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New upper bounds on the smallest size of a complete cap in the space $PG(3, q)$

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Dedicated to the memory of Professor Stefan Dodunekov

Abstract. In the projective spaces $PG(3, q)$, more than 370 new small complete caps are obtained for $q \leq 3109$, q prime. This implies new upper bounds on the smallest size $t_2(3, q)$ of a complete cap in $PG(3, q)$. From the new bounds it follows that the relation $t_2(3, q) < 6q$ holds for $q \in R$, where R is a set of 400 prime value in the interval $[2, 3109]$. The new upper bounds are obtained by finding new small complete caps in $PG(3, q)$ with the help of a computer search using FOP (Fixed Order of Points) algorithm.

Let $PG(3, q)$ be the projective 3-dimensional space over the Galois field F_q. An n-cap is a set of n points no three of which are collinear. An n-cap is called complete if it is not contained in an $(n + 1)$-cap of $PG(3, q)$.

In [10] the relationship among the theory of n-caps, coding theory and mathematical statistics is presented. In particular, a complete cap in $PG(3, q)$, points of which are treated as 4-dimensional q-ary columns, defines a parity check matrix of a q-ary linear code with codimension 4, Hamming distance 4, and covering radius 2. Caps in $PG(3, q)$ can be interpreted as linear almost maximum distance separable (AMDS) codes; see [10].

One of the most important problems in the study of projective spaces, which is also of interest in coding theory, is determining the smallest size $t_2(3, q)$ of a complete cap in $PG(3, q)$.

The trivial lower bound for the size of a complete cap in $PG(3, q)$ is $\sqrt{2}q$. The exact value of $t_2(3, q)$ is known only for $q \leq 7$; see [6, Table 3] and the references therein.

If q is even the trivial bound is substantially sharp. In the case q odd, the known constructions yield complete caps of size far from $\sqrt{2}q$; see the survey papers [6, 10] and the more recent works [1, 2, 7].

In 1959 Segre constructed complete caps in $PG(3,q)$ of size $3q+2$, consisting of three conics plus two points; see [13]. In 1995, Pambianco and Storme announced the following result, cited in [10, Table 4.8] and proven in [6, Theorem 6]: for q even, we have

$$t_2(3,q) \leq 2q + t_2(2,q),$$

where $t_2(2,q)$ is the smallest size of a complete arc in $PG(2,q)$. In [6, Theorem 6] this result was proven in a generalized form: for q even, if a complete k-arc in $PG(2,q)$ exists, then there exists a complete $2q+k$ cap in $PG(3,q)$. For q odd, infinite families of complete caps of size approximately $q^2/2$ and $q^2/3$ [8], see also references in [6], sharing many points with an elliptic quadric, have been obtained by generalizing a classical method by Segre and Lombardo Radice for constructing complete plane caps.

In general the problem of determining $t_2(3,q)$ remains still an open hard question.

According to the survey papers [6, 10], the smallest known complete caps in three-dimensional spaces of arbitrarily high odd order q have size approximately $q\sqrt{q/2}$. These constructions were presented by Pellegrino in 1998 [12]: unfortunately, as pointed out in [4], there are some gaps in the proofs and some counterexamples can be easily found also for small q.

In [6, Theorem 8, Table 7] it is proven that

$$t_2(3,q) \leq 3q \text{ for } 2 \leq q \leq 17, \quad t_2(3,q) < 4q \text{ for } 2 \leq q \leq 89.$$

In [4] the authors show the existence of complete caps in the affine three-dimensional space $AG(3,q)$ of size at most $10q$ for $q \leq 30000$, q odd prime, using ideas similar to those contained in [12] and a computer-assisted search. These results imply that

$$t_2(3,q) \leq 11q, \quad q \leq 30000, q \text{ odd}.$$

In this work we apply the same techniques that were used in [3] in the search for small complete arcs in $PG(2,q)$ to obtain estimations on the upper bounds on $t_2(3,q)$. In the following let $\bar{t}_2(3,q)$ be smallest size of a *known* complete cap in $PG(3,q)$ and R be a set of 400 prime values in the interval [2, 3109] (see Table 1).

Let

$$\theta_{up}(q) = \frac{1}{2\ln(0.01 \cdot q)} + 0.605, \qquad \phi_{up}(q) = \frac{1}{\ln 0.2 \cdot q} + 0.706. \tag{1}$$

Our results can be summarized in the following theorem.

Theorem 1. *In $PG(3,q)$ it holds that*

$$t_2(3,q) \leq \begin{cases} 5q & for \quad q \leq 223, \ q \in R \\ 6q & for \quad q \leq 3109, \ q \in R \\ q \ln q & for \quad 127 \leq q \leq 3109, \ q \in R \\ q \ln^{0.9} q & for \quad 787 \leq q \leq 3109, \ q \in R \\ \theta_{up}(q) q \ln q & for \quad 101 \leq q \leq 3109, \ q \in R \\ q \ln^{\phi_{up}(q)} q & for \quad q \leq 3109, \ q \in R \end{cases}. \qquad (2)$$

Complete caps with sizes satisfying (2) *can be obtained by Algorithm FOP (Fixed Order of Points) with lexicographical orders of points represented in homogenous coordinates.*

The **FOP algorithm** we used in our search is described in [3]: suppose that the points of $PG(3,q)$ are ordered as

$$A_1, A_2, \ldots, A_{q^3+q^2+q+1}.$$

Consider the empty set as root of the search and let $K^{(j)}$ be the partial solution obtained in the j-th step, as extension of the root. We put

$$K^{(0)} = \emptyset, \quad K^{(1)} = \{A_1\}, \quad K^{(2)} = \{A_1, A_2\}, \quad K^{(j+1)} = K^{(j)} \cup \{A_{m(j)}\},$$

$$m(j) = \min\{i \in [m(j-1), q^3+q^2+q+1] \mid \nexists P, Q \in K^{(j)} \ : \ A_i, P, Q \text{ are collinear}\},$$

i.e. $m(j)$ is the minimum index such that the corresponding point is not saturated by $K^{(j)}$. The process ends when a complete cap is obtained.

In our search we used the Lexicographical order. Let q be prime and let the elements of the field $\mathbb{F}_q = \{0, 1, \ldots, q-1\}$ be treated as integers modulo q. Let the points A_i of $PG(3,q)$ be represented in homogenous coordinates so that $A_i = (x_0^{(i)}, x_1^{(i)}, x_2^{(i)}, x_3^{(i)})$, $x_j^{(i)} \in \mathbb{F}_q$, where the leftmost non-zero element is 1. For A_i, we put $i = x_0^{(i)} q^3 + x_1^{(i)} q^2 + x_2^{(i)} q + x_3^{(i)}$. So, the homogenous coordinates of a point A_i coincide with i written in q-ary.

Let

$$d(q) = \frac{t_2(3,q)}{q \ln q}, \qquad c(q) = \frac{\ln(t_2(3,q)/q)}{\ln \ln q}. \qquad (3)$$

In Figure 1 the values $\frac{\bar{t}_2(3,q)}{q}$, $q \in R$, are shown; Figure 2 presents the comparison between the values $\bar{t}_2(3,q)$ and $6q$, $q \in R$. The functions $\phi_{up}(q)$ and $\theta_{up}(q)$ (see (1)) are good approximations of the functions $c(q)$ and $d(q)$. In Figures 3 and 4 the functions $c(q)$ and $d(q)$ (see (3)) are shown, respectively.

Table 1 summarizes the results obtained. New results are written in bold font.

Figure 1: The values $\frac{\bar{t}_2(3,q)}{q}$, $q \in R$

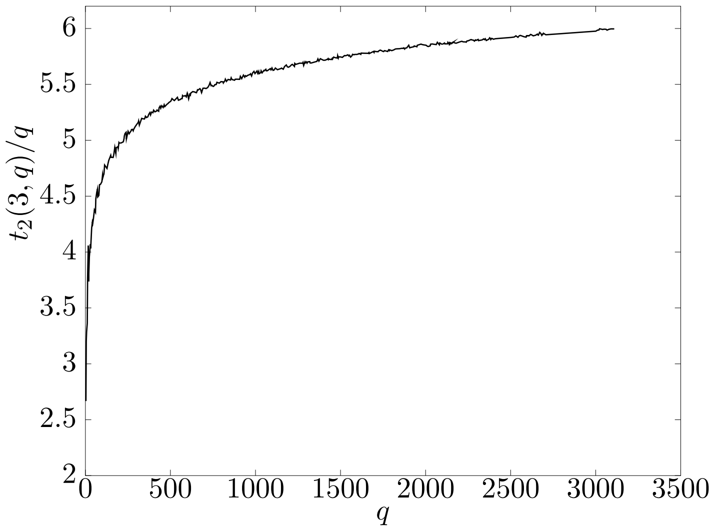

Figure 2: The values $\bar{t}_2(3,q)$ and $6q$, $q \in R$

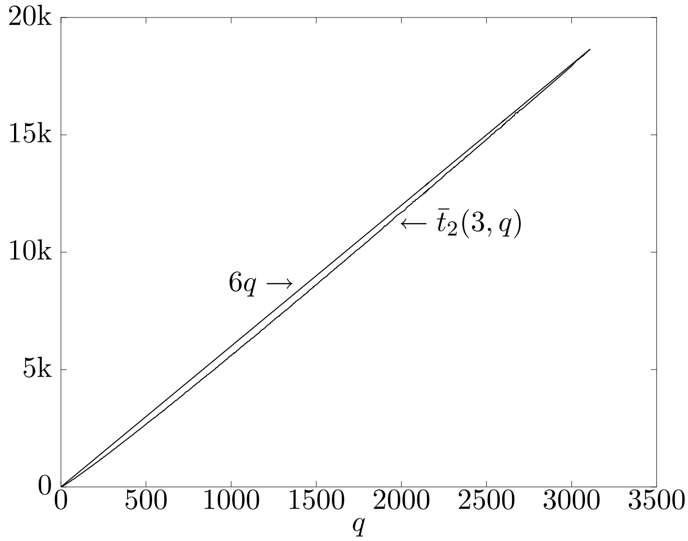

Figure 3: The function $c(q)$ (the bottom curve) and $\phi_{up}(q)$ (the top curve), $q \in R$

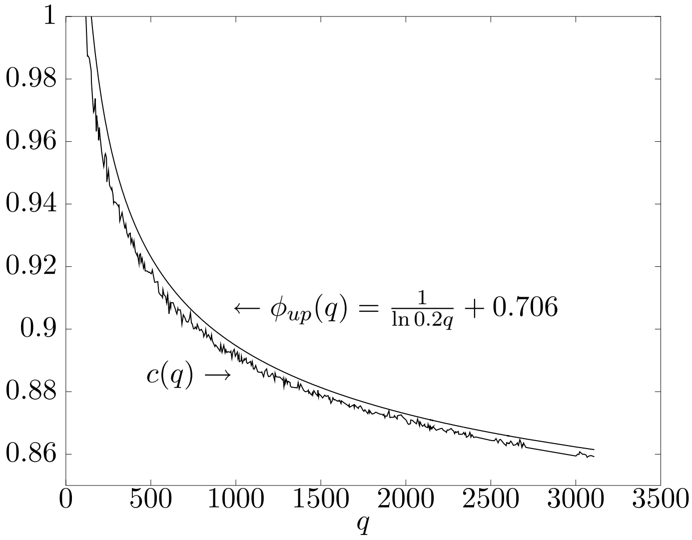

Figure 4: The functions $d(q)$ (the bottom curve) and $\theta_{up}(q)$ (the top curve), $q \in R$

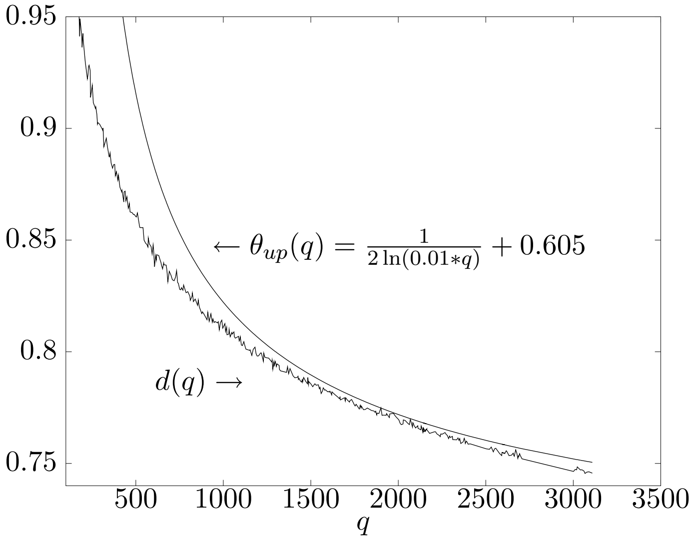

Table 1: $t_2(3,q)$, $q \in R$

q	$\bar{t}_2(3,q)$	q	$\bar{t}_2(3,q)$	q	$\bar{t}_2(3,q)$	q	$\bar{t}_2(3,q)$	q	$\bar{t}_2(3,q)$	q	$\bar{t}_2(3,q)$	q	$\bar{t}_2(3,q)$	q	$\bar{t}_2(3,q)$
3	8	5	16	7	23	11	37	13	49	17	69	19	71	23	91
29	118	31	125	37	156	41	175	43	183	47	202	53	232	59	257
61	273	67	304	71	324	73	328	79	356	83	382	89	410	97	449
101	**474**	103	**481**	107	**502**	109	**512**	113	**540**	127	**603**	131	**626**	137	**660**
139	**671**	149	**725**	151	**732**	157	**761**	163	**790**	167	**814**	173	**854**	179	**874**
181	**893**	191	**944**	193	**951**	197	**981**	199	**990**	211	**1050**	223	**1112**	227	**1138**
229	**1154**	233	**1179**	239	**1212**	241	**1208**	251	**1275**	257	**1300**	263	**1334**	269	**1368**
271	**1381**	277	**1412**	281	**1429**	283	**1442**	293	**1501**	307	**1582**	311	**1605**	313	**1619**
317	**1628**	331	**1720**	337	**1750**	347	**1801**	349	**1813**	353	**1841**	359	**1868**	367	**1912**
373	**1952**	379	**1989**	383	**2002**	389	**2043**	397	**2083**	401	**2113**	409	**2149**	419	**2205**
421	**2219**	431	**2267**	433	**2291**	439	**2329**	443	**2342**	449	**2384**	457	**2419**	461	**2447**
463	**2462**	467	**2476**	479	**2549**	487	**2596**	491	**2621**	499	**2668**	503	**2692**	509	**2735**
521	**2791**	523	**2801**	541	**2913**	547	**2931**	557	**2988**	563	**3024**	569	**3057**	571	**3084**
577	**3112**	587	**3170**	593	**3195**	599	**3248**	601	**3254**	607	**3260**	613	**3317**	617	**3334**
619	**3356**	631	**3430**	641	**3482**	643	**3493**	647	**3512**	653	**3543**	659	**3592**	661	**3601**
673	**3676**	677	**3693**	683	**3706**	691	**3777**	701	**3832**	709	**3873**	719	**3934**	727	**3992**
733	**4044**	739	**4056**	743	**4080**	751	**4117**	757	**4154**	761	**4184**	769	**4229**	773	**4266**
787	**4337**	797	**4403**	809	**4468**	811	**4471**	821	**4544**	823	**4565**	827	**4578**	829	**4582**
839	**4652**	853	**4725**	857	**4764**	859	**4769**	863	**4784**	877	**4856**	881	**4886**	883	**4897**
887	**4920**	907	**5030**	911	**5076**	919	**5102**	929	**5187**	937	**5214**	941	**5249**	947	**5277**
953	**5318**	967	**5404**	971	**5433**	977	**5447**	983	**5507**	991	**5566**	997	**5580**	1009	**5671**
1013	**5672**	1019	**5724**	1021	**5713**	1031	**5779**	1033	**5807**	1039	**5822**	1049	**5900**	1051	**5908**
1061	**5979**	1063	**5976**	1069	**6010**	1087	**6132**	1091	**6142**	1093	**6158**	1097	**6195**	1103	**6205**
1109	**6242**	1117	**6307**	1123	**6332**	1129	**6358**	1151	**6495**	1153	**6510**	1163	**6554**	1171	**6628**
1181	**6701**	1187	**6716**	1193	**6745**	1201	**6797**	1213	**6886**	1217	**6912**	1223	**6935**	1229	**6957**
1231	**7002**	1237	**7025**	1249	**7100**	1259	**7164**	1277	**7265**	1279	**7286**	1283	**7274**	1289	**7343**
1291	**7341**	1297	**7397**	1301	**7411**	1303	**7415**	1307	**7452**	1319	**7523**	1321	**7519**	1327	**7552**
1361	**7766**	1367	**7830**	1373	**7838**	1381	**7913**	1399	**7997**	1409	**8054**	1423	**8147**	1427	**8178**
1429	**8176**	1433	**8207**	1439	**8236**	1447	**8281**	1451	**8318**	1453	**8320**	1459	**8381**	1471	**8441**
1481	**8495**	1483	**8538**	1487	**8530**	1489	**8557**	1493	**8579**	1499	**8613**	1511	**8676**	1523	**8769**
1531	**8814**	1543	**8895**	1549	**8914**	1553	**8947**	1559	**8955**	1567	**9020**	1571	**9064**	1579	**9103**
1583	**9131**	1597	**9212**	1601	**9240**	1607	**9276**	1609	**9294**	1613	**9315**	1619	**9340**	1621	**9369**
1627	**9403**	1637	**9460**	1657	**9575**	1663	**9608**	1667	**9643**	1669	**9640**	1693	**9810**	1697	**9818**
1699	**9849**	1709	**9902**	1721	**9969**	1723	**9990**	1733	**10063**	1741	**10101**	1747	**10127**	1753	**10157**
1759	**10206**	1777	**10297**	1783	**10358**	1787	**10368**	1789	**10393**	1801	**10450**	1811	**10517**	1823	**10605**
1831	**10653**	1847	**10748**	1861	**10837**	1867	**10869**	1871	**10902**	1873	**10906**	1877	**10918**	1879	**10943**
1889	**11006**	1901	**11122**	1907	**11113**	1913	**11154**	1931	**11272**	1933	**11296**	1949	**11376**	1951	**11413**
1973	**11563**	1979	**11591**	1987	**11632**	2003	**11718**	2011	**11746**	2017	**11778**	2027	**11848**	2029	**11890**
2039	**11952**	2053	**12027**	2063	**12082**	2069	**12159**	2081	**12184**	2083	**12229**	2087	**12247**	2089	**12250**
2099	**12297**	2111	**12392**	2113	**12387**	2129	**12482**	2131	**12523**	2137	**12536**	2141	**12544**	2143	**12588**
2153	**12629**	2161	**12699**	2141	**12544**	2143	**12588**	2153	**12629**	2161	**12699**	2179	**12787**	2203	**12957**
2207	**12985**	2213	**13025**	2221	**13062**	2237	**13153**	2239	**13161**	2243	**13208**	2251	**13259**	2267	**13378**
2269	**13388**	2273	**13405**	2281	**13448**	2287	**13464**	2293	**13524**	2297	**13539**	2309	**13628**	2311	**13616**
2333	**13760**	2339	**13811**	2341	**13829**	2347	**13849**	2351	**13890**	2357	**13890**	2371	**13997**	2377	**14066**
2381	**14073**	2383	**14063**	2389	**14123**	2393	**14142**	2399	**14164**	2411	**14247**	2503	**14818**	2521	**14934**
2531	**15021**	2539	**15045**	2543	**15058**	2549	**15121**	2551	**15140**	2557	**15184**	2579	**15283**	2591	**15345**
2593	**15387**	2609	**15485**	2593	**15387**	2609	**15485**	2617	**15563**	2621	**15582**	2633	**15625**	2647	**15723**
2657	**15809**	2659	**15805**	2663	**15820**	2671	**15934**	2677	**15945**	2683	**15933**	2687	**15985**	2689	**15992**
2693	**16045**	2699	**16062**	2707	**16090**	3001	**17935**	3011	**18024**	3019	**18073**	3023	**18132**	3037	**18200**
3041	**18220**	3049	**18276**	3061	**18344**	3067	**18357**	3079	**18454**	3083	**18480**	3089	**18524**	3109	**18643**

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