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New types of estimates for the smallest size of complete arcs in a finite Desarguesian projective plane

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Abstract. New types of upper bounds for the smallest size $t_2(2,q)$ of a complete arc in the projective plane $PG(2,q)$ are proposed. The value $t_2(2,q) = d(q)\sqrt{q}\ln q$, where $d(q) < 1$ is a decreasing function of q, is used. The case $d(q) < \alpha/\ln \beta q + \gamma$, where α, β, γ are positive constants independent of q, is considered. It is shown that

$$t_2(2,q) < (2/\ln \frac{1}{10}q + 0.32)\sqrt{q}\ln q \text{ if } q \leq 67993, q \text{ prime, and } q \in R,$$

where R is a set of 27 values in the region 69997...110017.
Also, for $q \in [9311, 67993]$, q prime, and $q \in R$, it is shown that

$$\sqrt{q}(\ln q)^{a_1 - bq} < t_2(2,q) < \sqrt{q}(\ln q)^{a_2 - bq},$$

$a_1 = 0.771, a_2 = 0.752, b = 2.2 \cdot 10^{-7}$. In addition, our results allow us to conjecture that these estimates hold for all q. An algorithm FOP using any *fixed order* of *points* in $PG(2,q)$ is proposed for constructing complete arcs. The algorithm is based on an intuitive postulate that $PG(2,q)$ contains a sufficient number of relatively small complete arcs. It is shown that the type of order on the points of $PG(2,q)$ is not relevant.

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1. Introduction

Let $\mathrm{PG}(2,q)$ be the projective plane over the Galois field $\mathbb{F}_q$. An n-arc is a set of n points no three of which are collinear. An n-arc is called complete if it is not contained in an $(n+1)$-arc of $\mathrm{PG}(2,q)$. For an introduction to projective geometries over finite fields; see [25,45,47].

In [27,53] the close relationship among the theory of n-arcs, coding theory and mathematical statistics is presented. In particular, a complete arc in a plane $\mathrm{PG}(2,q)$, the points of which are treated as 3-dimensional q-ary columns, defines a parity check matrix of a q-ary linear code with codimension 3, Hamming distance 4, and covering radius 2. Arcs can be interpreted as linear maximum distance separable (MDS) codes [51, Sec. 7], [54] and they are related to optimal coverings arrays [23] and to superregular matrices [28].

One of the main problems in the study of projective planes, which is also of interest in coding theory, is the finding of the spectrum of possible sizes of complete arcs. In particular, the value of $t_2(2,q)$, the smallest size of a complete arc in $\mathrm{PG}(2,q)$, is interesting. Finding estimates of the minimum size $t_2(2,q)$ is a hard open problem.

This paper is devoted to *upper bounds* for $t_2(2,q)$.

Surveys of results on the sizes of plane complete arcs, methods of their construction, and comprehension of the relating properties can be found in [4,5, 7,8,11,12,17,24,25,27,29,32,38,40–42,45–52]. Some problems connected with small complete plane arcs are also considered in [1,3,13,14,18–21,24,25,27,29, 30,37,39,44,55].

The exact values of $t_2(2,q)$ are known only for $q \leq 32$; see [2,16,22,25,26, 31,33,34] and the work [9] where the equalities $t_2(2,31) = t_2(2,32) = 14$ are established.

There are the following lower bounds; see [3,17,44,45]:

$$t_2(2,q) > \begin{cases} \sqrt{2q}+1 & \text{for any } q, \\ \sqrt{3q}+\frac{1}{2} & \text{for } q=p^h,\ p \text{ prime},\ h=1,2,3, \\ \frac{q-13+\sqrt{8q^3-15q^2+6q+1}}{2(q-3)} & \text{for any } q>3, \end{cases}$$

Let $t(\mathcal{P}_q)$ be the size of the smallest complete arc in any (not necessarily Galois) projective plane $\mathcal{P}_q$ of order q. In [29], for sufficiently large q, the following result is proved by probabilistic methods (we give it in the form of [27, Tab. 2.6]):

$$t(\mathcal{P}_q) \leq D\sqrt{q}\log^C q, \quad C \leq 300, \tag{1.1}$$

where C and D are constants independent of q. The logarithm basis is natural, see [29, p. 10]. The authors of [29] conjecture that the constant can be reduced to $C = 10$ but this conjecture is not proved. A survey and an analysis of random constructions for geometrical objects can be found in [19]; see also the references therein.

Regarding complete arcs of sizes smaller $\frac{1}{2}q$ obtained by algebraic constructions, following [27, p. 209], complete arcs in $\mathrm{PG}(2,q)$ have been constructed with sizes approximately $\frac{1}{3}q$ (see [1,4,30,48,49,55]), $\frac{1}{4}q$ (see [4,30,50]), $2q^{0.9}$

(see [48] where such arcs are constructed for $q > 7^{10}$). It is noted in [19, Sec. 8], that the smallest size of a complete arc in $\mathrm{PG}(2,q)$ obtained via algebraic constructions is $cq^{3/4}$ where c is a universal constant; see [50, Sec. 3] and [51, Th. 6.8].

In [4,5], for large ranges of q, the form of the bound of (1.1) has been applied but the value of the constant C was essentially reduced to $C = 0.75$ [4] and to $C = 0.73$ [5] whereas $D < 1$. In particular, the following results are obtained in [4,5] using randomized greedy algorithms; see Sect. 2.1:

$$t_2(2,q) < \sqrt{q}\ln^{0.75} q \quad \text{for } \ 23 \le q \le \ 5107 \ [4], \tag{1.2}$$

$$t_2(2,q) < \sqrt{q}\ln^{0.73} q \quad \text{for } 109 \le q \le 13627 \ [5]. \tag{1.3}$$

In this work we improve the known upper bounds on $t_2(2,q)$ and propose new types of upper bounds on $t_2(2,q)$.

The main results of the paper are the following. In the first, we propose a new algorithm FOP (fixed order of points); see Sect. 2.2.

In addition, using this algorithm we obtained Theorems 1.1 and 1.3 below. Let

$$\begin{aligned} R = \{&69997, 70001, 79999, 80021, 81001, 82003, 83003, 84011, 85009,\\ &86011, 87011, 88001, 89003, 90001, 91009, 92003, 93001, 94007,\\ &95003, 96001, 97001, 98009, 99013, 99989, 99991, 109987, 110017\}. \end{aligned} \tag{1.4}$$

We denote

$$\theta_{up}(q) = \frac{2}{\ln(0.1q)} + 0.32. \tag{1.5}$$

Theorem 1.1. *Let $d(q) < 1$ be a decreasing function of q. Then*

$$t_2(2,q) = d(q)\sqrt{q}\ln q, \qquad d(q) < \theta_{up}(q), \tag{1.6}$$

where $q \le 67993$, q prime, and $q \in R$; see (1.4). Complete arcs with sizes satisfying (1.6) can be obtained by the algorithm FOP with lexicographical order *of points represented in homogenous coordinates.*

We also formulate the following conjecture, based on previous results.

Conjecture 1.2. *The upper bound (1.6) holds for all q.*

Let $t_2^L(2,q)$ be the minimum order of complete arcs in $PG(2,q)$ obtained using FOP algorithm with lexicographical order of points; see Sect. 2.2. In order to compare our results with a bound of type (1.1), we introduce the functions $c(q)$ and $c_L(q)$, defined as

$$t_2(2,q) = \sqrt{q}\ln^{c(q)} q, \qquad t_2^L(2,q) = \sqrt{q}\ln^{c_L(q)} q \tag{1.7}$$

or equivalently

$$c(q) = \frac{\ln \frac{t_2(2,q)}{\sqrt{q}}}{\ln\ln q}, \qquad c_L(q) = \frac{\ln \frac{t_2^L(2,q)}{\sqrt{q}}}{\ln\ln q}. \tag{1.8}$$

Let R be as in (1.4).

Theorem 1.3. *Let $c(q)$ be a decreasing function of q. Let $a_1 = 0.771, a_2 = 0.752, b = 2.2 \cdot 10^{-7}$ be absolute positive constants independent of q. Then*

$$t_2(2,q) = \sqrt{q}\ln^{c(q)} q, \quad c(q) \leq c_L(q), \quad a_1 - bq < c_L(q) < a_2 - bq, \quad (1.9)$$

for $q \in [9311, 67993]$, q prime, and $q \in R$. Complete arcs with sizes satisfying (1.9) can be obtained by algorithm FOP with lexicographical orders of points represented in homogenous coordinates.

Some results of this paper were briefly presented in [8].

The paper is organized as follows. In Sect. 2 the new algorithm FOP is presented, while Sect. 3 describes the results obtained with the algorithm. These results improve the upper bounds on $t_2(2,q)$ and allow us to propose new types of upper bounds on $t_2(2,q)$.

2. Randomized greedy algorithms and FOP algorithm

In [4,5,11,15], the authors used randomized greedy algorithms to obtain results on small complete arcs in $PG(2,q)$. In this work we propose and use a different type of algorithm; see Sect. 2.2. We called it FOP (fixed order of points). In this section we summarize the main features of the randomized greedy algorithms used in [4,5] and we give a description of the FOP algorithm we used in our search.

2.1. Randomized greedy algorithm

A *randomized greedy algorithm* maximizes at every step an objective function f but some steps are executed in a random manner. The number of these steps, their ordinal numbers and some other parameters of the algorithm (see, e.g. $d_{q,i}$ below) have been taken intuitively. Also, if the same maximum of f can be obtained in distinct ways, one way is chosen randomly.

We begin to construct a complete arc by using a starting set S_0 of points. As S_0 we can take the frame or a subset of points of an arc obtained in previous stages of the search. At the i-th step one point is added to the set S_{i-1} and we obtain a point set S_i. As the value of the objective function f we consider *the number of covered* points in $PG(2,q)$, that is, points that lie on bisecants of the set obtained.

On every "random" i-th step we take $d_{q,i}$ *randomly chosen points* of $PG(2,q)$ *uncovered by* S_{i-1} and compute the objective function f adding each of these $d_{q,i}$ points to S_{i-1}. The point providing the maximum of f is included into S_i. On every "non-random" j-th step we consider *all points uncovered by* S_{j-1} and add to S_{j-1} the point providing the maximum of f.

A generator of random numbers is used for a random choice. To get arcs with small sizes, usually a few attempts should be made with distinct starting conditions of the generator for the same set S_0.

2.2. Algorithm FOP

The new type of algorithm proposed in this work is described in the following. Consider the projective plane $PG(2,q)$ and fix a particular order on its points. The algorithm builds a complete arc *iteratively*.

Let $K^{(i-1)}$ be the arc obtained on the $(i-1)$-th step. On the next step, the first point in the fixed order not lying on the bisecants of $K^{(i-1)}$ is added to $K^{(i-1)}$. Suppose that the points of $PG(2,q)$ are ordered as $A_1, A_2, \ldots, A_{q^2+q+1}$. Consider the empty set as root of the search and let $K^{(j)}$ be the partial solution obtained in the j-th step, as extension of the root. We put

$$K^{(0)} = \emptyset,\ K^{(1)} = \{A_1\},\ K^{(2)} = \{A_1, A_2\},\ m(1) = 2,\ K^{(j+1)} = K^{(j)} \cup \{A_{m(j)}\},$$
$$m(j) = \min\{i \in [m(j-1)+1, q^2+q+1] \mid \nexists\ P, Q \in K^{(j)} : A_i, P, Q \text{ are collinear}\},$$

that is $m(j)$ is the minimum subscript i such that the corresponding point A_i is not saturated by $K^{(j)}$. The process ends when a complete arc is obtained.

Effectiveness of algorithm FOP is based on the following intuitive postulates that we formulated by experience on our previous works; see e.g. [4,5].

- **B1.** In $PG(2,q)$, there are relatively many complete k-arcs with size of order $k \approx \sqrt{q}\ln q$.
- **B2.** In $PG(2,q)$, a complete k-arc, chosen in arbitrary way close to the random way, has the size of order $k \approx \sqrt{q}\ln q$ with high probability.
- **B3.** The sizes of complete arcs obtained by algorithm FOP vary insignificantly with the respect to the order of points.

In our work we used two different types of order on the points of $PG(2,q)$, the lexicographical order and the Singer order.

2.2.1. Lexicographical order. Let q be prime and let the elements of the field $\mathbb{F}_q = \{0, 1, \ldots, q-1\}$ be treated as integers modulo q. Let the points A_i of $PG(2,q)$ be represented in homogenous coordinates so that $A_i = (x_0^{(i)}, x_1^{(i)}, x_2^{(i)})$, $x_j^{(i)} \in \mathbb{F}_q$, where the leftmost non-zero element is 1. For A_i, we put $i = x_0^{(i)}q^2 + x_1^{(i)}q + x_2^{(i)}$. So, the homogenous coordinates of a point A_i are treated as its number i written in the q-ary scale of notation.

2.2.2. Singer order. The plane $PG(2,q)$ has a cyclic Singer group of order q^2+q+1. The order associated to the Singer group is the following:

$$A_1 = (1,0,0),\ \ A_{i+1} = T(A_i),\ \ i = 1, 2, \ldots, q^2+q, \tag{2.1}$$

where $T \in PGL(3,q)$ is the collineation with associated matrix

$$\bar{T} = \begin{pmatrix} 0 & 0 & c \\ 1 & 0 & b \\ 0 & 1 & a \end{pmatrix}, \tag{2.2}$$

with $x^3 - ax^2 - bx - c$ minimal polynomial of a primitive element of $\mathbb{F}_{q^3}$.

Remark 2.1. In Coding Theory, *greedy codes* (or *lexicographical codes,* or *lexicodes*) are considered, see e.g. [10,35,36,43,56] and the references therein. These codes are constructed by two ways. The first kind of greedy codes is considered in most works on this topic. In order to obtain a q-ary code of length n with minimum distance d one writes all q-ary n-vector in a list using a certain order. The first vector of the list should be included to the code. Then step-by-step, one takes the next vector from the list which has distance d or more from all vectors already chosen.

In another kind, see [10,35,36], one creates a *parity check matrix* of a code with codimension r and parameters q, d as above. All q-ary r-vectors are written as columns in a list in some order. The first column of the list should be included into the matrix. Then step-by-step, one takes the next column from the list which cannot be represented as a linear combination of $d-2$ or smaller columns already chosen. The process is finished when no new column may be included to the matrix.

If a point of $PG(2,q)$ is treated as a column 3-vector then formally FOP algorithm is an algorithm of the second kind creating a parity check matrix with $r = 3, d = 4$. But this viewpoint is only formal. In Coding Theory, for given r, d the aim is to get a *long code* while our goal is to obtain a *short complete arc.* Moreover, our estimates and computer search show that for $r = 3, d = 4$, the FOP algorithm gives "bad" codes that are essentially shorter than "good" codes corresponding to ovals and hyperovals. Finally, note that we do not use the word "greedy" in a name of the FOP algorithm as in Projective Geometry the terms "greedy algorithm" and "randomized greedy algorithm" are traditionally connected with other approaches, see [4,5,11,15].

3. Results on the smallest size of complete arcs in $PG(2,q)$, $q \leq 67993$, q prime, and $q \in R$

In this paper we investigate the minimum size of complete arcs in $PG(2,q), q \leq 67993, q$ prime, and $q \in R$, where R is a set of 27 values in the interval $[69997, 110017]$. To do this we performed a computer-based search using the FOP algorithm described above. In [6] the sizes of complete arcs found are listed.

The main result of this paper is Theorem 1.1, where the values $t_2(2,q)$ are presented as $d(q)\sqrt{q}\ln q$. It is also possible to compare the results obtained with lexicographical order (see Sect. 2.2) with a bound of type (1.1) (here

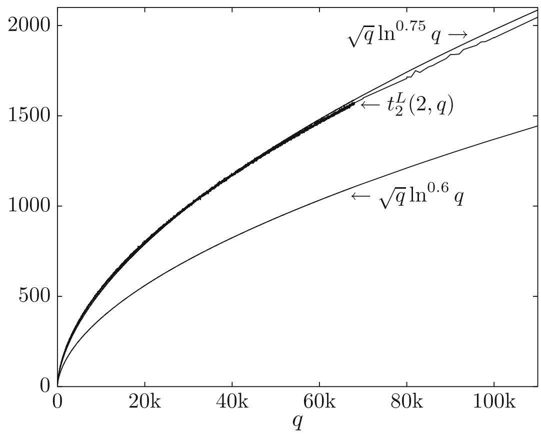

FIGURE 1 The functions $\sqrt{q}\ln^{0.75} q$ (*the top curve*), $t_2^L(2,q)$ (*the second curve*), and $\sqrt{q}\ln^{0.6} q$ (*the bottom curve*), for $q \leq 67993$, q prime, and $q \in R$

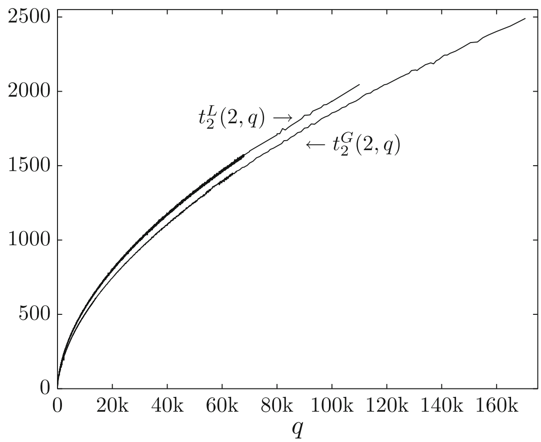

FIGURE 2 The functions $t_2^L(2,q)$ (*the top curve*) and $t_2^G(2,q)$ (*the bottom curve*)

we take the absolute constant D equal to 1): in particular Fig. 1 shows the comparison among the functions $t_2^L(2,q)$, $\sqrt{q}\ln^{0.75} q$, and $\sqrt{q}\ln^{0.6} q$, $q \leq 67993$, q prime, and $q \in R$. Note that our experimental results are very close to the function $\sqrt{q}\ln^{0.75} q$: this bound is similar to the bound obtained using randomized greedy algorithms; see [4,5].

In the following we use the notations $t_2^S(2,q)$ and $t_2^G(2,q)$ in order to represent the results obtained with Singer order (see Sect. 2.2) and greedy randomized algorithms (see [4,5]), respectively.

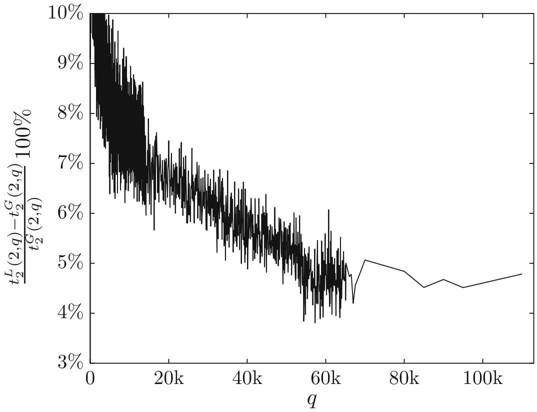

Figure 3 Difference in percentage between $t_2^G(2,q)$ and $t_2^L(2,q)$

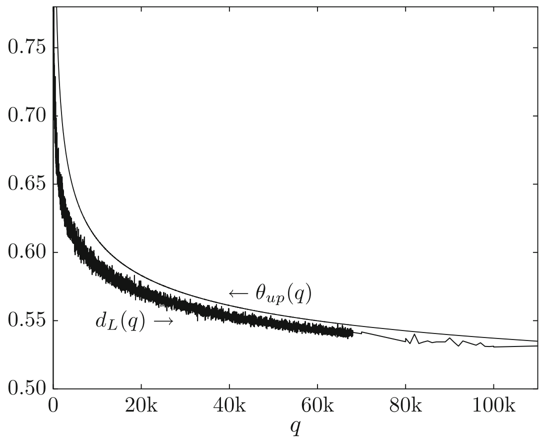

Figure 4 The functions $\theta_{up}(q)$ (*the top curve*) and $d_L(q)$ (*the bottom curve*), for $q \leq 67993$, q prime, and $q \in R$

The comparison with the execution time using randomized greedy algorithms justifies the use of FOP algorithm in order to investigate the behavior of the function $t_2(2,q)$: in Fig. 2 the functions $t_2^G(2,q)$ for $q \leq 14009$ or $q \in S$, where S is a set of 700 prime powers in the region $[14107, 170503]$, and $t_2^L(2,q)$ for $q \leq 67993$, q prime, and $q \in R$, are shown. The values of $t_2^G(2,q)$ for $q \leq 13627$ and a part of the set S are taken from [5], the rest of $t_2^G(2,q)$ values in Fig. 2 are obtained in this work via randomized greedy algorithms.

When q grows, the difference in percentage between the results obtained with algorithm FOP and the randomized greedy algorithms presented in [4,5] decreases; see Fig. 3: for q approximately equal to 14000 the difference is very close to 8 %, while for q approximately equal to 60000 the difference is close to 5 %.

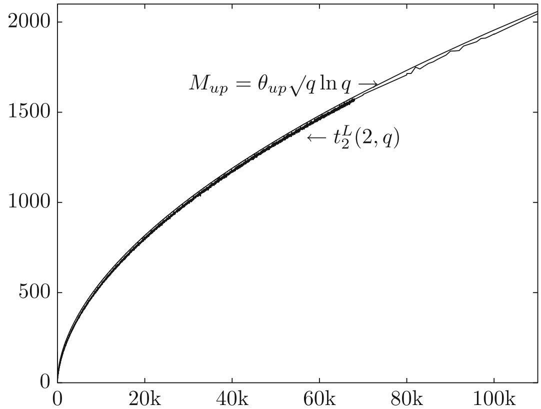

FIGURE 5 The functions $M_{up} = \theta_{up}\sqrt{q}\ln q$ (*the top curve*) and $t_2^L(2,q)$ (*the bottom curve*), for $q \leq 67993$, q prime, and $q \in R$

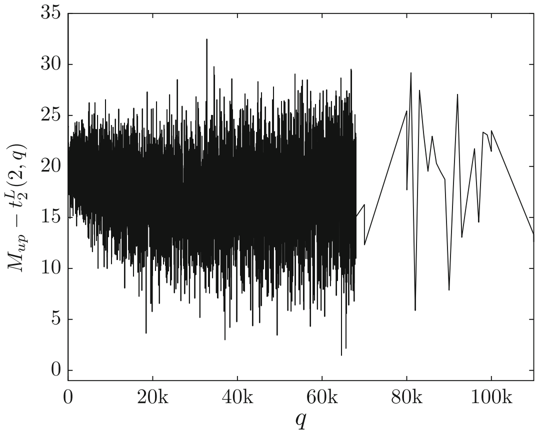

FIGURE 6 Difference between $M_{up} = \theta_{up}\sqrt{q}\ln q$ and $t_2^L(2,q)$, for $q \leq 67993$, q prime, and $q \in R$

In Remark 4.1 in [5] it is pointed out that the difference between $\sqrt{q}\ln^{0.73}$ and the results obtained with randomized greedy algorithms $t_2^G(2,q)$ increases when q grows. This fact allows the authors to suppose that the upper bound on $t_2(2,q)$ is more complicated than the bound of type (1.1), that is $D\sqrt{q}\ln^C q$, where C and D are absolute constants. In order to investigate the existence of different types of estimates for the smallest size of complete arcs in $PG(2,q)$, in this section we also compare our experimental results with other types of bounds. The functions $d(q)$, see also Theorem 1.1, and $d_L(q)$ are defined as

$$d(q) = \frac{t_2(2,q)}{\sqrt{q}\ln q}; \qquad d_L(q) = \frac{t_2^L(2,q)}{\sqrt{q}\ln q}. \tag{3.1}$$

It is clear that $t_2(2,q) \leq \min\{t_2^L(2,q), t_2^S(2,q)\}$ and $d(q) \leq d_L(q)$.

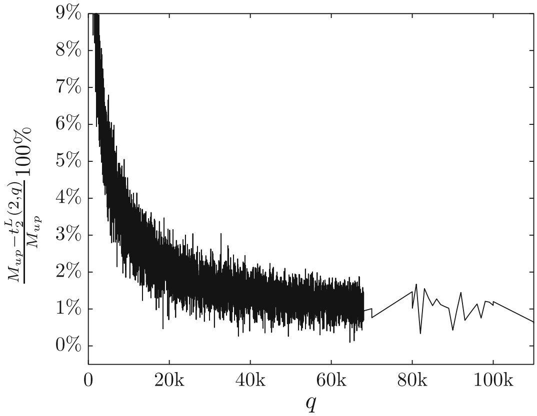

FIGURE 7 Difference in percentage between $M_{up} = \theta_{up}\sqrt{q}\ln q$ and $t_2^L(2,q)$, for $q \leq 67993$, q prime, and $q \in R$

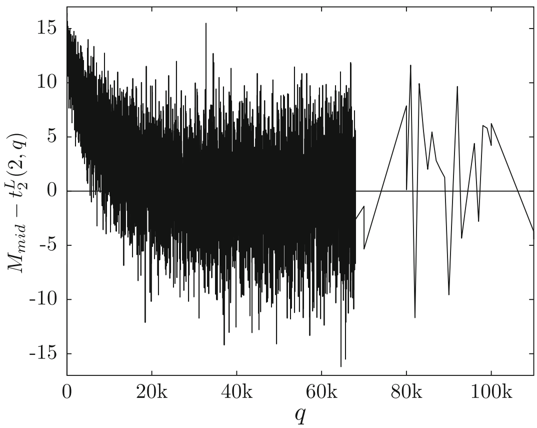

FIGURE 8 Difference between $M_{mid} = \theta_{mid}\sqrt{q}\ln q$ and $t_2^L(2,q)$, for $q \leq 67993$, q prime, and $q \in R$

The function θ_{up} defined in (1.5) is a good upper bound for $d_L(q)$. Figure 4 shows the comparison between the values $d_L(q) = \frac{t_2^L(2,q)}{\sqrt{q}\ln q}$, obtained with FOP algorithm using lexicographical order, and the function $\theta_{up}(q)$.

The curves $t_2^L(2,q)$ and $M_{up} = \theta_{up}\sqrt{q}\ln q$ practically coalesce with each other, but always $t_2^L(2,q) < M_{up}$; see Fig. 5: in Fig. 6 it is shown that the difference $M_{up} - t_2^L(2,q)$ is greater than 0 for each $q \leq 67993$, q prime, and $q \in R$; in Fig. 7 the values $\frac{M_{up}-t_2^L(2,q)}{M_{up}}100$ % are presented. Note that this difference is less that 3 % for $q \geq 20000$.

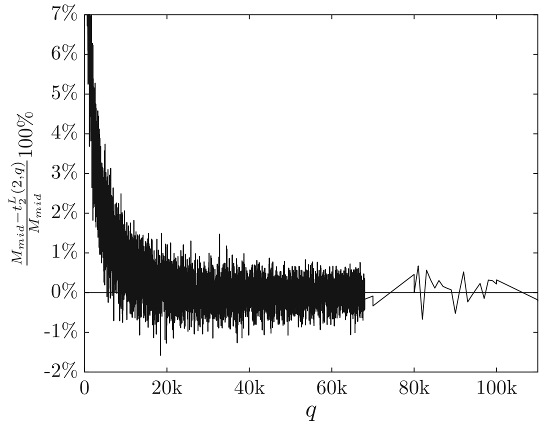

FIGURE 9 Difference in percentage between $M_{mid} = \theta_{mid}\sqrt{q}\ln q$ and $t_2^L(2,q)$, for $q \leq 67993$, q prime, and $q \in R$

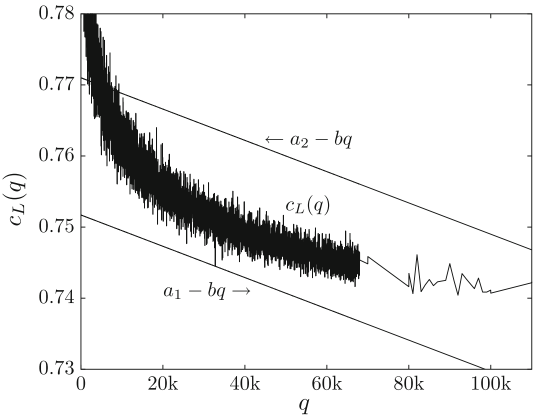

FIGURE 10 The functions $c_L(q)$, $a_1 - bq$, and $a_2 - bq$, $q \leq 67993$, q prime, and $q \in R$, $a_1 = 0.771$, $a_2 = 0.752$, $b = 2.2 \cdot 10^{-7}$

Since we are interested in an approximation for $d_L(q)$, the function θ_{mid} defined as

$$\theta_{mid}(q) = \frac{1.79}{\ln(0.121q)} + 0.342 \tag{3.2}$$

can represent a good example. Let $M_{mid} = \theta_{mid}\sqrt{q}\ln q$. Figure 8 shows that the difference $|M_{mid} - t_2^L(2,q)| \leq 17$ for each $q \leq 67993$, q prime, and $q \in R$; in Fig. 9 the values $\frac{M_{mid}-t_2^L(2,q)}{M_{mid}}100$ % are presented. Note that M_{mid} is a good approximation for $t_2^L(2,q)$, since their difference in percentage is approximately in the interval $[-1,+1]$ for $q \geq 20000$.

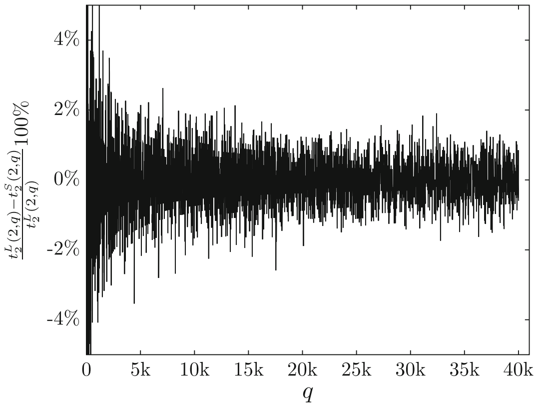

FIGURE 11 Difference in percentage between $t_2^L(2,q)$ and $t_2^S(2,q)$ for $q \leq 40009$, q prime

Figure 10 shows the linear approximations for the function $c_L(q)$; see (1.8) for the definition.

Assumption **B3** in Sect. 2.2 states that in general there exists no particular order on the points of $PG(2,q)$ that can be used to obtain better results than the others for every q. The goodness of this assumption is proven by our experimental results: we searched using FOP algorithm with Singer order for $q \leq 40009$. The results obtained using Singer order are close to those obtained using lexicographical order, since the difference in percentage is approximately in the interval $[-2,+2]$ for $q \geq 5000$; see Fig. 11. This fact strengthens our confidence in the validity of Assumption **B3**: every randomized order can be used and the results will not essentially be influenced by the choice.

In Table 1, the values $t_2^L(2,q)$, $c_L(q)$ and $d_L(q)$, with $q \in R$ are given.

TABLE 1 The values $t_2^L(2,q)$, $c_L(q)$ and $d_L(q)$, with $q \in R$

q	$t_2^L(2,q)$	$c(q)$	$d(q)$	q	$t_2^L(2,q)$	$c(q)$	$d(q)$	q	$t_2^L(2,q)$	$c(q)$	$d(q)$
69997	1595	0.7448	0.5403	70001	1599	0.7458	0.5417	79999	1707	0.7416	0.5345
80021	1715	0.7434	0.5369	81001	1715	0.7406	0.5331	82003	1750	0.7461	0.5401
83003	1740	0.7409	0.5332	84011	1756	0.7418	0.5343	85009	1771	0.7426	0.5351
86011	1779	0.7417	0.5338	87011	1793	0.7422	0.5344	88001	1805	0.7424	0.5344
89003	1817	0.7425	0.5344	90001	1839	0.7448	0.5373	91009	1840	0.7424	0.5341
92003	1842	0.7404	0.5313	93001	1867	0.7434	0.5351	94007	1875	0.7427	0.5340
95003	1883	0.7420	0.5330	96001	1891	0.7413	0.5319	97001	1909	0.7428	0.5338
98009	1911	0.7408	0.5311	99013	1922	0.7408	0.5310	99989	1934	0.7411	0.5312
99991	1932	0.7407	0.5306	109987	2046	0.7421	0.5314	110017	2048	0.7425	0.5318

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