

The Berger-Wang formula for the Markovian joint spectral radius[☆]

Victor Kozyakin

*Institute for Information Transmission Problems
Russian Academy of Sciences
Bolshoj Karetny lane 19, Moscow 127994 GSP-4, Russia*

Abstract

The Berger-Wang formula establishes equality between the joint and generalized spectral radii of a set of matrices. For matrix products whose multipliers are applied not arbitrarily but in accordance with some Markovian law, there are also known analogs of the joint and generalized spectral radii. However, the known proofs of the Berger-Wang formula hardly can be directly applied in the case of Markovian products of matrices since they essentially rely on the arbitrariness of appearance of different matrices in the related matrix products. Nevertheless, as has been shown by X. Dai [1] the Berger-Wang formula is valid for the case of Markovian analogs of the joint and the generalized spectral radii too, although the proof in this case heavily exploits the more involved techniques of multiplicative ergodic theory. In the paper we propose a matrix theory construction allowing to deduce the Markovian analog of the Berger-Wang formula from the classical Berger-Wang formula.

Keywords: Infinite matrix products, Joint spectral radius, Generalized spectral radius, Berger-Wang formula, Topological Markov chains

PACS: 02.10.Ud, 02.10.Yn

2010 MSC: 15A18, 15A60, 60J10

1. Introduction

Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$ be the field of real or complex numbers, and $\mathcal{A} = \{A_1, A_2, \dots, A_N\}$ be a finite set of $(d \times d)$ -matrices with the elements from \mathbb{K} . Given a sub-multiplicative norm¹ $\|\cdot\|$ on $\mathbb{K}^{d \times d}$, the limit

$$\rho(\mathcal{A}) := \limsup_{n \rightarrow \infty} \rho_n(\mathcal{A}) \quad \left(= \lim_{n \rightarrow \infty} \rho_n(\mathcal{A}) = \inf_{n \geq 1} \rho_n(\mathcal{A}) \right), \quad (1)$$

[☆]Supported by the Russian Foundation for Basic Research, Project No. 13-01-13105.

Email address: kozyakin@iitp.ru (Victor Kozyakin)

URL: <http://www.iitp.ru/en/users/46.htm> (Victor Kozyakin)

¹A norm $\|\cdot\|$ on a space of linear operators is called sub-multiplicative if $\|AB\| \leq \|A\| \cdot \|B\|$ for any operators A and B .

where

$$\rho_n(\mathcal{A}) := \sup \left\{ \|A_{i_n} \cdots A_{i_1}\|^{1/n} : i_j \in \{1, 2, \dots, N\} \right\},$$

is called the *joint spectral radius* of the set of matrices \mathcal{A} [2]. This limit always exists and does not depend on the norm $\|\cdot\|$. If \mathcal{A} is a singleton set then (1) turns into the known Gelfand formula for the spectral radius of a linear operator. By this reason sometimes (1) is called the generalized Gelfand formula [3].

The *generalized spectral radius* of the set of matrices \mathcal{A} is the quantity defined by a similar to (1) formula in which instead of the norm is taken the spectral radius $\rho(\cdot)$ of the corresponding matrices [4, 5]:

$$\hat{\rho}(\mathcal{A}) := \limsup_{n \rightarrow \infty} \hat{\rho}_n(\mathcal{A}) \quad \left(= \sup_{n \geq 1} \hat{\rho}_n(\mathcal{A}) \right), \quad (2)$$

where

$$\hat{\rho}_n(\mathcal{A}) := \sup \left\{ \rho(A_{i_n} \cdots A_{i_1})^{1/n} : i_j \in \{1, 2, \dots, N\} \right\}.$$

As has been noted by M. Berger and Y. Wang [6] the quantities $\rho(\mathcal{A})$ and $\hat{\rho}(\mathcal{A})$ for bounded sets of matrices \mathcal{A} in fact coincide with each other:

$$\hat{\rho}(\mathcal{A}) = \rho(\mathcal{A}). \quad (3)$$

This fundamental formula has numerous applications in the theory of joint/generalized spectral radius. In particular, it implies the continuous dependence of the joint/generalized spectral radius on the set of matrices \mathcal{A} . Another important consequence of the *Berger-Wang formula* (3) is the fact that the quantities $\hat{\rho}_n(\mathcal{A})$ and $\rho_n(\mathcal{A})$, for any n , form the lower and upper bounds respectively for the joint/generalized spectral radius of the set of matrices \mathcal{A} :

$$\hat{\rho}_n(\mathcal{A}) \leq \hat{\rho}(\mathcal{A}) = \rho(\mathcal{A}) \leq \rho_n(\mathcal{A}), \quad (4)$$

which may serve as the basis for estimating the accuracy of computation of the joint/generalized spectral radius.

The characteristic feature of the definitions (1) and (2) is that the matrix products $A_{i_n} \cdots A_{i_1}$ in them correspond to all the possible sequences of indices (i_1, \dots, i_n) . Much more complicated is the situation when the matrix products $A_{i_n} \cdots A_{i_1}$ in formulae (1) and (2) are subjected to some additional restrictions, for example, some combinations of matrices in them are forbidden. Let us describe in more details a situation of the kind.

Given an $(N \times N)$ -matrix $\Omega = (\omega_{ij})$ with the elements from the binary set $\{0, 1\}$ then the finite sequence (i_1, \dots, i_n) taking the values in $\{1, 2, \dots, N\}$ will be called Ω -*admissible* if $\omega_{i_{j+1}i_j} = 1$ for all $1 \leq j \leq n-1$ and there exists $i_* \in \{1, 2, \dots, N\}$ such that $\omega_{i_*i_n} = 1$. Denote by $W_{N,\Omega}$ the set of all Ω -admissible sequences (i_1, \dots, i_n) . The matrix products $A_{i_n} \cdots A_{i_1}$ corresponding to the Ω -admissible sequences (i_1, \dots, i_n) will be called *Markovian* since the products of matrices of the kind arise naturally in the theory of matrix cocycles over the topological Markov chains, see, e.g., [7, 8].

Now, define analogs of formulae (1) and (2) for the Ω -admissible products of matrices. The limit

$$\rho(\mathcal{A}, \Omega) := \limsup_{n \rightarrow \infty} \rho_n(\mathcal{A}, \Omega), \quad (5)$$

where

$$\rho_n(\mathcal{A}, \Omega) := \sup \left\{ \|A_{i_n} \cdots A_{i_1}\|^{1/n} : (i_1, \dots, i_n) \in W_{N, \Omega} \right\},$$

will be called the *Markovian joint spectral radius* of the set of matrices \mathcal{A} defined by the *matrix of admissible transitions* Ω . If, for some n , the set of all Ω -admissible sequences (i_1, \dots, i_n) is empty then put $\rho_n(\mathcal{A}, \Omega) = 0$. In this case for each $k \geq n$ the sets of all Ω -admissible sequences (i_1, \dots, i_k) will be also empty, and hence $\rho(\mathcal{A}, \Omega) = 0$. The question whether there exist arbitrarily long Ω -admissible sequences can be resolved in a finite number of steps. In particular, the set $W_{N, \Omega}$ has arbitrarily long sequences if each column of the matrix Ω contains at least one nonzero element.

The limit (5) always exists and does not depend on the norm $\|\cdot\|$. To justify this, let us note that the quantity $\rho_n^n(\mathcal{A}, \Omega)$ is sub-multiplicative in n . Then, like in the case of formula (1), by the Fekete Lemma [9] (see also [10, Ch. 3, Sect. 1]) there exist $\lim_{n \rightarrow \infty} \rho_n(\mathcal{A}, \Omega)$ and $\inf_{n \geq 1} \rho_n(\mathcal{A}, \Omega)$, and both of them are equal to the limit (5):

$$\rho(\mathcal{A}, \Omega) := \limsup_{n \rightarrow \infty} \rho_n(\mathcal{A}, \Omega) = \lim_{n \rightarrow \infty} \rho_n(\mathcal{A}, \Omega) = \inf_{n \geq 1} \rho_n(\mathcal{A}, \Omega).$$

The quantity

$$\hat{\rho}(\mathcal{A}, \Omega) := \limsup_{n \rightarrow \infty} \hat{\rho}_n(\mathcal{A}, \Omega), \quad (6)$$

where

$$\hat{\rho}_n(\mathcal{A}, \Omega) := \sup \left\{ \rho(A_{i_n} \cdots A_{i_1})^{1/n} : (i_1, \dots, i_n) \in W_{N, \Omega} \right\},$$

will be called the *Markovian generalized spectral radius* of the set of matrices \mathcal{A} defined by the matrix of admissible transitions Ω . Here again we put $\hat{\rho}_n(\mathcal{A}, \Omega) = 0$ if the set of Ω -admissible sequences of indices (i_1, \dots, i_n) is empty. Like in the case of formula (2), the limit (6) coincides with $\sup_{n \geq 1} \hat{\rho}_n(\mathcal{A}, \Omega)$.

For the Markovian products of matrices there are valid the inequalities

$$\hat{\rho}_n(\mathcal{A}, \Omega) \leq \hat{\rho}(\mathcal{A}, \Omega) \leq \rho(\mathcal{A}, \Omega) \leq \rho_n(\mathcal{A}, \Omega), \quad (7)$$

similar to (4). However the question whether there is valid the equality

$$\hat{\rho}(\mathcal{A}, \Omega) = \rho(\mathcal{A}, \Omega), \quad (8)$$

similar to the Berger-Wang equality (3), becomes more complicated. The reason is that the known proofs [3, 6, 11–13] of the classical Berger-Wang formula (3) essentially use the fact that different matrices in the related matrix products can be multiplied in an arbitrary order. Impossibility to multiply matrices in an arbitrary order, in the Markovian case, requires to develop a different approach. The arising difficulties have

been overcome by X. Dai in [1] by using the techniques of the multiplicative ergodic theory. To formulate the related assertion we need some auxiliary definitions.

An Ω -admissible finite sequence (i_1, \dots, i_n) will be referred to as *periodically extendable* if $\omega_{i_1 i_n} = 1$. In general, not every Ω -admissible finite sequence can be periodically extended. However, if there are arbitrarily long Ω -admissible sequences then there exist also arbitrarily long Ω -admissible periodically extendable sequences. The set of all Ω -admissible periodically extendable sequences will be denoted by $W_{N,\Omega}^{(\text{per})}$.

Define the quantity

$$\hat{\rho}_n^{(\text{per})}(\mathcal{A}, \Omega) := \sup \left\{ \rho(A_{i_n} \cdots A_{i_1})^{1/n} : (i_1, \dots, i_n) \in W_{N,\Omega}^{(\text{per})} \right\},$$

and set²

$$\hat{\rho}^{(\text{per})}(\mathcal{A}, \Omega) := \limsup_{n \rightarrow \infty} \hat{\rho}_n^{(\text{per})}(\mathcal{A}, \Omega). \quad (9)$$

Dai's Theorem [1]. $\hat{\rho}^{(\text{per})}(\mathcal{A}, \Omega) = \rho(\mathcal{A}, \Omega)$.

Since $W_{N,\Omega}^{(\text{per})} \subseteq W_{N,\Omega}$ then $\hat{\rho}_n^{(\text{per})}(\mathcal{A}, \Omega) \leq \hat{\rho}_n(\mathcal{A}, \Omega)$ for each $n \geq 1$, and therefore $\hat{\rho}^{(\text{per})}(\mathcal{A}, \Omega) \leq \hat{\rho}(\mathcal{A}, \Omega)$. This last inequality together with (7) by Dai's Theorem then implies the Markovian analog (8) of the Berger-Wang formula (3).

The goal of the paper is to propose a matrix theory approach, called for brevity in the next section the " Ω -lift of the set of matrices \mathcal{A} ", which will allow to reduce consideration of the Markovian products of matrices to consideration of arbitrary (all possible) products of some auxiliary matrices. Thereby we will be able deduce the Markovian analog of the Berger-Wang formula from the classical Berger-Wang formula.

2. Ω -lift of the set of matrices \mathcal{A}

Recall that $\mathcal{A} = \{A_1, A_2, \dots, A_N\}$ is a finite set of $(d \times d)$ -matrices with the elements from the field $\mathbb{K} = \mathbb{R}, \mathbb{C}$ of real or complex numbers, and $\Omega = (\omega_{ij})$ is an $(N \times N)$ -matrix with the elements 0 or 1 which defines admissibility of consecutive co-multipliers in the products of matrices from \mathcal{A} .

For each $i = 1, 2, \dots, d$ define the $(N \times N)$ -matrix

$$\Omega_i = \omega_i^T \delta_i, \quad (10)$$

where the row-vectors ω_i and δ_i are of the form

$$\omega_i = \{\omega_{1i}, \omega_{2i}, \dots, \omega_{Ni}\}, \quad \delta_i = \{\delta_{1i}, \delta_{2i}, \dots, \delta_{Ni}\}, \quad (11)$$

with δ_{ij} being the Kronecker symbol, and the upper index " T " denotes transposition of a vector. Then all the nonzero elements of the matrix Ω_i , if any, belong to the i -th column of the matrix, and what is more, these elements coincide with the corresponding elements of the matrix Ω .

²Like in the definitions of the Markovian joint and generalized spectral radii we put $\hat{\rho}_n^{(\text{per})}(\mathcal{A}, \Omega) = 0$ if the set of all the periodically extendable sequences of length n is empty.

Now, define the set $\mathcal{A}_\Omega \subset \mathbb{K}^{Nd \times Nd}$ of block $(N \times N)$ -matrices $A^{(i)} := \Omega_i \otimes A_i$ whose elements are $(d \times d)$ -matrices, i.e.,

$$\mathcal{A}_\Omega := \{\Omega_i \otimes A_i : i = 1, 2, \dots, N\},$$

where \otimes stands for the Kronecker products of matrices [14]. The set of matrices \mathcal{A}_Ω will be called the Ω -lift of the set of matrices \mathcal{A} .

Example 1. Let $N = 4$ and

$$\Omega = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

Then

$$\begin{aligned} \Omega_1 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & \Omega_2 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \\ \Omega_3 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, & \Omega_4 &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} A^{(1)} &= \begin{pmatrix} A_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ A_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & A^{(2)} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & A_2 & 0 & 0 \\ 0 & A_2 & 0 & 0 \end{pmatrix}, \\ A^{(3)} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & A_3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & A_3 & 0 \end{pmatrix}, & A^{(4)} &= \begin{pmatrix} 0 & 0 & 0 & A_4 \\ 0 & 0 & 0 & A_4 \\ 0 & 0 & 0 & A_4 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

Given a sub-multiplicative norm $\|\cdot\|$ on the space of $(d \times d)$ -matrices $\mathbb{K}^{d \times d}$, define the norm $\| \! \| \cdot \| \! \|$ on $\mathbb{K}^{Nd \times Nd}$ by setting, for a block $(N \times N)$ -matrix $M = (m_{ij})$ with the elements $m_{ij} \in \mathbb{K}^{d \times d}$,

$$\| \! \| M \| \! \| := \max_{1 \leq i \leq N} \sum_{j=1}^N \| m_{ij} \|.$$

The norm $\| \! \| \cdot \| \! \|$ is also sub-multiplicative which is seen from the following inequalities for

block matrices $B = (b_{ij})$ and $C = (c_{ij})$:

$$\begin{aligned}
\|BC\| &= \max_{1 \leq i \leq N} \sum_{j=1}^N \left\| \sum_{k=1}^N b_{ik} c_{kj} \right\| \leq \max_{1 \leq i \leq N} \sum_{j=1}^N \sum_{k=1}^N \|b_{ik}\| \|c_{kj}\| \\
&= \max_{1 \leq i \leq N} \sum_{k=1}^N \sum_{j=1}^N \|b_{ik}\| \|c_{kj}\| = \max_{1 \leq i \leq N} \sum_{k=1}^N \left(\|b_{ik}\| \sum_{j=1}^N \|c_{kj}\| \right) \\
&\leq \max_{1 \leq i \leq N} \sum_{k=1}^N (\|b_{ik}\| \cdot \|C\|) = \left(\max_{1 \leq i \leq N} \sum_{k=1}^N \|b_{ik}\| \right) \cdot \|C\| = \|B\| \cdot \|C\|.
\end{aligned}$$

Theorem 1. *The quantities*

$$\begin{aligned}
\rho_n(\mathcal{A}_\Omega) &:= \sup \left\{ \|A^{(i_n)} \cdots A^{(i_1)}\|^{1/n} : A^{(i_k)} \in \mathcal{A}_\Omega \right\}, \\
\hat{\rho}_n(\mathcal{A}_\Omega) &:= \sup \left\{ \rho(A^{(i_n)} \cdots A^{(i_1)})^{1/n} : A^{(i_k)} \in \mathcal{A}_\Omega \right\}, \\
\rho_n(\mathcal{A}, \Omega) &:= \sup \left\{ \|A_{i_n} \cdots A_{i_1}\|^{1/n} : (i_1, \dots, i_n) \in W_{N, \Omega} \right\}, \\
\hat{\rho}_n^{(per)}(\mathcal{A}, \Omega) &:= \sup \left\{ \rho(A_{i_n} \cdots A_{i_1})^{1/n} : (i_1, \dots, i_n) \in W_{N, \Omega}^{(per)} \right\},
\end{aligned}$$

for each $n \geq 1$ satisfy the equalities

$$\rho_n(\mathcal{A}_\Omega) = \rho_n(\mathcal{A}, \Omega), \quad \hat{\rho}_n(\mathcal{A}_\Omega) = \hat{\rho}_n^{(per)}(\mathcal{A}, \Omega).$$

This theorem together with the Berger-Wang formula (3) implies the claim of Dai's Theorem: $\hat{\rho}^{(per)}(\mathcal{A}, \Omega) = \rho(\mathcal{A}, \Omega)$. As was mentioned by X. Dai in a private communication to the author, from Theorem 1 it follows also the Lipschitz continuity of the Markovian joint spectral radius with respect to \mathcal{A} . Specifically, there is valid the following Markovian analog of Wirth's Theorem from [15].

Corollary 1. *The map $\mathcal{A} \mapsto \rho(\mathcal{A}, \Omega)$ is locally Lipschitz continuous on the set of all \mathcal{A} for which the set of matrices \mathcal{A}_Ω is irreducible³.*

To prove Corollary 1 it suffices to note that the classical joint spectral radius is locally Lipschitz continuous on the variety of all the irreducible matrix sets [15, 16], and the map $\mathcal{A} \mapsto \mathcal{A}_\Omega$ is also Lipschitz continuous. This, by Theorem 1, implies the claim of Corollary 1.

Let us remark that irreducibility of the set of matrices \mathcal{A}_Ω depends not only on irreducibility of \mathcal{A} but also on the structure of the matrix of admissible transitions Ω . In general, neither irreducibility of \mathcal{A}_Ω follows from irreducibility of \mathcal{A} nor irreducibility of \mathcal{A}_Ω implies irreducibility of \mathcal{A} .

³A set of matrices is called irreducible if all the matrices from this set do not have common invariant subspaces except the trivial zero space and the whole space.

3. Proof of Theorem 1

To prove Theorem 1 we need the following

Lemma 1. *Given matrices Ω_{i_k} , $k = 1, 2, \dots, n$, then*

(i) *it is valid the representation:*

$$\Omega_{i_n} \Omega_{i_{n-1}} \cdots \Omega_{i_1} = (\omega_{i_n i_{n-1}} \cdots \omega_{i_2 i_1}) \cdot \boldsymbol{\omega}_{i_n}^T \boldsymbol{\delta}_{i_1}, \quad (12)$$

where the vectors $\boldsymbol{\omega}_i$ and $\boldsymbol{\delta}_i$ are of the form (11);

(ii) $\Omega_{i_n} \Omega_{i_{n-1}} \cdots \Omega_{i_1} \neq 0$ if and only if the sequence (i_1, \dots, i_n) is Ω -admissible, i.e.,

$$\omega_{i_{k+1} i_k} = 1, \quad k = 1, 2, \dots, n-1, \quad (13)$$

and there exists i_* such that $\omega_{i_* i_n} = 1$. In this case only the (i_*, i_1) -th elements of the matrix $\Omega_{i_n} \Omega_{i_{n-1}} \cdots \Omega_{i_1}$ satisfying $\omega_{i_* i_n} = 1$ are other than zero;

(iii) if condition (13) holds then the matrix $\Omega_{i_n} \Omega_{i_{n-1}} \cdots \Omega_{i_1}$ has a diagonal nonzero element if and only if the sequence (i_1, \dots, i_n) is Ω -admissible and periodically extendable, i.e., $\omega_{i_1 i_n} = 1$. In this case the nonzero diagonal element of the matrix $\Omega_{i_n} \Omega_{i_{n-1}} \cdots \Omega_{i_1}$ belongs to the i_1 -th row and column.

Proof. By using representations (10) for the matrices Ω_{i_k} we get

$$\Omega_{i_n} \Omega_{i_{n-1}} \cdots \Omega_{i_1} = \boldsymbol{\omega}_{i_n}^T \boldsymbol{\delta}_{i_n} \boldsymbol{\omega}_{i_{n-1}}^T \boldsymbol{\delta}_{i_{n-1}} \cdots \boldsymbol{\omega}_{i_1}^T \boldsymbol{\delta}_{i_1},$$

where the vectors $\boldsymbol{\omega}_{i_k}$ and $\boldsymbol{\delta}_{i_k}$ are defined by equalities (11). Here $\boldsymbol{\delta}_{i_{k+1}} \boldsymbol{\omega}_{i_k}^T = \omega_{i_{k+1} i_k}$ for each $k = 1, 2, \dots, n-1$ from which equality (12) follows. Assertion (i) is proved.

By (12) $\Omega_{i_n} \Omega_{i_{n-1}} \cdots \Omega_{i_1} \neq 0$ if and only if condition (13) holds and besides $\boldsymbol{\omega}_{i_n}^T \boldsymbol{\delta}_{i_1} \neq 0$. But $\boldsymbol{\omega}_{i_n}^T \boldsymbol{\delta}_{i_1} \neq 0$ if and only if there exists i_* such that $\omega_{i_* i_n} = 1$. From here assertion (ii) follows.

At last, if condition (13) holds then from assertion (ii) it follows that the matrix $\Omega_{i_n} \Omega_{i_{n-1}} \cdots \Omega_{i_1}$ has a diagonal nonzero element if and only if $\omega_{i_1 i_n} = 1$. This last equality together with condition (13) means that the sequence of indices (i_1, \dots, i_n) is not only Ω -admissible but also periodically extendable. Assertion (iii) is proved. \square

Let us illustrate the statement of Lemma 1 by an example.

Example 2. *Let Ω be the same matrix as in Example 1. Then $\Omega_4 \Omega_2 \Omega_1 = 0$ because of $\omega_{21} = 0$ and therefore the sequence of indices $(1, 2, 4)$ is not Ω -admissible.*

At the same time $\omega_{31} = 1$, $\omega_{43} = 1$, $\omega_{14} = 1$. Then the sequence of indices $(1, 3, 4)$ is Ω -admissible and periodically extendable. Hence by Lemma 1 $\Omega_4 \Omega_3 \Omega_1 \neq 0$ and the diagonal element of the matrix $\Omega_4 \Omega_3 \Omega_1$ belonging to the 1-st row and column is other than zero. This conclusion is supported by the following expression:

$$\Omega_4 \Omega_3 \Omega_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Now, start proving Theorem 1. Fix an n , and for a sequence of indices (i_1, \dots, i_n) consider the matrix $A^{(i_n)}A^{(i_{n-1})} \dots A^{(i_1)}$. By using the identity

$$(P \otimes Q)(R \otimes S) \equiv (PR) \otimes (QS)$$

which, in particular, holds for any matrices P, R of dimensions $N \times N$ and matrices Q, S of dimensions $d \times d$ (see [14, Lemma 4.2.10]) we obtain

$$A^{(i_n)}A^{(i_{n-1})} \dots A^{(i_1)} = \Omega_{i_n} \Omega_{i_{n-1}} \dots \Omega_{i_1} \otimes A_{i_n} A_{i_{n-1}} \dots A_{i_1}. \quad (14)$$

First prove the inequality

$$\rho_n(\mathcal{A}_\Omega) \leq \rho_n(\mathcal{A}, \Omega). \quad (15)$$

Let us note that the products of matrices $A^{(i_k)}$ for which $A^{(i_n)}A^{(i_{n-1})} \dots A^{(i_1)} = 0$ do not contribute to the computation of the quantity $\rho_n(\mathcal{A}_\Omega)$. Therefore it suffices to consider the case when $A^{(i_n)}A^{(i_{n-1})} \dots A^{(i_1)} \neq 0$. By (14) the latter may happen only in the case when $\Omega_{i_n} \Omega_{i_{n-1}} \dots \Omega_{i_1} \neq 0$. But, by assertions (i) and (ii) of Lemma 1, $\Omega_{i_n} \Omega_{i_{n-1}} \dots \Omega_{i_1} \neq 0$ if and only if the sequence of indices (i_1, \dots, i_n) is Ω -admissible, that is, $(i_1, \dots, i_n) \in W_{N, \Omega}$. In this case equality (14) by assertion (i) of Lemma 1 can be rewritten in the form:

$$A^{(i_n)}A^{(i_{n-1})} \dots A^{(i_1)} = (\omega_{i_n}^T \delta_{i_1}) \otimes A_{i_n} A_{i_{n-1}} \dots A_{i_1}, \quad (16)$$

where $(\omega_{i_n}^T \delta_{i_1}) \neq 0$. So, in the right-hand part of (16) stands a block matrix all the block elements of which belong to a single column and coincide with $A_{i_n} A_{i_{n-1}} \dots A_{i_1}$. From here and from the definition of the norm $\|\cdot\|$ it follows that

$$\|A^{(i_n)}A^{(i_{n-1})} \dots A^{(i_1)}\|^{1/n} = \|A_{i_n} A_{i_{n-1}} \dots A_{i_1}\|^{1/n} \leq \rho_n(\mathcal{A}, \Omega).$$

These last relations hold for any set of block matrices $A^{(i_k)} \in \mathcal{A}_\Omega$, $k = 1, 2, \dots, n$, satisfying $\|A^{(i_n)}A^{(i_{n-1})} \dots A^{(i_1)}\| \neq 0$, and therefore they imply (15).

Prove the inequality reciprocal to (15). As has been already proved the product of matrices $A^{(i_k)}$ can be represented in the form (14). Then by assertions (i) and (ii) of Lemma 1 for an Ω -admissible sequence (i_1, \dots, i_n) (i.e., such that $(i_1, \dots, i_n) \in W_{N, \Omega}$) holds equality (16), where $(\omega_{i_n}^T \delta_{i_1}) \neq 0$. Then

$$\|A_{i_n} A_{i_{n-1}} \dots A_{i_1}\|^{1/n} = \|A^{(i_n)}A^{(i_{n-1})} \dots A^{(i_1)}\|^{1/n} \leq \rho_n(\mathcal{A}_\Omega).$$

But since these last relations are valid for any Ω -admissible sequence (i_1, \dots, i_n) then

$$\rho_n(\mathcal{A}, \Omega) \leq \rho_n(\mathcal{A}_\Omega). \quad (17)$$

From (15) and (17) we obtain the first claim Theorem 1: $\rho_n(\mathcal{A}_\Omega) = \rho_n(\mathcal{A}, \Omega)$.

To prove the second claim of Theorem 1 let us observe that by assertion (i) of Lemma 1 the block matrix in the right-hand part of (14) may have nonzero elements only in one (block) column. In this case its spectral radius may be nonzero only in the case when it has nonzero diagonal element. By assertion (iii) of Lemma 1 the latter may happen if

and only if the sequence (i_1, \dots, i_n) is Ω -admissible and periodically extendable, that is, $(i_1, \dots, i_n) \in W_{N,\Omega}^{(\text{per})}$. By (16) in this case it is valid the following equality

$$\rho(A^{(i_n)} A^{(i_{n-1})} \dots A^{(i_1)}) = \rho(A_{i_n} A_{i_{n-1}} \dots A_{i_1}). \quad (18)$$

Let us note that equality (18) is valid for any set of block matrices $A^{(i_k)} \in \mathcal{A}_\Omega$ satisfying $\|A^{(i_n)} A^{(i_{n-1})} \dots A^{(i_1)}\| \neq 0$ as well as for any sequence $(i_1, \dots, i_n) \in W_{N,\Omega}$ and the corresponding set of matrices $A^{(i_k)} \in \mathcal{A}_\Omega$. From here, like in the case of the first claim of the theorem, we get

$$\hat{\rho}_n(\mathcal{A}_\Omega) = \hat{\rho}_n(\mathcal{A}, \Omega).$$

Theorem 1 is proved. \square

4. Concluding remarks

In this section we shall discuss alternative definitions of the Markovian joint spectral radius, and also the possibility to apply the techniques of Ω -lifts of the set of matrices \mathcal{A} in the cases not mentioned above.

4.1. Alternative definitions of the Markovian joint spectral radius

The definition (5) of the Markovian joint spectral radius is essentially depends on the fact over which sets of matrices the norms of the products of matrices are maximized during computing the quantities $\rho_n(\mathcal{A}, \Omega)$, i.e., on how is the notion of Ω -admissible sequences defined.

For example, we could treat a finite sequence (i_1, \dots, i_n) Ω -admissible if $\omega_{i_{j+1}i_j} = 1$ was carried out for all $1 \leq j \leq n-1$ (not assuming existence of i_* such that $\omega_{i_*i_n} = 1$). The set of all the finite sequences (i_1, \dots, i_n) , Ω -admissible in this sense, will be denoted by $W_{N,\Omega}^{(0)}$.

Also, we could treat a finite sequence (i_1, \dots, i_n) Ω -admissible if it was a starting interval of some infinite to the right sequence (i_1, \dots, i_n, \dots) for which the relations $\omega_{i_{j+1}i_j} = 1$ were valid for all $j \geq 1$. The set of all the finite sequences (i_1, \dots, i_n) , Ω -admissible in this sense, will be denoted by $W_{N,\Omega}^{(\infty)}$.

Clearly,

$$W_{N,\Omega}^{(\text{per})} \subseteq W_{N,\Omega}^{(\infty)} \subseteq W_{N,\Omega} \subseteq W_{N,\Omega}^{(0)}.$$

Then

$$\rho_n^{(\text{per})}(\mathcal{A}, \Omega) \leq \rho_n^{(\infty)}(\mathcal{A}, \Omega) \leq \rho_n(\mathcal{A}, \Omega) \leq \rho_n^{(0)}(\mathcal{A}, \Omega), \quad n \geq 1, \quad (19)$$

where

$$\begin{aligned} \rho_n^{(\text{per})}(\mathcal{A}, \Omega) &:= \sup \left\{ \|A_{i_n} \dots A_{i_1}\|^{1/n} : (i_1, \dots, i_n) \in W_{N,\Omega}^{(\text{per})} \right\}, \\ \rho_n^{(\infty)}(\mathcal{A}, \Omega) &:= \sup \left\{ \|A_{i_n} \dots A_{i_1}\|^{1/n} : (i_1, \dots, i_n) \in W_{N,\Omega}^{(\infty)} \right\}, \\ \rho_n^{(0)}(\mathcal{A}, \Omega) &:= \sup \left\{ \|A_{i_n} \dots A_{i_1}\|^{1/n} : (i_1, \dots, i_n) \in W_{N,\Omega}^{(0)} \right\}, \end{aligned}$$

and the quantity $\rho_n(\mathcal{A}, \Omega)$ has been defined already as

$$\rho_n(\mathcal{A}, \Omega) := \sup \left\{ \|A_{i_n} \cdots A_{i_1}\|^{1/n} : (i_1, \dots, i_n) \in W_{N, \Omega} \right\}.$$

Observe that $\|A_{i_n} A_{i_{n-1}} \cdots A_{i_1}\| \leq \alpha \|A_{i_{n-1}} \cdots A_{i_1}\|$ with $\alpha = \max_{1 \leq i \leq N} \|A_i\|$, and besides $(i_1, \dots, i_{n-1}) \in W_{N, \Omega}$ for any $(i_1, \dots, i_{n-1}, i_n) \in W_{N, \Omega}^{(0)}$. Then

$$\begin{aligned} \sup \left\{ \|A_{i_n} \cdots A_{i_1}\| : (i_1, \dots, i_n) \in W_{N, \Omega}^{(0)} \right\} \\ \leq \alpha \sup \left\{ \|A_{i_{n-1}} \cdots A_{i_1}\| : (i_1, \dots, i_{n-1}) \in W_{N, \Omega} \right\}, \quad n > 1, \end{aligned}$$

and therefore

$$\rho_n^{(0)}(\mathcal{A}, \Omega) \leq \alpha^{1/n} (\rho_{n-1}(\mathcal{A}, \Omega))^{(n-1)/n}, \quad n > 1. \quad (20)$$

From (19) and (20), and from the evident inequality $\hat{\rho}_n^{(\text{per})}(\mathcal{A}, \Omega) \leq \rho_n^{(\text{per})}(\mathcal{A}, \Omega)$ it follows that

$$\hat{\rho}_n^{(\text{per})}(\mathcal{A}, \Omega) \leq \rho_n^{(\text{per})}(\mathcal{A}, \Omega) \leq \rho_n^{(\infty)}(\mathcal{A}, \Omega) \leq \rho_n^{(0)}(\mathcal{A}, \Omega) \leq \alpha^{1/n} (\rho_{n-1}(\mathcal{A}, \Omega))^{(n-1)/n}$$

for any $n > 1$. Then, by passing to the upper limits in these last inequalities, by Dai's Theorem we obtain the following generalization of the definition of the Markovian joint spectral radius:

$$\begin{aligned} \rho(\mathcal{A}, \Omega) &:= \limsup_{n \rightarrow \infty} \rho_n(\mathcal{A}, \Omega) = \\ &= \limsup_{n \rightarrow \infty} \rho_n^{(\text{per})}(\mathcal{A}, \Omega) = \limsup_{n \rightarrow \infty} \rho_n^{(\infty)}(\mathcal{A}, \Omega) = \limsup_{n \rightarrow \infty} \rho_n^{(0)}(\mathcal{A}, \Omega). \end{aligned} \quad (21)$$

At last, by observing that quantities $(\rho_n(\mathcal{A}, \Omega))^n$, $(\rho_n^{(\text{per})}(\mathcal{A}, \Omega))^n$, $(\rho_n^{(\infty)}(\mathcal{A}, \Omega))^n$ and $(\rho_n^{(0)}(\mathcal{A}, \Omega))^n$ are sub-multiplicative in n , we by the Fekete Lemma [9], like in the case of formula (1), can replace in (21) all upper limits by limits:

$$\begin{aligned} \rho(\mathcal{A}, \Omega) &:= \lim_{n \rightarrow \infty} \rho_n(\mathcal{A}, \Omega) = \\ &= \lim_{n \rightarrow \infty} \rho_n^{(\text{per})}(\mathcal{A}, \Omega) = \lim_{n \rightarrow \infty} \rho_n^{(\infty)}(\mathcal{A}, \Omega) = \lim_{n \rightarrow \infty} \rho_n^{(0)}(\mathcal{A}, \Omega). \end{aligned}$$

4.2. Products of matrices defined by subshifts of finite type

In the symbolic dynamics the topological Markov chains are a particular case of the so-called *k-step topological Markov chains* [7, § 1.9] or the *subshifts of finite type* [8]. Here, by a *k-step topological Markov chain* or a *subshift of finite type* is meant the restriction of the shift operator in the space $\{1, 2, \dots, N\}^{\mathbb{Z}}$ (or $\{1, 2, \dots, N\}^{\mathbb{N}}$) to the set of all the sequences (i_1, \dots, i_n, \dots) in which admissibility of appearance of the symbol i_n is defined not by the immediately preceding symbol i_{n-1} , as in the usual topological Markov shifts, but by the sequence of k preceding symbols $(i_{n-k}, \dots, i_{n-1})$.

The *k-step topological Markov chains* may be treated as usual (1-step) Markov chains over the alphabet $\{1, 2, \dots, N\}^k$ [7, § 1.9]. Therefore the described above approach of Ω -lifts of sets of matrices is applicable for consideration of the joint/generalized spectral radius of sets of matrices in which admissibility of matrix products is defined in accordance with some *k-step topological Markov chains* or *subshifts of finite type*.

4.3. Another characteristics of matrix products

It seems the approach of Ω -lifts of sets of matrices can be applied to analyze some other Markovian analogs of formulae for computing the joint spectral radius, e.g., for computing the Markovian joint spectral radius via the trace of matrix products [17].

At the same time the specific feature of the proposed approach is that all the matrices from \mathcal{A}_Ω are degenerate, and among the products of matrices from \mathcal{A}_Ω may occur zero matrices. This makes improbable applying the proposed approach to study, for example, the Markovian analogs of the lower joint spectral radius [18–22] introduced in [23].

4.4. Infinite sets of matrices

In the paper we confined ourselves to consideration of only finite sets of matrices \mathcal{A} although neither in the definitions of the joint or generalized spectral radii (both usual and Markovian) nor in the classical Berger-Wang Theorem [6] the finiteness of the set of matrices \mathcal{A} is not required. In connection with this there arises a question about possibility of applying the techniques of Ω -lifts to infinite sets of matrices \mathcal{A} .

The main difficulty here is that for infinite sets of matrices \mathcal{A} the related matrix of admissible transitions Ω is also infinite. Then the matrices' from \mathcal{A}_Ω are also become infinite, that is, more adequately they should be treated as linear operators in some infinite-dimensional spaces. But, when passing to linear operators in infinite-dimensional spaces, the relationship between the joint and generalized spectral radii becomes more complicated [24–26].

In connection with this the applicability of the techniques of Ω -lifts for infinite sets of matrices \mathcal{A} , as well as the validity of Dai's Theorem in this case, remains to be an open question.

4.5. Barabanov's norms

A prominent tool in analysis of convergence of matrix products is the so-called Barabanov's norm [27] which, for a finite set of matrices \mathcal{A} is defined to be a vector norm $\|\cdot\|$ such that for some ρ (necessary coinciding with $\rho(\mathcal{A})$) holds the identity $\rho\|x\| \equiv \max_i \|A_i x\|$.

Known proofs of existence of the Barabanov norm, see, e.g., [27–29], essentially use the fact of arbitrariness of appearance of different matrices in the related matrix products. To the best of the author's knowledge, for the case of Markovian joint spectral radius there are no analogs of the Barabanov norm. At the same time, by using the procedure of Ω -lifts, for matrices from \mathcal{A}_Ω all the products are allowed! Hence for such matrices it is possible to define the norm of Barabanov.

To which extent this consideration might be useful for investigation of the Markovian joint spectral radius will show the future.

Acknowledgments

The author is indebted to Prof. Xiongping Dai for inspirational discussions and numerous fruitful comments.

References

- [1] X. Dai, Robust periodic stability implies uniform exponential stability of Markovian jump linear systems and random linear ordinary differential equations, ArXiv.org e-Print archive, to appear in J. Franklin Institute (Dec. 2013). arXiv:1307.4209, doi:10.1016/j.jfranklin.2014.01.010.
URL <http://arxiv.org/abs/1307.4209>
- [2] G.-C. Rota, G. Strang, A note on the joint spectral radius, *Nederl. Akad. Wetensch. Proc. Ser. A* 63 = *Indag. Math.* 22 (1960) 379–381.
- [3] M.-H. Shih, J.-W. Wu, C.-T. Pang, Asymptotic stability and generalized Gelfand spectral radius formula, *Linear Algebra Appl.* 252 (1997) 61–70. doi:10.1016/0024-3795(95)00592-7.
URL <http://www.sciencedirect.com/science/article/pii/0024379595005927>
- [4] I. Daubechies, J. C. Lagarias, Sets of matrices all infinite products of which converge, *Linear Algebra Appl.* 161 (1992) 227–263. doi:10.1016/0024-3795(92)90012-Y.
URL <http://www.sciencedirect.com/science/article/pii/002437959290012Y>
- [5] I. Daubechies, J. C. Lagarias, Corrigendum/addendum to: “Sets of matrices all infinite products of which converge” [*Linear Algebra Appl.* 161 (1992), 227–263; MR1142737 (93f:15006)], *Linear Algebra Appl.* 327 (1-3) (2001) 69–83. doi:10.1016/S0024-3795(00)00314-1.
URL <http://www.sciencedirect.com/science/article/pii/S0024379500003141>
- [6] M. A. Berger, Y. Wang, Bounded semigroups of matrices, *Linear Algebra Appl.* 166 (1992) 21–27. doi:10.1016/0024-3795(92)90267-E.
URL <http://www.sciencedirect.com/science/article/pii/002437959290267E>
- [7] A. Katok, B. Hasselblatt, Introduction to the modern theory of dynamical systems, Vol. 54 of *Encyclopedia of Mathematics and its Applications*, Cambridge University Press, Cambridge, 1995, with a supplementary chapter by Katok and Leonardo Mendoza.
- [8] B. P. Kitchens, *Symbolic dynamics*, Universitext, Springer-Verlag, Berlin, 1998, one-sided, two-sided and countable state Markov shifts. doi:10.1007/978-3-642-58822-8.
URL <http://link.springer.com/book/10.1007/978-3-642-58822-8>
- [9] M. Fekete, Über die Verteilung der Wurzeln bei gewissen algebraischen Gleichungen mit ganzzahligen Koeffizienten, *Math. Z.* 17 (1) (1923) 228–249. doi:10.1007/BF01504345.
URL <http://link.springer.com/article/10.1007/2FBF01504345>
- [10] G. Pólya, G. Szegő, *Problems and theorems in analysis. I*, Classics in Mathematics, Springer-Verlag, Berlin, 1998, series, integral calculus, theory of functions, Translated from the German by Dorothee Aeppli, Reprint of the 1978 English translation.
- [11] L. Elsner, The generalized spectral-radius theorem: an analytic-geometric proof, *Linear Algebra Appl.* 220 (1995) 151–159, proceedings of the Workshop “Nonnegative Matrices, Applications and Generalizations” and the Eighth Haifa Matrix Theory Conference (Haifa, 1993). doi:10.1016/0024-3795(93)00320-Y.
URL <http://www.sciencedirect.com/science/article/pii/002437959300320Y>
- [12] J. Bochi, Inequalities for numerical invariants of sets of matrices, *Linear Algebra Appl.* 368 (2003) 71–81. arXiv:math/0206128, doi:10.1016/S0024-3795(02)00658-4.
URL <http://www.sciencedirect.com/science/article/pii/S0024379502006584>
- [13] X. Dai, Extremal and Barabanov semi-norms of a semigroup generated by a bounded family of matrices, *J. Math. Anal. Appl.* 379 (2) (2011) 827–833. doi:10.1016/j.jmaa.2010.12.059.
URL <http://dx.doi.org/10.1016/j.jmaa.2010.12.059>
- [14] R. A. Horn, C. R. Johnson, *Topics in matrix analysis*, Cambridge University Press, Cambridge, 1994, corrected reprint of the 1991 original.
- [15] F. Wirth, The generalized spectral radius and extremal norms, *Linear Algebra Appl.* 342 (2002) 17–40. doi:10.1016/S0024-3795(01)00446-3.
URL <http://www.sciencedirect.com/science/article/pii/S0024379501004463>
- [16] V. Kozyakin, An explicit Lipschitz constant for the joint spectral radius, *Linear Algebra Appl.* 433 (1) (2010) 12–18. arXiv:0909.3170, doi:10.1016/j.laa.2010.01.028.
URL <http://www.sciencedirect.com/science/article/pii/S0024379510000418>
- [17] Q. Chen, X. Zhou, Characterization of joint spectral radius via trace, *Linear Algebra Appl.* 315 (1-3) (2000) 175–188. doi:10.1016/S0024-3795(00)00149-X.
URL <http://www.sciencedirect.com/science/article/pii/S002437950000149X>
- [18] T. Bousch, J. Mairesse, Asymptotic height optimization for topical IFS, Tetris heaps, and the finiteness conjecture, *J. Amer. Math. Soc.* 15 (1) (2002) 77–111 (electronic). doi:10.1090/S0894-0347-01-00378-2.
URL <http://www.ams.org/journals/jams/2002-15-01/S0894-0347-01-00378-2/>

- [19] V. Y. Protasov, R. M. Jungers, V. D. Blondel, Joint spectral characteristics of matrices: a conic programming approach, *SIAM J. Matrix Anal. Appl.* 31 (4) (2009/10) 2146–2162. doi:10.1137/090759896.
URL <http://epubs.siam.org/doi/abs/10.1137/090759896>
- [20] R. M. Jungers, On asymptotic properties of matrix semigroups with an invariant cone, *Linear Algebra Appl.* 437 (5) (2012) 1205–1214. doi:10.1016/j.laa.2012.04.006.
URL <http://www.sciencedirect.com/science/article/pii/S0024379512002704>
- [21] N. Guglielmi, V. Protasov, Exact computation of joint spectral characteristics of linear operators, *Found. Comput. Math.* 13 (1) (2013) 37–97. arXiv:1106.3755, doi:10.1007/s10208-012-9121-0.
URL <http://link.springer.com/article/10.1007%2Fs10208-012-9121-0>
- [22] J. Bochi, I. D. Morris, Continuity properties of the lower spectral radius, ArXiv.org e-Print archive (Sep. 2013). arXiv:1309.0319.
URL <http://arxiv.org/abs/1309.0319>
- [23] L. Gurvits, Stability of discrete linear inclusion, *Linear Algebra Appl.* 231 (1995) 47–85. doi:10.1016/0024-3795(95)90006-3.
URL <http://www.sciencedirect.com/science/article/pii/0024379595900063>
- [24] V. S. Shulman, Y. V. Turovskii, Joint spectral radius, operator semigroups, and a problem of W. Wojtyński, *J. Funct. Anal.* 177 (2) (2000) 383–441. doi:10.1006/jfan.2000.3640.
URL <http://www.sciencedirect.com/science/article/pii/S0022123600936401>
- [25] V. S. Shulman, Y. V. Turovskii, Formulae for joint spectral radii of sets of operators, *Studia Math.* 149 (1) (2002) 23–37. doi:10.4064/sm149-1-2.
URL <http://journals.impan.pl/cgi-bin/doi?sm149-1-2>
- [26] I. D. Morris, The generalised Berger-Wang formula and the spectral radius of linear cocycles, *J. Funct. Anal.* 262 (3) (2012) 811–824. arXiv:0906.2915, doi:10.1016/j.jfa.2011.09.021.
URL <http://www.sciencedirect.com/science/article/pii/S002212361100365X>
- [27] N. E. Barabanov, On the Lyapunov exponent of discrete inclusions. I, *Automat. Remote Control* 49 (2) (1988) 152–157, translation from *Avtomat. i Telemekh.* (1988), no. 2, 40–46.
- [28] R. Jungers, The joint spectral radius, Vol. 385 of *Lecture Notes in Control and Information Sciences*, Springer-Verlag, Berlin, 2009, Theory and applications. doi:10.1007/978-3-540-95980-9.
URL <http://www.springerlink.com/content/16h325u40254/#section=70183&page=1>
- [29] F. Wirth, On the structure of the set of extremal norms of a linear inclusion, in: *Proceedings of the 44th IEEE Conference on Decision and Control, and the European Control Conference 2005* Seville, Spain, December 12–15, 2005, pp. 3019–3024. doi:10.1109/CDC.2005.1582624.
URL <http://ieeexplore.ieee.org/xpl/articleDetails.jsp?arnumber=1582624>