

Matrix products with constraints on the sliding block relative frequencies of different factors[☆]

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Abstract

One of fundamental results of the theory of joint/generalized spectral radius, the Berger–Wang theorem, establishes equality between the joint and generalized spectral radii of a set of matrices. Generalization of this theorem on products of matrices whose factors are applied not arbitrarily but are subjected to some constraints is connected with essential difficulties since known proofs of the Berger–Wang theorem rely on the arbitrariness of appearance of different matrices in the related matrix products. Recently, X. Dai [1] proved an analog of the Berger–Wang theorem for the case when factors in matrix products are formed by some Markov law.

We extend the concepts of joint and generalized spectral radii to products of matrices with constraints on the sliding block relative frequencies of occurrences of different factors, and prove an analog of the Berger–Wang theorem for this case.

Keywords: Infinite matrix products, Joint spectral radius, Generalized spectral radius, Berger–Wang formula, Topological Markov chains, Symbolic sequences

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1. Introduction

In various theoretic and applied problems there arise the matrix products

$$M_{\alpha_0} M_{\alpha_1} \cdots M_{\alpha_n}, \quad n \geq 0, \tag{1}$$

where M_i are $(d \times d)$ -matrices from a finite collection $\mathcal{M} = \{M_1, M_2, \dots, M_r\}$ with the elements from the field $\mathbb{K} = \mathbb{R}, \mathbb{C}$ of real or complex numbers, and $\alpha = (\alpha_n)$ is a sequence of symbols from the set $\mathcal{A} = \{1, 2, \dots, r\}$, see, e.g., [1–6] and the bibliography therein.

The question about the rate of growth of the matrix products (1) is relatively simple (at least theoretically) in edge cases, for example, when the sequence $\alpha = (\alpha_n)$ is periodic

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or in (1) all possible sequences $\alpha = (\alpha_n)$ with symbols from $\mathcal{A} = \{1, 2, \dots, r\}$ are considered. In the latter case the question about the rate of growth of all possible matrix products (1) can be answered in terms of the so-called joint or generalized spectral radii of the set of matrices \mathcal{M} [7–10].

In intermediate situations, when the sequences $\alpha = (\alpha_n)$ in (1) are relatively complex but still not totally arbitrary, the question about the rate of growth of the matrix products (1) becomes highly nontrivial and its resolution essentially depends on the structure of the index sequences $\alpha = (\alpha_n)$. In particular, one of the key features of the index sequence in (1) is the frequency p_i of occurrences of the index i .

As a rule, the frequency p_i is defined as the limit of the relative frequencies $p_{i,n}$ of occurrences of the symbol i among the first n members of a sequence. However, one should bear in mind that such a definition of frequency is rather subtle and not very constructive from the point of view of mathematical formalism. Already in the situation when one deals with a single sequence $\alpha = (\alpha_n)$, this definition is not enough informative since it does not answer the question of how often different symbols appear in intermediate, not tending to infinity, finite segments of a sequence. This definition becomes all the less satisfactory in situations when one should deal with not a single sequence but with an infinite collection of such sequences. The principal deficiency here is that the definition of frequency given above does not withstand transition to the limit with respect to different sequences which results in substantial theoretical and conceptual difficulties.

To give “good” properties (from the point of view of ability to use mathematical methods) to determination of frequency one often needs either to require some kind of uniformity of convergence of the relative frequencies $p_{i,n}$ to p_i or to treat appearance of the related symbols in a sequence as a realization of events generated by some random or deterministic ergodic system, and so on. In the latter case, for the analysis of behavior of the matrix products (1), the so-called multiplicative ergodic theorem (in a probabilistic or a measure theoretic setting) is most often used, see, e.g., [11, 12]. However, under such an approach one needs to impose rather strong restrictions on the laws of forming the index sequences $\alpha = (\alpha_n)$ which are often difficult to verify or confirm in applications. In all such cases the arising families of the index sequences and of the related matrix products can be rather attractive from the purely mathematical point of view but their description becomes less and less constructive. In applications, it often leads to emergence of an essential conceptual gap or of some kind strained interpretation at use of the related objects and constructions.

At the present, there exists an extensive literature devoted to studying the frequency properties of various classes of symbolical sequences, see, e.g., [13–18] and the references therein. The problem on constructive determination of classes of symbolic sequences with prescribed frequency properties and of the related matrix products has got lesser attention. Only recently, in [1, 19] the area of applicability of the methods of joint/generalized spectral radius was expanded on the matrix products (1) in which the index sequences $\alpha = (\alpha_n)$ are described by the topological Markov chains. In connection with this, the aim of the article is to describe a class of r -symbolic sequences for which it is possible to constructively define “approximate relative frequencies” of symbols in the *sliding ℓ -blocks*, that is, in each block of ℓ consecutive symbols. Such a description will be given in terms of the ℓ -step topological Markov chains (subshifts of finite type) which allows to use the approach developed in [1, 19] to define the joint/generalized spectral radius

for the products of matrices whose factors are applied not arbitrarily but are subjected to some frequency constraints.

Outline the structure of the work. In Section 2, we recall basic definitions related to symbolic sequences. Then we introduce the concept of symbolic sequences with constraints on the sliding ℓ -block relative frequencies, and study their principal properties. In Theorem 1 we show that the left shift on the set of symbolic sequences with constraints on the sliding ℓ -block relative frequencies is an ℓ -step topological Markov chain. In spite of its obviousness, Theorem 1 is of principal importance in further constructions. Properties of symbolic sequences with constraints on the sliding block relative frequencies obtained in Section 2 are used in Section 3 to analyze the rate of growth of norms of matrix products. For this purpose, we recall the basics of the theory of joint/generalized spectral radius and the results from [1, 19] related to the generalization of the Berger–Wang theorem on the case of Markovian products of matrices. In the final part of Section 3, we define the concepts of joint and generalized spectral radii for matrix products with constraints on the sliding ℓ -block relative frequencies, and deduce with the help of Theorem 1 from the results of [1, 19] an analog of the Berger–Wang theorem for this case, Theorem 2.

2. Symbolic sequences

Let us recall basic definitions related to symbolic sequences, see, e.g., [18]. Consider infinite sequences $\alpha = (\alpha_n)$ defined for $n \in \mathbb{N} := \{0, 1, 2, \dots\}$ ¹ and taking values from the alphabet (set of symbols) $\mathcal{A} = \{1, 2, \dots, r\}$, and denote by $\mathcal{A}^{\mathbb{N}}$ the totality of all such sequences. By \mathcal{A}_l we denote the set of all finite sequences $\alpha = (\alpha_n)$ of length l , and by $\mathcal{A}^* = \cup_{l \geq 1} \mathcal{A}_l$ we denote the set of all finite sequences (of all possible lengths) taking values in \mathcal{A} . The number of elements of a sequence $\alpha \in \mathcal{A}^*$, called the *length* of α , is denoted by $|\alpha|$, and $|\alpha|_i$ stands for the number of occurrences of the symbol i in α . The quantity $\frac{|\alpha|_i}{|\alpha|}$ is then the *relative frequency* of occurrences of the symbol i in α .

2.1. Sequences with constraints on the sliding block relative frequencies of symbols

Let $p = (p_1, p_2, \dots, p_r)$ be a set of positive numbers satisfying $\sum_{i=1}^r p_i = 1$, and let

$$p^- = (p_1^-, p_2^-, \dots, p_r^-), \quad p^+ = (p_1^+, p_2^+, \dots, p_r^+), \quad (2)$$

be sets of lower and upper bounds for p :

$$0 \leq p_i^- < p_i < p_i^+ \leq 1, \quad i = 1, 2, \dots, r. \quad (3)$$

Given a natural number ℓ , denote by $\mathcal{A}_\ell(p^\pm)$ the set of all finite sequences $\alpha \in \mathcal{A}_\ell$ for which the relative frequencies of occurrences of different symbols satisfy

$$p_i^- \leq \frac{|\alpha|_i}{|\alpha|} \leq p_i^+, \quad i = 1, 2, \dots, r. \quad (4)$$

¹In symbolic dynamics, see, e.g., [16, 18, 20], sequences defined for all integer values of i , that is, for $i \in \mathbb{Z}$, are also often considered. In this work sequences of such a type will not be needed.

By rewriting these last inequalities in the form

$$p_i^- |\alpha| \leq |\alpha|_i \leq p_i^+ |\alpha|, \quad i = 1, 2, \dots, r,$$

we can interpret them as constraints on the number of occurrences of different symbols in the sequences $\alpha \in \mathcal{A}_\ell$. By $\mathcal{A}_\ell^{\mathbb{N}}(p^\pm)$ we will denote the set of all sequences from $\mathcal{A}^{\mathbb{N}}$ each finite subsequence α of which of length ℓ satisfies the constraints (4).

Example 1. Let $r = 3, \ell = 10$ and $p = (0.23, 0.33, 0.44)$. Define the sets (2) of lower and upper bounds for p by setting $p_i^- = p_i - 0.1$ and $p_i^+ = p_i + 0.1$ for $i = 1, 2, 3$, that is,

$$p^- = (0.13, 0.23, 0.34), \quad p^+ = (0.33, 0.43, 0.54).$$

Then the set $\mathcal{A}_\ell^{\mathbb{N}}(p^\pm)$ contains the following sequences:

$$\begin{aligned} \alpha_1 &= (2, 1, 2, 3, 3, 3, 2, 3, 3, 1, 2, 1, 2, 3, 1, 3, 2, 3, 3, 3, 2, 1, 2, 2, 1, 3, 3, 1, 3, 3, 2, \dots), \\ \alpha_2 &= (3, 2, 2, 1, 3, 3, 3, 3, 2, 1, 2, 1, 2, 3, 3, 2, 3, 3, 2, 1, 2, 1, 3, 1, 3, 2, 3, 3, 2, 2, 1, \dots), \\ \alpha_3 &= (1, 1, 3, 3, 2, 2, 1, 2, 3, 3, 1, 3, 1, 3, 2, 2, 3, 2, 2, 3, 1, 3, 1, 3, 2, 1, 3, 2, 2, 3, 1, \dots). \end{aligned}$$

Remark 1. The set $\mathcal{A}_\ell(p^\pm)$ may be empty even in the case when inequalities (3) are satisfied. However if $\mathcal{A}_\ell(p^\pm) \neq \emptyset$ then $\mathcal{A}_\ell^{\mathbb{N}}(p^\pm) \neq \emptyset$, too. To show this it suffices to take an arbitrary sequence $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_{\ell-1}) \in \mathcal{A}_\ell(p^\pm)$ and to observe that its periodic extension to the right (with period ℓ) belongs to $\mathcal{A}_\ell^{\mathbb{N}}(p^\pm)$.

Remark 2. In general, the frequencies of occurrences of the symbols $i = 1, 2, \dots, r$ in the sequences from $\mathcal{A}_\ell^{\mathbb{N}}(p^\pm)$ are not well defined. More precisely it means the following. Denote by $\alpha_n = (\alpha_0, \alpha_1, \dots, \alpha_n)$ the initial interval of length n for an arbitrary sequence $\alpha = (\alpha_0, \alpha_1, \dots) \in \mathcal{A}_\ell^{\mathbb{N}}(p^\pm)$. For each $i = 1, 2, \dots, r$, the quantity $\frac{|\alpha_n|_i}{|\alpha_n|}$ is the relative frequency of occurrences of the symbol i among the first n members of α . Then the relative frequencies $\frac{|\alpha_n|_i}{|\alpha_n|}$ are “close” to the corresponding quantities p_i but, in general, they may have no limits as $n \rightarrow \infty$.

The next lemma answers the question when the set $\mathcal{A}_\ell(p^\pm)$ is nonempty. Recall that, for a real number x , the floor value $\lfloor x \rfloor$ is the largest integer not greater than x and the ceiling value $\lceil x \rceil$ is the smallest integer not less than x .

Lemma 1. $\mathcal{A}_\ell(p^\pm) \neq \emptyset$ if and only if

$$\lceil p_i^- \ell \rceil \leq \lfloor p_i^+ \ell \rfloor, \quad i = 1, 2, \dots, r, \tag{5}$$

$$\sum_{i=1}^r \lceil p_i^- \ell \rceil \leq \ell \leq \sum_{i=1}^r \lfloor p_i^+ \ell \rfloor. \tag{6}$$

Proof. Condition (5) means that, for each $i = 1, 2, \dots, r$, there must exist at least one sequence α of length ℓ the number $|\alpha|_i$ of occurrences of the symbol i in which satisfies the “frequency constraints” (4). Whereas condition (6), provided that condition (5) is satisfied, is equivalent to the existence of at least one sequence α of length ℓ the number $|\alpha|_i$ of occurrences of each symbol $i = 1, 2, \dots, r$ in which satisfies (4). \square

Example 2. Let $r = 3, \ell = 10$, and the sets (2) of lower and upper bounds for p be the same as in Example 2. Then

$$\lceil p_1^- \ell \rceil = 2, \quad \lceil p_2^- \ell \rceil = 3, \quad \lceil p_3^- \ell \rceil = 4, \quad \lfloor p_1^+ \ell \rfloor = 3, \quad \lfloor p_2^+ \ell \rfloor = 4, \quad \lfloor p_3^+ \ell \rfloor = 5,$$

and $\sum_{i=1}^r \lceil p_i^- \ell \rceil = 9 \leq \ell \leq 12 = \sum_{i=1}^r \lfloor p_i^+ \ell \rfloor$. So, both conditions (5) and (6) hold.

Let again $p = (0.23, 0.33, 0.44)$, but the sets of lower and upper bounds for p this time be defined by the equalities $p_i^- = p_i - 0.01$ and $p_i^+ = p_i + 0.01$ for $i = 1, 2, 3$, that is,

$$p^- = (0.22, 0.32, 0.43), \quad p^+ = (0.24, 0.34, 0.45).$$

In this case

$$\lceil p_1^- \ell \rceil = 3, \quad \lceil p_2^- \ell \rceil = 4, \quad \lceil p_3^- \ell \rceil = 5, \quad \lfloor p_1^+ \ell \rfloor = 2, \quad \lfloor p_2^+ \ell \rfloor = 3, \quad \lfloor p_3^+ \ell \rfloor = 4,$$

and conditions (5) are not valid for any $i = 1, 2, 3$.

At last, let again $p = (0.23, 0.33, 0.44)$, but the sets of lower and upper bounds for p this time be defined as $p_i^- = p_i - 0.05$ and $p_i^+ = p_i + 0.05$ for $i = 1, 2, 3$, that is,

$$p^- = (0.18, 0.28, 0.39), \quad p^+ = (0.28, 0.38, 0.49).$$

Then

$$\lceil p_1^- \ell \rceil = 2, \quad \lceil p_2^- \ell \rceil = 3, \quad \lceil p_3^- \ell \rceil = 4, \quad \lfloor p_1^+ \ell \rfloor = 2, \quad \lfloor p_2^+ \ell \rfloor = 3, \quad \lfloor p_3^+ \ell \rfloor = 4,$$

and conditions (5) hold for each $i = 1, 2, 3$ whereas condition (6) is not valid because $\sum_{i=1}^r \lfloor p_i^+ \ell \rfloor = 9 < \ell$.

From Lemma 1 and Example 2 it is seen that $\mathcal{A}_\ell(p^\pm) \neq \emptyset$ if and only if the ‘‘gap’’ between the related quantities p_i^- and p_i^+ is not ‘‘too small’’.

Lemma 2. *Let the sets of quantities p^- and p^+ satisfy (5), and also let one of the following conditions be valid:*

(i) *in (6) one of the inequalities turns to a strict equality, that is,*

$$\sum_{i=1}^r \lceil p_i^- \ell \rceil = \ell \quad \text{or} \quad \sum_{i=1}^r \lfloor p_i^+ \ell \rfloor = \ell;$$

(ii) *in (6) both of inequalities are strict, that is,*

$$\sum_{i=1}^r \lceil p_i^- \ell \rceil < \ell < \sum_{i=1}^r \lfloor p_i^+ \ell \rfloor, \tag{7}$$

while in (5) all inequalities, with the exception possibly of one, turn to equalities.

Then all the sequences from $\mathcal{A}_\ell^{\mathbb{N}}(p^\pm)$ are periodic.

Proof. Let α be a sequence from $\mathcal{A}_\ell(p^\pm)$ (such a sequence exists by Lemma 1). Then

$$|\alpha| = \sum_{i=1}^r |\alpha|_i = \ell, \quad (8)$$

and inequalities (4) imply

$$\lceil p_i^- \ell \rceil \leq |\alpha|_i \leq \lfloor p_i^+ \ell \rfloor, \quad i = 1, 2, \dots, r. \quad (9)$$

Now observe that, if in condition (i) the first equality holds, then (8) and (9) imply

$$|\alpha|_i = \lceil p_i^- \ell \rceil, \quad i = 1, 2, \dots, r, \quad (10)$$

whereas, if in condition (i) the second equality holds, then (8) and (9) imply

$$|\alpha|_i = \lfloor p_i^+ \ell \rfloor, \quad i = 1, 2, \dots, r. \quad (11)$$

At last, let condition (ii) be valid. Then in (5) $r - 1$ inequalities of r are, in fact, equalities. Without loss of generality one can think that in this case the first $r - 1$ of inequalities (5) are equalities, and then from (8) and (9) it follows that

$$|\alpha|_i = \lceil p_i^- \ell \rceil = \lfloor p_i^+ \ell \rfloor, \quad i = 1, 2, \dots, r - 1, \quad (12)$$

$$|\alpha|_r = \sum_{i=1}^{r-1} |\alpha|_i. \quad (13)$$

So, under any of conditions (i) or (ii), the quantities $|\alpha|_i$ are defined uniquely by one of relations (10)–(13) and do not depend on the choice of $\alpha \in \mathcal{A}_\ell(p^\pm)$. In this case in each sequence $\alpha' \in \mathcal{A}_\ell^\mathbb{N}(p^\pm)$, for any $\ell + 1$ successive symbols $(\alpha'_k, \dots, \alpha'_{k+\ell-1}, \alpha'_{k+\ell})$, the equality $\alpha'_k = \alpha'_{k+\ell}$ takes place, that is, the sequence α' is ℓ -periodic. \square

By Lemma 2 aperiodic behavior of sequences from $\mathcal{A}_\ell^\mathbb{N}(p^\pm)$ may happen only in the case when the “gap” between the related quantities p_i^- and p_i^+ is large enough to conditions (5) be valid and in conditions (6) both inequalities be strict.

Lemma 3. *Let the sets of quantities p^- and p^+ satisfy the system of inequalities (5) at least two of which are strict, and also let condition (7) hold. Then for each finite sequence $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_{\ell-1}) \in \mathcal{A}_\ell(p^\pm)$ there exist at least two different infinite sequences $\alpha' = (\alpha'_n)_{n \geq 0} \in \mathcal{A}_\ell^\mathbb{N}(p^\pm)$ and $\alpha'' = (\alpha''_n)_{n \geq 0} \in \mathcal{A}_\ell^\mathbb{N}(p^\pm)$ whose initial subsequences of length ℓ coincide with α , that is, $\alpha'_n = \alpha''_n = \alpha_n$ for $n = 0, 1, \dots, \ell - 1$.*

Proof. Denote by $I := \{i_1, \dots, i_m\}$ the set of all $i \in \{1, 2, \dots, r\}$ for which the inequalities $\lceil p_i^- \ell \rceil < \lfloor p_i^+ \ell \rfloor$ in (5) are strict, that is,

$$\begin{aligned} \lceil p_i^- \ell \rceil &< \lfloor p_i^+ \ell \rfloor, & i = i_1, \dots, i_m, \\ \lceil p_i^- \ell \rceil &= \lfloor p_i^+ \ell \rfloor, & i \neq i_1, \dots, i_m. \end{aligned}$$

Then by the lemma conditions $m \geq 2$.

Let $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_{\ell-1})$ be a sequence from $\mathcal{A}_\ell(p^\pm)$; such a sequence exists by Lemma 1. Observe that there exist $i', i'' \in I$ such that

$$\lceil p_{i'}^-\ell \rceil < |\alpha|_{i'} \leq \lfloor p_{i'}^+\ell \rfloor, \quad \lceil p_{i''}^-\ell \rceil \leq |\alpha|_{i''} < \lfloor p_{i''}^+\ell \rfloor. \quad (14)$$

Indeed, if for all $i \in I$ the equality $|\alpha|_i = \lceil p_i^-\ell \rceil$ was valid, then the equality $\sum_{i=1}^r \lceil p_i^-\ell \rceil = |\alpha| = \ell$ would also be valid, which contradicts to (7). And if for all $i \in I$ the equality $|\alpha|_i = \lfloor p_i^+\ell \rfloor$ was valid, then the equality $\sum_{i=1}^r \lfloor p_i^+\ell \rfloor = |\alpha| = \ell$ would be valid, which also contradicts to (7).

Now observe that without loss of generality we can suppose that the first symbol in $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_{\ell-1})$ coincides with i' , that is, $\alpha_0 = i'$. Indeed, in the opposite case we can extend the sequence α to the right by periodicity with period ℓ , and instead of α take such an interval $(\alpha_k, \alpha_1, \dots, \alpha_{k+\ell-1})$ of the extended sequence for which the equality $\alpha_k = i'$ holds.

So, let $\alpha_0 = i'$. Then construct two sequences

$$(\alpha'_0, \alpha'_1, \dots, \alpha'_{\ell-1}, \alpha'_\ell), \quad (\alpha''_0, \alpha''_1, \dots, \alpha''_{\ell-1}, \alpha''_\ell)$$

whose initial intervals of length ℓ coincide with α while the members with the indices ℓ are as follows: $\alpha'_\ell = i'$ and $\alpha''_\ell = i''$. In virtue of (14), both variants of extension of the sequence α lead to that the sequences $(\alpha'_1, \dots, \alpha'_{\ell-1}, \alpha'_\ell)$ and $(\alpha''_1, \dots, \alpha''_{\ell-1}, \alpha''_\ell)$ are distinct and belong to $\mathcal{A}_\ell(p^\pm)$. By extending them to the right to infinite sequences from $\mathcal{A}_\ell^\mathbb{N}(p^\pm)$ (for example, by ℓ -periodicity), we obtain the required sequences α', α'' . \square

Corollary 1. *Under the conditions of Lemma 3 the set $\mathcal{A}_\ell^\mathbb{N}(p^\pm)$ contains infinitely many different aperiodic sequences.*

Denote by $\mathcal{A}_\ell^*(p^\pm)$ the set of all finite sequences each of which is an initial interval of a sequence from $\mathcal{A}_\ell^\mathbb{N}(p^\pm)$. Clearly, the set $\mathcal{A}_\ell^*(p^\pm)$ consists either of sequences of length lesser than ℓ allowing extension to the right to sequences from $\mathcal{A}_\ell(p^\pm)$ or of sequences of length greater than or equal to ℓ each finite subsequence of which of length ℓ belongs to $\mathcal{A}_\ell(p^\pm)$.

Lemma 4. *The following assertions are valid:*

- (i) *If $\alpha = (\alpha_n)_{n=0}^\infty \in \mathcal{A}_\ell^\mathbb{N}(p^\pm)$, then $\alpha_m = (\alpha_{n+m})_{n=0}^\infty \in \mathcal{A}_\ell^\mathbb{N}(p^\pm)$ for any $m \geq 0$.*
- (ii) *If $\alpha = (\alpha_0, \dots, \alpha_k) \in \mathcal{A}_\ell^*(p^\pm)$, then for any $m = 1, 2, \dots, k$ the inclusions $\alpha' = (\alpha_0, \dots, \alpha_{m-1}) \in \mathcal{A}_\ell^*(p^\pm)$ and $\alpha'' = (\alpha_m, \dots, \alpha_k) \in \mathcal{A}_\ell^*(p^\pm)$ hold.*

The assertions of Lemma 4 are obvious and so their proofs are omitted. Assertion (i) is equivalent to the *invariance of the set $\mathcal{A}_\ell^\mathbb{N}(p^\pm)$ with respect to the left shifts*, that is, to the fact that together with each sequence $\alpha = (\alpha_n)_{n \in \mathbb{N}}$ the set $\mathcal{A}_\ell^\mathbb{N}(p^\pm)$ contains also the sequence $\alpha' = (\alpha'_n)_{n \in \mathbb{N}}$ defined by the equalities $\alpha'_n = \alpha_{n+1}$ for $n \in \mathbb{N}$. Assertion (i) implies that, in definition of the set $\mathcal{A}_\ell^*(p^\pm)$, one could consider all finite intervals (not only the initial ones) of the sequences from $\mathcal{A}_\ell^\mathbb{N}(p^\pm)$. The property expressed by assertion (ii) will be referred to as the *sub-additivity of the set $\mathcal{A}_\ell^*(p^\pm)$* .

Remark 3. The author failed to find in the literature any explicit references to the symbolic sequences with the allowed ℓ -blocks of type $\mathcal{A}_\ell(p^\pm)$. Nevertheless, symbolic

sequences with similar properties arise in various applications and theoretical studies. The symbolic sequences with the allowed ℓ -blocks of type $\mathcal{A}_\ell(p^\pm)$ may be treated as the “constrained sequences” arising in the theory of the so-called constrained noiseless channels [13, Ch. 17]. The frequency properties of such sequences with specific allowed forbidden patterns of symbols was studied, e.g., in [21], and in [14, 15] the theory of joint spectral radius was used for these goals. Conceptually close are also the so-called (d, k) -constrained sequences arising in the method of RLL coding (runlength-limited coding) used for data processing in mass data storage systems such as hard drives, compact discs and so on, see, e.g., [16, 17].

The symbolic sequences with the allowed ℓ -blocks of type $\mathcal{A}_\ell(p^\pm)$ are similar to the so-called k -balanced sequences [16, 18], although the former constitute a broader class. The symbolic sequences with the allowed ℓ -blocks of type $\mathcal{A}_\ell(p^\pm)$ may have no limiting frequencies for some symbols which differentiate them from the symbolic sequences whose relative frequencies are uniformly convergent [22].

2.2. Subshifts of finite type

In this section we show that the set of infinite sequences $\mathcal{A}_\ell^\mathbb{N}(p^\pm)$ can be naturally treated as the so-called *subshifts of finite type* (or *ℓ -step topological Markov chains*). Recall the necessary definitions following to [18, 20].

As usual the operator $\sigma : \alpha \mapsto \alpha'$ transferring a sequence $\alpha = (\alpha_n)_{n \in \mathbb{N}} \in \mathcal{A}^\mathbb{N}$ to the sequence $\alpha' = (\alpha'_n)_{n \in \mathbb{N}} \in \mathcal{A}^\mathbb{N}$ defined by the equalities $\alpha'_n = \alpha_{n+1}$ for $n \in \mathbb{N}$ is called the *left shift* or simply *shift* on $\mathcal{A}^\mathbb{N}$. Given a square matrix $\omega = (\omega_{ij})_{i,j=1}^r$ of order r with the elements from the set $\{0, 1\}$, we define

$$\mathcal{A}_\omega^\mathbb{N} := \{\alpha = (\alpha_n) \in \mathcal{A}^\mathbb{N} : \omega_{\alpha_n \alpha_{n+1}} = 1, n \in \mathbb{N}\}.$$

In other words, the matrix ω determines all transitions between the symbols of the alphabet $\mathcal{A} = \{1, 2, \dots, r\}$ in the sequences from $\mathcal{A}_\omega^\mathbb{N}$. The restriction of the shift operator σ to $\mathcal{A}_\omega^\mathbb{N}$ is called the *topological Markov chain* defined by the matrix ω of allowed transitions [18, 20]. In this case σ is also often referred to as a *subshift of finite type*².

There is a natural class of symbolic systems more general than the topological Markov chains. Given a map $\Omega : \mathcal{A}^{\ell+1} \rightarrow \{0, 1\}$, define

$$\mathcal{A}_\Omega^\mathbb{N} := \{\alpha = (\alpha_n) \in \mathcal{A}^\mathbb{N} : \Omega(\alpha_n, \dots, \alpha_{n+\ell}) = 1, n \in \mathbb{N}\}.$$

The restriction of the shift map σ to $\mathcal{A}_\Omega^\mathbb{N}$ is called the *ℓ -step topological Markov chain* defined by the function Ω of allowed transitions.

From the point of view of dynamics, the ℓ -step topological Markov chains are the same as the usual (1-step) topological Markov chains since the former can be described as the topological Markov chains with the alphabet $\mathcal{A} = \{1, 2, \dots, r\}^\ell$ and the transition matrix ω such that

$$\omega_{(\alpha_1, \dots, \alpha_\ell), (\alpha'_1, \dots, \alpha'_\ell)} = 1, \tag{15}$$

if $\alpha'_k = \alpha_{k+1}$ for $k = 1, \dots, \ell - 1$ and $\Omega(\alpha_1, \dots, \alpha_\ell, \alpha'_\ell) = 1$, see, e.g., [20, Sect. 1.9].

²Sometimes the terms topological Markov chain or subshift of finite type are applied not to the shift operator σ but to the set of the sequences $\mathcal{A}_\Omega^\mathbb{N}$.

Theorem 1. *Let, for some sets of quantities $p^- = (p_1^-, \dots, p_r^-)$ and $p^+ = (p_1^+, \dots, p_r^+)$ satisfying (3), the conditions of Lemma 1 hold. The restriction of the shift σ to the set $\mathcal{A}_\ell^{\mathbb{N}}(p^\pm)$ is an ℓ -step topological Markov chain.*

Proof. It suffices to note that $\mathcal{A}_\ell^{\mathbb{N}}(p^\pm) = \mathcal{A}_\Omega^{\mathbb{N}}$, where the map $\Omega : \mathcal{A}^{\ell+1} \rightarrow \{0, 1\}$ is such that $\Omega(\alpha_n, \dots, \alpha_{n+\ell}) = 1$ if and only if $(\alpha_n, \dots, \alpha_{n+\ell-1}), (\alpha_{n+1}, \dots, \alpha_{n+\ell}) \in \mathcal{A}_\ell(p^\pm)$, and $\Omega(\alpha_n, \dots, \alpha_{n+\ell}) = 0$ in the opposite case. \square

Despite its evidence, Theorem 1 is of principal importance since it allows to treat the symbolic sequences with constraints on the sliding block relative frequencies of symbols as the (ℓ -step) topological Markov chains.

3. Matrix products with constraints on the sliding block relative frequencies of factors

In this section we strengthen the results of [1, 19] on the joint/generalized spectral radius for the Markovian matrix products for the matrix products (1) whose index sequences $\alpha = (\alpha_n)$ are subjected to constraints on the sliding block relative frequencies of symbols.

3.1. The joint and generalized spectral radii

Recall basic concepts of the theory of joint/generalized spectral radius. Given a sub-multiplicative norm³ $\|\cdot\|$ on $\mathbb{K}^{d \times d}$, the limit

$$\rho(\mathcal{M}) := \limsup_{n \rightarrow \infty} \rho_n(\mathcal{M}) \quad \left(= \lim_{n \rightarrow \infty} \rho_n(\mathcal{M}) = \inf_{n \geq 1} \rho_n(\mathcal{M}) \right), \quad (16)$$

where

$$\rho_n(\mathcal{M}) := \sup \left\{ \|M_{\alpha_n} \cdots M_{\alpha_1}\|^{1/n} : \alpha_j \in \mathcal{A} \right\},$$

is called the *joint spectral radius* of the set of matrices \mathcal{M} [10]. This limit always exists, finite and does not depend on the norm $\|\cdot\|$. If \mathcal{M} consists of a single matrix, then (16) turns into the known Gelfand formula for the spectral radius of a linear operator. By this reason sometimes (16) is called the generalized Gelfand formula [23].

The *generalized spectral radius* of the set of matrices \mathcal{M} is the quantity defined by a similar to (16) formula in which instead of the norm it is taken the spectral radius $\rho(\cdot)$ of the corresponding matrices [8, 9]:

$$\hat{\rho}(\mathcal{M}) := \limsup_{n \rightarrow \infty} \hat{\rho}_n(\mathcal{M}) \quad \left(= \sup_{n \geq 1} \hat{\rho}_n(\mathcal{M}) \right), \quad (17)$$

where

$$\hat{\rho}_n(\mathcal{M}) := \sup \left\{ \rho(M_{\alpha_n} \cdots M_{\alpha_1})^{1/n} : \alpha_j \in \mathcal{A} \right\}.$$

³A norm $\|\cdot\|$ on a space of linear operators is called sub-multiplicative if $\|AB\| \leq \|A\| \cdot \|B\|$ for any operators A and B .

As has been proved by M. Berger and Y. Wang [7] the quantities $\rho(\mathcal{M})$ and $\hat{\rho}(\mathcal{M})$ coincide with each other for bounded sets of matrices \mathcal{M} :

$$\hat{\rho}(\mathcal{M}) = \rho(\mathcal{M}). \quad (18)$$

This formula has numerous applications in the theory of joint/generalized spectral radius. It implies the continuous dependence of the joint/generalized spectral radius on \mathcal{M} . From (18) it also follows that the quantities $\hat{\rho}_n(\mathcal{M})$ and $\rho_n(\mathcal{M})$, for any n , form the lower and upper bounds respectively for the joint/generalized spectral radius of \mathcal{M} :

$$\hat{\rho}_n(\mathcal{M}) \leq \hat{\rho}(\mathcal{M}) = \rho(\mathcal{M}) \leq \rho_n(\mathcal{M}), \quad (19)$$

which is often used to evaluate the accuracy of computation of the joint/generalized spectral radius.

3.2. The Markovian joint and generalized spectral radii

The distinctive feature of the definitions (16) and (17) is that the products of matrices $M_{\alpha_n} \cdots M_{\alpha_1}$ in them correspond to all possible sequences of indices $\alpha = (\alpha_1, \dots, \alpha_n)$. A more complicated situation is when these matrix products are somehow constrained, for example, some combinations of matrices in them are forbidden. One of situations of the kind was investigated in [1, 19], where the concepts of the Markovian joint and generalized spectral radii were introduced to analyze the matrix products with constraints of the Markovian type on the neighboring matrices. Another situation of the kind is described in what follows.

Let $\omega = (\omega_{ij})$ be an $(r \times r)$ -matrix with the elements from the set $\{0, 1\}$. A finite sequence $(\alpha_1, \dots, \alpha_n)$ with the elements from \mathcal{A} will be called ω -admissible if $\omega_{\alpha_{j+1}\alpha_j} = 1$ for all $1 \leq j \leq n-1$ and besides there exists $\alpha_* \in \mathcal{A}$ such that $\omega_{\alpha_*\alpha_n} = 1$. Denote by $W_{r,\omega}$ the set of all ω -admissible finite sequences. The matrix products $M_{\alpha_n} \cdots M_{\alpha_1}$ corresponding to the ω -admissible sequences $(\alpha_1, \dots, \alpha_n)$ will be called *Markovian* since such products of matrices arise naturally in the theory of matrix cocycles over the topological Markov chains, see, e.g., [16, 20].

Define analogs of formulae (16) and (17) for the ω -admissible products of matrices. The limit

$$\rho(\mathcal{M}, \omega) := \limsup_{n \rightarrow \infty} \rho_n(\mathcal{M}, \omega), \quad (20)$$

where

$$\rho_n(\mathcal{M}, \omega) := \sup \left\{ \|M_{\alpha_n} \cdots M_{\alpha_1}\|^{1/n} : (\alpha_1, \dots, \alpha_n) \in W_{r,\omega} \right\},$$

is called the *Markovian joint spectral radius* of the set of matrices \mathcal{M} defined by the *matrix of admissible transitions* ω . If, for some n , the set of all ω -admissible sequences $(\alpha_1, \dots, \alpha_n)$ is empty then we put $\rho_n(\mathcal{M}, \omega) = 0$. In this case, the sets of all ω -admissible sequences $(\alpha_1, \dots, \alpha_k)$ will be also empty for each $k \geq n$, and therefore $\rho(\mathcal{M}, \omega) = 0$. The question on existence of arbitrarily long ω -admissible sequences can be answered algorithmically in a finite number of steps. In particular, the set $W_{r,\omega}$ has arbitrarily long sequences if each column of the matrix ω contains at least one nonzero element.

Likewise to formula (16), the limit (20) always exists, finite and does not depend on the norm $\|\cdot\|$. Moreover, by the Fekete Lemma [24] (see also [25, Ch. 3, Sect. 1]) the submultiplicativity in n of the quantity $\rho_n^n(\mathcal{M}, \omega)$ implies the existence of $\lim_{n \rightarrow \infty} \rho_n(\mathcal{M}, \omega)$ and of $\inf_{n \geq 1} \rho_n(\mathcal{M}, \omega)$ and their equality to the limit (20):

$$\rho(\mathcal{M}, \omega) := \limsup_{n \rightarrow \infty} \rho_n(\mathcal{M}, \omega) = \lim_{n \rightarrow \infty} \rho_n(\mathcal{M}, \omega) = \inf_{n \geq 1} \rho_n(\mathcal{M}, \omega).$$

The quantity

$$\hat{\rho}(\mathcal{M}, \omega) := \limsup_{n \rightarrow \infty} \hat{\rho}_n(\mathcal{M}, \omega), \quad (21)$$

where

$$\hat{\rho}_n(\mathcal{M}, \omega) := \sup \left\{ \rho(M_{\alpha_n} \cdots M_{\alpha_1})^{1/n} : (\alpha_1, \dots, \alpha_n) \in W_{r, \omega} \right\},$$

is called the *Markovian generalized spectral radius* of the set of matrices \mathcal{M} defined by the matrix of admissible transitions ω . Here again we are putting $\hat{\rho}_n(\mathcal{M}, \omega) = 0$ if the set of ω -admissible sequences of indices $(\alpha_1, \dots, \alpha_n)$ is empty. Like in the case of formula (17), the limit (21) coincides with $\sup_{n \geq 1} \hat{\rho}_n(\mathcal{M}, \omega)$.

For the Markovian products of matrices there are valid inequalities

$$\hat{\rho}_n(\mathcal{M}, \omega) \leq \hat{\rho}(\mathcal{M}, \omega) \leq \rho(\mathcal{M}, \omega) \leq \rho_n(\mathcal{M}, \omega), \quad (22)$$

similar to (19). However the question whether there is a valid equality

$$\hat{\rho}(\mathcal{M}, \omega) = \rho(\mathcal{M}, \omega), \quad (23)$$

similar to the Berger–Wang equality (18), becomes more complicated. To answer it we will need one more definitions.

An ω -admissible finite sequence $(\alpha_1, \dots, \alpha_n)$ will be called *periodically extendable* if $\omega_{\alpha_1 \alpha_n} = 1$. Not every ω -admissible finite sequence can be periodically extended. However, if arbitrarily long ω -admissible sequences exist, then periodically extendable ω -admissible sequences exist too. The set of all periodically extendable ω -admissible sequences is denoted by $W_{r, \omega}^{(\text{per})}$.

Define the quantity

$$\hat{\rho}_n^{(\text{per})}(\mathcal{M}, \omega) := \sup \left\{ \rho(M_{\alpha_n} \cdots M_{\alpha_1})^{1/n} : (\alpha_1, \dots, \alpha_n) \in W_{r, \omega}^{(\text{per})} \right\},$$

and set⁴

$$\hat{\rho}^{(\text{per})}(\mathcal{M}, \omega) := \limsup_{n \rightarrow \infty} \hat{\rho}_n^{(\text{per})}(\mathcal{M}, \omega).$$

Dai's Theorem. (See [1]) $\hat{\rho}^{(\text{per})}(\mathcal{M}, \omega) = \rho(\mathcal{M}, \omega)$.

Since $W_{r, \omega}^{(\text{per})} \subseteq W_{r, \omega}$ then $\hat{\rho}_n^{(\text{per})}(\mathcal{M}, \omega) \leq \hat{\rho}_n(\mathcal{M}, \omega)$ for each $n \geq 1$, and therefore $\hat{\rho}^{(\text{per})}(\mathcal{M}, \omega) \leq \hat{\rho}(\mathcal{M}, \omega)$. This last inequality together with (22) by Dai's Theorem then implies the Markovian analog (23) of the Berger–Wang formula (18).

⁴Like in the definitions of the Markovian joint and generalized spectral radii we put $\hat{\rho}_n^{(\text{per})}(\mathcal{M}, \omega) = 0$ if the set of all the periodically extendable sequences of length n is empty.

3.3. The constrained joint and generalized spectral radii

Let, likewise in the previous section, $\mathcal{M} = \{M_1, M_2, \dots, M_r\}$ be a finite set of matrices with the elements from the field $\mathbb{K} = \mathbb{R}, \mathbb{C}$ of real or complex numbers. Let also ℓ be a natural number and $p_i, p_i^\pm, i = 1, 2, \dots, r$, be sets of quantities satisfying (3).

A finite sequence $\alpha = (\alpha_1, \dots, \alpha_n)$ with the elements from \mathcal{A} will be referred to as $\mathcal{A}_\ell(p^\pm)$ -admissible if $\alpha \in \mathcal{A}_\ell^*(p^\pm)$, that is, if each its subsequence $(\alpha_j, \dots, \alpha_{j+\ell-1})$ of length ℓ belongs to $\mathcal{A}_\ell(p^\pm)$, and each its subsequence of length lesser than ℓ allows extension to the right to a sequence from $\mathcal{A}_\ell(p^\pm)$. Denote by $W_{\mathcal{A}_\ell(p^\pm)}$ the set of all finite $\mathcal{A}_\ell(p^\pm)$ -admissible sequences $\alpha = (\alpha_1, \dots, \alpha_n)$. The problem on existence of arbitrarily long $\mathcal{A}_\ell(p^\pm)$ -admissible sequences has been resolved in Lemmata 1–4. Products of matrices $M_{\alpha_n} \cdots M_{\alpha_1}$ corresponding to the $\mathcal{A}_\ell(p^\pm)$ -admissible sequences $(\alpha_1, \dots, \alpha_n)$ will be referred to as $\mathcal{A}_\ell(p^\pm)$ -admissible.

Now, the concepts of joint and generalized spectral radii for $\mathcal{A}_\ell(p^\pm)$ -admissible products of matrices from \mathcal{M} can be defined by almost literal repetition of the related definitions from the previous section. The limit

$$\rho(\mathcal{M}, \mathcal{A}_\ell(p^\pm)) := \limsup_{n \rightarrow \infty} \rho_n(\mathcal{M}, \mathcal{A}_\ell(p^\pm)), \quad (24)$$

where

$$\rho_n(\mathcal{M}, \mathcal{A}_\ell(p^\pm)) := \sup \left\{ \|M_{\alpha_n} \cdots M_{\alpha_1}\|^{1/n} : (\alpha_1, \dots, \alpha_n) \in W_{\mathcal{A}_\ell(p^\pm)} \right\},$$

is called the $\mathcal{A}_\ell(p^\pm)$ -constrained joint spectral radius of the set of matrices \mathcal{M} .

Like in the cases of formulae (16) or (20), the limit (24) always exists, finite and does not depend on the norm $\|\cdot\|$. Moreover, from assertion (ii) of Lemma 4 it follows that the sets $\mathcal{A}_\ell^*(p^\pm)$ possess the property of sub-additivity and then the quantity $\rho_n^n(\mathcal{M}, \mathcal{A}_\ell(p^\pm))$ is sub-multiplicative in n . Therefore, by the already mentioned Fekete Lemma there exists $\lim_{n \rightarrow \infty} \rho_n(\mathcal{M}, \mathcal{A}_\ell(p^\pm))$ coinciding with $\rho(\mathcal{M}, \mathcal{A}_\ell(p^\pm))$ as well with $\inf_{n \geq 1} \rho_n(\mathcal{M}, \mathcal{A}_\ell(p^\pm))$. This means that the $\mathcal{A}_\ell(p^\pm)$ -constrained joint spectral radius may be defined by each of the following equalities:

$$\rho(\mathcal{M}, \mathcal{A}_\ell(p^\pm)) := \limsup_{n \rightarrow \infty} \rho_n(\mathcal{M}, \mathcal{A}_\ell(p^\pm)) = \lim_{n \rightarrow \infty} \rho_n(\mathcal{M}, \mathcal{A}_\ell(p^\pm)) = \inf_{n \geq 1} \rho_n(\mathcal{M}, \mathcal{A}_\ell(p^\pm)).$$

Similarly, the quantity

$$\hat{\rho}(\mathcal{M}, \mathcal{A}_\ell(p^\pm)) := \limsup_{n \rightarrow \infty} \hat{\rho}_n(\mathcal{M}, \mathcal{A}_\ell(p^\pm)),$$

where

$$\hat{\rho}_n(\mathcal{M}, \mathcal{A}_\ell(p^\pm)) := \sup \left\{ \rho(M_{\alpha_n} \cdots M_{\alpha_1})^{1/n} : (\alpha_1, \dots, \alpha_n) \in W_{\mathcal{A}_\ell(p^\pm)} \right\},$$

can be called the $\mathcal{A}_\ell(p^\pm)$ -constrained generalized spectral radius of the set of matrices \mathcal{M} .

For the $\mathcal{A}_\ell(p^\pm)$ -admissible products of matrices the inequalities

$$\hat{\rho}_n(\mathcal{M}, \mathcal{A}_\ell(p^\pm)) \leq \hat{\rho}(\mathcal{M}, \mathcal{A}_\ell(p^\pm)) \leq \rho(\mathcal{M}, \mathcal{A}_\ell(p^\pm)) \leq \rho_n(\mathcal{M}, \mathcal{A}_\ell(p^\pm)), \quad (25)$$

similar to (19) or (22), hold. However the question about the validity of the equality

$$\hat{\rho}(\mathcal{M}, \mathcal{A}_\ell(p^\pm)) = \rho(\mathcal{M}, \mathcal{A}_\ell(p^\pm)), \quad (26)$$

analogous to the Berger–Wang equality (18), like in the cases of the Markovian joint and generalized spectral radii, is not so evident. Let us call an $\mathcal{A}_\ell(p^\pm)$ -admissible finite sequence $(\alpha_0, \dots, \alpha_{n-1})$ *periodically extendable* if it may be extended to the right to an n -periodic sequence belonging to $\mathcal{A}_\ell^{\mathbb{N}}(p^\pm)$. The set of all periodically extendable $\mathcal{A}_\ell(p^\pm)$ -admissible sequences will be denoted by $W_{\mathcal{A}_\ell(p^\pm)}^{(\text{per})}$. Clearly, this set is nonempty provided that $\mathcal{A}_\ell(p^\pm) \neq \emptyset$ because each sequence from $\mathcal{A}_\ell(p^\pm)$ admits an ℓ -periodic extension to the right.

Define, for each $n \geq 1$, the quantity

$$\hat{\rho}_n^{(\text{per})}(\mathcal{M}, \mathcal{A}_\ell(p^\pm)) := \sup \left\{ \rho(M_{\alpha_n} \cdots M_{\alpha_1})^{1/n} : (\alpha_1, \dots, \alpha_n) \in W_{\mathcal{A}_\ell(p^\pm)}^{(\text{per})} \right\},$$

and set

$$\hat{\rho}^{(\text{per})}(\mathcal{M}, \mathcal{A}_\ell(p^\pm)) := \limsup_{n \rightarrow \infty} \hat{\rho}_n^{(\text{per})}(\mathcal{M}, \mathcal{A}_\ell(p^\pm)).$$

Then the following generalization of the Berger–Wang formula for the case of matrix products with constraints on the sliding block relative frequencies of factors or, what is the same, for the case of $\mathcal{A}_\ell(p^\pm)$ -admissible matrix products is valid.

Theorem 2. $\hat{\rho}^{(\text{per})}(\mathcal{M}, \mathcal{A}_\ell(p^\pm)) = \rho(\mathcal{M}, \mathcal{A}_\ell(p^\pm))$.

Proof. By Theorem 1 the restriction of the shift σ on the set $\mathcal{A}_\ell^{\mathbb{N}}(p^\pm)$ is an ℓ -step topological Markov chain. But, as was mentioned in Section 2.2, the ℓ -step topological Markov chains are the same as the usual (1-step) topological Markov chains with the alphabet $\mathcal{A} = \{1, 2, \dots, r\}^\ell$ and appropriate matrices of allowed transitions (15). Then the claim of theorem follows from Dai’s Theorem. \square

Since $W_{\mathcal{A}_\ell(p^\pm)}^{(\text{per})} \subseteq W_{\mathcal{A}_\ell(p^\pm)}$ then $\hat{\rho}_n^{(\text{per})}(\mathcal{M}, \mathcal{A}_\ell(p^\pm)) \leq \hat{\rho}_n(\mathcal{M}, \mathcal{A}_\ell(p^\pm))$ for each $n \geq 1$, and therefore $\hat{\rho}^{(\text{per})}(\mathcal{M}, \mathcal{A}_\ell(p^\pm)) \leq \hat{\rho}(\mathcal{M}, \mathcal{A}_\ell(p^\pm))$. From here and from (25), by Theorem 2, then follows the analog (26) of the Berger–Wang formula (18) for the matrix products with constraints on the sliding block relative frequencies of factors.

Remark 4. In this section the admissible sequences were defined as follows. It was given the set $\mathcal{P} = \mathcal{A}_\ell(p^\pm)$ of the sequences of length ℓ , and a finite sequence was treated admissible in two cases: either when its length was less than ℓ and the sequence had a right-extension to a sequence from \mathcal{P} or when its length was greater than or equal to ℓ and each its subsequence of length ℓ belonged to \mathcal{P} .

Under such a treatment of admissible sequences specific type of the set \mathcal{P} is not important. Therefore all the constructions of this section retain their validity for an arbitrary set \mathcal{P} and accordingly defined admissible sequences, cf. [21]. Of course, under such an approach the question about non-emptiness of the set of all admissible sequences should be addressed separately. The Markov sequences described in Section 3.2 keep within this, wider scheme of definition of admissible sequences, in particular.

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