots, \tau_z\}.$$

If in $\theta_1(E)$ we have $\omega \leq \nu$ then $\tau_i = \omega + i - 1$, $i = \overline{1, z}$, where the numbers $\tau_1, \cdots, \tau_{\nu-\omega+1}$ correspond to elements $\rho_\omega, \cdots, \rho_\nu$ of $\theta_1(E)$, the numbers $\tau_{\nu-\omega+2}, \cdots, \tau_{\nu-\omega+1+\Lambda}$ correspond to the submatrix $0^{m\Lambda}$, and the numbers $\tau_{\nu-\omega+2+\Lambda}, \cdots, \tau_z$ correspond to elements $\rho_{\nu+1}, \cdots, \rho_{\nu+\xi}$ of $\theta_1(E)$. If $\omega = \nu + 1$ then $\tau_i = \nu + i$, $i = \overline{1, z}$, where the numbers $\tau_1, \cdots, \tau_\Lambda$ correspond to $0^{m\Lambda}$ and the numbers $\tau_{\Lambda+1}, \cdots, \tau_z$ correspond to elements $\rho_\omega, \cdots, \rho_{\nu+\xi}$ of $\theta_1(E)$.

Let $l_0 = 0$. Then $\Lambda = 0$ and $D_\rho^{mR}(g)$ is a parity check matrix of a code with covering radius R. For $\pi = 0$ we have in (17) $z = 0$, $Q \leq R$.

Conditions A2–A4 can be considered similarly to Condition A1. We consider some distinctive situations.

Condition A2: Let in (17) $b_{j_1} = 0$. We should take $\tau_1 = 1$. Since $g = \overline{g}_\rho$ we always can take $\theta(E) = \{\rho_1, \cdots, \rho_E\}$. We put $\tau_i = i$, $i = \overline{1, z}$, where the numbers $\tau_1, \cdots, \tau_\Lambda$ correspond to the submatrix $0^{m\Lambda}$ and the numbers $\tau_{\Lambda+1}, \cdots, \tau_z$ correspond to elements of the subset $\theta(E)$.

Condition A3: Here $E = z$. Let $b_{j_1} = 0$, $b_{j_z} = *$. We should put $\tau_1 = 1$, $\tau_z = R$. We take $\theta(z) = \{\rho_1, \cdots, \rho_{z-1}, \rho_\gamma\}$, $\tau_i = i$, $i = \overline{1, z-1}$, $\tau_z = R$. We add the column $(0, \cdots, 0, 1)^T$ to the Vandermonde matrix.

Condition A4: We have in (17) $z = R$, $Q = 0$.

In most cases, Conditions C can be proved similarly to Conditions A. We consider distinctive situations when an element $b_\xi \in \mathrm{GF}(q^m)$ is calculated. In the matrix H^r one always can find a submatrix $B^{mR}(b_\xi)$ with the indicators b_ξ since for Conditions C we have $\beta = \overline{B}$. The notation d^{mR}, U, U^*, $u = (u_1, \cdots, u_R)^T$, $\tau_1, \cdots, \tau_z$, Δ is the same as above.

Condition C1: Let in (17) $\pi = 0$, $u_1 \neq 0$. We take $b_\xi = u_2/u_1$, $z = 2$, $j_1 = j_2 = \xi$, $\{\tau_1, \tau_2\} = \{1, 2\}$ and

$$U^* = (\varphi_\xi, u_1, u_1 b_\xi, \cdots, u_1 b_\xi^{R-1})^T - (\varphi_\xi, 0, \cdots, 0)^T.$$

The matrix d^{mR} contains submatrices A_ε, $\varepsilon = \overline{2, R-1}$.

Condition C2: Let in (17), $\pi = a_1\varphi_{j_1}$, $b_{j_1} = *$, $u_1 \neq 0$. We take $b_\xi = u_2/u_1$, $z = 3$, $j_2 = j_3 = \xi$, $\{\tau_1, \tau_2, \tau_3\} = \{1, 2, R\}$, $a_1 w = u_R - u_1 b_\xi^{R-1}$ and

$$U^* = a_1(\varphi_{j_1}, 0, 0, \cdots, w)^T + (\varphi_\xi, u_1, u_1 b_\xi, \cdots, u_1 b_\xi^{R-1})^T - (\varphi_\xi, 0, 0, \cdots, 0)^T.$$

Condition C3: If $(q^m - 1)/3$ is not an integer then $b_1^3 \neq b_2^3$ for $b_1, b_2 \in GF(q^m)$, $b_1 \neq b_2$. Let in (17) $\pi = a_1\varphi_{j_1} + a_2\varphi_{j_2}$. We take $\{\tau_1, \tau_2\} = \{1, 4\}$, $z = 2$. Then $\Delta = b_{j_2}^3 - b_{j_1}^3 \neq 0$. The matrix d^{mR} contains a submatrix A_1. If $\pi = 0$, $u_1 \neq 0$, then $b_\xi^3 = u_4/u_1$, $z = 2$, $j_1 = j_2 = \xi$, $\{\tau_1, \tau_2\} = \{1, 4\}$, and

$$U^* = (\varphi_\xi, u_1, u_1 b_\xi, u_1 b_\xi^2, u_1 b_\xi^3)^T - (\varphi_\xi, 0, 0, 0, 0)^T.$$

Condition C4: In (17) let $\pi = a_1\varphi_{j_1}$.

i) $b_{j_1} \neq \#$, $u_2 \neq u_1 b_{j_1}$, $u_3 \neq u_1 b_{j_1}^2$. We put $Q = 0$, $z = 3$, $j_2 = j_3 = \xi$, and

$$\begin{aligned} b_\xi &= (u_3 + u_2 b_{j_1})/(u_2 + u_1 b_{j_1}) \neq b_{j_1} \\ a_1 t &= (u_1 b_\xi + u_2)/(b_{j_1} + b_\xi) \\ y &= (u_1 b_{j_1} + u_2)/(b_{j_1} + b_\xi). \end{aligned}$$

Then

$$U = a_1(\varphi_{j_1}, t, t b_{j_1}, t b_{j_1}^2)^T + (\varphi_\xi, y, y b_\xi, y b_\xi^2)^T + (\varphi_\xi, 0, 0, 0)^T.$$

ii) $b_{j_i} \neq \#$, $u_2 \neq u_1 b_{j_1}$, $u_3 = u_1 b_{j_1}^2$. We take $b_\xi = \#$, $w = u_2 + a_1 u_1 b_{j_1}$, $z = 3$, and

$$U = a_1(\varphi_{j_1}, u_1, u_1 b_{j_1}, u_1 b_{j_1}^2)^T + (\varphi_\xi, 0, w, 0)^T + (\varphi_\xi, 0, 0, 0)^T.$$

iii) $b_{j_1} = \#$, $u_1 \neq 0$. For $q = 2^i$ we always can find $b_\xi = (u_3/u_1)^{1/2}$.

$$U = a_1(\varphi_{j_1}, 0, w, 0)^T + (\varphi_\xi, u_1, u_1 b_\xi, u_1 b_\xi^2)^T + (\varphi_\xi, 0, 0, 0)^T$$
$$a_1 w = u_2 + u_1 b_\xi.$$

Now let in (17) $\pi = 0$, $u_1 \neq 0$. We take $b_\xi = u_2/u_1$, $w = u_3 + u_1 b_\xi^2$. If $w \neq 0$ then

$$U = (\varphi_\xi, u_1, u_1 b_\xi, u_1 b_\xi^2)^T + (\varphi_\xi, 0, 0, 0)^T + (0, 0, 0, w)^T.$$ □

IV. Estimates of l and $h(H^r, l)$ for New Codes

The estimates are important for an iterative process when the new $[n, n-r]_q R, l$ code V with $l \geq l_0$ is a starting code for the next step.

Saying "$l = a$" we treat V as an $[n, n-r]_q R, a$ code. The case $l = l_0$ is always possible. Cases $l > l_0$ are possible since the matrix H^r contains θ-groups (see (7)). When for H^r we design an R, l-partition with $l > l_0$ thereby we show that the new code V can be treated as an $[n, n-r]_q R, l$ code. Submatrices Ψ^r, Ω_u^r, Ξ_t^r, and H_Ω^r of H^r are defined in (15). Let H_A^r and $H_{\Omega A}^r$ (resp. H_C^r and $H_{\Omega C}^r$) be the matrix H^r and its submatrix H_Ω^r obtained under Sufficient Conditions A (resp., Conditions C). Let $\Omega(f)$ be the union of columns of all submatrices Ω_u^r having the same indicator $b_u = f$, where $u \in \{\overline{1, Y}\}$, $f \in \beta$. We assume that for each submatrix A_λ of the matrix $D_\rho^{mR}(g)$ in (11) there exists a ρ_λ, 0-partition K_λ with $h(A_\lambda, 0; K_\lambda) = \eta_\lambda$. We denote

$$\Sigma \triangleq \eta_1 + \cdots + \eta_\gamma.$$

A vector g of $G(\rho)$ has a component $\rho_\lambda = 1$. Hence Ψ^r contains a submatrix Ξ_t^r. To form θ-groups we treat columns of the submatrix W_q^m from Ξ_t^r as points of the projective geometry PG$(m-1, q)$ (see Section I). We use, e.g., the fact that three points of the same line are a 3-group.

We can also form θ-groups in submatrices Ω_u^r. For $j = \overline{1, M}$, $M = q^m$, we denote a column of Ω_u^r

$$S_{uj} \triangleq (\varphi_u, e_j, e_j b_u, \cdots, e_j b_u^{R-1})^T$$

if $b_u \in$ GF(q^m)

$$S_{uj} \triangleq (\varphi_u, 0, \cdots, 0, e_j)^T$$

if $b_u = *$

$$S_{uj} \triangleq (\varphi_u, 0, e_j, 0)^T$$

if $b_u = \#$. Further, $e_1 = 0$, $e_2 = 1$. Columns S_{u1}, S_{u2}, S_{uj}, $j \in \{\overline{3, M}\}$, are a 3-group.

Remark 4.1: From Proof of Theorem 3.1 we see that for all Conditions A and C we use in (17) at most one column from each submatrix Ξ_t^r. For Conditions A two columns from the same submatrix Ω_u^r do not occur in (17). For Conditions C the column $(\varphi_\xi, 0, \cdots, 0)^T$ of Ω_ξ^r can occur in (17) together with another column from the same submatrix Ω_ξ^r.

We consider Conditions A. Let $l = l_0$. To construct an R, l_0-partition K_A for H_A^r we partition columns of $H_{\Omega A}^r$ into $|\beta|$ subsets so that each subset is $\Omega(f)$. Columns of Ψ^r are partitioned into Σ subsets in accordance with the partitions K_λ of submatrices A_λ. We put $|\beta| = h_0$. Then

$$h(H_A^r, l_0) \leq h_0 + \Sigma, \qquad q \geq 2. \tag{19}$$

Now $l = R - p > l_0$, $p \geq 2$, $R \geq 3$, $q \geq 3$. For an $(R, R-p)$-partition K_A^p we use 3-groups with $\Gamma \leq 1$, $\Delta(X) = 3 - \Gamma$ (see Example 2.1). In J submatrices Ξ_t^r, Ω_u^r we form N 3-groups that do not mutually intersect. The maximal number of 3-groups with $\Gamma = 1$ is equal to J (see Remark 4.1). We put $N \geq \lceil (R-p-l_0+J)/3 \rceil$. Then $(R-p)-l_0 \leq 3N - J$. To construct K_A^p we form $3N$ additional subsets in the R, l_0-partition K_A so that each new subset consists of one column from the formed 3-groups. So

$$h(H_A^r, R-p) \leq h_0 + \Sigma + 3\left\lceil \frac{R-p-l_0+J}{3} \right\rceil, \qquad p \geq 2,\ R \geq 3,\ q \geq 3. \tag{20}$$

Let $l = R \geq 3$, $q \geq 3$. If in (7) $z = R-2$ or $R-1$ we put $\theta = 3$, $\Gamma = 1$ with $\Delta(X) = 2$ or $\Delta(X) = 1$ (see Example 2.1). For H_A^r we construct an R, R-partition K_{tA}^* with the following property Π: one always can add a 3-group to the representation (17) so that $\Gamma = 1$ and all columns of the 3-group belong to distinct subsets of this partition. We put $|\beta| = h_0$ and obtain $3h_0$ subsets in $H_{\Omega A}^r$ partitioning every union $\Omega(f)$ into three subsets so that the first (resp., second) subset consists of all columns of the form S_{u1} (resp., S_{u2}). Columns S_{u1}, S_{u2}, S_{uj}, $j \geq 3$, are a 3-group. So, if a column from any submatrix Ω_u^r is present in (17) (i.e., $z \geq 1$) then the property Π holds. In all we can use h_0 3-groups from submatrices Ω_u^r. For z of these groups we have $\Gamma \leq 1$, where z is taken from (17). For $h_0 - z$ groups we have $\Gamma = 0$. We take $\Delta(X) \geq 1$ for one group, $\Delta(X) \geq 2$ for $z-1$ groups, and $\Delta(X) = 3$ for $h_0 - z$ groups. The sum of all $\Delta(X)$ is no less than $3h_0 - z - 1 > R - z$ (since $h_0 \geq R$). So, the number of the formed 3-groups is sufficient to get $l = R$.

Let $l_0 \geq 1$. Then in (17) $z \geq 1$. We partition columns of Ψ^r into Σ subsets (as for K_A) and obtain an R, R-partition K_{1A}^* so that

$$h(H_A^r, R) \leq 3h_0 + \Sigma, \qquad \text{for } R > l_0 \geq 1,\ R \geq 3,\ q \geq 3. \tag{21}$$

Let $l_0 = 0$, $m \geq 2$. In (17) the case $z = 0$ is possible. Let $g = \overline{g}_\rho$. Then the submatrix Ψ^r consists of R submatrices Ξ_t^r. For an R, R-partition K_{2A}^* we form $f_{m-1,q} + 2$ subsets in each submatrix Ξ_t^r treating columns of its submatrix W_q^m as points of PG$(m-1, q)$ and passing $f_{m-1,q}$ lines from some point P. One subset consists of the point P, $f_{m-1,q}$ singleton subsets contain one point of each line, and the last subset contains the rest of the columns of Ξ_t^r. The property Π holds since three points of a line are a 3-group. So, for $g = \overline{g}_\rho$ we have

$$h(H_A^r, R) \leq 3h_0 + R(f_{m-1,q} + 2) \qquad \text{if } R \geq 3,\ l_0 = 0,\ m \geq 2,\ q \geq 3. \tag{22}$$

Now we consider Conditions C. Let $l = l_0 = 0$. To construct an R, 0-partition K_C for H_C^r we obtain $2|\beta|$ subsets in $H_{\Omega C}^r$ partitioning every union $\Omega(f)$ into two subsets so that the first subset consists of all columns of the form S_{u1}. In the submatrix Ψ^r we obtain Σ subsets. So

$$h(H_C^r, 0) \leq 2|\beta| + \Sigma, \quad q \geq 2. \tag{23}$$

Now for H_C^r we construct a partition $\overline{K}_C$ adding to K_C one subset consisting of the column S_{13}. For $q \geq 3$ columns S_{11}, S_{12}, S_{13} form a 3-group and belong to distinct subsets of $\overline{K}_C$. By Example 2.2, we have

$$h(H_C^r, Z) \leq 2|\beta| + \Sigma + 1,\ q \geq 3, Z = 1 \text{ if } R = 3,\quad Z = 2 \text{ if } R \geq 4. \tag{24}$$

Let $l = R \geq 3$, $q \geq 3$, $m \geq 2$. For Conditions C1, C2, C4 we design an R, R-partition K_C^*. Similarly to the partition K_{2A}^* we form $f_{m-1,q}+2$ subsets from columns each submatrix Ξ_t^r and obtain $3|\beta|$ subsets in $H_{\Omega C}^r$ partitioning every union $\Omega(f)$ into 3 subsets. The property Π holds. So,

$$h(H_C^r, R) \leq 3|\beta| + Q(f_{m-1,q}+2), \quad \text{for } R \geq 3,\ q \geq 3,\ m \geq 2 \quad (25)$$

where $Q = R-1$ for Conditions C1 and C2; $Q = 1$ for Condition C4.

V. Saturated Sets of Points in Projective Geometry

We consider saturated sets in accordance with Definition 1.1 and Fact 1.2. $(R-1)$-saturated sets of n points in the projective geometry $PG(r-1, q)$ with parameters (26)–(28) are given in [18], [19], [24], [28], [32, pp. 329, 332]. Values of q_0, q_1 are sufficiently large.

$$R = 2,\ q > q_0,\ r = 3,\ n = 2(q-1)/v + 2,\ v \mid (q-1). \quad (26)$$

$$R = 3,\ q > q_1,\ r = 4,\ n \leq 2(q-1)/s,\ s \mid (q-1). \quad (27)$$

$$\begin{gathered} R = 2,\ r = 3,\ n = \lfloor 0.5q + 2 \rfloor, \text{ for } q \geq 8, \\ n = 4, 6, 6, 6, 12, 14, 16, 18, 18, 22, 22, \text{ for} \\ q = 3, 4, 5, 7, 23, 29, 31, 37, 41, 43, 47, \text{ respectively.} \end{gathered} \quad (28)$$

Sets of (28) are complete caps [17], [18] (see [12], [14] for $q = 2$).

A *d-secant* of a point set H^r is a line L such that $|L \cap H^r| = d$ [18]. A point set H^r in PG $(r-1, q)$ is 1-saturated if any point of PG $(r-1, q)\backslash H^r$ lies on d-secant of H^r, where $d \geq 2$ [1], [32].

A 1-saturated set of $2q+1$ points in a geometry PG $(3, 2^i)$ is obtained in [4, p. 104]. We generalize the construction of [4] for odd q.

Theorem 5.1: We consider the projective geometry PG $(3, q)$ for even and odd q. Let $q \geq 4$. Let H^4 be the following set of $2q+1$ points:

$$H^4 = \left\{ \begin{array}{ccccc|c|ccccc} 1 & 1 & 1 & \cdots & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & a_3 & \cdots & a_q & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & a_3^2 & \cdots & a_q^2 & 0 & 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 1 & a_3 & \cdots & a_q \end{array} \right\} = \{\varphi_1 \ldots \varphi_{2q+1}\} \quad (29)$$

where $0, 1, a_i \in GF(q)$, $a_i \neq 0$, $a_i \neq 1$, $i = \overline{3, q}$; $a_i \neq a_j$ if $i \neq j$, i.e.

$$\{0, 1, a_3, \cdots, a_q\} = \mathrm{GF}(q)$$

φ_j is a point of PG $(3, q)$, $j = \overline{1, 2q+1}$.

Then H^4 is a 1-saturated set in PG $(3, q)$ for even and odd q. One can also treat H^4 as a parity check matrix of a $[2q+1, 2q-3]_q 2, 0$ code with $h(H^4, 0) \leq 5$ for even q and $h(H^4, 0) \leq 7$ for odd q.

Proof: Denote by π the plane of points of the form $(a, b, c, 0)^T$. Let S be a point set in π. Denote by P_S a point of $\pi \backslash S$. Let B be a collection of lines in π. Denote by $N(P_S, B)$ the number of lines of B containing the point P_S. Let $B(S)$ be the collection of all 2-secants of S. Then $N(P_S, B(S)) \leq \lfloor |S|/2 \rfloor$. Let

$$\begin{aligned} L^* &\triangleq \{\varphi_{q+2}, \cdots, \varphi_{2q+1}\} \\ L &\triangleq L^* \cup A \\ A &\triangleq (0, 0, 1, 0)^T \\ \Psi &\triangleq \mathrm{PG}(3, q) \backslash \{\pi \cup L\}. \end{aligned}$$

Then L is a line, A is a point, $A = \pi \cap L$. Let Q be a point of Ψ. Let W be a plane spanned by Q and L.

Let $q = 2k$ with integer k

$$\begin{aligned} C^* &\triangleq \{\varphi_1, \cdots, \varphi_{q+1}\} \\ C_1 &\triangleq \{\varphi_1, \cdots, \varphi_{q-1}\} \\ F &\triangleq B(C^*) \backslash B(C_1). \end{aligned}$$

Then $C = C^* \cup A$ is an oval. We have

$$N(P_C, B(C)) = k + 1$$

[18, p. 163],

$$N(P_C, B(C^*)) = k$$
$$N(P_C, F) \geq k - \lfloor |C_1|/2 \rfloor = 1.$$

So, all points of $\pi \backslash C$ lie on 2-secants of C^* belonging to F. Besides $A \in L$, L is a q-secant of L^*. The oval C has not 1-secants. The plane W intersects the plane π on a 2-secant of C containing A and a point X of C^*. The line through Q and X is a 2-secant ofH^4 since it intersects L^*. So all points of Ψ lie on 2-secants of H^4. By the construction, we form a 2, 0-partition for the *matrix* H^4 as follows:

$$C_1,\ \{\varphi_q\},\ \{\varphi_{q+1}\},\ \{\varphi_{q+2}\},\ \{\varphi_{q+3}, \cdots, \varphi_{2q+1}\}.$$

We need two last subsets to cover A.

Now $q = 2k + 1$

$$\begin{aligned} D^* &\triangleq \{\varphi_1, \cdots, \varphi_q\} \\ D_1 &\triangleq \{\varphi_1, \cdots, \varphi_{q-4}\} \\ D &= D^* \cup A \\ G &\triangleq B(D^*) \backslash B(D_1). \end{aligned}$$

D is an oval. We have $N(P_D, B(D)) \geq k$ [18, Table 8.2], $N(P_D, B(D^*)) \geq k-1$, $N(P_D, G) \geq k-1-\lfloor |D_1|/2 \rfloor = 1$. So all points of $\pi \backslash D$ lie on 2-secants of D^* belonging to G. In π there is *the only* 1-secant of D (J, say) containing A [18, Table 8.2]. The point φ_{q+1} lies on J. The plane W intersects π on a line M containing A and a point $X \in H^4$. M is either J (then $X = \varphi_{q+1}$) or a 2-secant of D (then $X \in D^*$). The line through Q and X is a 2-secant of H^4. A 2, 0-partition for the *matrix* H^4 has the form D_1, $\{\varphi_{q-3}\}$, $\{\varphi_{q-2}\}$, $\{\varphi_{q-1}\}$, $\{\varphi_q, \varphi_{q+1}\}$, $\{\varphi_{q+2}\}$, $\{\varphi_{q+3}, \cdots, \varphi_{2q+1}\}$. We do not need the 2-secant through φ_q and φ_{q+1}. □

An 1-saturated set of $3p$ points in a plane PG $(2, p^2)$ is obtained in [32, p. 326]. We construct a 1-saturated set of $3p-1$ points.

Theorem 5.2: Let $q = p^2 \geq 4$. Then in the plane PG $(2, p^2)$ a 1-saturated set H^3 of $3p-1$ points $\varphi_1, \cdots, \varphi_{3p-1}$ can be formed as follows (see (30) at the bottom of this page): where α is a primitive element of GF (p^2); $0, 1, c_i \in \mathrm{GF}(p)$, $c_i \neq 0, 1$, $i = \overline{3, p}$; $c_i \neq c_j$ if $i \neq j$, $i, j \in \{\overline{3, p}\}$, i.e., $\{0, 1, c_3, \cdots, c_p\} = \mathrm{GF}(p)$.

$$H^3 = \left\{ \begin{array}{ccccc|cccc|ccccc} 1 & 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & c_3 & \cdots & c_p & \alpha & c_3\alpha & \cdots & c_p\alpha & 0 & 1 & c_3 & \cdots & c_p \end{array} \right\} = \{\varphi_1 \ldots \varphi_{3p-1}\} \quad (30)$$

Proof: We denote $B = b_1\alpha + b_0$, $F = f_1\alpha + f_0$, $x = f_1 - b_1 f_0 b_0^{-1}$, where $B, F \in GF(p^2)$, $b_1, b_0, f_1, f_0 \in GF(p)$. Let L_1 (resp., L_2) be the line consisting of points of the form $(1, 0, B)^T$ (resp., $(0, 1, B)^T$). Let $B \neq 0$. Then

$$(1, B, F)^T = (1, 0, x\alpha)^T + B(0, 1, f_0 b_0^{-1})^T$$

if $b_0 \neq 0$

$$(1, B, F)^T = (1, 0, f_0)^T + B(0, 1, f_1 b_1^{-1})^T$$

if $b_0 = 0$, $b_1 \neq 0$. So, any point of $PG(2, q)\backslash\{L_1 \cup L_2\}$ lies on a 2-secant of H^3. Any point of the lines L_1, L_2 is also covered since $\varphi_i \in L_1$, $i < 2p$, $\varphi_j \in L_2$, $j \geq 2p$. □

VI. CODES WITH COVERING RADIUS $R = 2$

Notation 6.1: We denote by Construction AiR_ρ (resp., Construction CjR_ρ) the construction of Theorem 3.1 for the covering radius R with the vector $g = \overline{g}_\rho$ of $G(\rho)$ under Condition Ai (resp., Cj). For Condition A4 we put $\rho = 0$. For Construction AiR_ρ or CjR_ρ we have $n = Yq^m + \rho f_{m,q}$ in (16) and $\Sigma = \rho$ in (19)–(21), (23), (24). For Construction $C12_1$, e.g., it holds that $h(H_C^r, 0) \leq 2q^m + 1$ in (23).

Definition 6.1: A q-ary linear $[Y, Y-s]_q 2, 0$ code of codimension s, covering radius 2 with a parity check matrix Φ^s is called an AL2-code if for any *nonzero* q-ary s-dimensional column π of F_q^s there exists a linear combination $L(\pi)$ of two distinct columns of Φ^s such that $\pi = L(\pi)$. Both coefficients in the combination $L(\pi)$ are q-ary and nonzero.

The term "AL2" means "almost $l = 2$." Spherical 2, 2-capsules centered at vectors of an AL2-code C with length Y cover all vectors of $E_q^Y \backslash C$.

Definition 6.2: Let Φ^s be a parity check matrix of a q-ary AL2-code of codimension s. A partition of the column set of the matrix Φ^s into nonempty subsets is called an *AL2-partition* if for any *nonzero* q-ary s-dimensional column π of F_q^s there exists a linear combination $L(\pi)$ of two columns from *distinct subsets* such that $\pi = L(\pi)$. Both coefficients in the combination $L(\pi)$ are q-ary and nonzero. Denote by $h(\Phi^s, \text{AL2})$ the minimal number of subsets in an AL2-partition for a matrix Φ^s.

For a matrix H_A^r formed by Construction $A32_2$ the partition K_{2A}^* (see (22)) is an AL2-partition since any column of H_A^r is involved in a 3-group such that all its columns belong to distinct subsets of K_{2A}^*. So, Construction $A32_2$ for $m \geq 2$, $q \geq 3$, gives an AL2-code with

$$h(H_A^r, \text{AL2}) \leq 3h_0 + 2(f_{m-1,q} + 2).$$

The partition K_C^* (see (25)) is an AL2-partition for a matrix H_C^r formed by Construction $C12_1$. Hence Construction $C12_1$ for $m \geq 2$, $q \geq 3$, gives an AL2-code with

$$h(H_C^r, \text{AL2}) \leq 3q^m + f_{m-1,q} + 2.$$

Theorem 6.1: We introduce Construction AL2. Let $q \geq 2$. Let a *starting code* V_0 be a q-ary $[Y, Y-s]_q 2, 0$ AL2-code of length Y, codimension s, covering radius 2, with a parity check matrix Φ^s of (13), let K_0 be an AL2-partition of the column set of the matrix Φ^s into h_0 subsets, and let a *new code* V be a code with a parity check matrix

$$H_L^r = \begin{bmatrix} P_*^s(\varphi_1) & \vline & P^s(\varphi_1) & \cdots & P^s(\varphi_Y) \\ D_1^{2m}(g) & \vline & B^{2m}(b_1) & \cdots & B^{2m}(b_Y) \end{bmatrix}, \quad r = s + 2m, \; g = (1) \quad (31)$$

where H_L^r is a matrix H^r of (14) with a submatrix $P_*^s(\varphi_1)$ instead of 0^s; notations $D_1^{2m}(g)$, $P^s(\varphi_i)$, $B^{2m}(b_i)$, φ_i, β, and assignment of indicators b_i are the same as in (14), (15); $P_*^s(\varphi_1)$ is the $s \times f_{m,q}$ matrix of equal columns φ_1; $b_1 \neq *$; $\beta \subseteq GF(q^m) \cup \{*\}$; $q^m + 1 \geq h_0$. Then V is an $[n, n-(s+2m)]_q R, 0$ code with $R = 2$, $n = Yq^m + f_{m,q}$.

Proof: Let $U = (\pi, u_1, u_2)^T$, $\pi \in F_q^s$, $u_1, u_2 \in F_q^m$. If $\pi \neq 0$ then $\pi = a\varphi_j + c\varphi_k$, $a, c \in GF^*(q)$, $b_j \neq b_k$. If $b_j, b_k \neq *$ then

$$U = a(\varphi_j, x, xb_j)^T + c(\varphi_k, y, yb_k)^T$$

where $ax + cy = u_1$, $axb_j + cyb_k = u_2$. If $b_k = *$ then

$$U = a(\varphi_j, x, xb_j)^T + c(\varphi_k, 0, y)^T$$

where $ax = u_1$, $axb_j + cy = u_2$.

Let $\pi = 0$. If $u_2 \neq u_1 b_1$ then there exists a column ω of W_q^m with $d\omega = u_1 b_1 - u_2$, $d \in GF^*(q)$. We have

$$U = d(\varphi_1, u_1 d^{-1}, u_1 d^{-1} b_1)^T - d(\varphi_1, 0, \omega)^T.$$

If $u_2 = u_1 b_1$ then

$$U = (\varphi_1, u_1, u_1 b_1)^T - (\varphi_1), 0, 0)^T.$$

Example 6.1: From the $[11, 6]_3 2$ Golay code Construction $A32_2$, $m = \overline{3, 5}$, forms three codes. The first $[323, 312]_3 2$ code with $h(H_A^r, 0) \leq 13$, $h(H_A^r, \text{AL2}) \leq 45$, is V_0 for Construction $C12_1$, $m = 3$, and Construction AL2, $m \geq 4$. We obtained a family A_1 of $[n, n-r]_3 2$ codes with

$$\begin{aligned} &R = 2, \; q = 3, \; r = 2t + 1, \; \overline{\mu}_3(2, A_1) \approx 1.1816 < 2, \\ &n = 324 \times 3^{t-5} - 1, \quad \text{for } t = \overline{5, 7}, \\ &n = 323.5 \times 3^{t-5} - 0.5, \quad \text{for } t \geq 8. \end{aligned} \quad (32)$$

By (28), in $PG(2, 9)$ there is a 1-saturated set of six points. We treat the points as columns λ_i of F_9^3 and get a $[24, 18]_3 2$ code C by the method of [4, p. 103]. We consider columns of W_3^2 as elements a_j of $GF(9)$ and form a matrix

$$\Psi = [\Psi_1, \cdots, \Psi_6]$$

with

$$\Psi_i = [\varphi_{i1}, \cdots, \varphi_{i4}], \quad \varphi_{ij} = a_j \lambda_i, \; i = \overline{1, 6}, \; j = \overline{1, 4}.$$

Now in Ψ we change elements of $GF(9)$ by two elements of $GF(3)$ and obtain a parity check matrix Φ^6 of the code C. Since any three columns $\varphi_{ij}, \varphi_{ik}, \varphi_{ip}$ are a 3-group, C is an AL2-code. By design, $h(\Phi^6, 0) = 6$. Using C as V_0 for Construction $C12_1$, $m = 2$, and Construction AL2, $m \geq 3$, we obtain a family A_2 of $[n, n-r]_3 2$ codes with

$$R = 2, \; q = 3, \; r = 2t \geq 10, \; n = 24.5 \times 3^{t-3} - 0.5, \quad \overline{\mu}_3(2, A_2) \approx 1.65. \quad (33)$$

From the above we have entries in Table I for $R = 2$, $r = 5, 6$, $r \geq 10$. Besides, using the $[4, 1]_3 2$ repetition code Construction $A32_2$ forms a $[14, 9]_3 2$ code D and a $[44, 37]_3 2$ code. From D Construction $C12_1$ produces an $[130, 121]_3 2$ code. Using Hamming codes Construction DS gives an $[8, 4]_3 2$ AL2-code E. From E Construction AL2 forms a $[76, 68]_3 2$ code.

TABLE I
UPPER BOUNDS ON THE LENGTH FUNCTION $l(r, R; 3)$ FOR $R = 2, 3$

r	$R=2$	$R=3$	r	$R=2$	$R=3$	r	$R=2$	$R=3$
			9	130	37	17	8734	674
2	2	-	10	220	57	18	17860	971
3	4	3	11	323	76	19	26203	1295
4	8	5	12	661	107	20	53581	2024
5	11	8	13	971	170	21	78610	2915
6	24	11	14	1984	229	22	160744	3887
7	44	15	15	2915	323	23	235831	6074
8	76	24	16	5953	431	24	482233	8667

Example 6.2: Let $q \geq 7$. From the $[2q+1, 2q-3]_q 2$ code Σ of Theorem 5.1 with $h(H^4, 0) \leq 7 \leq q$ Construction $C12_1$, $m = 1$, forms a code with $n = 2q^2 + q + 1$, $r = 6$, $h(H_C^r, 0) \leq 2q + 1$. Then Construction $C12_1$, $m = 2$, gives an AL2-code with $r = 10$, $h(H_C^r, \text{AL2}) \leq 3q^2 + 3$. Using Construction AL2, $m \geq 3$, we get a family A_3 of $[n, n-r]_q 2$ codes with

$$R = 2,\ q \geq 7,\ r = 2t,\ \overline{\mu}_q(2, A_3) \approx 2 - 2q^{-1} + 0.5q^{-2} < 2,$$
$$n = 2f_{t,q} - f_{t-1,q}, \quad \text{for } t \geq 2,\ t \neq 4, 6, 7. \quad (34)$$

If $q = 4, 5$ then $h(H^4, 0) > q$. We use the code Σ as V_0 for Construction $A32_2$, $m = 2$, and get an AL2-code. Then we use Construction AL2 and obtain a family A_4 with

$$\overline{\mu}_q(2, A_4) \approx 2 - 2q^{-1} + 2.5q^{-2} < 2.$$

Example 6.3: V_0 is a $[Y(q), Y(q)-3]_q 2$ code, $Y(q) < q$ (see, e.g., Section V). Construction $A32_2$, $m = 2$, gives an AL2-code. From this code Construction AL2, $m \geq 2$, forms a family of $[n, n-r]_q 2$ codes with

$$R = 2,\ r = 2t + 1 \geq 11,$$
$$n = Y(q)q^{t-1} + 2f_{t-1,q} - f_{t-3,q} < n^*_{r,2} \quad (35)$$

where $Y(q) = 3p - 1$ if $q = p^2 \geq 4$, $Y(q) = \lfloor 0.5q + 2 \rfloor$ if $q \geq 8$, $Y(q) = 2(q-1)v^{-1} + 2$, $v \mid (q-1)$, if q is large (see (30), (28), (26)).

If $q = p^2 \geq 9$ we form an AL2-partition K for the matrix H^3 in (30) as follows: $\{\varphi_1\}, \{\varphi_2\}, \{\varphi_3, \cdots, \varphi_{2p-1}\}, \{\varphi_{2p}\}, \{\varphi_{2p+1}\}, \{\varphi_{2p+2}, \cdots, \varphi_{3p-1}\}$. Construction AL2, $m \geq 1$, produces a family of $[n, n-r]_q 2$ codes with

$$R = 2,\ q = p^2 \geq 9,\ r = 2t + 1 \geq 5,$$
$$n = (3p-1)q^{t-1} + f_{t-1,q} < n^*_{r,2}.$$

The parameters of codes of Examples 6.2 and 6.3 are better than ones in [32, formulas (5), (9)] and in the relation (5) of Section I.

VII. Codes with Covering Radius $R = 3$

We use Constructions $A23_1$, $A23_2$, $A43_0$, $C13_2$, $C43_1$ (see Notation 6.1). For Construction $C13_2$ we have $h(H_C^r, 1) \leq 2q^m + 3$ in (24). For $C43_1$ we have $h(H_C^r, 0) \leq 2q^m + 3$ in (23), $h(H_C^r, 3) \leq 3q^m + f_{m-1,q} + 5$ in (25).

Example 7.1: From the trivial $[3, 0]_3 3$ code Construction $C13_2$, $m = 1$, forms an $[11, 5]_3 3, 1$ code F. From F Construction $A23_2$, $m = \overline{2, 5}$, produces four codes. The first $[107, 95]_3 3, 3$ code is a code V_0 for Construction $A43_0$, $m \geq 4$. We obtained a family B_1 of $[n, n-r]_3 3, 3$ codes with

$$R = 3,\ q = 3,\ r = 3t,\ \overline{\mu}_3(3, B_1) \approx 3.074 < 4.5,$$
$$n = 12 \times 3^{t-2} - 1, \quad \text{for } t = \overline{4, 7},$$
$$n = 107 \times 3^{t-4}, \quad \text{for } t \geq 8. \quad (36)$$

From the $[4, 2]_3 1$ Hamming code and the $[11, 6]_3 2$ Golay code Construction DS forms a $[15, 8]_3 3, 1$ code G with $d = 3$ (see Example 2.2). From G (in the same way as for (36)) Construction $A23_2$, $m = \overline{3, 6}$, and Construction $A43_0$, $m \geq 4$, produce a $[431, 415]_3 3, 3$ code and a code family B_2 with

$$R = 3,\ q = 3,\ r = 3t + 1,\ \overline{\mu}_3(3, B_2) \approx 2.480 < 4.5,$$
$$n = 16 \times 3^{t-2} - 1, \quad \text{for } t = \overline{5, 8},$$
$$n = 431 \times 3^{t-5}, \quad \text{for } t \geq 9. \quad (37)$$

Similarly, using the $[13, 10]_3 1$ code instead of the $[4, 2]_3 1$ code we get a $[24, 16]_3 3, 1$ code, a $[674, 657]_3 3, 3$ code, and a code family B_3 with

$$R = 3,\ q = 3,\ r = 3t + 2,\ \overline{\mu}_3(3, B_3) \approx 3.162 < 4.5,$$
$$n = 25 \times 3^{t-2} - 1, \quad \text{for } t = \overline{5, 8},$$
$$n = 674 \times 3^{t-5}, \quad \text{for } t \geq 9. \quad (38)$$

From the above we have entries in Table I for $R = 3$, $r = 3, 6, 7, 8, 12$, $r \geq 15$. For $r = 4, 5, 9, 10, 13$ we use Hamming codes and codes with $R = 2$ in Construction DS. From the $[8, 3]_3 3, 2$ code of (8) Construction $A23_1$, $m = 2, 3$, forms a $[76, 65]_3 3, 3$ code and a $[229, 215]_3 3, 3$ code.

Example 7.2: Let $q \geq 4$. From the $[2q+1, 2q-3]_q 2$ code of Theorem 5.1 and the $[q+1, q-1]_q 1$ Hamming code Construction DS produces a $[3q+2, 3q-4]_q 3, 1$ code V_0 with $h_0 \leq 7 + 2 = 9$. To design a 3,1-partition for a parity check matrix of the code V_0 we add two subsets to the 2, 0-partition from Proof of Theorem 5.1. The first subset is $\{\varphi_{q+3}\}$, the second subset contains columns of a parity check matrix of the Hamming code. Columns $\varphi_{q+2}, \varphi_{q+3}, \varphi_{q+4}$ form a 3-group. Construction $A23_2$, $m = 2$, produces an $[L, L-12]_q 3, 3$ code, $L = 3q^3 + 2q^2 + 2q + 2$. Construction $A43_0$, $m \geq 3$, gives a family B_4 of $[n, n-r]_q 3, 3$ codes with

$$q \geq 4,\ r = 3t \geq 21,\ n = Lq^{t-4},$$
$$\overline{\mu}_q(3, B_4) \approx 4.5 \times (1 - q^{-1} + q^{-2}/3).$$

If $q = 2^i \geq 8$ then $h_0 \leq 5 + 2 = 7$. We iteratively use Construction $C43_1$ with $m = 1$, $m = 2$, and get a $[P, P-15]_q 3, 3$ code $P = 3q^4 + 2q^3 + q^2 + q + 1$, $h(H_C^{15}, 3) \leq 3q^2 + 6$. Then we use $A43_0$, $m \geq 3$, and obtain a family B_5 with

$$r = 3t,\ n = Pq^{t-5},\ \overline{\mu}_q(3, B_5) \approx 4.5 \times (1 - q^{-1} - 2q^{-2}/3).$$

Example 7.3: V_0 is an $[N, N-4]_q 3, 1$ code with $N \leq 2(q-1)/s$, $s \mid (q-1)$, $h_0 \leq N < q$, and large q (see (27)). Here $l_0 = 1$ since the code V_0 has $d = 3$ (see Example 2.2). Construction $A23_2$, $m = 1$, and Construction $A43_0$, $m \geq 2$, produce a family of $[n, n-r]_q 3, 3$ codes with

$$r = 3t + 1 \geq 16,\ s \mid (q-1),$$
$$n = 2q^{t-1}(q-1)/s + 2q^{t-2} < n^*_{r,3}.$$

VIII. Codes with Covering Radius $R \geq 4$

Remark 8.1: In Sections VI and VII for $R = 2, 3$ and *arbitrary* $q \geq 3$ we obtained families of $[n_{r,R}, n_{r,R} - r]_q R$ codes with $n_{r,R} < n^*_{r,R}$. Using codes of Sections VI and VII in Construction DS we can design *infinite* families of codes with $n_{r,R} < n^*_{r,R}$ for *arbitrary* $q \geq 3$, $R \geq 4$. To improve parameters of new families with $R \geq 4$ we can use Construction DS for a starting code and then apply Constructions AiR_ρ, CjR_ρ. In this case the parameter l of the starting code is near to R.

Example 8.1: Let $R = 2\tau \geq 4$, $q \geq 4$. Using τ the $[2q+1, 2q-3]_q 2$ codes of (29) Construction DS forms a $[2q\tau + \tau, 2q\tau - 3\tau]_q R, l$ code C with $l \leq R - 2$. To get an $(R, R-2)$-partition for a parity check matrix of C we form at most eight subsets in each matrix H^4 of (29) adding the subset $\{\varphi_{q+3}\}$ to the 2, 0-partition from Proof of Theorem 5.1. Columns $\varphi_{q+2}, \varphi_{q+3}, \varphi_{q+4}$ form a 3-group with $\Gamma \leq 2$ (see Example 2.1). Construction $A2R_2$ produces a $[\Pi, \Pi - s]_q R, R$ code with $s = 2R + mR$, $q^m \geq 8\tau$, $\Pi = (2q+1)\tau q^m + 2f_{m,q}$, $h(H_A^s, R) \leq 24\tau + 2$. Now Construction $A4R_0$, with $m = M$, $q^M > 24\tau + 2$, gives a family of $[n, n-r]_q R, R$ codes with $R = 2\tau$, $q \geq 4$, $r = R(m + M + 2)$, $n = (Rq^{m+1} + \tau q^m + 2f_{m,q})q^M < n^*_{r,R}$.

If $R = 2\tau + 1 \geq 5$ we can use the codes of Example 7.2 for DS.

Example 8.2: Let $R = 2\tau \geq 4$, $q \geq 8$, $\nu = \lfloor 0.5q + 2 \rfloor$. A parity check matrix of the $[\nu, \nu - 3]_q 2$ code B of (28) has a 4-group since it corresponds to a complete cap [18]. From τ the codes B Construction DS forms a $[\nu\tau, \nu\tau - 3\tau]_q R$, l_0 code V_0 with $l_0 \leq R - 3$ (see Example 2.1 and (7)). We use the trivial R, l_0-partition, Condition A1, $q^m - 1 \geq \nu\tau$, $g = (1, 2)$, and a parity check matrix of the code in (34) as submatrix $A^{m \cdot 2}$ of the matrix $D_\rho^{mR}(g)$. We get a family of $[n, n - r]_q R$ codes with

$$R = 2\tau,\ q \geq 8,\ r = 3\tau + mR,$$
$$n = \lfloor 0.5q + 2 \rfloor \tau q^m + 3f_{m,q} - f_{m-1,q} < n^*_{r,R}.$$

Example 8.3: Let $R = 2\tau \geq 4$, $q = p^2 \geq 9$, $\omega = 3p - 1$. Using (30) Construction DS forms a $[\omega\tau, \omega\tau - 3\tau]_q R$, R code. We can form an R, R-partition with $h_0 = 6\tau$ using the partition K of Example 6.3. Construction $A4R_0$ gives $[n, n - r]_q R$ codes with $r = 3\tau + mR$, $n = \omega\tau q^m < n^*_{r,R}$.

IX. Conclusion

Developing results from the author's recent paper [8] this work gives constructions of q-ary codes with arbitrary covering radius $R \geq 2$ and arbitrary basis $q \geq 2$. To construct an *infinite family* of codes with covering radius R it is sufficient to have a *starting* code with the same covering radius. Saturated sets of points in projective geometries over finite fields are effectively used as starting codes. New infinite families of covering codes obtained here have relatively "good" parameters. Described constructions seem to be quite clear and regular, and algorithms of "decoding" (for finding a codeword at distance at most R to any vector of the space) can be easy designed.

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