Finding solutions of parabolic Monge-Ampère equations by using the geometry of Cartan fields

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Abstract

In a series of papers we have described normal forms of parabolic Monge-Ampère equations (PMAEs) by means of their characteristic distribution. In particular, PMAEs with two independent variables are associated with Lagrangian (or Legendrian) subdistributions of the contact distribution of a 5-dimensional contact manifold. The geometry of Cartan fields (sections of the contact distribution) allowed us to get the aforementioned normal forms. In the present work, for a distinguished class of PMAEs, we will construct 3-parametric families of solutions starting from a particular type of Cartan field contained in the characteristic distribution. We will illustrate the method by several concrete computations. Moreover, we will see that for some linear PMAEs (including the 1-dimensional heat equation) each solution produces a Cartan field of the aforementioned type; as a result, a recursive process for obtaining new solutions will be offered. At the end, after showing that some classical equations on affine connected 3-dimensional manifolds are PMAEs, we will apply the integration method to some particular examples.

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1 Introduction

1.1 Notation and conventions

Throughout this paper, everything is supposed to be C^∞ and local. For simplicity, when X is a vector field and $\mathcal{P}$ is a distribution on the same manifold, we write "$X \in \mathcal{P}$" to indicate that X is a smooth (local) section of tangent subbundle $\mathcal{P}$. We will use $X(T)$ to denote the Lie derivative of a tensor T along X. We denote by $\langle v_1, v_2\rangle$ the linear span of vectors v_1, v_2. Finally, first and second order jet coordinates will be indifferently denoted either by $z_x, z_y, z_{xx}, z_{xy}, z_{yy}$ or p, q, r, s, t, respectively.

1.2 Introduction to the main results

In this paper we present a method to obtain families of solutions of *parabolic* Monge-Ampère equations (PMAEs). Such method is based on the geometrical approach to PMAEs developed in several previous papers (see [2, 3, 4, 5]).

MAEs are second order PDEs of the form

$$N(z_{xx}z_{yy} - z_{xy}^2) + Az_{xx} + Bz_{xy} + Cz_{yy} + D = 0\,, \tag{1}$$

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in the unknown function $z = z(x,y)$, with coefficients A, B, C, D and N depending on x, y, z, z_x, z_y. As it is well known, any contact transformation maps a MAE into another one. As we said, in what follows we will be concerned with PMAEs, which form the following subclass of (1):

$$N(z_{xx}z_{yy} - z_{xy}^2) + Az_{xx} + Bz_{xy} + Cz_{yy} + D = 0\,, \quad B^2 - 4AC + 4ND = 0. \tag{2}$$

Geometrically, this means that characteristic directions at any point of the contact manifold $M = \{(x, y, z, z_x, z_y)\}$ fill a 2-dimensional subdistribution $\mathcal{D}$ of the contact distribution $\mathcal{C}$ of M:

$$\mathcal{C} = \{\theta = 0\}, \text{ with } \theta = dz - z_x dx - z_y dy.$$

We call $\mathcal{D}$ the *characteristic distribution* of the PMAE. As $\mathcal{D}$ is generally non integrable, it is necessary to consider also its derived flag

$$\mathcal{D} \subset \mathcal{D}' = \mathcal{D} + [\mathcal{D}, \mathcal{D}] \subset \mathcal{D}'' = \mathcal{D}' + [\mathcal{D}', \mathcal{D}']. \tag{3}$$

Generically, $\mathcal{D}'' = TM$. Symmetry properties of (3) allow to obtain important classification results on PMAEs in a simple and straightforward way. In fact, such a study is based, to a large extent, on the geometry of *Cartan fields*, i.e. sections of $\mathcal{C}$. Cartan fields are not all contact-equivalent: there exists a simple invariant, called *type*, which is an integer number belonging to $\{2, 3, 4\}$ that measures the degree of "genericity" of a Cartan field X: the higher the type, the less symmetric X is with respect to $\mathcal{C}$. More precisely, X is of type $2, 3$ or 4 if it is contained in many, one or no integrable 2-dimensional subdistribution of $\mathcal{C}$, respectively. As we shall see, the existence of a Cartan field contained in the characteristic distribution of a PMAE gives information about the existence of various kinds of intermediate integrals (classical, non-holonomic, complete).
Roughly speaking, a complete integral of (2) is any 3-parametric family of solutions: its existence is equivalent to the existence of a *generalized intermediate integral*, i.e. a Cartan field of type less than 4 contained in the characteristic distribution $\mathcal{D}$. This is a generalization of both the classical ([7]) and the non-holonomic ([10]) notion of intermediate integral; in fact, a classical intermediate integral $f \in C^\infty(M)$ of (2) can be identified with a *hamiltonian field* X_f (a special kind of type 2 Cartan field) contained in $\mathcal{D}$, whereas a non-holonomic intermediate integral is any Cartan field of type 2 in $\mathcal{D}$.
The existence of intermediate integrals of equation (2) is strictly linked to integrability properties of the derived flag (3). In particular, we will focus on the following case: when $\mathcal{D}''$ is 4-dimensional and integrable, the associated PMAE is contact equivalent to

$$z_{yy} = b(x, y, z, z_x, z_y), \quad \partial_{z_x}(b) \neq 0. \tag{4}$$

The characteristic distribution of above PMAE is

$$\mathcal{D} = \langle \partial_p, \widehat{\partial}_y + b\partial_q \rangle\,, \quad \widehat{\partial}_y := \partial_y + q\partial_z$$

The method for finding complete integrals of equation (4) consist of 4 steps:

1. for an arbitrary coefficient $F \in C^\infty(M)$, consider the Cartan field
$$X := F\partial_p + \widehat{\partial}_y + b\partial_q; \tag{5}$$
2. impose on X the condition of being of type 3:
$$\operatorname{rank}\{\theta, X(\theta), X^2(\theta), X^3(\theta)\} = 3; \tag{6}$$
3. find the only integrable 2-dimensional distribution containing X:
$$\widehat{\mathcal{D}} := \{\theta = 0, X(\theta) = 0, X^2(\theta) = 0\};$$
4. integrate $\widehat{\mathcal{D}}$, thus obtaining ∞^3 (generalized) solutions of (5); if M is the first order jet bundle of $\mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ (which, locally, is always the case), then check whether such surfaces in M are prolongations of classical solutions $\mathbb{R}^2 \to \mathbb{R}$ or not.

The above method will be illustrated by several examples of linear and non-linear MAEs, including (linear and non-linear) heat and Burgers' equations.

We underline that in the case of the heat equation, provided that $F = F(x, y)$, a straightforward calculation shows that F produces a Cartan field of type 3 if and only if

$$F_{yy} = F_x,$$

i.e. if F is a solution to the heat equation. This implies that, from each solution of the heat equation, we get a 3-parametric family of solutions. These, in turn, generate new families of solutions, and so on. This recursive procedure allows us to obtain some hierarchies of solutions starting from simple ones. In particular, we obtain all polynomial solutions of the heat equation by starting from the constant solution. Of course this procedure applies to all MAEs that are contact-equivalent to the heat equation.

Finally, it is easily seen that, given a torsion-free linear connection on $\mathbb{R}^3$, the system of second order PDEs describing totally geodesic surfaces defines a section of $J^2(\tau) \to J^1(\tau)$, where $\tau : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ is the trivial bundle. But such a section is just a Lagrangian distribution on the contact manifold $J^1(\tau)$, so that it corresponds, for what was said above, to a PMAE: in the case of a metric connection, such PMAE turns out to be the equation of the zero-Gaussian curvature for surfaces. Thus, such PMAEs, if non-generic, can be put in some more simple form (such forms are listed in Theorem 4) by a contact transformation. We apply this observation to a simple example: indeed we see which kind of zero-Gaussian curvature equation corresponds to the heat equation via a partial Legendre transformation. Then, we exploit the knowledge of some solutions to the heat equation to get solutions of the corresponding zero-Gaussian curvature equation.

2 Preliminary notions

In this section we briefly survey and sum up, without proofs, the main results of [4, 5] which will be necessary for our purposes.

2.1 Contact manifolds and scalar PDEs with two independent variables

Although most of what follows is generalizable to higher dimensions (see [2] for an introduction to the field), we will limit ourselves to MAEs with two independent variables. In order to define such MAEs, we need to consider a 5-dimensional contact manifold.

Definition 1. *A 5-dimensional smooth manifold M endowed with a 4-dimensional distribution $\mathcal{C} = \mathrm{Ker}\,\theta$, where*

$$d\theta \wedge d\theta \wedge \theta \neq 0$$

is called a (5-dimensional) contact manifold. A diffeomorphism of M which preserves $\mathcal{C}$ is called a contact transformation.

Note that θ induces a conformal symplectic structure on the contact distribution $\mathcal{C}$ by means of the 2-form $d\theta$. We will denote by $W^\perp$ the orthogonal complement of $W \subset \mathcal{C}$ w.r.t $d\theta$.

Example 1. *Let V be a 3-dimensional manifold. The manifold whose points are the 2-dimensional tangent planes of V will be denoted $J^1(V, 2)$ (the classical space of contact elements in the sense of Sophus Lie, i.e. PT^*V)and called* space of 1-jets of 2-submanifolds *of V. Each jet $\mathfrak{p} \in J^1(V, 2)$ represents a plane $L_\mathfrak{p}$ of T_pV; so, we have a projection $\pi_{10}\colon J^1(V, 2) \to V$, $\mathfrak{p} \mapsto p$. If x, y, z are local coordinates on V, each tangent plane $\mathfrak{p}$ is described by*

$$\widetilde{\theta}(\mathfrak{p}) := (dz)_p - p_0(dx)_p - q_0(dy)_p\,,\ \pi_{10}(\mathfrak{p}) = p$$

(the parameters p_0 and q_0 are the directional coefficients); in other words, we are considering first order Taylor expansion $z = z_0 + p_0(x - x_0) + q_0(y - y_0)$; then, by dropping 0 subindexes, (x, y, z, p, q) represents a set of coordinates on $J^1(V, 2)$. The contact distribution $\mathcal{C}$ can be defined in this way: $X_\mathfrak{p} \in \mathcal{C}_\mathfrak{p}$ if and only of $(\pi_{10})_*(X_\mathfrak{p}) \in L_\mathfrak{p}$. Alternatively, $\theta_\mathfrak{p} := (\pi_{10})^*\widetilde{\theta}(\mathfrak{p})$ (defined as above) and $\mathcal{C} = \mathrm{Ker}(\theta)$. As a consequence, $\theta = dz - pdx - qdy$.

The local expression obtained in the above example holds in general: there always exists a set of coordinates (x, y, z, p, q) w.r.t. the contact form θ (which is defined up to a conformal factor) is given by

$$\theta = dz - pdx - qdy.$$

Such coordinates are called *contact (or Darboux) coordinates.* Locally defined vector fields

$$\widehat{\partial}_x \overset{\text{def}}{=} \partial_x + p\partial_z, \quad \widehat{\partial}_y \overset{\text{def}}{=} \partial_y + q\partial_z, \quad \partial_p, \quad \partial_q$$

span the contact distribution $\mathcal{C}$. The 2-form $d\theta$ is non-degenerate on $\mathcal{C}_m$, $\forall\, m \in M$, so that it is a conformal symplectic structure $d\theta|_{\mathcal{C}}$ in the distribution $\mathcal{C}$.

Recall that a *Legendre transformation* is a local contact transformation $(x, y, z, p, q) \to (x', y', z', p', q')$ defined by

$$x' = p\,,\ y' = q\,,\ z' = z - px - qy\,,\ p' = -x\,,\ q' = -y. \tag{7}$$

The action of such transformation on vector fields interchanges the roles of variables x, y and p, q, respectively; in fact

$$\partial_z \mapsto \partial_{z'}, \quad \widehat{\partial}_x \mapsto -\partial_{p'}, \quad \widehat{\partial}_y \mapsto -\partial_{q'}, \quad \partial_p \mapsto \widehat{\partial}_{x'}, \quad \partial_q \mapsto \widehat{\partial}_{y'}.$$

Sometimes it is useful to define a "partial" Legendre transformation; for instance, we can take

$$x' = p\,,\ y' = y\,,\ z' = z - px\,,\ p' = -x\,,\ q' = q;$$

in this case, only the coordinates x and p are interchanged (joint the corresponding partial derivatives).

Recall also that an integrable subdistribution of $\mathcal{C}$ is $d\theta$-isotropic, hence it has dimension ≤ 2. As it is well know, any 2-dimensional integral submanifold of $\mathcal{C}$, that is to say, a *Lagrangian submanifold*, if parametrizable by (x, y), is of the form:

$$z = \phi(x, y), \quad p = \frac{\partial \phi}{\partial x}, \quad q = \frac{\partial \phi}{\partial y}. \tag{8}$$

Definition 2. *A scalar first order partial differential equation (1^{st} order PDE) with two independent variables is a hypersurface $\mathcal{F}$ of a 5-dimensional contact manifold $(M, \mathcal{C})$. A solution of $\mathcal{F}$ is, by definition, a Lagrangian submanifold contained in $\mathcal{F}$.*

If $\mathcal{F} = \{f(x^i, z, p_i) = 0\}$, $f \in C^\infty(M)$, a solution parametrized by (x, y) can be written as (8), where the function ϕ satisfies

$$f\left(x, y, z, \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}\right) = 0,$$

which coincides with the classical notion of solution.

Remark 1. *The role of coordinates "x, y", as independent variables is purely external. A contact transformation can change the aforesaid role (see (7)).*

Definition 3. *The prolongation of a contact manifold $(M, \mathcal{C})$ is the fiber bundle $\pi : M^{(1)} \to M$ where*

$$M^{(1)} = \bigcup_{m \in M} \mathcal{L}(\mathcal{C}_m)$$

with $\mathcal{L}(\mathcal{C}_m)$ the Lagrangian Grassmannian of $(\mathcal{C}_m, (d\theta)_m)$. A point of $M^{(1)}$ will be denoted either by m^1 or by L_{m^1}.

The tangent space to the fibres of π has a special structure. Namely, $T_{m^1}\mathcal{L}(\mathcal{C}_m) \simeq L^*_{m^1} \odot L^*_{m^1}$, where $\odot$ is the symmetric product. This implies that the notion of *rank* of a vertical tangent vector, i.e. a vector belonging to $T_{m^1}\mathcal{L}(\mathcal{C}_m)$ is well defined. In the next section we will see that vectors of rank 1 are strictly related with characteristic directions of a given PDE.

Example 2. *If $M = J^1(V, 2)$, we can consider the space $J^2(V, 2)$ of* 2-jets of 2-submanifolds of V. *Its elements are the equivalence classes of 2-submanifolds having a contact of order 2: if (x, y, z) are local coordinates on V, then theses classes correspond to second order Taylor expansions $z = z_0 + p_0(x - x_0) + q_0(y - y_0) + \frac{1}{2}r_0(x - x_0)^2 + s_0(x - x_0)(y - y_0) + \frac{1}{2}t_0(y - y_0)^2$. The space defined in this way can be identified with the dense open set of $M^{(1)}$ comprised by Lagrangian planes which project on V without degeneracy.*

A system of contact coordinates (x, y, z, p, q) on M induces coordinates

$$(x, y, z, p, q, r, s, t), \tag{9}$$

on $M^{(1)}$ as follows: a point $m^1 \equiv L_{m^1} \in M^{(1)}$ has coordinates (9) iff the corresponding Lagrangian plane L_{m^1} is given by:

$$L_{m^1} = \langle \widehat{\partial}_x + r\partial_p + s\partial_q, \widehat{\partial}_y + s\partial_p + t\partial_q \rangle \subset \mathcal{C}_m,$$

where all vectors are taken in the point $m = \pi(m^1)$.

Each Lagrangian submanifold $\Sigma \subset M$ can be naturally lifted to $M^{(1)}$ by considering the set of tangent planes $\Sigma^{(1)} := \{T_m\Sigma \in M^{(1)} \,|\, m \in \Sigma\}$. If Σ can be parametrized by (8) then $\Sigma^{(1)}$ is

$$z = \phi(x,y), \quad p = \frac{\partial \phi}{\partial x}, \quad q = \frac{\partial \phi}{\partial y}, \quad r = \frac{\partial^2 \phi}{\partial x^2}, \quad s = \frac{\partial^2 \phi}{\partial x \partial y}, \quad t = \frac{\partial^2 \phi}{\partial y^2}. \tag{10}$$

Definition 4. *Let $(M, \mathcal{C})$ be a 5-dimensional contact manifold and $M^{(1)}$ its prolongation. A hypersurface $\mathcal{E}$ of $M^{(1)}$ is called a scalar second order partial differential equation (2^{nd} order PDE) with two independent variables. A solution of $\mathcal{E}$ is a Lagrangian submanifold $\Sigma \subset M$ whose prolongation $\Sigma^{(1)}$ is contained in $\mathcal{E}$.*

As in the first order case, if $\mathcal{E} = \{F(x,y,z,p,q,r,s,t) = 0\}$, then a solution Σ parametrized by x, y can be written as (10) where the function $\phi = \phi(x,y)$ satisfies the equation

$$F\left(x, y, \phi, \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial^2 \phi}{\partial x^2}, \frac{\partial^2 \phi}{\partial x \partial y}, \frac{\partial^2 \phi}{\partial y^2}\right) = 0,$$

which coincides with the classical notion of solution.

2.2 Monge-Ampère equations and characteristic distributions

The following definition is due to V.V. Lychagin (see [10, 11]).

Definition 5. *Let $\omega \in \Lambda^2(M)$ not in the differential ideal generated by θ. We associate with ω the hypersurface $\mathcal{E}_\omega$ of $M^{(1)}$ defined by*

$$\mathcal{E}_\omega \stackrel{def}{=} \{m^1 \in M^{(1)} \text{ s.t. } \omega|_{L_{m^1}} = 0\}, \tag{11}$$

where $L_{m^1} \subset T_{\pi(m^1)}M$ is the Lagrangian plane associated with m^1 (recall that π is the projection of $M^{(1)}$ onto M). Equations of this form are called Monge-Ampère equations *(MAEs).*

A generic MAE takes the form

$$N(rt - s^2) + Ar + Bs + Ct + D = 0, \tag{12}$$

with $N, A, B, C, D \in C^\infty(M)$ and, conversely, each equation (12) is a MAE.

There exists a one-to-one correspondence between MAEs (11) and 2-dimensional subdistributions of the contact distribution $\mathcal{C}$, up to $d\theta$-orthogonality. In fact, if $\mathcal{D} = \langle X, Y \rangle \subset \mathcal{C}$, then $\mathcal{E}_{X(\theta)\wedge Y(\theta)}$ is a MAE equation. Also, a direct computation shows that if we had started from $\mathcal{D}^\perp$ we would have obtained the same MAE. Conversely, if $\mathcal{E}_\omega$ is a MAE, we can canonically reconstruct the above distributions $\mathcal{D}$ and $\mathcal{D}^\perp$ in the following way. Let us consider a point $m^1 \in \mathcal{E}_\omega$ and a vector v of rank 1 at m^1 tangent to $\mathcal{E}_\omega$. Let us consider an adapted curve to v at m^1. The points of such curve are Lagrangian planes at $m = \pi(m^1)$: in view of the fact that v is of rank 1, all such Lagrangian planes have a common 1-dimensional intersection ℓ_{m^1}, called a *characteristic direction*. Since, up to a factor, there are only 2 vectors of rank 1 at m^1, then to such point two lines $\ell^1_{m^1}$ and $\ell^2_{m^1}$ of T_mM are associated. If we let the point m^1 vary on the fibre of $\mathcal{E}_\omega \to M$, then $\ell^1_{m^1}$ (resp. $\ell^2_{m^1}$) fills a 2-dimensional distribution $\mathcal{D}$ (resp. $\mathcal{D}^\perp$) such that, if $\mathcal{D} = \langle X, Y \rangle \subset \mathcal{C}$, then $\mathcal{E}_{X(\theta)\wedge Y(\theta)} = \mathcal{E}_\omega$. Distributions $\mathcal{D}$ and $\mathcal{D}^\perp$ are called the *characteristic distributions* of the MAE $\mathcal{E}_\omega$.

Note that the sign of

$$\Delta := B^2 - 4AC + 4ND$$

determines if equation (12) is hyperbolic ($\Delta > 0$, with two families of characteristic directions), elliptic ($\Delta < 0$, without characteristic directions) or parabolic ($\Delta = 0$, with one family of characteristic directions). Thus, in the case in which the MAE is parabolic, the characteristic distributions $\mathcal{D}$ and $\mathcal{D}^\perp$ coincide, i.e. we obtain a Lagrangian distribution. The above reasoning proves the following proposition (see [2, 3, 4, 5] for full details and a multidimensional generalization).

Proposition 1. *There is a one-to-one correspondence between MAEs (resp. PMAEs) and* 2*-dimensional (resp. Lagrangian) subdistributions of* $\mathcal{C}$ *up to* $d\theta$*-orthogonality: if* $\mathcal{D} = \langle X, Y\rangle \subset \mathcal{C}$ *then* $\mathcal{E}_{X(\theta)\wedge Y(\theta)}$ *is a MAE (parabolic if* $\mathcal{D}$ *is Lagrangian). Conversely, if* $\mathcal{E}$ *is a MAE (resp. PMAE), then, their characteristic distributions are* 2*-dimensional* $d\theta$*-orthogonal subdistributions of* $\mathcal{C}$ *(resp. coincide, thus giving a Lagrangian distribution). In particular, a PMAE is equivalent to a section of* $M^{(1)} \to M$*.*

In view of above proposition, we will indifferently write $\mathcal{E}_{X(\theta)\wedge Y(\theta)}$ or $\mathcal{E}_{\mathcal{D}}$. From now on we will take into account only PMAEs. The above reasoning suggests the following alternative way of defining PMAEs:

$$m^{(1)} \in \mathcal{E}_{\mathcal{D}} \quad \text{if and only if} \quad L_{m^{(1)}} \bigcap \mathcal{D}_m \neq 0. \tag{13}$$

Property (13) allows us to say that a solution of a PMAE is a Lagrangian submanifold which non-trivially intersects its characteristic (Lagrangian) distribution. If the generators of $\mathcal{D}$ are locally expressed by

$$X = \widehat{\partial}_x + R\partial_p + S\partial_q\,, \quad Y = \widehat{\partial}_y + S\partial_p + T\partial_q, \tag{14}$$

with R, S, $T \in C^\infty(M)$, then

$$X(\theta) = dp - Rdx - Sdy\,, \quad Y(\theta) = dq - Sdx - Tdy\,,$$

from which it follows that equation $\mathcal{E}_{X(\theta)\wedge Y(\theta)} \subset M^{(1)}$ is

$$(s - S)^2 - (r - R)(t - T) = 0$$

(a particular case of (12)). By using total or partial Legendre transformations it can easily be showed that expression (14) is the most general coordinate representation of $\mathcal{D}$, up to contact transformations.

Remark 2. *In view Section 5, note that the above construction of a PMAE starting from its Lagrangian characteristic distribution can be immediately generalized to multidimensional MAEs (see [2] for full details). Namely, let* $(M^{2n+1}, \mathcal{C}^{2n})$ *be a* $(2n+1)$*-dimensional contact manifold with Darboux coordinates* $(x_1, \ldots, x_n, z, p_1, \ldots, p_n)$ *and let, as above,* $M^{(1)} = \{$*Lagrangian subspaces of* $\mathcal{C}\}$ *be the prolongation of* M*, locally described by coordinates* $(x_i, z, p_i, r_{ij} = r_{ji} = z_{x_ix_j})$*; then, any Lagrangian distribution* $\mathcal{D} = \mathcal{D}^\perp \subset \mathcal{C}$ *is of the form* $\mathcal{D} = \langle X_1, \ldots, X_n\rangle$*, with* $X_i = \partial_{x_i} + p_i\partial_z + R_{i1}(x, z, p)\partial_{p_1} + \cdots + R_{in}(x, z, p)\partial_{p_n}$*,* $R_{ij} = R_{ji}$*, and, hence, corresponds to the section*

$$r_{ij} = R_{ij}(x, z, p)$$

of bundle $M^{(1)} \to M$*. Therefore, exactly as in the two variables case, one can associate with* $\mathcal{D}$ *the* 2^{nd} *order scalar MAE* $\mathcal{E}_{\mathcal{D}} = \{$*Lagrangian subspaces non trivially intersecting* $\mathcal{D}\} \subset M^{(1)}$*, whose coordinate expression is*

$$\det \|r_{ij} - R_{ij}\| = 0$$

2.3 Geometry of Cartan fields

A *Cartan field* is by definition, a section X of the contact distribution $\mathcal{C}$. There is an integer that is canonically associated with the direction spanned by such a field.

Definition 6. *The* type *of a Cartan field* $X \in \mathcal{C}$ *is the rank of system*

$$\theta\,,\ X(\theta)\,,\ X^2(\theta)\,,\ X^3(\theta). \tag{15}$$

The type of a Cartan field can take the values 2, 3 or 4 (1 is not possible as θ has no characteristic symmetries. Furthermore, we do not consider higher order derivatives because $X^4(\theta)$ is always dependent on (15) since all of them are incident with X. Any 1-form $\alpha \in \Lambda^1(M)$ determines a Cartan vector field $X_\alpha \in \mathcal{C}$ by the relation

$$X_\alpha(\theta) = X_\alpha \lrcorner d\theta = \alpha - \alpha(Z)\theta\,, \quad \theta(X_\alpha) = 0\,,$$

where Z is the Reeb vector field (associated with θ) defined by conditions $\theta(Z) = 1$, $Z \lrcorner d\theta = 0$. In particular $X = X_{X(\theta)}$ for any Cartan field X. Although X_α depends on the choice of θ, its direction does not change. In the case $\alpha = df$, we simply write X_f instead of X_{df} and it will be called the *Hamiltonian vector field* associated with f. X_f coincides with the classical characteristic vector field of the first order equation $f = 0$. Its local expression in a contact coordinate system (x, y, z, p, q) on M is

$$X_f = \partial_p(f)\,\widehat{\partial}_x + \partial_q(f)\,\widehat{\partial}_y - \widehat{\partial}_x(f)\,\partial_p - \widehat{\partial}_y(f)\,\partial_q.$$

In particular: $X_x = -\partial_p$, $X_y = -\partial_q$, $X_z = -p\partial_p - q\partial_q$, $X_p = \widehat{\partial}_x$, $X_q = \widehat{\partial}_y$.
The following theorem is a consequence of the Darboux theorem.

Theorem 1 (structure of integrable distributions)**.** *Let $\mathcal{D}$ be a 2-dimensional distribution in $\mathcal{C}$. Then $\mathcal{D}$ is integrable if and only if it is spanned by two hamiltonian fields X_f e X_g, with f and g independent and in involution: $X_f(g) = X_g(f) = 0$. In this case, there is a contact chart x, y, z, p, q, such that $\mathcal{D} = \langle \partial_p, \partial_q \rangle$.*

The normal forms for Cartan fields are described in the following statement.

Theorem 2 ([4])**.** *Let $X \in \mathcal{C}$. Then, up to a factor,*

1) X is of type 2 if and only if, in a suitable contact chart, it takes the form

$$X = \partial_p \quad (\textit{Hamiltonian}) \quad \textit{or} \quad X = \partial_p + z\partial_q \quad (\textit{non-Hamiltonian}).$$

In this case, X is contained in the family of integrable distributions $\langle X, X_f \rangle$, where $X(f) = 0$.

2) X is of type 3 if and only if in a suitable contact chart, it takes the form

$$X = \partial_p + b\partial_q\,, \quad \textit{with } X(b) \neq 0;$$

In this case X is contained in just one integrable distribution, namely $\langle \partial_p, \partial_q \rangle$ which intrinsically can be defined as $\{\theta = 0, X(\theta) = 0, X^2(\theta) = 0\}$.

3) X is of type 4 if and only if it is not contained in any integrable distribution.

2.4 Normal forms of parabolic Monge-Ampère equations

Normal forms of PMAEs are derived from the corresponding normal forms of the associated characteristic distributions.

Let $\mathcal{E}$ be a second order PDE. An *intermediate integral* of $\mathcal{E}$ is a function $f \in C^\infty(M)$ such that solutions of 1^{st} order PDE $f = k$, $k \in \mathbb{R}$, are also solutions of $\mathcal{E}$. It can be showed that for PMAE this is equivalent to $X_f \in \mathcal{D}$, i.e. X_f is characteristic for the equation. Furthermore, as $\mathcal{D}$ is lagrangian, f is a first integral of any characteristic field of $\mathcal{E}_\omega$. It follows that there exist PMAEs without intermediate integrals and it is interesting to consider possible extensions of the classical notion:

Definition 7. *A* generalized intermediate integral *of a PMAE $\mathcal{E}_\omega$ is a field $X \in \mathcal{D}$ of type less than 4 (i.e. 2 or 3).*

Definition 8. *A complete integral of $\mathcal{E}_\omega = \mathcal{E}_\mathcal{D}$ is a 2-dimensional foliation of M whose leaves are solutions of $\mathcal{E}_\mathcal{D}$ or, equivalently, a 2-dimensional integrable distribution $\widehat{\mathcal{D}} \subset \mathcal{C}$ such that $\mathcal{D}_m \cap \widehat{\mathcal{D}}_m \neq 0$ for any $m \in M$.*

Let us now show the (almost) equivalence of the two above definitions: if $X \in \mathcal{D}$ is a generalized intermediate integral, it has type 2 or 3. Then, according to Theorem 2, X is included in, at least, an integrable distribution, that is a complete integral. Conversely, if $\widehat{\mathcal{D}}$ is a complete integral, the same theorem shows that each field in $\mathcal{D} \cap \widehat{\mathcal{D}}$ cannot be of type 4. This reasoning suggests the scheme proposed in the Introduction to obtain complete integrals of a PMAE. Indeed, once a vector field of type 3 is found, one can construct the only integrable distribution $\widehat{\mathcal{D}}$ containing it.

Theorem 3 ([4, 6])**.** *Let $\mathcal{D} \subset \mathcal{C}$ be a non generic lagrangian distribution. Then, there exist local contact coordinates on M in which $\mathcal{D}$ takes one of the following normal forms:*

a) $\mathcal{D} = \langle \partial_p, \widehat{\partial}_y \rangle$;

b) $\mathcal{D} = \langle \partial_p, \widehat{\partial}_y + b\partial_q \rangle$, $b \in C^\infty(M)$, $\partial_p(b) \neq 0$;

c) $\mathcal{D} = \langle \partial_p + z\partial_q, \widehat{\partial}_y - z\widehat{\partial}_x + b\partial_q \rangle$, $b \in C^\infty(M)$, $\partial_p(b) + z\partial_z(b) \neq 0$.

For self-consistency, we give a sketch of the proof of previous theorem. If $\mathcal{D}$ is integrable, then by a contact transformation we can put it in the form $\langle \partial_p, \partial_q \rangle$ in view of Theorem 1. Then we obtain the normal form *a)* by applying a partial Legendre transformation. When $\mathcal{D}$ is not integrable, $\mathcal{D}'$ has rank 3 and it is completely determined by a generator X of its $d\theta$-orthogonal complement. Normal forms for X are listed in Theorem 2. Consequently, we get normal forms *b)* and *c)*.

Finally, in order to obtain normal forms for PMAEs, it is sufficient to compute $\mathcal{E}_\omega$ where $\omega = X(\theta) \wedge Y(\theta)$ with $\mathcal{D} = \langle X, Y \rangle$. This observation, together Theorem 3, leads to the following

Theorem 4. *Let $\mathcal{E}$ be a PMAE and $\mathcal{D}$ be the corresponding characteristic distribution. If $\dim \mathcal{D}'' < 5$, then there exist local contact coordinates on M in which $\mathcal{E}$ takes one of the following normal forms:*

1) $z_{yy} = 0$, *when $\mathcal{D}$ is integrable;*

2) $z_{yy} = b, \ \ b \in C^\infty(M)$, $\partial_{z_x}(b) \neq 0$, *when $\mathcal{D}''$ is 4-dimensional and integrable;*

3) $z_{yy} - 2zz_{xy} + z^2 z_{xx} = b, \ \ b \in C^\infty(M)$ *with* $\partial_{z_x}(b) + z\partial_z(b) \neq 0$, *when $\mathcal{D}''$ is 4-dimensional and non-integrable.*

3 Construction of a complete integral starting from a vector field of type 3

In this section we implement the scheme exposed in the introduction and we will apply it to several PMAEs of type (4) including the heat and Burgers equations.

We recall that the first step consists in determining a vector field of type 3 inscribed in the characteristic distribution of the given PMAE. Then, in view of 2) of Theorem 2, we compute the only integrable distribution containing such Cartan field. Such integrable distribution is by definition a complete integral of the given PMAE. In more concrete words, to find a Cartan field of type 3, we must impose (6) with X given by (5). A priori we have 5 PDEs in the unknown F (i.e. the vanishing of all minor of order 4 of matrix obtained by $\{\theta, X(\theta), X^2(\theta), X^3(\theta)\}$). Actually such matrix turns out to be of rank one, so that the condition of X to be of type 3 reduces to the following single PDE:

$$\sum_{i,j=1}^{5} A_{ij} F_{x_i x_j} + \sum_{i,j=1}^{5} B_{ij} b_{x_i x_j} + \\ + b_p(b_z F + b_q F_y + p b_p F_z - b_x F_p - b_y F_q - b F_z + q b_q F_z - p b_z F_p - q b_z F_q + b_p F_x) = 0 \quad (16)$$

where F_{ij} and b_{ij} are the second derivative w.r.t. $(x_1, x_2, x_3, x_4, x_5) = (x, y, z, p, q)$ and A, B are the following symmetric matrices

$$A = \frac{1}{2}\begin{pmatrix} 0 & b_p & qb_p & b_pF & bb_p \\ b_p & 0 & pb_p & G & b_pF \\ qb_p & pb_p & 2pqb_p & qG + pb_pF & pbb_p + qb_pF \\ b_pF & G & qG + pb_pF & 2FG & bG + b_pF^2 \\ bb_p & b_pF & pbb_p + qb_pF & bG + b_pF^2 & 2bb_pF \end{pmatrix}, \quad B = -b_p \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & q & F & b \\ 0 & q & q^2 & qF & qb \\ 0 & F & qF & F^2 & bF \\ 0 & b & qb & bF & b^2 \end{pmatrix}$$

where $G := \widehat{\partial}_y(F) - \widehat{\partial}_x(b) + bF_q + FF_p - b_qF$. As one can see, finding solutions to equation (16) could be a quite difficult task. For this reason, for the computations of complete integrals of PMAEs, we will consider specific assumptions on the form of F.

Once a Cartan field X of type 3 is obtained, we have to find three linearly independent functions f_1, f_2, f_3 on the contact manifold M such that they satisfy the following system

$$h_0^i \theta + h_1^i X(\theta) + h_2^i X^2(\theta) = df_i \quad (17)$$

for some smooth functions h_0^i, h_1^i, h_2^i on M, where X is given by (5). This essentially means that the (integrable) distribution $\widehat{\mathcal{D}} = \{\theta = 0, \, X(\theta) = 0, \, X^2(\theta) = 0\}$ is described by $\{df_1 = 0, \, df_2 = 0, \, df_3 = 0\}$. For the sake of clarity, we will refer to the following equation

$$h_0 \theta + h_1 X(\theta) + h_2 X^2(\theta) = df. \quad (18)$$

Note that (18) implies

$$dh_0 \wedge \theta + h_0 \wedge d\theta + dh_1 \wedge X(\theta) + h_1 \wedge dX(\theta) + dh_2 \wedge X^2(\theta) + h_2 \wedge dX^2(\theta) = 0. \tag{19}$$

Equation (19) gives an overdetermined system of 10 PDEs in 3 unknown variables. After obtaining h_0, h_1, h_2 from (19), we can solve (18) to obtain three linearly independent functions f_1, f_2, f_3. The submanifold $\{f_1 = K_1\,, f_2 = K_2\,, f_3 = K_3\}$, $K_i \in \mathbb{R}$, is a complete integral of (4). In many cases, from a complete integral, classical solutions can be also obtained.

3.1 Some computations of complete integrals of the heat equation

We now illustrate the procedure described above by computing some examples of complete integrals of the heat equation

$$z_{yy} = z_x \tag{20}$$

The characteristic distribution of (20) is

$$\langle \partial_p\,, \widehat{\partial}_y + p\partial_q \rangle \tag{21}$$

and a vector field of type 3 contained in (21) is of the form

$$X := F\partial_p + \widehat{\partial}_y + p\partial_q\,, \quad F \in C^\infty(M). \tag{22}$$

To start with, we will consider the "most simple case", i.e., as a vector field of type 3 we take that obtained by putting $F = K_4$ in (22), with $K_4 \in \mathbb{R}$. In fact, it is to see that, in this case, the constant function is a solution of (16). After writing this case in full details, we will consider some less trivial examples. Namely, we will examine the cases when F ia a linear combination of exponential functions and when F depends on "first derivative" p and q. As we will see in Section 4 the computation of simple complete integrals of the heat equation is the starting point for generating new solutions.

So, let $F = K_4 \in \mathbb{R}$. In this case a set of solutions to (17) is provided by

$$\begin{array}{lll} h_0^1 = -1, & h_1^1 = y, & h_2^1 = \frac{1}{2}(2x - y^2) \\ h_0^2 = 0, & h_1^2 = 1, & h_2^2 = -y \\ h_0^3 = 0, & h_1^3 = 0, & h_2^3 = 1. \end{array} \tag{23}$$

Then, by integrating (17), we obtain

$$f_1 = \frac{1}{6}K_4 y^3 - \frac{1}{2}py^2 + qy - K_4 xy - z + xp\,,\ f_2 = \frac{1}{2}K_4 y^2 - py - K_4 x + q\,,\ f_3 = -K_4 y + p.$$

Therefore a complete integral is

$$f_1 = K_1\,,\ f_2 = K_2\,,\ f_3 = K_3\,, \quad K_i \in \mathbb{R}$$

so that, by solving the second the third equations w.r.t. p, q and then substitute them in the first one, we obtain the following 3-parametric family of solutions to (20):

$$z = \frac{1}{6}K_4 y^3 + \frac{1}{2}K_3 y^2 + K_4 xy + K_2 y + K_3 x - K_1 \tag{24}$$

Now we analyze the case in which the F the vector field of type 3 is (22) depend only on (x, y). In view of equation (16), in this case we have that

$$F_{yy} - F_x = 0, \tag{25}$$

i.e. F is a solution of the heat equation (in Section 4 we shall use this property for generating solutions starting from a known one). A particular solution of (25) is provided by

$$F = K_1 e^{K_3^2 x + K_3 y} + K_2 e^{K_3^2 x - K_3 y}\,, \quad K_1, K_2, K_3 \in \mathbb{R}. \tag{26}$$

In this case, a set of solutions to (19) is still given by (23). By integrating (17), we obtain the following three linearly independent functions

$$f_1 = \tfrac{1}{2K_3^3}\Big(K_1 e^{K_3^2 x + K_3 y}(K_3^2 y^2 - 2K_3^2 x - 2K_3 y + 2) + K_2 e^{K_3^2 x - K_3 y}(-K_3^2 y^2 + 2K_3^2 x - 2K_3 y - 2) \\ + K_3^3(-2z + 2qy + 2px - py^2)\Big)$$

$$f_2 = \tfrac{1}{K_3^2}\left(K_1 e^{K_3^2 x + K_3 y}(K_3 y - 1) - K_2 e^{K_3^2 x - K_3 y}(K_3 y + 1) - K_3^2(py - q)\right)$$

$$f_3 = -\tfrac{1}{K_3}\left(K_1 e^{K_3^2 x + K_3 y} - K_2 e^{K_3^2 x - K_3 y} - K_3 p\right)$$

Thus, a complete integral is $\{f_1 = K_4\,, f_2 = K_5\,, f_3 = K_6\}$, $K_i \in \mathbb{R}$. If we compute p and q from the second and third equations and substitute them in the first one, once fixed K_1, K_2 and K_3, we obtain the following 3-parametric family of solutions of the heat equation

$$z = 2K_1 e^{K_3^2 x + K_3 y} - 2K_2 e^{K_3^2 x + K_3 y} + K_3^3(K_6 y^2 + 2K_5 y + 2K_6 x - 2K_4) \tag{27}$$

We note that, in some sense, we have already obtained the above complete integral. Indeed (26) is actually a solution of the heat equation (see also Section 4 to see how to exploit this observation) and the above complete integral is a linear combination of (26) and of polynomials coming from (24).

Now we analyze the case in which the vector field of type 3 is (22) with

$$F = k_1 p + k_2 q + k_3\,, \quad k_1, k_3 \in \mathbb{R},\ k_2 \in \mathbb{R}\setminus\{0\}.$$

It straightforward to check that the above F satisfies (16). In this case, equation (19) is satisfied by the following functions

$$h_0 = c_1$$

$$h_1 = -\tfrac{c_1 k_1^2 pq - c_1 k_1 p^2 + c_1 k_1 k_2 q^2 + c_2 k_2 q + c_3 k_2 p}{(-k_1 pq + p^2 - k_2 q^2) k_2}$$

$$h_2 = \tfrac{c_1 k_2 q^2 + c_1 k_1 pq - c_2 k_1 q + c_3 k_2 q - c_1 p^2 + c_2 p}{(-k_1 pq + p^2 - k_2 q^2) k_2}$$

By using the above functions, by putting $h_j^i := h_j$ with $c_k = \delta_k^i$, we can integrate (17) and finding functions f_1, f_2 and f_3 such that $\{f_1 = K_1, f_2 = K_2, f_3 = K_3\}$ is a complete integral of the heat equation. We limited ourself to take into consideration the case $k_1 = k_2 = 1$, $k_3 = 0$. In this case, solutions to (17) are

$$f_1 = z - p + q\,, \quad f_2 = \frac{1}{2}\ln(p^2 - pq - q^2) + \frac{\sqrt{5}}{5}\operatorname{arctanh}\left(\frac{\sqrt{5}(p+2q)}{5p}\right) - 2x - y\,,$$

$$f_3 = -\frac{2\sqrt{5}}{5}\operatorname{arctanh}\left(\frac{\sqrt{5}(p+2q)}{5p}\right) + x + y.$$

Instead of the above f_2, we can consider the as a new f_2 the following linear combination:

$$f_{2\text{new}} := 2f_2 + f_3 = \ln(p^2 - pq - q^2) - 3x - y.$$

By solving the system $\{f_1 = K_1, f_{2\text{new}} = K_2, f_3 = K_3\}$, we obtain the following 3-parametric family of solutions to the heat equation:

$$z = \frac{3\sqrt{e^{3x+y+K_2}} \pm \sqrt{5}\sqrt{e^{3x+y+K_2}}\tanh\left(\frac{\sqrt{5}}{2}(x+y-K_3)\right) \mp \sqrt{5}K_1\sqrt{1 - \tanh^2\left(\frac{\sqrt{5}}{2}(x+y-K_3)\right)}}{\sqrt{1 - \tanh^2\left(\frac{\sqrt{5}}{2}(x+y-K_3)\right)}}$$

3.2 A complete integral of the Burger equation

In this and in the next section we apply the procedure of computing complete integrals described in the beginning of the section to non-linear MAEs of type (4). To start with, we consider the Burgers' equation

$$z_{yy} = z_x + z z_y. \tag{28}$$

We underline that Burgers' equation cannot be linearized by a contact transformation. In particular, it is not contact-equivalent to the heat equation. In fact, it is well known that the algebra of contact symmetries of a linear PDE contains an abelian subalgebra that is isomorphic to the linear space of its solutions. This is not the case of Burgers' equation as its algebra of contact symmetries is 5-dimensional (see for instance [9] for a general introduction to the argument). It is worth noting that the well-known Hopf-Cole transformation links the Burgers' equation to the heat equation by a non-local transformation in the sense that it maps a solution of the former to a solution of the latter, but it is not a map between the two equations.

A vector field of type 3 contained in the characteristic distribution of (28) is given by (5) with

$$F = \frac{1}{2}pz + \frac{pq}{z}$$

and $b = p + zq$. Let us put

$$F_1 = -2qz + 4p + z^3\,, \quad F_2 = qz^2 - 2q^2 + 2pz.$$

After a computation, we realize that a set of solutions of (17) is

$$\begin{cases} h_0^1 = \dfrac{2\sqrt{z^3F_1}\,\mathrm{arctanh}\left(\frac{F_1-4p}{\sqrt{z^3F_1}}\right)F_2(-F_1+6p) + 2z^2(3pz^2+2pq-4zq^2+2z^3q)F_1}{z^2F_1^2F_2}\,, \\[2ex] h_1^1 = \dfrac{6\,\mathrm{arctanh}\left(\frac{F_1-4p}{\sqrt{z^3F_1}}\right)\sqrt{z^3F_1}F_2 - zF_1(-F_1+12p)}{F_2F_1^2}\,,\ h_2^1 = -2\dfrac{2\,\mathrm{arctanh}\left(\frac{F_1-4p}{\sqrt{z^3F_1}}\right)\sqrt{z^3F_1}F_2 + zF_1(F_1-4p)}{zF_2F_1^2}\,, \end{cases}$$

$$\begin{cases} h_0^2 = \dfrac{4pF_2(F_1-6p)(\ln(z^2F_2)-2\ln(p)) - 4F_1(q^2F_1+3pF_2-3zp^2)}{F_1^2F_2p}\,, \\[2ex] h_1^2 = \dfrac{-12pz^2F_2(\ln(z^2F_2)-2\ln(p)) - 2z(3pz^2-4pq-4zq^2+2z^3q)F_1}{F_2F_1^2p}\,, \\[2ex] h_2^2 = -2\dfrac{-4pzF_2(\ln(z^2F_2)-2\ln(p)) - 2z(F_2-pz)F_1}{F_2F_1^2p}\,, \end{cases} \qquad \begin{cases} h_0^3 = \dfrac{F_1-6p}{2z^2}\,, \\[2ex] h_1^3 = -\dfrac{3}{2}\,, \\[2ex] h_2^3 = \dfrac{1}{z}\,, \end{cases}$$

and the corresponding f_i's are

$$f_1 = \frac{2z^2\,\mathrm{arctanh}\left(\frac{F_1-4p}{\sqrt{z^3F_1}}\right)}{\sqrt{z^3F_1}} + y\,, \quad f_2 = \frac{2z(-\ln(z^2F_2)+2\ln(p))}{F_1} + x\,, \quad f_3 = \frac{F_1}{4z}.$$

Thus, $\{f_1 = K_1\,,\ f_2 = K_2\,,\ f_3 = K_3^2\}$ (we put on the right hand side term of third equation K_3^2 for simplicity reasons) is a complete integral of equation (28). If we solve the second and third equations w.r.t. p and q and substitute in the first one, we obtain the following 3-parametric family of solutions of the Burgers' equation:

$$z = K_3\frac{-2\tanh(yK_3-K_1K_3) \mp 4\sqrt{2e^{-2K_3^2(x-K_2)} - 2\tanh^2(yK_3-K_1K_3)e^{-2K_3^2(x-K_2)}}}{(-8e^{-2K_3^2(x-K_2)} + \tanh^2(yK_3-K_1K_3) + 8\tanh^2(yK_3-K_1K_3)e^{-2K_3^2(x-K_2)})}$$

3.3 Complete integrals of other non-linear equations

In this section we apply our method to compute complete integrals of some non-linear PMAE. We start by considering

$$z_{yy} = z_x^2. \tag{29}$$

Even in this simple case, equation (16) is rather complicated. Anyway it is easy to see that a vector field of type 3 contained in the characteristic distribution of (29) is given by (5) with

$$F = p^2\,, \quad b = p^2$$

and a set of solutions to system (17) is

$$\begin{cases} h_0^1 = 0\,, \\ h_1^1 = 0\,, \\ h_2^1 = -\dfrac{1}{2p^3}\,, \\ f_1 = x + y + \dfrac{1}{p}\,, \end{cases} \qquad \begin{cases} h_0^2 = 0\,, \\ h_1^2 = 1\,, \\ h_2^2 = -\dfrac{1}{2p}\,, \\ f_2 = q - p\,, \end{cases} \qquad \begin{cases} h_0^3 = 1\,, \\ h_1^3 = -y\,, \\ h_2^3 = \dfrac{yp-1}{2p^2}\,, \\ f_3 = z - \ln(-p) - y(q-p)\,, \end{cases}$$

from which we obtain the following complete integral of (29):

$$z = -\ln(x + y + K_1) + K_2 y + K_3.$$

A more interesting example is provided by the so-called "non-linear heat equation"

$$z_x = \frac{\partial}{\partial y}(f(z)z_y).$$

We analyze the case in which $f = z$, so that the above equation becomes

$$z_x = \frac{\partial}{\partial y}(zz_y) = z_y^2 + zz_{yy} \tag{30}$$

Also in this case, PDE (16) is not easy to handle. However, we were able to find a solution: a vector field of type 3 contained in the characteristic distribution of (30) is given by (5) with

$$F = 3\frac{pq - q^3}{z}\,, \quad b = \frac{p - q^2}{z}.$$

A straightforward computation shows that a set of solutions to system (17) is

$$\begin{cases} h_0^1 = 0\,, \\ h_1^1 = \dfrac{z}{q^2 - p}\,, \\ h_2^1 = \dfrac{qz^2}{(q^2-p)^2}\,, \\ f_1 = y + \dfrac{zq}{q^2 - p}\,, \end{cases} \qquad \begin{cases} h_0^2 = 0\,, \\ h_1^2 = 0\,, \\ h_2^2 = -\dfrac{1}{3}\dfrac{z^2}{(q^2-p)^2}\,, \\ f_2 = x - \dfrac{1}{3}\dfrac{z}{q^2 - p}\,, \end{cases} \qquad \begin{cases} h_0^3 = 2\left(\dfrac{z}{p-q^2}\right)^{\frac{1}{3}}, \\ h_1^3 = -2q\left(\dfrac{z}{p-q^2}\right)^{\frac{4}{3}}, \\ h_2^3 = \dfrac{2}{3}(3q^2 - p)\left(\dfrac{z}{p-q^2}\right)^{\frac{7}{3}}, \\ f_3 = (2p - 3q^2)\left(\dfrac{z}{p-q^2}\right)^{\frac{4}{3}}, \end{cases}$$

from which we obtain the following complete integral of (30):

$$z = \frac{1}{6}\frac{(3K_2 - 3x)^{\frac{1}{3}}(K_1 - y)^2 - K_3 x + K_3 K_2}{(3K_2 - 3x)^{\frac{1}{3}}(K_2 - x)}.$$

4 Generation of solutions for MAEs equivalent to the heat equation

In this section we construct an algorithmic recursive procedure for generating solutions (starting from a known one) of MAEs which are equivalent to the heat equation (up to a contact transformation). We observe that among such MAEs we find those belonging to the following class:

$$z_{yy} = k_1 z_x + k_2 z_y + k_3 z, \quad k_1 \in \mathbb{R} \setminus \{0\},\ k_2, k_3 \in \mathbb{R}$$

In fact, equation (20) is transformed into equation $w_{yy} = w_x + 2\beta w_y + (-\alpha - \beta^2)w$ by considering the new unknown $w = e^{\alpha x + \beta y} z$ (see, for example [15], Ch.III, sec. 3)). Moreover, a change $s = \lambda x$, transforms w_x into λw_s.

The following propositions are useful for applying the aforementioned recursive procedure for generating solutions of the heat equation.

Proposition 2. *Solutions to equation* (20) *and vector fields of type* 3

$$X := F\partial_p + \widehat{\partial}_y + p\partial_q\,, \quad F = F(x, y)$$

contained in its characteristic distribution are in one-to-one correspondence.

Proof. In the case $F = F(x, y)$ and $b = p$, PDE (16) reduces to (25). □

Proposition 3. *Equation* (19) *is always satisfied by the following functions:*

$$\begin{aligned} h_0 &= c_1 \\ h_1 &= -c_1 y + c_2 \\ h_2 &= \tfrac{1}{2}(c_1 y^2 - 2c_2 y + 2c_3) - \tfrac{c_1}{k_1} x \end{aligned} \tag{31}$$

where X is given by (5) *with $F = F(x, y)$ satisfying $F_{yy} - F_x = 0$.*

Proof. Straightforward. □

Description of an algorithmic procedure for constructing solutions to (20) **starting from a known one.** Here we describe how to use the above proposition for obtaining a hierarchy of solutions of (20) (and, of course, to all equation contact-equivalent to it) starting from a know one. Let us consider a solution $z = z(x, y)$ to equation (20). In view of Proposition 2, such solution gives also a vector field of type 3 inscribed in the characteristic distribution of (20). We have already seen in Section 3 how to construct a complete integral starting from this kind of vector field. Such integrals are by definition a 3-parametric family of solutions of the considered MAE, thus by using them it is possible to construct a 3-parametric family of vector fields of type 3 inscribed in the characteristic distribution of (20). Then, we can start again the procedure. We note that the results contained in Proposition (3) permit to skip the discussion of the overdetermined system coming from (19) and go directly to compute the function f of equation (18).

4.1 Some hierarchies of solutions for the heat equation

In this section we compute some hierarchies of solutions for the heat equation (20) starting from some very elementary solutions. We have already seen in Section 3.1 that, if we consider as Cartan field of type 3 that obtained with $F = K_4 \in \mathbb{R}$ in (5), then we can construct the complete integral (24). As pointed out in Proposition 2, if we substitute F of (22) with function (24), we obtain a 3-parametric family of Cartan fields of type 3 contained in the characteristic distribution of the heat equation. Then, by using h_0, h_1, h_2 of (31), by putting $h^i_j := h_j$ with $c_k = \delta^i_k$, we can integrate (17). and by making the same procedure as above, we obtain the following solution to the heat equation:

$$\begin{aligned} z = \frac{1}{720}K_4 y^6 + \frac{1}{24}K_4 x y^4 + \frac{1}{120}K_3 y^5 + \frac{1}{4}K_4 x^2 y^2 + \frac{1}{6}K_3 x y^3 + \frac{1}{24}K_2 y^4 + \frac{1}{6}K_4 x^3 + \frac{1}{2}K_3 x^2 y \\ + \frac{1}{2}K_2 x y^2 - \frac{1}{6}K_1 y^3 + \frac{1}{2}K_2 x^2 - K_1 x y + \frac{1}{2}K_7 y^2 + K_7 x + K_6 y - K_5. \end{aligned} \tag{32}$$

If we iterate once again the recursive procedure, we obtain a solution of the heat equation which is a linear combination of the following polynomial:

$$1,\ y,\ y^2+2x,\ y^3+6yx,\ y^4+12y^2x+12x^2,\ y^5+20xy^3+60yx^2,\ y^6+30y^4x+180y^2x^2+120x^3,$$
$$y^7+42y^5x+420y^3x^2+840yx^3,\ y^8+56y^6x+840y^4x^2+3360y^2x^3+1680x^4,$$
$$y^9+72y^7x+1512y^5x^2+10080y^3x^3+15120yx^4.$$

Of course, we can repeat the recursive procedure an arbitrary number of times, obtaining to the n^{th} step a solution depending on $3n+1$ parameters. In this way we obtain the following proposition

Proposition 4. *The constant solution of the heat equation generates the hierarchy formed by all and only the polynomial solutions.*

Proof. Let us call "0 step" when we consider the constant solution, then 1^{st} step when we obtain (24), and so on. At the n^{th} step of the recursive procedure we obtain a solution which is a linear combination of the following polynomials:

$$p_m(x,y) := y^m + \sum_k \binom{m}{2k} \frac{(2k)!}{k} y^{m-2k} x^k\ ,\ 0 \le m \le 3n.$$

Linear combination of $p_m(x,y)$ are the only polynomial solutions of the heat equation (see for instance [8]). □

We note that, if we consider, in the 0 step, the F given by (26), the recursive procedure do not give new solutions. Indeed, after the 1^{st} step, we obtain (27), so that if we iterate the procedure, we obtain (26) plus the polynomial solutions. A similar phenomenon appears if we consider in the 0 step $F = K_1\cos(n\pi y)e^{-n^2\pi^2 x} + K_2\sin(n\pi y)e^{-n^2\pi^2 x}$.

5 Zero-Gaussian curvature equation, totally geodesic equation and PMAEs

Let (V^{n+1}, ∇) be a smooth manifold with a torsionless affine connection ∇, and let $S \subset V$ be a smooth hypersurface, locally described by equation

$$F = 0,$$

with $F \in C^\infty(V)$ such that $d_xF \neq 0$ for any $x \in S$. With S one can associate the *second fundamental form*:

$$h : D(S) \times D(S) \to C^\infty(S),$$

defined (up to a factor) by

$$h(X,Y) \stackrel{\text{def}}{=} \nabla_X(Y)(F), \tag{33}$$

for any vector fields $X, Y \in D(S)$ (for a more standard, but equivalent definition of h, see [13, 14]). In the case when V is a semi-Riemannian manifold, with metric g, and ∇ is its Levi-Civita connection, definition (33) reduces to the standard one by choosing a function F whose gradient field coincides, on S, with the normal vector field N.
S is said to be *degenerate* when such is its second fundamental form, i.e.,

$$\operatorname{rank} h < n = \dim S, \tag{34}$$

at any point of S. For an analogous notion in projective differential geometry, related with various degeneracies of the Gauss Map, see [1] In local coordinates, condition (34) is expressed as follows. Let $x_1, \ldots, x_n, x_{n+1} = z$ be a local chart on V and let

$$F(x_1, \ldots, x_n, z) = f(x_1, \ldots, x_n) - z = 0$$

be the local expression of S in such chart; then, vector fields

$$\widehat{\partial}_{x_i} \stackrel{\text{def}}{=} \partial_{x_i} + f_{x_i}\partial_z,$$

$i=1,\dots,n$, form a basis of $T(S)$ with respect to which h is represented by matrix $\|h_{ij}\|$, with

$$h_{ij} = h(\widehat{\partial}_{x_i},\widehat{\partial}_{x_j}) = \nabla_{\widehat{\partial}_{x_i}}(\widehat{\partial}_{x_i})(F)$$

$$= \nabla_{\partial_{x_i}}(\partial_{x_j})(F) - f_{x_ix_j} + f_{x_j}\nabla_{\partial_{x_i}}(\partial_z)(F) + f_{x_i}\{\nabla_{\partial_z}(\partial_{x_j})(F) + f_{x_j}\nabla_{\partial_z}(\partial_z)(F)\},$$

i.e.,

$$h_{ij} = -f_{x_ix_j} + R_{ij}(x,z,f_x) \tag{35}$$

with

$$R_{ij} = \sum_{k=1}^{n} f_{x_k}\left(\Gamma^k_{ij} + f_{x_j}\Gamma^k_{i,n+1} + f_{x_i}\Gamma^k_{n+1,j} + f_{x_i}f_{x_j}\Gamma^k_{n+1,n+1}\right) - \left(\Gamma^{n+1}_{ij} + f_{x_j}\Gamma^{n+1}_{i,n+1} + f_{x_i}\Gamma^{n+1}_{n+1,j} + f_{x_i}f_{x_j}\Gamma^{n+1}_{n+1,n+1}\right),$$

the Γ^r_{ij} being the Christoffel symbols of ∇ in the chart $(x_1,\dots,x_n,z)$, $z=x_{n+1}$. Hence, condition (34) translates into

$$\det\|h_{ij}\| = \det\left\|f_{x_ix_j} - R_{ij}\right\| = 0, \tag{36}$$

which is a scalar PMAE in the unknown function $f(x_1,\dots,x_{n-1})$. If the rank of h is zero, i.e. h vanishes on S, then we obtain a system of PDEs described by the vanishing of (35):

$$f_{x_ix_j} - R_{ij} = 0 \tag{37}$$

Below we comment equation (36) and system (37).

System (37) can be interpreted as a section of $J^2(V,n)\to J^1(V,n)$ (see also Remark 2 and [2, 12]), i.e. as a Lagrangian distribution on $J^1(V,n)$. By construction, such Lagrangian distribution is the characteristic distribution of (36), and viceversa.

System (37) describes totally geodesic hypersurfaces of V. Indeed, this is the case when the connection ∇ is the Levi-Civita connection of some (semi-Riemannian) metric. In fact, a submanifold S is called *totally geodesic* when any geodesic curve of S with respect to the induced metric $g|_S$ is also a geodesic of V. As is well known, such property is equivalent to the identically vanishing of the second fundamental form h on S. Now, the latter condition makes sense also for an affinely connected manifold (V^{n+1},∇) (in fact, the definition of (33) was given only for hypersurfaces, but it can be easily extended to submanifolds of any codimension, see also [12, 14]); so, totally geodesic submanifolds of (V,∇), with ∇ symmetric, are by definition those whose second form h identically vanishes.

For what concerns (36), we call it the equation of *zero-Gaussian curvature*. The reason for choosing this name is that, in the case of a Levi Civita connection, (36) describes hypersurfaces with vanishing Gauss-Kronecker curvature. We note that, even in the case in which we are dealing only with an affinely connected manifold (V^{n+1},∇), equation (36) can be canonically obtained. In the 3-dimensional case ($n=2$), this is a classical PMAE (i.e. with two independent variables). To sum up, we have obtained the following result

Proposition 5. *The equation of zero-Gaussian curvature of hypersurfaces of an affinely connected manifold (V,∇) is a MAE (PMAE in the case when $\dim(V)=3$) whose characteristic distribution is the Lagrangian distribution defining totally geodesic hypersurfaces, and viceversa.*

Now we show how the system of PDEs describing m-dimensional totally geodesic submanifolds can be obtained via the exponential map. Recall that, for any point $p\in V$, one can define the exponential map

$$\exp\colon T_p(V)\ni\xi\longmapsto\gamma_\xi(1)\in V,$$

where $t\mapsto\gamma_\xi(t)$ is the parametrized geodesic passing through p with velocity ξ; such map is a diffeomorphism between a neighborhood of $0\in T_p(V)$ and a neighborhood of p in V. Now, for any $\mathfrak{p}^1\subset J^1(V,m)$, let $L_{\mathfrak{p}^1}\subset T_pV$ be the m-dimensional linear subspace representing $\mathfrak{p}^1$; then, we can associate with $\mathfrak{p}^1$ the 2-jet of the submanifold image of $L_{\mathfrak{p}^1}$ via the exponential map:

$$\sigma_\nabla(\mathfrak{p}^1) := j^2_p\left(\exp(L_{\mathfrak{p}^1})\right).$$

In this way a section $\sigma_\nabla\colon J^1(V,m)\to J^2(V,m)$ is defined; the image of σ_∇ is, by construction, the second order PDE of the totally geodesic m-submanifolds.

5.1 A complete integral of a zero Gaussian curvature equation via a contact transformation

Let us consider the case when V is 3-dimensional. In this case, using coordinates $(x, y, z, z_x, z_y, z_{xx}, z_{xy}, z_{yy})$, system (37) reads as follows:

$$\begin{cases} z_{xx} + \Gamma^3_{11} + (2\Gamma^3_{13} - \Gamma^1_{11})z_x - \Gamma^2_{11} z_y + (\Gamma^3_{33} - 2\Gamma^1_{13})z_x^2 - 2\Gamma^2_{13} z_x z_y - \Gamma^1_{33} z_x^3 - \Gamma^2_{33} z_x^2 z_y = 0 \\ z_{xy} + \Gamma^3_{12} + (\Gamma^3_{23} - \Gamma^1_{12})z_x + (\Gamma^3_{13} - \Gamma^2_{12})z_y - \Gamma^1_{23} z_x^2 + (\Gamma^3_{33} - \Gamma^1_{13} - \Gamma^2_{23})z_x z_y \\ \qquad\qquad -\Gamma^2_{13} z_y^2 - \Gamma^1_{33} z_x^2 z_y - \Gamma^2_{33} z_x z_y^2 = 0 \\ z_{yy} + \Gamma^3_{22} - \Gamma^1_{22} z_x + (2\Gamma^3_{23} - \Gamma^2_{22})z_y - 2\Gamma^1_{23} z_x z_y + (\Gamma^3_{33} - 2\Gamma^2_{23})z_y^2 - \Gamma^1_{33} z_x z_y^2 - \Gamma^2_{33} z_y^3 = 0 \end{cases} \tag{38}$$

As we said before, system (38) describes a characteristic distribution of a PMAE of the form (36). It is of interest to find for which connections ∇ does such equation fall into one of the three non-generic classes of PMAEs listed in Theorem 4; this will be the subject of a future work. For the moment let us consider the following example. Let ∇ be the connection that has vanishing Christoffel symbols except $\Gamma^3_{22} = -x$. The corresponding PMAE (36), which reads:

$$z_{xx} z_{yy} - z_{xy}^2 - x z_{xx} = 0, \tag{39}$$

turns out to be contact-equivalent to the heat equation via the partial Legendre transformation

$$x' = p,\ y' = y,\ z' = z - xp,\ p' = -x,\ q' = q, \tag{40}$$

which we recall that is a contact transformation. Taking into account (38), equation (39) is the one describing surfaces $z = z(x, y)$ with zero-Gaussian curvature for the connection ∇.

If we have a complete integral of the heat equation, then we automatically get a 3-parametric family of surfaces with zero Gaussian curvature by means of the partial Legendre transformation.

As an example, take the integral (32) of the heat equation with $K_4 = 1$, $K_1 = K_2 = K_3 = 0$ and free K_5, K_6, K_7:

$$z = \frac{1}{720} y^6 + \frac{1}{24} x y^4 + \frac{1}{4} x^2 y^2 + \frac{1}{6} x^3 + K_7 \left(\frac{y^2}{2} + x \right) + K_6 y - K_5. \tag{41}$$

In order to apply transformation (40) to the above surface, we have also to consider the equations giving its extension to the space of 1-jets:

$$p = \frac{\partial z}{\partial x} = \frac{1}{24} y^4 + \frac{1}{2} x y^2 + \frac{1}{2} x^2 + K_7, \tag{42}$$

$$q = \frac{\partial z}{\partial y} = \frac{1}{120} y^5 + \frac{1}{6} x y^3 + \frac{1}{2} x^2 y + K_7 y + K_6. \tag{43}$$

Now, applying (40) to (41)–(43) and drop the prime symbols, we obtain

$$\begin{aligned} z - xp &= \frac{1}{720} y^6 - \frac{1}{24} p y^4 + \frac{1}{4} p^2 y^2 - \frac{1}{6} p^3 + K_7 \left(\frac{y^2}{2} - p \right) + K_6 y - K_5 \\ x &= \frac{1}{24} y^4 - \frac{1}{2} p y^2 + \frac{1}{2} p^2 + K_7 \\ q &= \frac{1}{120} y^5 - \frac{1}{6} p y^3 + \frac{1}{2} p^2 y + K_7 y + K_6 \end{aligned}$$

The second equation gives

$$p = \frac{1}{2} y^2 \mp \frac{1}{6} \sqrt{6y^4 + 72x - 72K_7} =: \phi(x, y, K_7)$$

so that

$$z = x\phi + \frac{1}{720} y^6 - \frac{1}{24} \phi y^4 + \frac{1}{4} \phi^2 y^2 + \frac{1}{6} \phi^3 + K_7 \left(\frac{y^2}{2} + \phi^2 \right) + K_6 y - K_5$$

represents a 3-parametric family of solutions of the zero Gaussian curvature equation (for the particular choice of linear connection in $\mathbb{R}^3$).

Remark 3. *Note that complete integrals obtained by this method are not always of classical type: for example, if we considered solution* (24)*, we would obtain "generalized" solutions in the sense that they cannot be parametrized by* x, y.

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