P. Diamond, P. Kloeden, V. Kozyakin,
A. Pokrovskii

Semi-Hyperbolicity and Bi-Shadowing

(preliminary version)

February 10, 2007

Springer

Preface

In the beginning of 90s the authors of this monograph proposed a generalization of the concept of hyperbolicity, first for differentiable mappings and later for Lipschitz mappings, which they called ‘semi-hyperbolicity’. This arose indirectly in the context of their research at that time on the effect of spatial discretization on the behavior of a dynamical system, in particular that of finite machine arithmetic in a computer representation of a dynamical system, and rapidly broadened into a series of papers in which differing aspects and applications of the concept were explored. These papers form the basis of this monograph, the aim of which is to present a more thorough and systematic development of the concept of semi-hyperbolicity, as well as to illustrate its practicality. While the connection with the theory of hyperbolic systems is important and will not be neglected, much of the motivation of the authors comes from their interest and background in applications of dynamical systems and this has naturally influenced the types of questions asked and investigated.

As it often happens, our everyday duties and new interests at long last distracted us from semi-hyperbolicity, bi-shadowing and all such things. So, the manuscript, almost completed, remained not finished formally.

To our surprise and pleasure, the whole theme of semi-hyperbolicity and bi-shadowing did not die. There appears a number of brilliant investigations. So, we decided to finalize, at least formally, the manuscript and make it available on the Web.

Brisbane–Frankfurt–Moscow–Cork
1992–2007

Phil Diamond
Peter Kloeden
Victor Kozyakin
Alexei Pokrovskii

Contents

1

Introduction

1.1 Modeling Dynamical Processes

Real world dynamical processes can be extremely complicated, yet numerous quite simple idealized mathematical models often appear to capture the essence of what is happening and allow useful predictions to be made. This fact, which is to some extent justifiable mathematically, has been central to the resounding success of rationalist scientific thinking over the past four hundred years. Moreover, a large amount of mathematical analysis owes its origin to the development of concepts and tools needed to formulate and investigate such idealized mathematical models, which, though relatively simple, are by no means trivial.

Traditionally mathematical models have involved differential equations, both ordinary and partial, thus when relevant representing continuous time dynamical systems on appropriately chosen state spaces. In contrast, much of the modern theory of dynamical systems has focussed on discrete time dynamical systems generated by iteration of a mapping f of a given state space X into itself, that is on difference equations

$$x_{n+1} = f\left(x_n\right), \qquad n = 0, 1, 2, \dots \; . \tag{1.1}$$

Though seemingly less complicated, their dynamical behavior can often be richer without the restricting constraint of continuous time, but in any case many results can often be transferred to the differential context by means of suspensions, time–1 maps or Poincaré return maps. Indeed, motivated by the properties of solutions of smooth differential equations, the mapping f in (1.1) is often assumed to be a diffeomorphism and the state space X a compact manifold.

Investigations of the long term or asymptotic behavior have preoccupied the development of the theory of dynamical systems. Not only has the behavior of individual systems been of interest, especially the existence of attractors, but also the classification of classes of systems that share common behavioral characteristics or retain them under transformation or perturbation. Stability is the crucial concept here, both within a given system and across a class of systems, and can correspondingly be distinguished as either dynamical stability or structural stability. Both types of stability here have connotations of robustness of dynamical behavior.

An idealized mathematical model f is typically taken as a given and as the starting point of a mathematical investigation, for which it remains a fixed reference point. Approximation methods may be required, but an approximating system f_h, for example difference equation (1.1) in the finite arithmetic field of a computer, and its behavior are then usually always compared back to that of the given idealized system.

In reality, an idealized mathematical model f is precisely that, a conveniently selected idealization of some nearby, but only imprecisely known system $\widetilde{f}$. In applications it is thus desirable that the properties of an approximation f_h to an idealized system f also say something about those of possible underlying systems $\widetilde{f}$ on which the idealization is based. Robustness of dynamical behavior is thus a particulary important issue in mathematical modeling.

1.1.1 Hyperbolicity

The concept of hyperbolicity in dynamical systems provides an elegant means of addressing the fundamental problem raised in mathematical modeling, that all systems close to an idealized system should share its essential dynamical properties. In a sense it combines and subsumes the idea of total stability, where all systems close to one with an asymptotically stable steady state have a small attracting set about this steady state, and the analogous idea of total instability. It is based on the elementary observation that a linear mapping with a saddle point, hence with both contracting stable and expanding unstable directions, is robust under small parameter changes of the linear mapping itself as well as under small nonlinear perturbations. This extends easily to a hyperbolic cycle with the proviso that the corresponding stable and unstable linear manifolds are successively mapped onto their counterparts under the linearized mapping. The profound insight of Anosov [5, 7, 6] was to generalize this idea to an arbitrary compact invariant subset (in his case a manifold) for each point of which there is a splitting (of the tangent space) into such stable and unstable linear manifolds.

Hyperbolicity and its implications have been intensively investigated since the 1960s led by Smale and his coworkers, with interest being heightened by the presence of chaos generating mechanisms within noncyclic hyperbolic sets (see, e.g. [38, 39] and bibliography therein). Identifying and classifying such sets have been fundamental tasks, as has been establishing their structural stability in which all systems in some neighborhood of a system with such a hyperbolic set are homeomorphic to it, thus generalizing the robustness asserted by the Hartman–Grobman theorem [39, 63, 64] for a system with a simple saddle point. Symbolic dynamics [72, 39] has been a key tool here, in which the existence of a homeomorphism between a system restricted to a hyperbolic set and the shift operator on a space of symbol sequences, that is the conjugacy of these systems, is established, thus allowing the more transparent complicated behavior of the latter system to be transferred back to the original system. Shadowing [9, 15, 14, 16, 18, 17, 49, 74], which asserts the existence of a true trajectory close to any approximate, pseudo-trajectory, is another useful property, suggesting that computer simulations do indeed reflect the dynamics of chaotic hyperbolic systems.

The importance of hyperbolicity in the theory of dynamical systems cannot be understated, yet there remains a very wide gap between the deep theoretical understanding that it provides and the application of these mathematical results to specific, practical dynamical systems. There are two reasons for this, one concerning the assumptions made about the systems under investigation and the other the nature of the mathematical assertions made and their proofs. Much of the theory of hyperbolic systems deals with diffeomorphisms on compact manifolds, yet many interesting systems lack the invertibility (eg. difference equations in population modeling) or the smoothness (eg. systems with hysteresis or control switchings) of diffeomorphisms. Of course, there have been attempts to generalize hyperbolicity to homeomorphisms or to noninvertible maps (eg. Ruelle's pre-hyperbolicity), but then the second issue of practicality becomes paramount. Hyperbolicity is an extremely difficult property to verify for specific systems generating diffeomorphisms and even more so for its more abstract generalizations to homeomorphisms and noninvertible maps. Indeed, nearly all mathematically confirmed examples of hyperbolic systems are artificial constructs. The nature of many theorems on hyperbolic systems and their proofs is another obstacle to their applicability to specific systems, particular as many proofs are nonconstructive or lack clear, tight estimates. Moreover, many assertions in theorems hold only 'generically', that is for systems belonging to some residual subset of systems; though 'typical' in such a sense, it says little about a specific model at hand.

1.1.2 Semi-Hyperbolicity

As restrictive as it may seem, hyperbolicity is nevertheless an ubiquitous characteristic of dynamical systems, at least in the sense that many dynamical systems satisfy some if not all of its defining properties and enjoy many of the dynamical consequences. In particular, smoothness and invertibility of the system are not essential, nor are the continuity and equivariance of the tangent space splitting at each point of a hyperbolic set, nor in fact is the invariance of the hyperbolic set itself. As mentioned above, various generalizations of the definition of hyperbolicity have been proposed, but their actual applicability to specific systems is problematical. What is urgently required is a pragmatic reformulation of the concept of hyperbolicity that is both widely and effectively applicable and at the same time directly addresses both the practical and philosophical issues raised by the use and analysis of mathematical modeling.

Several years ago the authors of this monograph proposed such a reformulation, first for differentiable mappings and later for Lipschitz mappings, which they called ‘semi-hyperbolicity’. This arose indirectly in the context of their research at that time on the effect of spatial discretization on the behavior of a dynamical system, in particular that of finite machine arithmetic in a computer representation of a dynamical system, and rapidly broadened into a series of papers in which differing aspects and applications of the concept were explored. These papers [2, 1, 3, 4, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 42, 43, 48] form the basis of this monograph, the aim of which is to present a more thorough and systematic development of the concept of semi-hyperbolicity, as well as to illustrate its practicality. While the connection with the theory of hyperbolic systems is important and will not be neglected, much of the motivation of the authors comes from their interest and background in applications of dynamical systems and this has naturally influenced the types of questions asked and investigated.

1.2 Outline of Book

The book consists of eight Chapters, List of Notation, Index and Bibliography.

In Chapter 1 we explain the necessity of introducing notions of semi-hyperbolicity and bi-shadowing from the point of view of mathematical modeling of real processes.

In Chapter 2 we briefly review background material on dynamical systems, particularly that involving differentiable hyperbolic mappings, and introduce terminology and results that will be required later.

In Chapter 3 the concept of semi-hyperbolicity is introduced and some examples are considered. Here different variants of definitions of semi-hyperbolicity are given aimed to generalize definition of hyperbolicity to differentiable mappings, which need not be invertible, so as to differentiable mappings on a general compact subset of $\mathbb{R}^d$ and then for a Lipschitz mappings. In both latter cases the subset on which the mapping is considered is not required to be invariant. Further generalizations of the concept of semi-hyperbolicity to mappings in Banach spaces will are given in Chapter 8.

The usefulness of first approximation methods to investigate the properties of semi-hyperbolic mappings and their trajectories leads us naturally to consider linear operators in spaces of sequences generated in one way or another by derivatives of semi-hyperbolic mappings. Some elementary properties of such linear operators that will be needed throughout the book are considered in Chapter 4.

In Chapter 4 it is also shown that a semi-hyperbolic sequence of matrices is hyperbolic, i.e. in the linear case semi-hyperbolicity implies hyperbolicity. This is generally not true for the nonlinear mappings, which will be shown by an example in Section 5.1 which demonstrates that a semi-hyperbolic mapping may not possess an invariant splitting and so cannot be a hyperbolic. Nevertheless, it is shown in Section 5.2 that a semi-hyperbolic mapping which is smooth and invertible in a neighborhood of a compact invariant set is, in fact, hyperbolic on that set. The proofs depend substantially on background material and results of Chapter 4.

In Section 2.2 it was mentioned that hyperbolic systems possess some rather strong and useful properties such as expansivity (see Definition 2.7 and Theorem 2.8) and shadowing (see Theorem 2.9). In Chapter 6 we will show that these and other properties remain valid for semi-hyperbolic systems. Moreover, explicit values or sharp estimates of relevant parameters and intervals of validity are obtained.

In Chapter 7 we consider properties of semi-hyperbolic mappings which can be conditionally qualified as structure-stability-properties. More precisely, it will be shown that topological entropy can only increase following to continuous perturbation of a Lipschitz semi-hyperbolic mapping. Then problems of conjugation and factorization of semi-hyperbolic mappings will be studied. And, at last, investigation of chaotic phenomena for semi-hyperbolic mappings will be discussed.

In the last Chapter 8 we briefly consider several applications of semi-hyperbolicity and its consequences, in particular to delay differential equations, systems with hysteresis and the numerical approximation of chaotic attractors.

2

Smooth Dynamical Systems

In this chapter we briefly review background material on dynamical systems, particularly that involving differentiable hyperbolic mappings, and introduce terminology and results that will be required later.

2.1 Hyperbolic Mappings

Major theoretical advances in the theory of dynamical systems have resulted from a proposal of D. Anosov in the early 1960s to extend the concept of a hyperbolic set from a finite cyclic set to a general compact invariant subset K. In particular, it was subsequently realized by S. Smale that such a generalization of hyperbolicity could be used to describe highly non-trivial recurrent behavior in dynamical systems as well as the robustness of such behavior. Their work and that of many others was in the context of diffeomorphisms on smooth compact manifolds, motivated in part by the fact that time-one mappings and return mappings of smooth differential equations are diffeomorphisms, and is now known as *differentiable dynamics*. An objective of this book is to show that complicated dynamical behavior is also possible under far less stringent assumptions.

In what follows $\|\cdot\|$ will denote a fixed, but otherwise arbitrary norm on $\mathbb{R}^d$.

2.1.1 Hyperbolic Cycles

Let X be an open subset of $\mathbb{R}^d$ and $f : X \mapsto X$ a differentiable mapping. Such a mapping generates a discrete time dynamical system through successive iteration, that is with

$$x_{n+1} = f(x_n), \qquad n = 0, 1, 2, \ldots ,$$

determining the trajectory of the system starting at the point $x_0 \in X$.

A trajectory starting at x_0 is called a *cycle* of f with period $p \geq 1$ if the successive points $x_0, x_1, \ldots, x_{p-1}$ are distinct and $x_p = x_0$. A point x_k of such a cycle is called a *periodic point* of f and a *hyperbolic periodic point*[1] if all of the eigenvalues $\lambda_j^{(k)}$ of the derivative Df_{x_k} satisfy the non-unit modulus condition:

$$|\lambda_j^{(k)}| \neq 1.$$

The eigenvalues of Df_{x_k} are, in fact, then the same at each of the points $\{x_0, x_1, \ldots, x_{p-1}\}$ of the cycle, so the superindex k can be omitted.

Suppose that the eigenvalues, after possible rearrangement, satisfy

$$|\lambda_j| < 1, \quad 0 \leq j \leq n_s, \quad |\lambda_j| > 1, \quad n_s + 1 \leq j \leq d \tag{2.1}$$

for some $0 \leq n_s \leq d$. Then the linear subspaces $E^s_{x_k}$ and $E^u_{x_k}$ spanned by the eigenvectors, and generalized eigenvectors if necessary, of Df_{x_k} corresponding to the eigenvalues with modulus less than 1 and modulus greater than 1, respectively, form a *splitting* or *decomposition* of $\mathbb{R}^d$, that is with

$$\mathbb{R}^d = E^s_{x_k} \oplus E^u_{x_k}.$$

with dimensions

$$\dim E^s_{x_k} = n_s, \quad \dim E^u_{x_k} = n_u := d - n_s \tag{2.2}$$

and projection operators

$$P^s_{x_k} : \mathbb{R}^d \mapsto E^s_{x_k}, \quad P^u_{x_k} : \mathbb{R}^d \mapsto E^u_{x_k}.$$

Note that an equivalent norm $\|\cdot\|_{x_k}$, which may depend on x_k, can be found such that $Df_{x_k}|_{E^s_{x_k}}$ is contractive and $Df_{x_k}|_{E^s_{x_k}}$ is expansive in the sense that there is a constant $\lambda > 1$ such that

$$\|Df_{x_k} u\|_{f(x_k)} \leq \lambda^{-1} \|u\|_{x_k}, \quad \|Df_{x_k} v\|_{f(x_k)} \geq \lambda \|v\|_{x_k}$$

for any $u \in E^s_{x_k}$ and $v \in E^u_{x_k}$. The linear subspaces $E^s_{x_k}$ and $E^u_{x_k}$ are thus usually called the *stable* and *unstable subspaces* at x_k, respectively. Condition (2.1) is in fact satisfied at each point of the cycle with the same n_s, so the corresponding stable and unstable spaces for each point of the cycle have the same dimensions (2.2). They are related by the *equivariance property:*

$$Df_{x_k}(E^s_{x_k}) = E^s_{f(x_k)}, \quad Df_{x_k}(E^u_{x_k}) = E^u_{f(x_k)}$$

for each k.

It is a trivial, yet nevertheless an important observation that a set $K = \{x_0, \ldots, x_{p-1}\}$ of the points of a cycle of f is *invariant* under f, or f-*invariant*, that is $f(K) = K$. In addition, the behavior of the dynamical system generated by the mapping f is robust with respect to small perturbations in the vicinity of such a cyclic a set K when its points are hyperbolic. This robustness manifests itself globally when the mapping f has only a finite number of cycles which are all hyperbolic.

[1] A hyperbolic periodic point of period 1 is called a *hyperbolic fixed point.*

2.1.2 Anosov Systems

Anosov began the investigation of non-cyclic hyperbolic sets with the concept of a *U-cascade*, which is now called an *Anosov system*, for a diffeomorphism f on an m-dimensional closed differentiable manifold M in $\mathbb{R}^d$. Recall that a diffeomorphism $f : M \mapsto M$ is an invertible mapping for which both f and its inverse f^{-1} are continuously differentiable on M.

In an Anosov system the tangent space $T_xM \cong \mathbb{R}^m$ at each point $x \in M$ is split into stable and unstable subspaces of the tangent mapping Tf_x at $x \in M$ with the separation

$$\delta(E_x^s, E_x^u) = \inf\{\|u - v\| : \ u \in E_x^s, \ v \in E_x^u, \ \|u\| = \|v\| = 1\}$$

between these linear subspaces being assumed uniformly bounded from below.

Definition 2.1 (Anosov System). *A diffeomorphism $f : M \mapsto M$ where M is an m-dimensional closed differentiable manifold in $\mathbb{R}^d$ generates an* Anosov system *on M if*

A1: *for each $x \in M$ there exists a splitting $T_xM = E_x^s \oplus E_x^u$ such that*

$$\dim E_x^s \equiv k \neq 0, \quad \dim E_x^u \equiv m - k \neq 0,$$

for some integer k independent of $x \in M$, and

$$Tf_x E_x^s = E_{f(x)}^s, \quad Tf_x E_x^u = E_{f(x)}^u;$$

A2: *there exist constants $\lambda > 1$ and $\varrho_0 > 0$ independent of $x \in M$ such that:*

$$\|Tf_x^n u\| \leq \varrho_0 \lambda^{-n} \|u\|, \quad \|Tf_x^n v\| \geq \varrho_0^{-1} \lambda^n \|v\|, \qquad n \geq 0,$$

$u \in E_x^s$ and $v \in E_x^u$;

A3: *there exists a constant $\varrho_1 > 0$ independent of $x \in M$ such that the separation*

$$\delta\left(E_x^s, E_x^u\right) \geq \varrho_1$$

for all $x \in M$.

An example of linear mapping f which generates an Anosov system and is called an *Anosov automorphism* will be considered in Chapter 3.

Definition 2.1 does not assume that the splitting depends continuously on x, by which it is meant that in a neighborhood of any point $x_0 \in M$ a set of vectors $e_1(x)$, $e_2(x)$, …, $e_m(x) \in T_xM$ which depend continuously on $x \in M$ can be chosen such that the vectors $e_1(x)$, $e_2(x)$, …, $e_k(x)$ form a basis of the subspace $E_x^s \subseteq T_xM$ and the vectors $e_{k+1}(x)$, $e_{k+2}(x)$, …, $e_m(x)$ form a basis of the subspace $E_x^u \subseteq T_xM$. This continuity is in fact a consequence of Condition A3 (cf. [6, 13]).

Theorem 2.2. *For an Anosov system generated by a diffeomorphism on a closed differentiable manifold M of $\mathbb{R}^d$ the splitting $T_xM = E_x^s \oplus E_x^u$ depends continuously on $x \in M$.*

Note that only compact Riemannian manifolds M were considered in [6, 13], but the proof of Theorem 2.2 there does not make use of this extra structure and in fact holds for a closed differentiable manifold.

2.1.3 Hyperbolic Diffeomorphisms

In an Anosov system the whole manifold M is a non-cyclic hyperbolic set. Smale extended the idea to an f-invariant subset K of the manifold. The term *hyperbolic mapping* will often be used loosely to refer to a mapping f that possesses such a hyperbolic invariant set K.

The usual definition of hyperbolicity found in the literature now refers to a diffeomorphism $f : M \mapsto M$ where M is a compact manifold and a compact f-invariant subset K of M. As for an Anosov system, the splitting is of the tangent space T_xM at each point $x \in K$, but the expansion and contraction on the stable and unstable subspaces of the tangent mapping Tf_x are expressed in terms of a Riemannian norm $\| \cdot \|_x$ on T_xM, which may vary with on the point $x \in K$, thus allowing the constant ϱ_0 in Condition A2 of an Anosov system to be set conveniently equal to 1.

Definition 2.3 (Hyperbolicity on a Manifold). *A closed subset K of a compact manifold M which is invariant for a diffeomorphism $f : M \mapsto M$ is said to be* hyperbolic *if there exists a splitting $T_xM = E_x^s \oplus E_x^u$ for each $x \in K$, which varies continuously in $x \in K$, a constant $\lambda > 1$ and a Riemannian norm $\| \cdot \|_x$ on T_xM such that*

H1*: *For all $x \in K$:*

$$Tf_x\left(E_x^s\right) = E_{f(x)}^s, \quad Tf_x\left(E_x^u\right) = E_{f(x)}^u,$$

H2*: *For all $x \in K$, $u \in E_x^s$ and $v \in E_x^u$:*

$$\|Tf_x u\|_{f(x)} \le \lambda^{-1} \|u\|_x, \quad \|Tf_x v\|_{f(x)} \ge \lambda \|v\|_x.$$

Conditions H1* and H2* in Definition 2.3 together imply that the splitting is continuous as in Theorem 2.2, which thus need not have been assumed, as well as the constancy of the dimension of the splitting subspaces along a trajectory of the mapping f.

Remark 2.4. The use of an equivalent adapted norm in Condition H2* in Definition 2.3 allows the constant ϱ_0 in Condition A2 of an Anosov system to be set equal to 1. This is convenient for many theoretical purposes, but can be cumbersome for specific practical examples. The Whitney Embedding Theorem provides one way of avoiding the problem and using the same norm at each point of $x \in K$ in the inequalities in Condition H2*. Essentially the manifold M is embedded in a higher dimensional space $\mathbb{R}^{d'}$ and the norm of this space can be used everywhere. Alternatively, Smale has shown that there is an iterate f^l of a hyperbolic diffeomorphism f in the sense of Definition 2.3 such that f^l is hyperbolic on the invariant set K with the inequalities in Condition H2* satisfied with respect to a norm that does not depend on the point of K.

The previous remark motivates the following variation of the definition of hyperbolicity for a mapping $f : X \mapsto \mathbb{R}^d$, where X is an open subset of $\mathbb{R}^d$ containing the f-invariant compact set K under consideration, which is a diffeomorphism on a neighborhood of the set K.

Definition 2.5 (Hyperbolicity). *A compact subset $K \subset X$ which is invariant for a diffeomorphism $f : X \mapsto \mathbb{R}^d$ is said to be* hyperbolic *if there exist a splitting $\mathbb{R}^d = E_x^s \oplus E_x^u$ for each $x \in K$, which varies continuously in $x \in K$, constants $\lambda > 1$ and $\varrho_0 > 0$, and a norm $\|\cdot\|$ on $\mathbb{R}^d$ such that*

H1: *For all $x \in K$:*

$$Df_x\left(E_x^s\right) = E_{f(x)}^s, \quad Df_x\left(E_x^u\right) = E_{f(x)}^u,$$

H2: *For all $u \in E_x^s$ and $v \in E_x^u$:*

$$\|Df_x^n u\| \leq \varrho_0 \lambda^{-n} \|u\|, \quad \|Df_x^n v\| \geq \varrho_0^{-1} \lambda^n \|v\|, \qquad n \geq 0.$$

2.1.4 Generalizations

Various generalizations of hyperbolicity to mappings other than diffeomorphisms have been proposed, in particular to homeomorphisms and to not necessarily invertible continuous mappings. Instead of a splitting into stable and unstable linear spaces, dynamically defined nonlinear stable and unstable sets are used.

Let (X, ϱ) be a compact metric space and let $f : X \to X$ be a homeomorphism of X onto itself. For an arbitrary point $x \in X$ and any $\varepsilon > 0$, the *local stable* and *unstable sets* of f of size ε at x are defined as

$$W_\varepsilon^s(x) = \{y \in X : \ \varrho(f^n x, f^n y) \leq \varepsilon, n \geq 0\}$$

and

$$W_\varepsilon^u(x) = \{y \in X : \ \varrho(f^n x, f^n y) \leq \varepsilon, n \leq 0\},$$

respectively, while the *stable* and *unstable sets* of f at x are defined as

$$W^s(x) = \{y \in X : \ \varrho(f^n x, f^n y) \to 0, \ n \to \infty\}$$

and

$$W^u(x) = \{y \in X : \ \varrho(f^n x, f^n y) \to 0, \ n \to -\infty\},$$

respectively.

Note that $\#\left(W_\varepsilon^s(x) \cap W_\varepsilon^u(x)\right) \geq 1$ for any $\varepsilon > 0$, where $\#A$ denotes the cardinality of the set A.

Definition 2.6 (Hyperbolic Homeomorphism). *A homeomorphism f of X onto itself is* hyperbolic *if there exist constants $\varepsilon_0 > 0$, $K > 0$ and $\lambda > 1$ such that*

$$\varrho(f^n x, f^n y) \leq K\lambda^{-n} \quad \text{for all} \quad x \in X, y \in W^s_{\varepsilon_0}(x) \quad \text{and} \quad n \geq 0,$$
$$\varrho(f^{-n} x, f^{-n} y) \leq K\lambda^{-n} \quad \text{for all} \quad x \in X, y \in W^u_{\varepsilon_0}(x) \quad \text{and} \quad n \geq 0,$$

and there exists $\delta_0 > 0$ such that

$$\# W^s_{\varepsilon_0}(x) \cap W^u_{\varepsilon_0}(y) = 1,$$

for all $x, y \in X$ with $\varrho(x, y) \leq \delta_0$.

The case of noninvertible f can be handled by a suggestion of D. Ruelle. Let $f : X \to X$ be a continuous mapping and for each $x \in X$ define sequence set

$$X^\dagger = \left\{ \{x_n\}_{n=-\infty}^{0} \in \prod_{n=-\infty}^{0} X : \ f(x_{-n}) = x_{-n+1}, \ n = 1, 2, \ldots \right\}$$

and the mapping $f^\dagger : X^\dagger \to X^\dagger$ by $f^\dagger(\{x_n\}) = \{f(x_n)\}$. Then $f^\dagger$ is a homeomorphism on $X^\dagger$ with respect to the metric $D(\{x_n\}, \{y_n\}) = \sup_{n \leq 0} \varrho(x_n, y_n)$ and f is said to be hyperbolic on X if $f^\dagger$ is hyperbolic on $X^\dagger$.

2.2 Fundamental Properties

The robustness of behavior of a dynamical system in the vicinity of a hyperbolic cycle is retained by dynamical systems with nontrivial hyperbolic sets, but now considerably more complicated dynamical behavior is possible. Some of the fundamental properties of hyperbolic diffeomorphisms will be summarized here without proof, focusing on those properties that carry over (possibly in modified form) to semi-hyperbolic mappings for which proofs will be presented in later Chapters. To simplify the exposition, these results will be stated in terms of a diffeomorphism $f : X \mapsto X \subset \mathbb{R}^d$ with a compact invariant hyperbolic subset K. Homeomorphisms will also be considered below.

2.2.1 Expansivity and Shadowing

An immediate consequence of hyperbolicity and portender of complicated dynamics within a hyperbolic set given its compactness and invariance is that a diffeomorphism is expansive on a hyperbolic set.

Definition 2.7 (Expansivity). *A homeomorphism $f : K \mapsto K$ is* expansive *on an invariant set K if there exists an $\varepsilon > 0$ such that for all x, $y \in K$ condition*

$$\|f^n(x) - f^n(y)\| \leq \varepsilon \qquad \text{for all} \quad n \in \mathbb{Z}$$

implies that $x = y$.

Theorem 2.8. *A diffeomorphism f with a hyperbolic set K is expansive on K.*

Thus no two distinct trajectories in a hyperbolic set can remain forever within a certain threshold of each other. Nevertheless, within the hyperbolic set there is always a true trajectory close to an approximate or pseudo-trajectory. A sequence $\{y_n\} \subset X$ is called a *δ-pseudo-trajectory* of a dynamical system generated by a mapping f if

$$\|y_{n+1} - f(y_n)\| < \delta \qquad \text{for all} \quad n \in \mathbb{Z}$$

and a true trajectory $\{x_n\}$ is said to *ε-shadow* a pseudo-trajectory $\{y_n\}$ if

$$\|y_n - x_n\| < \varepsilon \qquad \text{for all} \quad n \in \mathbb{Z}.$$

Theorem 2.9 (Shadowing Theorem). *If $f : X \mapsto X$ is a diffeomorphism with an f-invariant hyperbolic set $K \subset X$, then for every $\varepsilon > 0$ there exists a $\delta > 0$ and an open neighborhood U of K in X such that every δ-pseudo-trajectory of f in U is ε-shadowed by a true trajectory of f in K.*

A dynamical system satisfying the assertion of Theorem 2.9 will be said to have the *shadowing property*. The following characterization of hyperbolic homeomorphisms holds.

Theorem 2.10. *A homeomorphism f on a compact metric space (X, ϱ) is hyperbolic if and only if f is expansive and has the shadowing property.*

The pseudo-trajectories in the Shadowing Theorem are often interpreted as being the true trajectories of a nearby approximating system, for example generated by computation of the given system on a digital computer. In fact hyperbolic diffeomorphisms enjoy a stronger relationship with systems generated by nearby mappings.

2.2.2 Conjugate Systems

Let $\mathscr{H}(K)$ denote the space of homeomorphisms $f : K \mapsto K$, where K is a compact subset of $\mathbb{R}^d$. Note that $(\mathscr{H}(K), \varrho_0)$ is a complete metric space with the metric

$$\varrho_0(f, g) = \max_{x \in K} \left\{ \|f(x) - g(x)\|, \left\|f^{-1}(x) - g^{-1}(x)\right\| \right\}.$$

A mapping $g \in \mathscr{H}(K)$ is said to be topologically *semi-conjugate* to a mapping $f \in \mathscr{H}(K)$ if there exists a continuous mapping h of K onto K such that $f \circ h = h \circ g$ and topologically *conjugate* when the mapping $h : K \mapsto K$ is a homeomorphism. Many useful dynamical properties are preserved under topological conjugacy or semi-conjugacy, as is illustrated by the Hartman–Grobman theorem in which the comparison is made between a mapping and its linearization about a hyperbolic point. In fact, for a hyperbolic diffeomorphism f there is a neighborhood in $(\mathscr{H}(K), \varrho_0)$ of homeomorphisms g which are semi-conjugate to f under a common semi-conjugacy mapping h which is close to the identity mapping on K.

Theorem 2.11 (Topological Stability). *Let $f \in (\mathscr{H}(K), \varrho_0)$ be diffeomorphism which is hyperbolic on K. Then given $\varepsilon > 0$ there exists a unique continuous mapping $h : K \mapsto K$ with $\|x - h(x)\| < \varepsilon$ for all $x \in K$ and a $\delta = \delta(\varepsilon) > 0$ such that $h \circ g = f \circ h$ for all $g \in (\mathscr{H}(K), \varrho_0)$ with $\varrho_0(f, g) < \delta$. Moreover, if ε is small enough and K is a compact manifold, then h maps K onto K.*

The assertion of this theorem is known as the *topological stability* of the mapping f when ε is small enough to ensure that h is an onto mapping. The neighboring dynamical systems g in $(\mathscr{H}(K), \varrho_0)$ are then also expansive mappings with the shadowing property. Whether or not the *diffeomorphisms* g in this δ-neighborhood of f in the space $(\mathscr{H}(K), \varrho_0)$ are also hyperbolic on K or a subset of K is a much deeper question and has lead to an extensive structural theory of differential dynamics which involves a stronger concept of *structural stability* in the space $\mathscr{D}(K)$ of diffeomorphisms on K with a C^1-metric ϱ_1. In this theory *generic* properties possessed by diffeomorphisms in a residual subset, that is in a countable intersection of open dense subsets, of $(\mathscr{D}(K), \varrho_1)$ are of primary interest.

Topological conjugacies between mappings defined on different compact sets K_i, which need not be subsets of a common space, are also useful in comparing the dynamics of dynamical systems when that of one of them is simple enough to be well understood. For example, Smale established a topological conjugacy between his horseshoe diffeomorphism on the sphere restricted to the hyperbolic horseshoe subset with the shift operator σ on a space Σ of bi-infinite sequences $\boldsymbol{x} = \{x_n\}_{n \in \mathbb{Z}}$ where $x_n \in \{0, 1\}$ with the metric

$$\varrho(\boldsymbol{x}, \widetilde{\boldsymbol{x}}) = \sum_{n \in \mathbb{Z}} 2^{-|n|} \|x_n - \widetilde{x}_n\|,$$

i.e. where $(\sigma \boldsymbol{x})_n = x_{n+1}$ for each $n \in \mathbb{Z}$. The dynamics of the system generated on Σ by the homeomorphism σ is such that σ is expansive on Σ, the set $\mathrm{Per}(\sigma)$ of all periodic points of σ is dense in Σ and there is a trajectory of σ which is also dense in Σ. These properties are carried over to the horseshoe diffeomorphism on its hyperbolic horseshoe set by the conjugacy homeomorphism. *Symbolic dynamics* has become an indispensable tool in the theory of dynamical systems.

2.2.3 Hyperbolic Sets

Often a hyperbolic subset K of a mapping f defined on the set X is not known in advance and needs to be determined. There are several natural candidates for such a subset, the most obvious of which is the set $\mathrm{Per}(f, X)$ of all periodic points $x \in X$ of f all images $f^n(x)$ of which belong to X. More useful are the sets $\Omega(f, X)$ of non-wandering points of f and $\mathrm{CR}(f, X)$ of chain recurrent points of f. A point $x \in X$ is a *non-wandering point* of f if for every neighborhood U of x in X there exists an integer $n \geq 1$ such that $U \cap U_n \neq \emptyset$ for a sequence of nonempty sets $\{U_k\}$ defined recurrently

$$U_0 = U, \quad U_{k+1} = X \cap f(U_k), \qquad k = 0, 1, \ldots .$$

A point $x \in X$ is *chain recurrent* for f if for every $\varepsilon > 0$ there exists an ε-pseudo-trajectory $\{x_n\} \subseteq X$ of f with $x_0 = x$ and $\|x_N - x\| \leq \varepsilon$ for some $N > 0$.[2] The sets $\mathrm{Per}(f, X)$, $\Omega(f, X)$ and $\mathrm{CR}(f, X)$ are closed in X, and

$$\mathrm{Per}(f, X) \subseteq \Omega(f, X) \subseteq \mathrm{CR}(f, X),$$

where each subset is f-invariant.

Much of the structural theory of differential dynamics has focussed on the *Axiom A diffeomorphisms* introduced by Smale, for which $\Omega(f)$ is a hyperbolic set and $\mathrm{Per}(f)$ is dense in $\Omega(f)$, since it turns out that a diffeomorphism is structurally stable if and only if it is an Axiom A diffeomorphism and its (nonlinear) stable and unstable manifolds satisfy a strong transversality condition. Note that $\overline{\mathrm{Per}(f)} = \Omega(f) = \mathrm{CR}(f)$ for a generic system $f \in \mathscr{H}(\overline{X})$ and the mapping $f \mapsto \Omega(f)$ is *upper-semicontinuous* on $(\mathscr{H}(\overline{X}), \varrho_0)$ at such an f in the sense that for every $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that

$$\Omega(g) \subset \mathscr{O}_\varepsilon(\Omega(f)) \qquad \text{for all} \quad \varrho_0(g, f) < \delta,$$

where $\mathscr{O}_\varepsilon(\Omega(f))$ is the open ε-neighborhood of the subset $\Omega(f)$, i.e. the set

$$\mathscr{O}_\varepsilon(\Omega(f)) = \bigcup_{x \in K} \mathrm{Int}\, B(\varepsilon, x)$$

with $\mathrm{Int}\, B(\varepsilon, x)$ being the open ball of radius ε centered at x.

A variation of structural stability is *Ω-stability* in which a topological conjugacy is established between the restricted mappings $f|_{\Omega(f)}$ and $g|_{\Omega(g)}$. For nonsmooth mappings to be considered later in this book conjugacies between $f|_{\mathrm{CR}(f)}$ and $g|_{\mathrm{CR}(g)}$, and also between the sets of their trajectories, will be important.

---

[2] Sometimes (see, e.g. [68]) a slightly different, though equivalent, definition of a chain recurrent point is used: a point $x \in X$ is called chain recurrent for f if for every $\varepsilon, N > 0$ there exists an ε-pseudo-trajectory $\{x_n\} \subseteq X$ of f with $x_0 = x_K = x$ for some $K \geq N$.

2.2.4 Entropy

The word ‘entropy’ refers to a variety of related concepts in mathematics. In the book we are mainly concerned with that of topological entropy first introduced by Adler, Konheim and McAndrew as an invariant of topological conjugacy. Like other concepts of entropy, the topological entropy provides an index of how complicated the dynamics of a system are. There are several equivalent definitions [53, 80] of topological entropy and that used here is based on Bowen’s scheme which allows the entropy to be defined for a uniformly continuous mapping on a subset of a metric space that need not be compact (cf. [80]).

Let $f : X \mapsto X$ be a continuous mapping where X is an open bounded subset of $\mathbb{R}^d$ and let K be a compact subset of X. For a fixed positive integer N denote by $\mathrm{Tr}_{\pm N}(f, K)$ the totality of finite trajectories $\boldsymbol{x} = \{x_{-N} \ldots, x_0, \ldots, x_N\}$ of f that are contained entirely in K and introduce on $\mathrm{Tr}_{\pm N}(f, K)$ the metric

$$\varrho_N(\boldsymbol{x}, \widetilde{\boldsymbol{x}}) = \sup_{-N \le n \le N} \|x_n - \widetilde{x}_n\|.$$

Let $\varepsilon > 0$ and denote by $C_\varepsilon(\mathrm{Tr}_{\pm N}(f, K))$ the binary logarithm of maximal number of elements $\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(p)}$ in $\mathrm{Tr}_{\pm N}(f, K)$ such that[3]

$$\varrho_N(\boldsymbol{x}^{(i)}, \boldsymbol{x}^{(j)}) \ge \varepsilon \qquad \text{for all} \quad i \ne j.$$

Definition 2.12. *The* topological ε-entropy $h_\varepsilon(f, K)$ *of* $f : X \mapsto X$ *on a compact subset* K *of* X *is defined by*

$$h_\varepsilon(f, K) = \limsup_{N \to \infty} \frac{1}{2N + 1} C_\varepsilon(\mathop{\mathrm{Tr}}_{\pm N}(f, K)).$$

The topological entropy $h(f, K)$ *of* f *on* K *is defined to be the limit*[4]

$$h(f, K) = \lim_{\varepsilon \to 0} h_\varepsilon(f, K) = \sup_{\varepsilon > 0} h_\varepsilon(f, K).$$

One of the most difficult problems in investigating topological entropy is its explicit evaluation. For an ξ-expansive mapping f with $f(K) = K$ the topological entropy in some cases is given by the following formula

$$h(f, K) = h_\theta(f, K), \qquad \theta < \xi$$

which will be proved in Lemma 7.1 of Chapter 7.

The following theorem establishing the relation between entropy and the number of periodic points of a mapping is worth to mentioning here.

[3] If (Y, d) is a metric space and S is a compact subset of Y, then the ε-*capacity* of the set S, the quantity $C_\varepsilon(S) = \log s_\varepsilon(S)$, is the binary logarithm of the maximal number $s = s_\varepsilon(S)$ of elements $y_1, y_2, \ldots, y_s \in S$ satisfying $y_i \ne y_j$ for $i \ne j$. Hence $C_\varepsilon(\mathrm{Tr}_{\pm N}(f, K))$ is ε-capacity of the compact metric space $\mathrm{Tr}_{\pm N}(f, K)$.

[4] The second equality in the definition of topological entropy follows from the obvious fact that ε-entropy $h_\varepsilon(f, K)$ is non-increasing in ε.

Theorem 2.13. *If f is an expansive homeomorphism of a compact metric space K, then*

$$h(f, K) \geq \limsup_{n\to\infty} \frac{1}{n} \log N_n(f)$$

where $N_n(f)$ is the number of periodic points of f of period n.

If f is a diffeomorphism of a compact manifold M onto itself which satisfies Axiom A, then

$$h(f, K) = \limsup_{n\to\infty} \frac{1}{n} \log N_n(f).$$

2.2.5 Chaos

The term 'chaotic' is usually used when one wants to describe rather complicated behavior of a dynamical system that includes sensitive dependence on initial conditions, an abundance of unstable periodic trajectories and an irregular mixing effects. Although the term is frequently used by many authors, there it seems no commonly accepted definition of 'chaos'. Often the behavior of the shift mapping σ on the space Σ of binary sequences, described in Section 2.2.2, is used to provide a 'pattern' for chaotic behavior, which could be called *symbolic dynamics chaos* or *shift-mapping chaos.* The behavior of the Smale 'horseshoe' system and the closely related behavior of a dynamical system near a transversal homoclinic point are famous examples of this kind of chaos.

Definition 2.14. *Let $f : X \mapsto X$ be a diffeomorphism where X is an open subset of $\mathbb{R}^d$. A point $y \in X$ is called* homoclinic *if there exists a fixed point x of f, $x \neq y$, such that $y \in W^s(x) \cap W^u(x)$. A homoclinic point $y \in X$ is called* transversal homoclinic *if $T_y W^s(x) + T_y W^u(x) = \mathbb{R}^d$.*

The following theorem shows that one can observe 'chaotic' behavior of this kind in a neighborhood of any homoclinic point.

Theorem 2.15 (Smale's Homoclinic Theorem). *Let f be a C^1 diffeomorphism and y a transversal homoclinic point for a fixed point x of f. Then there is an integer n such that $g = f^n$ has a hyperbolic compact invariant set K which is homeomorphic to a Cantor set and contains x and y. Moreover, the mapping $g|_K$ is topologically conjugate to the shift σ acting on the space Σ of binary sequences.*

Chaotic behavior of this shift mapping kind is, however, quite restrictive. A more descriptive a broadly applicable definition which retains the essential features of shift mapping chaos was suggested by Li and Yorke [50] and generalized to higher dimensions by Marotto [52] and Shiraiwa and Kurata [71].

Definition 2.16 (Li–Yorke Chaos). *Let X be a subset of $\mathbb{R}^d$. A continuous mapping $f : X \mapsto X$ is called* chaotic *if*

CH1: *There exists a positive integer N such f has a periodic point of minimal period p for any integer $p \geq N$;*

CH2: *There exists an uncountable f-invariant set $S \subseteq X$ containing no periodic points (called a* scrambled set*) such that*

$$\limsup_{n\to\infty} \|f^n(x) - f^n(y)\| > 0$$

for every $x, y \in S$ with $x \neq y$, and for every $x \in S$ and a periodic point y;

CH3: *There exists an uncountable subset S_0 of S such that*

$$\liminf_{n\to\infty} \|f^n(x) - f^n(y)\| = 0$$

for every $x, y \in S_0$.

The following theorem due to Shiraiwa and Kurata [71] expands 'period 3 implies chaos' result of Li and York to higher dimensional mappings.

Theorem 2.17. *Let X be a subset of $\mathbb{R}^d$ and $f : X \mapsto X$ be a C^1-map. Let $x_0 \in X$ be a hyperbolic fixed point of f and assume that the following three conditions are satisfied:*

(i) $\dim E^u_{x_0} > 0$*;*

(ii) there exist an $\varepsilon > 0$, a point $x_1 \in W^u_\varepsilon(x_0)$, $x_1 \neq x_0$, and a positive integer m such that $f^m(x_1) \in W^s_\varepsilon(x_0$;

(iii) there exists a disk[5] *$D^u \subset W^u_\varepsilon(x_0)$ such that D^u is a neighborhood of x_1 in $W^u_\varepsilon(x_0)$, $f^m|_{D^u} : D^u \mapsto X$ is an embedding, and $f^m(D^u)$ intersects $W^u_\varepsilon(x_0)$ transversally at $f^m(x_1)$.*

Then the mapping f is chaotic.

[5] A *disk* is a homeomorphic image of a ball from a coordinate space.

3

Semi-Hyperbolic Mappings

The concept of semi-hyperbolicity is introduced in this Chapter and some examples are considered. The first definition generalizes Definition 2.3 to a differentiable mapping, which need not be invertible, on a compact manifold and involves variable norms at each point of the compact invariant subset of the manifold. The second and third definitions apply to a general compact subset of $\mathbb{R}^d$ as in Definition 2.5, first for a differentiable mapping and then for a Lipschitz mapping; in both cases a fixed norm is used and the subset is not required to be invariant.

Further generalizations of the concept of semi-hyperbolicity to mappings in Banach spaces will be given in Chapter 8.

3.1 Definitions

Consider a mapping $f : X \mapsto \mathbb{R}^d$ where X is an open subset of $\mathbb{R}^d$ and let K a nonempty compact subset of X such that $K \cap f(K) \neq \emptyset$, or consider a mapping $f : M \mapsto M$ where M is be a compact manifold and let K be an f-invariant subset of M. The following definition is stated for a splitting of $\mathbb{R}^d$. An analogous definition holds for a splitting $T_xM = E_x^s \oplus E_x^u$ of the tangent space T_xM of a compact manifold M, in which case the projectors $P_x^s : T_xM \mapsto E_x^s$ and $P_x^u : T_xM \mapsto E_x^u$.

Definition 3.1. *A* splitting *or* decomposition $\mathbb{R}^d = E_x^s \oplus E_x^u$ *with corresponding projectors* $P_x^s : \mathbb{R}^d \mapsto E_x^s$ *and* $P_x^u : \mathbb{R}^d \mapsto E_x^u$ *defined by*

$$P_x^s \mathbb{R}^d = E_x^s, \quad P_x^s E_x^u = 0, \tag{3.1}$$

$$P_x^u \mathbb{R}^d = E_x^u, \quad P_x^u E_x^s = 0, \tag{3.2}$$

is said to be uniform *on the compact subset* K *with respect to a mapping* f *if*

SH0: $\dim E_x^u = \dim E_{f(x)}^u$ *for* $x \in K$ *with* $f(x) \in K$*;*

and there exists a positive real number h such that

SH1: $\sup_{x\in K}\{\|P_x^s\|, \|P_x^u\|\} \le h$.

Note that neither the invariance of the set K with respect to the mapping f nor the continuity in x of the splitting subspaces E_x^s, E_x^u or of the projectors P_x^s, P_x^u are assumed in Definition 3.1.

Parameters measuring expansion, contraction and other rates in the definition of a semi-hyperbolic mapping will be given in terms of a 4-tuple called a split.

Definition 3.2. *A 4-tuple $\boldsymbol{s} = (\lambda_s, \lambda_u, \mu_s, \mu_u)$ of nonnegative real numbers is called a* split *if*

$$\lambda_s < 1 < \lambda_u, \quad (1-\lambda_s)(\lambda_u - 1) > \mu_s\mu_u.$$

The split $\boldsymbol{s}$ is called positive *if all the values λ_s, λ_u, μ_s and μ_u are positive.*

Clearly, for any given λ_s and λ_u satisfying $\lambda_s < 1 < \lambda_u$ a 4-tuple of nonnegative numbers $\boldsymbol{s} = (\lambda_s, \lambda_u, \mu_s, \mu_u)$ will be a split if the product $\mu_s\mu_u$ is small enough.

An alternative definition of a split with a geometrical interpretation is possible: a 4-tuple $(\lambda_s, \lambda_u, \mu_s, \mu_u)$ is a split if and only if the eigenvalues Δ_1 and Δ_2 of every matrix

$$\boldsymbol{\Delta} = \begin{bmatrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22} \end{bmatrix}$$

with components satisfying $|\delta_{11}| \le \lambda_s$, $|\delta_{12}| \le \mu_s, |\delta_{21}| \le \mu_u$, $|\delta_{22}| \ge \lambda_u$ are real and satisfy $|\Delta_1| < 1 < |\Delta_2|$.

The problem how much the elements of a split can be perturbed in order to resulting four-tuple of numbers remain be a split is of practical interest. Set

$$\nu(\boldsymbol{s}) := \frac{(1-\lambda_s)(\lambda_u - 1) - \mu_s\mu_u}{\lambda_u - \lambda_s + \mu_s + \mu_u} \tag{3.3}$$

and note that

$$\nu(\boldsymbol{s}) \le \frac{(1-\lambda_s)(\lambda_u - 1)}{\lambda_u - \lambda_s} \le \min\{1-\lambda_s, \lambda_u - 1\}, \tag{3.4}$$

and so $\nu(\boldsymbol{s}) < 1$.

Lemma 3.3. *If $\boldsymbol{s} = (\lambda_s, \lambda_u, \mu_s, \mu_u)$ is a split, then for $\widetilde{\lambda}_s = \lambda_s + \delta_1$, $\widetilde{\lambda}_u = \lambda_u - \delta_2$, $\widetilde{\mu}_s = \mu_s + \delta_3$, $\widetilde{\mu}_u = \mu_u + \delta_4$ and real δ_1, δ_2, δ_3, δ_4 satisfying $0 \le \delta_1, \delta_2, \delta_3, \delta_4 < \nu(\boldsymbol{s})$ the 4-tuple $\widetilde{\boldsymbol{s}} = (\widetilde{\lambda}_s, \widetilde{\lambda}_u, \widetilde{\mu}_s, \widetilde{\mu}_u)$ is also a split.*

Proof. Clearly the elements of the 4-tuple $\widetilde{\boldsymbol{s}}$ are non-negative and by (3.3), (3.4) we have $\delta_1 < 1 - \lambda_s$ and $\delta_2 < \lambda_u - 1$. Hence

$$\widetilde{\lambda}_s = \lambda_s + \delta_1 < \lambda_s + 1 - \lambda_s = 1, \quad \widetilde{\lambda}_u = \lambda_u - \delta_2 > \lambda_u - (\lambda_u - 1) = 1,$$

and by (3.3)

$$\begin{aligned}(1-\widetilde{\lambda}_s)(\widetilde{\lambda}_u-1)-\widetilde{\mu}_s\widetilde{\mu}_u \\ > (1-\lambda_s-\nu(\boldsymbol{s}))(\lambda_u-1-\nu(\boldsymbol{s}))-(\mu_s+\nu(\boldsymbol{s}))(\mu_u+\nu(\boldsymbol{s})) \\ = (1-\lambda_s)(\lambda_u-1)-\mu_s\mu_u-\nu(\boldsymbol{s})(\lambda_u-\lambda_s+\mu_s+\mu_u)=0.\end{aligned}$$

The lemma is proved. □

3.1.1 Differentiable Mappings on Compact Manifolds

Here the variable norm version of hyperbolicity on a manifold in Definition 2.3 is generalized to differentiable mappings on a manifold.

Definition 3.4. *Let $\boldsymbol{s}=(\lambda_s,\lambda_u,\mu_s,\mu_u)$ be a split and let M be a compact manifold in $\mathbb{R}^d$. A differentiable mapping $f: M \mapsto M$ is said to be* $\boldsymbol{s}$-semi-hyperbolic *on a compact invariant subset K of M if there exists a uniform splitting $T_xM = E_x^s \oplus E_x^u$ with projectors P_x^s, P_x^u and a norm $\|\cdot\|_x$ on T_xM for each $x \in K$ such that*

SH2(Diff)*: *For all $x \in K$:*

$$\|P^s_{f(x)}Tf_xu\|_{f(x)} \le \lambda_s\|u\|_x, \qquad u \in E_x^s, \tag{3.5}$$
$$\|P^s_{f(x)}Tf_xv\|_{f(x)} \le \mu_s\|v\|_x, \qquad v \in E_x^u, \tag{3.6}$$
$$\|P^u_{f(x)}Tf_xu\|_{f(x)} \le \mu_u\|u\|_x, \qquad u \in E_x^s, \tag{3.7}$$
$$\|P^u_{f(x)}Tf_xv\|_{f(x)} \ge \lambda_u\|v\|_x, \qquad v \in E_x^u, \tag{3.8}$$

where Tf_x denotes the tangent mapping of f at the point x.

Nonzero values of μ_s and μ_u allow for a possible leakage away from the strict equivariance of the splitting subspaces E_x^s and E_x^s (i.e. Condition H1), respectively, that holds when the f is a hyperbolic diffeomorphism on K in the sense of Definition 2.3.

Comparing Definition 3.4 with Definition 2.3 of a hyperbolic diffeomorphism, it is clear that a hyperbolic mapping on K with expansivity parameter $\lambda > 1$ is $\boldsymbol{s}$-semi-hyperbolic on K as in Definition 3.4 with the same splitting and the split $\boldsymbol{s}=(\lambda^{-1},\lambda,0,0)$, that is with $\mu_s=\mu_u=0$.

3.1.2 Differentiable Mappings on Compact Sets

Consider now a differentiable mapping $f: X \mapsto \mathbb{R}^d$ where X is an open subset of $\mathbb{R}^d$. Definition 2.5 of a hyperbolic diffeomorphism on a compact subset K of $\mathbb{R}^d$ will be generalized to a differentiable mapping, which need not be invertible, and to a nonempty subset K of X such that only satisfies $K \cap f(K) \neq \emptyset$ rather than being required to be f-invariant.

Definition 3.5. *Let* $\boldsymbol{s} = (\lambda_s, \lambda_u, \mu_s, \mu_u)$ *be a split and let* K *a compact subset of an open subset* X *of* $\mathbb{R}^d$. *A differentiable mapping* $f : X \mapsto \mathbb{R}^d$ *is said to be* $\boldsymbol{s}$-semi-hyperbolic *on* K *if there exists a uniform splitting* $\mathbb{R}^d = E_x^s \oplus E_x^u$ *with projectors* P_x^s, P_x^u *for each* $x \in K$ *and a norm* $\|\cdot\|$ *on* $\mathbb{R}^d$ *such that*

SH2(Diff): *For all* $x \in K$*:*

$$\|P_{f(x)}^s Df_x u\| \le \lambda_s \|u\|, \qquad u \in E_x^s, \tag{3.9}$$

$$\|P_{f(x)}^s Df_x v\| \le \mu_s \|v\|, \qquad v \in E_x^u, \tag{3.10}$$

$$\|P_{f(x)}^u Df_x u\| \le \mu_u \|u\|, \qquad u \in E_x^s, \tag{3.11}$$

$$\|P_{f(x)}^u Df_x v\| \ge \lambda_u \|v\|, \qquad v \in E_x^u, \tag{3.12}$$

where Df_x *denotes the derivative of the mapping* f *at the point* x.

Comparing Definition 3.5 with Definition 2.5, it is clear here too that a hyperbolic diffeomorphism on K with expansivity parameter $\lambda > 1$ is $\boldsymbol{s}$-semi-hyperbolic on K as in Definition 3.5 with the same splitting and the split $\boldsymbol{s} = (\lambda^{-1}, \lambda, 0, 0)$, that is with $\mu_s = \mu_u = 0$.

Since the splitting $\mathbb{R}^d = E_x^s \oplus E_x^u$ in Definition 3.5 is uniform then for projectors P_x^s, P_x^u condition SH1 from Definition 3.1 holds with some constant h. To stress this dependency on h, the $\boldsymbol{s}$-semi-hyperbolic map $f : X \mapsto \mathbb{R}^d$ may be called also $(\boldsymbol{s}, h)$*-semi-hyperbolic*.

3.1.3 Lipschitz Mappings on Compact Sets

To allow for nonsmooth applications, the following generalization of Definition 2.5 to Lipschitz mappings $f : X \mapsto \mathbb{R}^d$ is proposed. A generalization of Definition 2.3 to Lipschitz mappings on a manifold is also possible, but would be rather cumbersome and will be omitted as it will not be used elsewhere in this book.

As in Definition 3.5, the subset K is a nonempty compact subset of X such that $K \cap f(K) \ne \emptyset$ and a fixed norm is used.

Definition 3.6. *Let* $\boldsymbol{s} = (\lambda_s, \lambda_u, \mu_s, \mu_u)$ *be a split and* K *a compact subset of an open set* $X \subseteq \mathbb{R}^d$. *A Lipschitz mapping* $f : X \mapsto \mathbb{R}^d$ *is said to be* $\boldsymbol{s}$-semi-hyperbolic *on* K *if there exists a splitting* $\mathbb{R}^d = E_x^s \oplus E_x^u$ *on* K *with projectors* P_x^s *and* P_x^u *for each* $x \in K$, *a norm* $\|\cdot\|$ *on* $\mathbb{R}^d$ *and a positive real number* δ *such that*

SH0(Lip): $\dim E_x^u = \dim E_y^u$ *for all* $x, y \in K$ *with* $\|f(x) - y\| \le \delta$*;*
SH1(Lip): $\sup_{x \in K}\{\|P_x^s\|, \|P_x^u\|\} \le h$*;*
SH2(Lip): *The inclusion*

$$x + u + v \in X$$

and the inequalities

$$\|P_y^s\left(f(x+u+v)-f(x+\widetilde{u}+v)\right)\| \le \lambda_s\|u-\widetilde{u}\|, \quad (3.13)$$
$$\|P_y^s\left(f(x+u+v)-f(x+u+\widetilde{v})\right)\| \le \mu_s\|v-\widetilde{v}\|, \quad (3.14)$$
$$\|P_y^u\left(f(x+u+v)-f(x+\widetilde{u}+v)\right)\| \le \mu_u\|u-\widetilde{u}\|, \quad (3.15)$$
$$\|P_y^u\left(f(x+u+v)-f(x+u+\widetilde{v})\right)\| \ge \lambda_u\|v-\widetilde{v}\|. \quad (3.16)$$

hold for all $x, y \in K$ *with* $\|f(x)-y\| \le \delta$ *and all* $u, \widetilde{u} \in E_x^s$ *and* $v, \widetilde{v} \in E_x^u$ *such that* $\|u\|, \|\widetilde{u}\|, \|v\|, \|\widetilde{v}\| \le \delta$.

The first three inequalities in Condition SH2(Lip) of Definition 3.6 are just local Lipschitz conditions on the projections of the mapping f while the last one is an expansivity condition which implies a local invertibility in the unstable direction of f at x. As in the two differentiable cases above (Definitions 3.4 and 3.5), nonzero values of μ_s and μ_u allow a possible leakage away from the equivariance of the splitting subspaces E_x^s and E_x^s.

Remark 3.7. Condition SH0(Lip) in Definition 3.6 is more restrictive that in uniform splitting Definition 3.1 while SH1(Lip) is the same as SH1 there.

Comparing inequalities (3.13)–(3.16) in Condition SH2(Lip) with corresponding inequalities (3.5)–(3.8) or (3.9)–(3.12), we see that in the first case, for Lipschitz mappings on compact manifolds, projectors P_y^s and P_y^u are considered for y from some neighborhood of the point $f(x)$, while in (3.5)–(3.8) or (3.9)–(3.12) they are considered at the single $y = f(x)$. This additional restriction can be regarded as a weak form of continuous dependence of splitting $\mathbb{R}^d = E_x^s \oplus E_x^u$ in x (see also Section 3.1.4 and Remark there).

3.1.4 Continuous Splittings

The continuous dependence on x of the subspaces E_x^s and E_x^u of a splitting $T_xM = E_x^s \oplus E_x^u$ of a compact smooth manifold M or of a splitting $\mathbb{R}^m = E_x^s \oplus E_x^u$ of the space $\mathbb{R}^m$ is a fundamental property of hyperbolic mappings, but does not follow automatically for semi-hyperbolic mappings.

Definition 3.8. *A splitting* $T_xM = E_x^s \oplus E_x^u$ *of a compact smooth manifold* M *is said to* depend continuously on x *if in a neighborhood of any point* $x_0 \in M$ *a set of vectors* $e_1(x), e_2(x), \ldots, e_m(x) \in T_xM$ *in local coordinates which depends continuously on* x *can be chosen such that the vectors* $e_1(x), e_2(x), \ldots, e_k(x)$ *form a basis of the subspace* $E_x^s \subseteq T_xM$ *and the vectors* $e_{k+1}(x), e_{k+2}(x), \ldots, e_m(x)$ *form a basis of the subspace* $E_x^u \subseteq T_xM$.

For a splitting of the space $\mathbb{R}^m$ there is an alternative definition, which is sometimes more convenient to use, is possible.

Definition 3.9. *The splitting* $\mathbb{R}^m = E_x^s \oplus E_x^u$ *is said to* depend continuously on x *if the projectors* P_x^s *and* P_x^u *defined by* (3.1) *and* (3.2)*, respectively, are continuous in* x *as linear-operator-valued functions.*

Neither the invariance of the set K with respect to the mapping f nor the continuity in x of the splitting subspaces E^s_x, E^u_x or of the projectors P^s_x, P^u_x are assumed in Definition 3.1 of a uniform splitting.

For hyperbolic mappings the continuity of the splitting is a direct consequence of the definition, but this does not always follow from the weaker assumptions of semi-hyperbolicity. In some cases such continuity may hold and, as will be seen in later chapters, this allows stronger results to be obtained. The following definition applies for semi-hyperbolic mappings with respect to either a fixed norm or variable adapted norms.

Definition 3.10. *A mapping f is said to be* continuously semi-hyperbolic *on K if it is $\boldsymbol{s}$-semi-hyperbolic on K as in any of Definitions 3.4, 3.5 or 3.6 for some split $\boldsymbol{s}$ for which the uniform splitting is continuous in x on K in the sense of Definitions 3.8 or 3.9.*

Remark 3.11. A continuously $\boldsymbol{s}$-semi-hyperbolic differentiable mapping as in Definition 3.5 is $\boldsymbol{s}_\varepsilon$-semi-hyperbolic as in Definition 3.6 for any split $\boldsymbol{s}_\varepsilon = (\lambda_s+\varepsilon, \lambda_u-\varepsilon, \mu_s+\varepsilon, \mu_u+\varepsilon)$ for any sufficiently small $\varepsilon > 0$ and an appropriate choice of $\delta = \delta(\varepsilon) > 0$.

Continuously semi-hyperbolic mappings form an open set in the variety of all semi-hyperbolic mappings. To be more specific, we shall formulate the corresponding stronger statement for the case of Lipschitz semi-hyperbolic mappings.

Denote by $C(X, \mathbb{R}^d)$ with $X \subseteq \mathbb{R}^d$ the Banach space of all bounded continuous mappings $f : X \mapsto \mathbb{R}^d$ with the usual C-norm

$$\|f\|_C = \sup_{x\in X} \|f(x)\|.$$

Denote by $\mathrm{Lip}(X, \mathbb{R}^d)$ with $X \subseteq \mathbb{R}^d$ the Banach space of all bounded Lipschitz mappings $f : X \mapsto \mathbb{R}^d$ endowed by the norm

$$\|f\|_{\mathrm{Lip}} = \|f\|_C + \sup_{x,y\in X,\ x\neq y} \frac{\|f(x) - f(y)\|}{\|x - y\|}.$$

Denote by $\mathscr{O}_\varepsilon(K)$ the open ε-neighborhood of the subset K, i.e. the set

$$\mathscr{O}_\varepsilon(K) = \bigcup_{x\in K} \mathrm{Int}\, B(\varepsilon, x),$$

where $\mathrm{Int}\, B(\varepsilon, x)$ is the open ball of radius ε centered at x.

Lemma 3.12. *Let $X \subseteq \mathbb{R}^d$ be an open set and let the mapping $f \in \mathrm{Lip}(X, \mathbb{R}^d)$ be continuously semi-hyperbolic on a compact set $K \subset X$. Then there exists an $\eta = \eta(f, X) > 0$, a split $\boldsymbol{s}$, constants h and δ and a uniform splitting $\mathbb{R}^d = E^s_x \oplus E^u_x$ such that $\overline{\mathscr{O}_\eta(K)} \subset X$ and every mapping from the set*

$$F_\eta = \{g \in \mathrm{Lip}(X, \mathbb{R}^d): \ \|g - f\|_{\mathrm{Lip}} \le \eta\}$$

is semi-hyperbolic on $\overline{\mathscr{O}_\eta(K)}$ *with the split* $\boldsymbol{s}$*, constants* h *and* δ *and the splitting* $\mathbb{R}^d = E_x^s \oplus E_x^u$*.*

Proof. Let $E_x^s \oplus E_x^u$ be a continuous splitting of $\mathbb{R}^d$ for the mapping f such that Conditions SH0(Lip)–SH2(Lip) of Definition 3.6 hold for f with the split $\boldsymbol{s}^*$ and positive constants h^* and δ^*.

For each $x \in X$ let $\pi(x)$ be any one of nearest points in K to x. Since K is compact, such a function $\pi(x)$ is well defined, but possibly not in a unique way, and generally it will be not continuous. For what follows it will be important that

$$\pi(x) = x, \qquad x \in K,$$

and

$$\|\pi(x) - x\| < \eta, \qquad x \in \mathscr{O}_\eta(K).$$

Prolongate the splitting $\mathbb{R}^d = E_x^s \oplus E_x^u$ on the whole set X by

$$E_x^s := E_{\pi(x)}^s, \quad E_x^u := E_{\pi(x)}^u, \qquad x \in X.$$

Then the splitting

$$\mathbb{R}^d = E_x^s \oplus E_x^u \equiv E_{\pi(x)}^s \oplus E_{\pi(x)}^u, \qquad x \in X, \tag{3.17}$$

so obtained will satisfy Conditions SH0(Lip)–SH1(Lip) of Definition 3.6 with the constant $h = h^*$.

Let $\eta_0 > 0$ be such that $\mathscr{O}_{\eta_0}(K) \subset X$, choose

$$\eta < \min\left\{\eta_0, \ \frac{1}{2}\delta^*, \ h^{-1}\nu(\boldsymbol{s})\right\},$$

where $\nu(\boldsymbol{s})$ is as in (3.3), and set

$$\delta = \delta^* - 2\eta, \quad \boldsymbol{s} = \{\lambda_s^* + h\eta, \lambda_u^* - h\eta, \mu_s^* + h\eta, \mu_u^* + h\eta\}$$

Then each mapping g in the set F_η satisfies Condition SH2(Lip) of Definition 3.6 on the compact set $\overline{\mathscr{O}_\eta(K)}$ with the splitting (3.17), the split $\boldsymbol{s}$ and constants h, δ just introduced. □

Remark 3.13. In spite of continuity of the splitting $\mathbb{R}^d = E_x^s \oplus E_x^u$ for the mapping f in Lemma 3.12, the splitting (3.17) is generally not continuous. Continuity of the splitting (3.17) may be possible, for example, when the compact K is a retract of a neighborhood of itself.

3.2 Examples

All diffeomorphisms with a hyperbolic invariant set are semi-hyperbolic on that set provided the appropriate corresponding definitions are used. Of particular interest here therefore are examples of mappings which are semi-hyperbolic but not hyperbolic, and especially when the set K is not countable.

3.2.1 Elementary Examples

Differential dynamics concentrates on the behavior of a diffeomorphism *within* an invariant set. When one tries to approximate the dynamical behavior of a mapping numerically, what is happening in the neighboring exterior of such a set is often also of interest, particularly when the set itself is quite simple. Consider, for example, the problem of replicating a phase portrait about a saddle point using finite machine arithmetic. The shadowing results for semi-hyperbolic mappings in Chapter 6 are useful in such contexts, while those for hyperbolic diffeomorphisms are not applicable.

Example 3.14. Let A be a hyperbolic matrix, that is with eigenvalues $\lambda_1, \ldots, \lambda_d$ satisfy $|\lambda_i| \neq 1$, and consider the splitting

$$\mathbb{R}^d = E^s \oplus E^u \tag{3.18}$$

where E^s is the eigenspace of the matrix A corresponding to the eigenvalues satisfying $|\lambda_i| < 1$ and E^u is the eigenspace corresponding to the eigenvalues satisfying $|\lambda_i| > 1$. The linear mapping $f : \mathbb{R}^2 \mapsto \mathbb{R}^2$ defined by $f(x) = Ax$ is hyperbolic on the singleton set $\{0\}$ with splitting (3.18), but is only semi-hyperbolic on, say, the unit disk D_2 since D_2 is not invariant under f. The splitting on D_2 is obtained by translating that at $\{0\}$ to each point $x \in D_2$, and is thus both uniform and continuous.

Nonsmooth perturbations of hyperbolic mappings are also another source of examples of semi-hyperbolic mappings. These are more easily described as in the following example where the hyperbolic set is a saddle point.

Example 3.15. Let A be a 2×2 hyperbolic matrix as in Example 3.14 and consider the nonlinear mapping $f_\varepsilon : \mathbb{R}^2 \mapsto \mathbb{R}^2$ obtained as a perturbation of the linear mapping $f_0(x) = Ax$, such as

$$f_\varepsilon \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} \varepsilon |x_1| \\ \varepsilon |x_2| \end{pmatrix}$$

for all $x = (x_1, x_2)^T \in \mathbb{R}^2$ and some $\varepsilon > 0$. This mapping is Lipschitz everywhere, but is not differentiable at the origin. It is semi-hyperbolic in any set containing the origin in its interior with $E^s_x = \{0\} \oplus \mathbb{R}^1$ and $E^u_x = \mathbb{R}^1 \oplus \{0\}$ for all x.

Mappings formed by patching together other, often linear, mappings on adjoining subsets are ubiquitous in electronic and control systems and can behave very erratically. The switching points or sets can be problematical for semi-hyperbolicity.

Example 3.16. The piecewise linear mapping $f : \mathbb{R}^1 \mapsto \mathbb{R}^1$ with

$$f(x) = \begin{cases} \frac{1}{2}x, & x < 0 \\ 4x, & x \geq 0 \end{cases}$$

is not semi-hyperbolic on any set K containing the switching point $x = 0$ because there is no appropriate splitting of $\mathbb{R}^1$ at $x = 0$ for which the inequalities in Condition SH2(Lip) are satisfied. Each component mapping is semi-hyperbolic on its subset of definition, with the splitting $E_x^s \oplus E_x^u = \{0\} \oplus \mathbb{R}^1$ for $x \leq 0$ and with $E_x^s \oplus E_x^u = \mathbb{R}^1 \oplus \{0\}$ for $x \geq 0$.

The piecewise linear mapping $f : \mathbb{R}^1 \mapsto \mathbb{R}^1$ with

$$f(x) = \begin{cases} -\frac{1}{2}x, & x < 0 \\ 4x, & x \geq 0 \end{cases}$$

is also not semi-hyperbolic on any set K containing the switching point $x = 0$, but any iterate f^j for $j \geq 2$ is. An admissible splitting is $E_x^s \oplus E_x^u = \{0\} \oplus \mathbb{R}^1$ for $x \in K$ and split $\boldsymbol{s}$ is $(\lambda_s, \lambda_u, \mu_s, \mu_u) = (0, 2^{2j-3}, 0, 0)$. Here P_x^s is the zero mapping and P_x^u the identity mapping.

Note that the tent mapping $f(x) = 1 - |1 - 2x|$, which is Lipschitz, is not semi-hyperbolic on any subset of $\mathbb{R}^1$ containing the point $x = \frac{1}{2}$.

3.2.2 Anosov Endomorphisms

Examples of semi-hyperbolic mappings with a uniform splitting that is either not continuous or lacks the equivariance of a splitting for a hyperbolic mapping are more difficult to describe explicitly. In the following examples, algebraic Anosov automorphisms and noninvertible Anosov endomorphisms are introduced and ultimately an example with the desired properties obtained as a differentiable perturbation of an Anosov endomorphism on a torus. Structural stability is not contradicted here as neither the endomorphism which is being perturbed nor the perturbations themselves, while differentiable, are diffeomorphisms.

Write the elements of $\mathbb{R}^d$ as vectors x with coordinates $x_1, x_2, \ldots, x_d$ and let $\mathbb{T}^d$ be the standard d-dimensional torus, that is the factorization of $\mathbb{R}^d$ by the integer lattice. This torus $\mathbb{T}^d$ is a closed differentiable manifold in $\mathbb{R}^d$ with respect to the locally Euclidean metric

$$\varrho(x, y) = \sqrt{|x_1 - y_1|^2_{\text{mod}\,1} + |x_2 - y_2|^2_{\text{mod}\,1} + \cdots + |x_d - y_d|^2_{\text{mod}\,1}}$$

on $\mathbb{T}^d$ where

$$|t - s|_{\text{mod}\,1} = \min\{|t - s + 2k| : \quad k = 0, \pm 1\}, \qquad 0 \leq t, s < 1.$$

The tangent space $T_x\mathbb{T}^d$ can then be identified with $\mathbb{R}^d$ by an appropriate choice of natural coordinates generated by those of $\mathbb{R}^d$, so $T_x\mathbb{T}^d = \mathbb{R}^d$ for each $x \in \mathbb{T}^d$.

Denote the natural projection from $\mathbb{R}^d$ onto $\mathbb{T}^d$ by

$$\Pi(x) = (x_1 \bmod 1,\, x_2 \bmod 1,\, \ldots,\, x_d \bmod 1), \qquad x = (x_1, x_2, \ldots, x_d)$$

and associate the mapping $f : \mathbb{T}^d \mapsto \mathbb{T}^d$ defined by

$$f(x) = \Pi(Ax) \tag{3.19}$$

with a given $d \times d$ matrix A with integer components a_{ij}.

In addition, suppose that the matrix A is hyperbolic and $\mathbb{R}^d = E^s \oplus E^u$ is the splitting described in Example 3.14. Since the eigenvalues of the restriction $A|_{E^s}$ lie inside the unit circle while those of the restriction $A|_{E^u}$ lie outside the unit circle, there exist constants $C > 0$ and $\lambda > 1$ such that

$$\|A^n u\| \leq C\lambda^{-n}\|u\|, \quad \|A^n v\| \geq C^{-1}\lambda^n\|v\| \tag{3.20}$$

for all $u \in E^s$ and $v \in E^u$, where $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^d$.

Example 3.17 (Anosov Automorphisms and Endomorphisms). Let A be an invertible hyperbolic matrix with integer components. The mapping f defined by (3.19) is called an *algebraic hyperbolic automorphism* of the torus $\mathbb{T}^d$ if $|\det A| = 1$ and an the *algebraic hyperbolic endomorphism* of $\mathbb{T}^d$ otherwise.

An algebraic hyperbolic automorphism of a torus $\mathbb{T}^d$ is clearly globally invertible mapping and is a diffeomorphism on $\mathbb{T}^d$, but an algebraic hyperbolic endomorphism is generally not a diffeomorphism since it is only locally invertible.

Since $T_x\mathbb{T}^d$ has been identified with $\mathbb{R}^d$, the tangent mapping Tf_x can be identified with A and the linear subspaces $E^s_x := E^s$, $E^u_x := E^u$ for $x \in \mathbb{T}^d$ are invariant under $Tf_x \equiv A$. Thus

$$T_x\mathbb{T}^d = \mathbb{R}^d = E^s \oplus E^u = E^s_x \oplus E^u_x \tag{3.21}$$

is an equivariant splitting for the tangent mapping Tf_x.

Let P^s_x and P^u_x be projectors corresponding to the splitting (3.21) (see Definition 3.1) and for each $x \in \mathbb{T}^d$ define a norm

$$\|v\|_x = \max_{n \geq 0} \lambda^n \|A^n P^s_x v\| + \min_{n \geq 0} \lambda^{-n} \|A^n P^u_x v\|,$$

on $T_x\mathbb{T}^d = \mathbb{R}^d$, where λ is as in (3.20) (in fact, this norm $\|\cdot\|_x$ does not depend on x). Then

$$\|Tf_x u\|_x \leq \lambda^{-1}\|u\|_x, \qquad \|Tf_x v\|_x \geq \lambda\|v\|_x \tag{3.22}$$

for all $u \in E^s_x, v \in E^u_x$ and $x \in K$, that is Conditions H1* and H2* of Definition 2.3 are valid for the endomorphism f on the torus $\mathbb{T}^d$; if the mapping f is an automorphism, then it is a hyperbolic diffeomorphism on $\mathbb{T}^d$.

The relations (3.21) and (3.22) for the Anosov hyperbolic endomorphism f defined in Example 3.17 can be rewritten in the form

$$\begin{aligned}
&\|P^s_{f(x)} Tf_x u\|_{f(x)} \leq \lambda^{-1}\|u\|_x, && u \in E^s_x,\\
&\|P^s_{f(x)} Tf_x v\|_{f(x)} = 0, && v \in E^u_x,\\
&\|P^u_{f(x)} Tf_x v\|_{f(x)} = 0, && u \in E^s_x,\\
&\|P^u_{f(x)} Tf_x v\|_{f(x)} \geq \lambda\|v\|_x, && v \in E^u_x.
\end{aligned}$$

By Definition 3.4 this means that every Anosov hyperbolic endomorphism f is semi-hyperbolic on the torus $\mathbb{T}^d$. This property of semi-hyperbolicity is robust under small C^1-perturbations, as will be proved in a more general context in Theorem 5.2 in Chapter 5.

Lemma 3.18. *Given an Anosov hyperbolic endomorphism $f : \mathbb{T}^d \mapsto \mathbb{T}^d$, there exists an $\varepsilon > 0$ such that every differentiable mapping $f_\varepsilon : \mathbb{T}^d \mapsto \mathbb{T}^d$ which is ε-close to the f in C^1-metric is semi-hyperbolic.*

This extends a classical result of Anosov on the robustness of the hyperbolicity property when f is an automorphism (i.e. invertible endomorphism), in which case a nearby diffeomorphism $f_\varepsilon : \mathbb{T}^d \mapsto \mathbb{T}^d$ is hyperbolic and has, in particular, a hyperbolic splitting. Theorem 5.1 will show that invertibility is essential in this case.

It is sometimes more convenient to consider semi-hyperbolic mappings on a compact subset of $\mathbb{R}^d$ rather than on a manifold. This is also possible for Anosov mappings and their perturbations, but the transition from one approach to the other is often not trivial in practice. The following example illustrates how it can be done for one particular mapping which will be used later in the proof of Theorem 5.1.

Example 3.19. Consider the 3×3 matrix

$$A = \begin{bmatrix} 2 & 3 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

which has eigenvalues $\lambda_1 = 2 - \sqrt{3}, \lambda_2 = 2 + \sqrt{3}$ and $\lambda_3 = 2$ with corresponding eigenvectors $v_1 = (1, -1/\sqrt{3}, 0), v_2 = (1, 1/\sqrt{3}, 0)$ and $v_3 = (0, 0, 1)$, respectively. Denote by $f : \mathbb{T}^3 \mapsto \mathbb{T}^3$ the continuously differentiable mapping defined by $f(x) = \Pi(Ax)$.

Interpret a point $z \in \mathbb{R}^6$ as an ordered triplet (z_1, z_2, z_3) of complex numbers with the norm $\|z\| = \sqrt{|z_1|^2 + |z_2|^2 + |z_3|^2}$ and let $\imath : \mathbb{T}^3 \mapsto \mathbb{R}^6$ be the immersion which maps the point $\varphi = (\varphi_1, \varphi_2, \varphi_3) \in \mathbb{T}^3$ to the point $z = (\mathrm{e}^{i\varphi_1}, \mathrm{e}^{i\varphi_2}, \mathrm{e}^{i\varphi_2}) \in K$, where

$$K = \left\{ z \in \mathbb{R}^6 : \ |z_1| = |z_2| = |z_3| = 1 \right\}.$$

Consider the mapping

$$F(z) = f(P(z)), \qquad z_1, z_2, z_3 \neq 0,$$

where P is the natural projection on K defined by

$$P(z) = \left(\frac{z_1}{|z_1|}, \frac{z_2}{|z_2|}, \frac{z_3}{|z_3|} \right), \qquad z_1, z_2, z_3 \neq 0.$$

The restriction of the mapping F to K is then topologically conjugate by the immersion $\imath$ to the mappings f and f_ε, respectively.

It is not easy to find an explicit expression for F, but its restriction to K has the form

$$F(z) = \left(z_1^{a_{11}} z_2^{a_{12}} z_3^{a_{13}},\ z_1^{a_{21}} z_2^{a_{22}} z_3^{a_{23}},\ z_1^{a_{31}} z_2^{a_{32}} z_3^{a_{33}}\right), \qquad z \in K,$$

where the a_{ij} are the components of the matrix A generating the Anosov endomorphism f.

For each point $z \in K$ the sets of vectors

$$e_1^t(z) = \left(iz_1, -\frac{i}{\sqrt{3}}z_2, 0\right), \quad e_2^t(z) = \left(iz_1, \frac{i}{\sqrt{3}}z_2, 0\right), \quad e_3^t(z) = (0, 0, iz_3),$$

$$e_1^n(z) = (z_1, 0, 0), \quad e_2^n(z) = (0, z_2, 0), \quad e_3^n(z) = (0, 0, z_3)$$

are a basis for $\mathbb{R}^6$, for which the differential DF_z of F satisfies

$$\begin{aligned} DF_z e_1^t(z) &= (2-\sqrt{3})e_1^t(F(z)), \\ DF_z e_2^t(z) &= (2+\sqrt{3})e_2^t(F(z)), \\ DF_z e_3^t(z) &= 2e_3^t(F(z)), \\ DF_z e_1^n(z) &= DF_z e_2^n(z) = DF_z e_3^n(z) = 0. \end{aligned} \tag{3.23}$$

Consider the splitting $T_z\mathbb{R}^6 = E_z^s \oplus E_z^u$ for $z \in K$ with subspaces

$$\begin{aligned} E_z^s &= \operatorname{span}\left\{e_1^t(z),\ e_1^n(z),\ e_2^n(z),\ e_3^n(z)\right\}, \\ E_z^u &= \operatorname{span}\left\{e_2^t(z),\ e_3^t(z)\right\} \end{aligned}$$

(i.e. spanned by the vectors indicated) with corresponding projectors P_z^s and P_z^u satisfying

$$P_z^s\mathbb{R}^6 = E_z^s, \quad P_z^s E_z^u = 0, \quad P_z^u\mathbb{R}^6 = E_z^u, \quad P_z^u E_z^s = 0.$$

The relations (3.23) imply that the splitting $T_z\mathbb{R}^6 = E_z^s \oplus E_z^u$ is invariant for the differential DF_z. On the other hand, the inequalities

$$\begin{aligned} \|P_{F(z)}^s DF_z u\| &\le (2-\sqrt{3})\|u\|, && u \in E_z^s, \\ \|P_{F(z)}^s DF_z v\| &= 0, && v \in E_z^u, \\ \|P_{F(z)}^u DF_z v\| &= 0, && u \in E_z^s, \\ \|P_{F(z)}^u DF_z v\| &\ge 2\|v\|, && v \in E_z^u, \end{aligned}$$

hold, so the differentiable mapping F is both hyperbolic and semi-hyperbolic on the compact subset set K of $\mathbb{R}^6$ in the sense of Definitions 2.5 and 3.5.

4

Semi-Hyperbolic Sequences of Matrices

The usefulness of first approximation methods to investigate the properties of semi-hyperbolic mappings and their trajectories leads us naturally to consider linear operators in spaces of sequences generated in one way or another by derivatives of semi-hyperbolic mappings. Some elementary properties of such linear operators that will be needed throughout the book are considered here.

Throughout this Chapter $\|\cdot\|$ will denote a fixed but otherwise arbitrary norm on $\mathbb{R}^d$ and $\ell^\infty(\mathbb{I}, \mathbb{R}^d)$ will denote the space of all bounded sequences of vectors $x_n \in \mathbb{R}^d$ with indices n taking values in 'interval' $\mathbb{I}$ in $\mathbb{Z}$ which can be finite, uni- or bi-directionally infinite depending on context. The norm on $\ell^\infty(\mathbb{I}, \mathbb{R}^d)$ is defined as

$$\|\{x_n\}\|_\infty = \sup_{n\in\mathbb{I}} \|x_n\|.$$

4.1 The Split Matrix

Recall that by Definition 3.2 a *split* is a four-tuple $\boldsymbol{s} = (\lambda_s, \lambda_u, \mu_s, \mu_u)$ of nonnegative real numbers for which

$$\lambda_s < 1 < \lambda_u, \quad (1 - \lambda_s)(\lambda_u - 1) > \mu_s \mu_u. \tag{4.1}$$

Recall also, that the split $\boldsymbol{s}$ is *positive* if all of the numbers λ_s, λ_u, μ_s and μ_u are positive. Clearly, without loss of generality, the split in Definitions 3.4, 3.5 and 3.6 can be assumed to be positive.

For a given positive split $\boldsymbol{s}$ define the 2×2 *split matrix* $M(\boldsymbol{s})$ by

$$M(\boldsymbol{s}) := \begin{bmatrix} \lambda_s & \mu_s \\ \mu_u/\lambda_u & 1/\lambda_u \end{bmatrix}. \tag{4.2}$$

Its spectral radius $\sigma(\boldsymbol{s})$ is given by

$$\sigma(\boldsymbol{s}) = \frac{1}{2}\left(\frac{1}{\lambda_u} + \lambda_s + \sqrt{\left(\frac{1}{\lambda_u} - \lambda_s\right)^2 + \frac{4\mu_s\mu_u}{\lambda_u}}\right) \tag{4.3}$$

and, as simple but cumbersome calculation show,

$$\sigma(\boldsymbol{s}) < 1 - \nu(\boldsymbol{s}) < 1 \tag{4.4}$$

where $\nu(\boldsymbol{s})$ was defined in (3.3). Moreover, since the entries of the matrix $M(\boldsymbol{s})$ are positive for a positive split $\boldsymbol{s}$, it follows by the Perron–Frobenius Theorem that $\sigma(\boldsymbol{s})$ is the maximal eigenvalue of $M(\boldsymbol{s})$ and that the corresponding eigenvector has positive components. This eigenvector will be written here as $(1, \gamma(\boldsymbol{s}))^T$, where

$$\gamma(\boldsymbol{s}) := \frac{1}{2\mu_s}\left(\frac{1}{\lambda_u} - \lambda_s + \sqrt{\left(\frac{1}{\lambda_u} - \lambda_s\right)^2 + \frac{4\mu_s\mu_u}{\lambda_u}}\right). \tag{4.5}$$

When the split $\boldsymbol{s}$ is fixed, we shall write $M = M(\boldsymbol{s})$, $\gamma = \gamma(\boldsymbol{s})$ and $\sigma = \sigma(\boldsymbol{s})$, so

$$M\begin{pmatrix}1\\ \gamma\end{pmatrix} = \begin{bmatrix}\lambda_s & \mu_s\\ \mu_u/\lambda_u & 1/\lambda_u\end{bmatrix}\begin{pmatrix}1\\ \gamma\end{pmatrix} = \sigma\begin{pmatrix}1\\ \gamma\end{pmatrix}, \tag{4.6}$$

and introduce a new norm $\|\cdot\|_*$ on $\mathbb{R}^2$ defined by

$$\|(y_1, y_2)\|_* := \max\{\gamma|y_1|, |y_2|\}.$$

The corresponding norm $\|M\|_*$ of the linear operator with the matrix (4.2) clearly coincides with the spectral radius σ of M, so $\|M\|_* = \sigma < 1$ by (4.4) and $\|Mx\|_* \le \sigma\|x\|_*$ for all $x \in \mathbb{R}^2$.

Lemma 4.1. *The following inequalities are valid.*

$$\min\{\gamma, 1\}\,\max\{|y_1|, |y_2|\} \le \|(y_1, y_2)\|_* \le \max\{\gamma, 1\}\,\max\{|y_1|, |y_2|\}$$

Proof. By direct calculation. □

The next lemma will play in what follows an important role.

Lemma 4.2. *For $m = 1, 2, \ldots$ let the following inequalities be satisfied:*

$$\begin{aligned} a_m^s &\le \lambda_s a_{m-1}^s + \mu_s a_{m-1}^u + h^s,\\ a_m^u &\le \frac{\mu_u}{\lambda_u} a_{m-1}^s + \frac{1}{\lambda_u} a_{m-1}^u + h^u, \end{aligned} \tag{4.7}$$

with $a_m^s, a_m^u, h^s, h^m \ge 0$ and $a_0^s = a_0^u = 0$. Then

$$\limsup_{m\to\infty}(a_m^s + a_m^u) \le \frac{\max\{h^s,\ \lambda_u h^u\}}{\nu(\boldsymbol{s})}, \tag{4.8}$$

with $\nu(\boldsymbol{s})$ defined by (3.3).

Proof. Define vectors $\boldsymbol{a}_m = (a_m^s, a_m^u)^T$ for $m = 0, 1, \dots$ and $\boldsymbol{h} = (h^s, h^u)^T$. Then the inequalities (4.7) can be rewritten in the vector form

$$\boldsymbol{a}_m \le M\boldsymbol{a}_{m-1} + \boldsymbol{h}, \qquad m = 1, 2, \dots \tag{4.9}$$

where $\boldsymbol{a}_0 = 0$ and the inequality between vectors is interpreted componentwise. Since the entries of the matrix $M = M(\boldsymbol{s})$ are non-negative, we preserve inequality direction when applying the matrix M to the both sides of inequality (4.9).Hence

$$\boldsymbol{a}_m \le M^m\boldsymbol{a}_0 + (I + M + \cdots + M^{m-1})\boldsymbol{h}, \qquad m = 1, 2, \dots ,$$

from which, in view of equality $\boldsymbol{a}_0 = 0$, we obtain

$$\limsup_{m\to\infty} \boldsymbol{a}_m \le (I - M)^{-1}\boldsymbol{h} \tag{4.10}$$

where the lim sup is applied componentwise.

By direct calculation

$$(I - M)^{-1} = \frac{1}{(1-\lambda_s)(\lambda_u - 1) - \mu_s\mu_u} \begin{bmatrix} \lambda_u - 1 & \mu_s\lambda_u \\ \mu_u & (1-\lambda_s)\lambda_u \end{bmatrix},$$

so (4.10) can be rewritten componentwise as

$$\limsup_{m\to\infty} a_m^s \le \frac{(\lambda_u - 1)h^s + \mu_s\lambda_u h^u}{(1-\lambda_s)(\lambda_u - 1) - \mu_s\mu_u},$$
$$\limsup_{m\to\infty} a_m^u \le \frac{\mu_u h^s + (1-\lambda_s)\lambda_u h^u}{(1-\lambda_s)(\lambda_u - 1) - \mu_s\mu_u},$$

from which inequality (4.8) is an immediate consequence. □

4.1.1 A Perturbation Theorem

Various proofs for semi-hyperbolic mappings will involve a matrix or a sequence of matrices that have similar semi-hyperbolic properties. This motivates the next definition where $\boldsymbol{s} = (\lambda_s, \lambda_u, \mu_s, \mu_u)$ is a given split.

Definition 4.3 (Semi-Hyperbolic Matrix). *A $d \times d$ matrix A is called a* $(\boldsymbol{s}, h)$-semi-hyperbolic matrix *if the corresponding linear mapping is $\boldsymbol{s}$-semi-hyperbolic on $\mathbb{R}^d$ with respect to a splitting $\mathbb{R}^d = E^s \oplus E^u$ with corresponding projectors P^s and P^u satisfying $\|P^s\| \le h$ and $\|P^u\| \le h$.*

The following Perturbation Theorem, which asserts that the class of semi-hyperbolic matrices is an open set in the space of all matrices, indicates the robustness of semi-hyperbolic matrices and provides explicit estimates of possible perturbations which do not affect the semi-hyperbolicity of a matrix.

Theorem 4.4 (Perturbation Theorem). *Let a $d \times d$ matrix A be $(\boldsymbol{s}, h)$-hyperbolic. Then every $d \times d$ matrix $\widetilde{A}$ satisfying*

$$\|\widetilde{A} - A\| \le \delta < \frac{\nu(\boldsymbol{s})}{h}$$

is $(\widetilde{\boldsymbol{s}}, h)$-hyperbolic with the split $\widetilde{\boldsymbol{s}} = (\lambda_s + \delta h, \lambda_u - \delta h, \mu_s + \delta h, \mu_u + \delta h)$.

Proof. In view of $(\boldsymbol{s}, h)$-hyperbolicity of A there is a decomposition $\mathbb{R}^d = E^s \oplus E^u$ with corresponding projections $P^s : \mathbb{R}^d \mapsto E^s$ and $P^u = I - P^s : \mathbb{R}^d \mapsto E^u$ such that

$$\|P^s\| \le h, \quad \|P^u\| \le h$$

and

$$\|P^s A P^s x\| \le \lambda_s \|P^s x\|, \quad \|P^s A P^u x\| \le \mu_s \|P^u x\|,$$
$$\|P^u A P^s x\| \le \mu_u \|P^s x\|, \quad \|P^u A P^u x\| \ge \lambda_u \|P^u x\|,$$

for all $x \in \mathbb{R}^d$. Hence

$$\|P^s \widetilde{A} P^s x\| \le \|P^s(\widetilde{A} - A) P^s x\| + \|P^s A P^s x\| \le (\delta h + \lambda_s)\|P^s x\|$$

and

$$\|P^u \widetilde{A} P^u x\| \ge \|P^u A P^u x\| - \|P^u(\widetilde{A} - A) P^u x\| \ge (\lambda_u - \delta h)\|P^u x\|.$$

Similarly,

$$\|P^s \widetilde{A} P^u x\| \le (\mu_s + \delta h)\|P^u x\|, \qquad \|P^u \widetilde{A} P^s x\| \le (\mu_u + \delta h)\|P^s x\|.$$

Define

$$\widetilde{\lambda}_s = \lambda_s + \delta h, \quad \widetilde{\lambda}_u = \lambda_u - \delta h, \quad \widetilde{\mu}_s = \mu_s + \delta h, \quad \widetilde{\mu}_u = \mu_u + \delta h.$$

By Lemma 3.3 these numbers form a split $\widetilde{\boldsymbol{s}}$, which completes the proof of Theorem 4.4. □

4.2 Semi-Hyperbolicity Implies Hyperbolicity

In this Section an investigation of the properties of sequences of semi-hyperbolic mappings, required later, will be started. Throughout this Section let $\{A_n\}$ be a sequence of $d \times d$ matrices defined for $n \in \mathbb{I}$, a given interval of $\mathbb{Z}$ which could be finite or infinite.

Definition 4.5 (Semi-Hyperbolic Sequence of Matrices). *A bounded sequence of $d \times d$ matrices $\{A_n\}$ will be called $(\boldsymbol{s}, h)$-semi-hyperbolic if for each matrix A_n, $n \in \mathbb{I}$, there is a splitting $\mathbb{R}^d = E_n^s \oplus E_n^u$ with*

$$\dim E_n^s = \dim E_{n+1}^s, \quad \dim E_n^u = \dim E_{n+1}^u \tag{4.11}$$

and projections $P_n^s : \mathbb{R}^d \mapsto E_n^s$, $P_n^u = I - P_n^s : \mathbb{R}^d \mapsto E_n^u$, *with the uniform matrix norm bounds*

$$\|P_n^s\| \le h, \quad \|P_n^u\| \le h, \qquad n \in \mathbb{I}, \tag{4.12}$$

such that

$$\begin{aligned}
\|P_{n+1}^s A_n P_n^s x\| &\le \lambda_s \|P_n^s x\|, \\
\|P_{n+1}^s A_n P_n^u x\| &\le \mu_s \|P_n^u x\|, \\
\|P_{n+1}^u A_n P_n^s x\| &\le \mu_u \|P_n^s x\|, \\
\|P_{n+1}^u A_n P_n^u x\| &\ge \lambda_u \|P_n^u x\|,
\end{aligned} \tag{4.13}$$

for all $n \in \mathbb{I}$ *and* $x \in \mathbb{R}^d$.

Conditions (4.11)–(4.13) are the obvious analogs of Conditions SH0–SH1 in Definition 3.1 of a uniform splitting and of Condition SH2 in the Definitions 3.4 and 3.5 for a semi-hyperbolic mapping. With the index n dropped, the projection bounds (4.12) and the inequalities (4.13) apply for single a semi-hyperbolic matrix as in Definition 4.3.

Henceforth let $\{A_n\}$ be an $(\boldsymbol{s}, h)$-semi-hyperbolic bi-sequence of uniformly bounded and invertible $d \times d$ matrices (cf. Definition 4.5, with $\mathbb{I} = \mathbb{Z}$ here).

We can interpret inequalities (4.13) in Definition 4.5 as conditions that ensure the existence of 'almost equivariant' splitting $\mathbb{R}^d = E_n^s \oplus E_n^u$ for the sequence $\{A_n\}$ with contracting subspaces E_n^s and expanding subspaces E_n^u. Here the word 'almost' means that the subspaces $A_n E_{n+1}^s$ and $A_n E_{n+1}^u$ are, in fact, only close to E_n^s and E_n^u, respectively, but generally do not coincide with them.

The main objective of Section is to show that semi-hyperbolicity of a matrix sequence $\{A_n\}$ implies existence of a genuinely equivariant splitting $\mathbb{R}^d = \widehat{E}_n^s \oplus \widehat{E}_n^u$.

4.2.1 Sequence Operators $\mathscr{R}$ and $\mathscr{L}$

For each integer $k \in \mathbb{Z}$ we define on the sequence space $\ell^\infty([k, \infty), \mathbb{R}^d)$ the norm

$$\|\boldsymbol{x}\|_\infty^* = \sup_{n \ge k} \left\{ \|P_n^s x_n\|, \|P_n^u x_n\| \right\},$$

which is clearly equivalent to the norm $\|\cdot\|_\infty$. Similarly, on the sequence space $\ell^\infty((-\infty, k], \mathbb{R}^d)$ we define the norm

$$\|\boldsymbol{x}\|_\infty^* = \sup_{n \le k} \left\{ \|P_n^s x_n\|, \|P_n^u x_n\| \right\}.$$

We now introduce an operator

$$\mathscr{R} := \mathscr{R}_{[\alpha,k,v]} : \ell^\infty([k,\infty),\mathbb{R}^d) \mapsto \ell^\infty([k,\infty),\mathbb{R}^d)$$

with parameters α a positive number, k an integer and $v \in P_k^s\mathbb{R}^d$ that maps a sequence $\boldsymbol{x} = \{x_n\} \in \ell^\infty([k,\infty),\mathbb{R}^d)$ to the sequence $\boldsymbol{v} = \{v_n\} \in \ell^\infty([k,\infty),\mathbb{R}^d)$ defined by

$$\begin{aligned} P_k^s v_k &= v, \\ P_n^s v_n &= \alpha P_n^s A_{n-1} x_{n-1}, && n \geq k+1, \\ P_n^u v_n &= \alpha^{-1}(P_{n+1}^u A_n P_n^u)^{-1} P_{n+1}^u x_{n+1} \\ &\quad - (P_{n+1}^u A_n P_n^u)^{-1} P_{n+1}^u A_n P_n^s x_n, && n \geq k. \end{aligned} \tag{4.14}$$

In view of the split inequalities (4.1) there exists a positive number ω such that for all $\alpha \in [1-\omega, 1+\omega]$ the strict inequalities

$$\alpha\lambda_s < 1 < \alpha\lambda_u, \quad (1-\alpha\lambda_s)(\alpha\lambda_u - 1) > \alpha^2\mu_s\mu_u \tag{4.15}$$

hold. Hence the quantity

$$\varrho = \sup_{\alpha\in[1-\omega,1+\omega]} \max\left\{\frac{\alpha\mu_u + \alpha\lambda_u - 1}{\alpha\lambda_u - 1}, \ \frac{1-\alpha\lambda_s + \alpha\mu_s}{1-\alpha\lambda_s}\right\}$$

is well defined and for all $\alpha \in [1-\omega, 1+\omega]$ the spectral radius $\sigma(M_\alpha)$ of the matrix

$$M_\alpha = \begin{bmatrix} \alpha\lambda_s & \alpha\mu_s \\ \mu_u/\lambda_u & 1/(\alpha\lambda_u) \end{bmatrix}$$

is equal (cf. (4.3)) to

$$\sigma(M_\alpha) = \frac{1}{2}\left(\frac{1}{\alpha\lambda_u} + \alpha\lambda_s + \sqrt{\left(\frac{1}{\alpha\lambda_u} - \alpha\lambda_s\right)^2 + \frac{4\alpha\mu_s\mu_u}{\lambda_u}}\right).$$

and is less than 1.

Since the entries of the matrix M_α are positive (recall that we have restricted ourselves to positive splits $\boldsymbol{s}$), it follows by the Perron-Frobenius Theorem that its spectral radius $\sigma(M_\alpha)$ is an eigenvalue and that the corresponding eigenvector has positive coordinates. Without loss of generality this vector has the form $(1,\gamma_\alpha)^T$ but the exact expression for the value of γ_α is not important here so is omitted (cf. (4.5)). What is important is that for each $\alpha \in [1-\omega, 1+\omega]$ the norm

$$\|x\|_\infty^\alpha = \sup_{n\geq k}\{\gamma_\alpha\|P_n^s x_n\|, \|P_n^u x_n\|\}$$

on the space $\ell^\infty([k,\infty),\mathbb{R}^d)$ is equivalent to the norm $\|\cdot\|_\infty$.

Lemma 4.6. *For any $\alpha \in [1-\omega, 1+\omega]$, any integer k and any $v \in P_k^s\mathbb{R}^d$ the operator $\mathscr{R}_{[\alpha,k,v]}$ is contracting with the constant $\sigma(M_\alpha)$ in the norm $\|\cdot\|_\infty^\alpha$. Moreover, the set*

$$\mathscr{V}_{[\alpha,k,v]} = \left\{ \boldsymbol{x} : \ \|P_n^s x_n\| \le \|v\|, \ \|P_n^u x_n\| \le \frac{\alpha\mu_u}{\alpha\lambda_u - 1}\|v\|, \quad n \ge k \right\} \tag{4.16}$$

is invariant under $\mathscr{R}_{[\alpha,k,v]}$.

Corollary 4.7. *For any* $v \in P^s\mathbb{R}^d$ *the operator* $\mathscr{R}_{[\alpha,k,v]}$ *has a unique fixed point* $\boldsymbol{x}^* = \boldsymbol{x}_{[\alpha,k,v]} = \{x_n^*\}$ *for which* $P_k^s x_k^* = v$ *and*

$$\|P_n^s x_n^*\| \le \|v\|, \quad \|P_n^u x_n^*\| \le \frac{\alpha\mu_u}{\alpha\lambda_u - 1}\|v\|, \qquad n \ge k.$$

Corollary 4.7 follows immediately from Lemma 4.6 by Banach Contraction Mapping Theorem, so it is needed only to prove the Lemma 4.6.

Proof. Fix a value of $\alpha \in [1-\omega, 1+\omega]$. The definition of the vector $(1,\gamma_\alpha)^T$ gives the equality

$$\begin{bmatrix} \alpha\lambda_s & \alpha\mu_s \\ \mu_u/\lambda_u & 1/(\alpha\lambda_u) \end{bmatrix} \begin{pmatrix} 1 \\ \gamma_\alpha \end{pmatrix} = \sigma(M_\alpha) \begin{pmatrix} 1 \\ \gamma_\alpha \end{pmatrix},$$

from which it follows that

$$\alpha\lambda_s + \alpha\mu_s\gamma_\alpha = \sigma(M_\alpha), \frac{\mu_u}{\lambda_u} + \frac{1}{\alpha\lambda_u}\gamma_\alpha = \sigma(M_\alpha)\gamma_\alpha. \tag{4.17}$$

Given arbitrary $\boldsymbol{x}$, $\widetilde{\boldsymbol{x}} \in \ell^\infty([k,\infty),\mathbb{R}^d)$ write $\boldsymbol{v} = \mathscr{R}_{[\alpha,k,v]}(\boldsymbol{x})$ and $\widetilde{\boldsymbol{v}} = \mathscr{R}_{[\alpha,k,v]}(\widetilde{\boldsymbol{x}})$. Then from (4.14) it follows that

$$\begin{aligned} &P_k^s(v_k - \widetilde{v}_k) = 0, \\ &P_n^s(v_n - \widetilde{v}_n) = \alpha P_n^s A_{n-1}(x_{n-1} - \widetilde{x}_{n-1}), && n \ge k+1, \\ &P_n^u(v_n - \widetilde{v}_n) = \alpha^{-1}(P_{n+1}^u A_n P_n^u)^{-1} P_{n+1}^u (x_{n+1} - \widetilde{x}_{n+1}) \\ &\qquad - (P_{n+1}^u A_n P_n^u)^{-1} P_{n+1}^u A_n P_n^s (x_n - \widetilde{x}_n), && n \ge k. \end{aligned}$$

In view of (4.13) the following relations are thus valid

$$\begin{aligned} \|P_k^s(v_k - \widetilde{v}_k)\| &= 0, \\ \|P_n^s(v_n - \widetilde{v}_n)\| &\le \alpha\lambda_s \|P_n^s(x_{n-1} - \widetilde{x}_{n-1})\| \\ &\quad + \ \alpha\mu_s \|P_n^u(x_{n-1} - \widetilde{x}_{n-1})\|, && n \ge k+1, \\ \|P_n^u(v_n - \widetilde{v}_n)\| &\le \frac{\mu_u}{\lambda_u}\|P_n^s(x_n - \widetilde{x}_n)\| \\ &\quad + \ \frac{1}{\alpha\lambda_u}\|P_{n+1}^u(x_{n+1} - \widetilde{x}_{n+1})\|, && n \ge k, \end{aligned}$$

from which and (4.17) it is clear that

$$\|\mathscr{R}_{[\alpha,k,v]}(\boldsymbol{x}) - \mathscr{R}_{[\alpha,k,v]}(\widetilde{\boldsymbol{x}})\|_\infty^\alpha \le \sigma(M_\alpha)\|\boldsymbol{x} - \widetilde{\boldsymbol{x}}\|_\infty^\alpha,$$

i.e. the operator $\mathscr{R}_{[\alpha,k,v]}$ is contracting in the norm $\|\cdot\|_\infty^\alpha$ with contraction constant $\sigma(M_\alpha)$.

It remains only to show that the set (4.16) is invariant for $\mathscr{R}_{[\alpha,k,v]}$. Given an arbitrary $\boldsymbol{x} = \{x_n\} \in \mathscr{V}_{[\alpha,k,v]}$, denote $\boldsymbol{v} = \mathscr{R}_{[\alpha,k,v]}(\boldsymbol{x})$. From (4.13) and (4.14) it is apparent that

$$\begin{aligned}
&\|P_k^s v_k\| = \|v\|, \\
&\|P_n^s v_n\| \le \alpha\lambda_s \|P_n^s x_{n-1}\| + \alpha\mu_s \|P_n^u x_{n-1}\|, && n \ge k+1, \\
&\|P_n^u v_n\| \le \frac{\mu_u}{\lambda_u}\|P_n^s x_n\| + \frac{1}{\alpha\lambda_u}\|P_{n+1}^u x_{n+1}\|, && n \ge k.
\end{aligned}$$

Hence, in view of the definition (4.16) of the set $\mathscr{V}_{[\alpha,k,v]}$, we have

$$\begin{aligned}
&\|P_k^s v_k\| = \|v\|, \\
&\|P_n^s v_n\| \le \alpha\lambda_s \|v\| + \frac{\alpha^2\mu_s\mu_u}{\alpha\lambda_u - 1}\|v\|, && n \ge k+1, \\
&\|P_n^u v_n\| \le \frac{\mu_u}{\lambda_u}\|v\| + \frac{\alpha\mu_u}{\alpha\lambda_u(\alpha\lambda_u - 1)}\|v\|, && n \ge k,
\end{aligned}$$

and thus, from (4.15), can conclude that

$$\begin{aligned}
&\|P_k^s v_k\| = \|v\|, \\
&\|P_n^s v_n\| \le \|v\| + \frac{\alpha^2\mu_s\mu_u - (1-\alpha\lambda_s)(\alpha\lambda_u - 1)}{\alpha\lambda_u - 1}\|v\| \le \|v\|, && n \ge k+1, \\
&\|P_n^u v_n\| \le \frac{\alpha\mu_u}{\alpha\lambda_u - 1}\|v\|, && n \ge k.
\end{aligned}$$

This completes the proof of invariance of the set (4.16) under $\mathscr{R}_{[\alpha,k,v]}$. Lemma 4.6 is proved. □

The next lemma follows immediately from Lemma 4.6 and from the definition of the operator $\mathscr{R}_{[\alpha,k,v]}$.

Lemma 4.8. *The only bounded solution of the difference equation*

$$x_{n+1} = \alpha A_n x_n, \qquad n \ge k, \tag{4.18}$$

which satisfies the condition $P_k^s x_k = v$ *is the fixed point* $\boldsymbol{x}_{[\alpha,k,v]}$ *of the operator* $\mathscr{R}_{[\alpha,k,v]}$.

We can combine Lemmata 4.6 and 4.8 to show that the fixed point sequence $\boldsymbol{x}_{[1,k,v]}$ of $\mathscr{R}_{[1,k,v]}$ is not only bounded one but is also exponentially convergent to zero.

Corollary 4.9. *The fixed point* $\boldsymbol{x}^* = \boldsymbol{x}_{[\alpha,k,v]} = \{x_n^*\}$ *of the operator* $\mathscr{R}_{[1,k,v]}$ *satisfies*

$$\|x_n^*\| \le \frac{\varrho}{(1+\omega)^{|n-k|}}\|v\|, \qquad n \ge k. \tag{4.19}$$

Proof. According to the definition of the constant $\omega > 0$ the operator $\mathscr{R}_{[1+\omega,k,v]}$ has a fixed point $\boldsymbol{x}^*_{1+\omega} = \{x^*_{1+\omega,n}\}$. By Lemma 4.8 $\{x^*_{1+\omega,n}\}$ is the bounded solution of equation (4.18) for $\alpha = 1+\omega$. Hence the sequence $\boldsymbol{x}^* = \{x^*_n\}$, where $x^*_n = (1+\omega)^{k-n} x^*_{1+\omega,n}$, $n \geq k$, is the solution of equation (4.18) for $\alpha = 1$. In view of Corollary 4.7 we have for any $n \geq k$

$$\|x^*_{1+\omega,n}\| \leq \|P^s_n x^*_{1+\omega,n}\| + \|P^u_n x^*_{1+\omega,n}\| \leq \frac{\alpha\mu_u + \alpha\lambda_u - 1}{\alpha\lambda_u - 1}\|v\| \leq \varrho\|v\|$$

and so by the definition of $\boldsymbol{x}^*$ the inequalities (4.19) are valid.

Thus, $\boldsymbol{x}^*$ is the bounded solution of equation (4.18) for $\alpha = 1$ and by Lemma 4.8 it is the fixed point of the operator $\mathscr{R}_{[1,k,v]}$ for which estimate (4.19) is valid. □

We now define an operator

$$\mathscr{L} := \mathscr{L}_{[\alpha,k,w]} : \ell^\infty((-\infty,k],\mathbb{R}^d) \mapsto \ell^\infty((-\infty,k],\mathbb{R}^d)$$

with parameters α a positive number, k an integer and $w \in P^u_k\mathbb{R}^d$ that maps a sequence $\boldsymbol{y} = \{y_n\} \in \ell^\infty((-\infty,k],\mathbb{R}^d)$ to the sequence $\boldsymbol{w} = \{w_n\} \in \ell^\infty((-\infty,k],\mathbb{R}^d)$ defined by

$$\begin{aligned}
P^s_n w_n &= \alpha P^s_n A_{n-1} y_{n-1}, && n \leq k,\\
P^u_k w_k &= w,\\
P^u_n w_n &= \alpha^{-1}(P^u_{n+1}A_nP^u_n)^{-1}P^u_{n+1}y_{n+1}\\
&\quad - (P^u_{n+1}A_nP^u_n)^{-1}P^u_{n+1}A_nP^s_n y_n, && n \leq k-1.
\end{aligned}$$

The proofs of the following two lemmata and corollaries are essentially repetitions of those for Lemmata 4.6 and 4.8 and for Corollaries 4.7 and 4.9, so will be omitted.

Lemma 4.10. *For any $\alpha \in [1-\omega, 1+\omega]$, any integer k and any $w \in P^u_k\mathbb{R}^d$ the operator $\mathscr{L}_{[\alpha,k,w]}$ is contracting in the norm $\|\cdot\|^\alpha_\infty$ with constant $\sigma(M_\alpha)$. Moreover, the set*

$$\mathscr{W}_{[\alpha,k,w]} = \left\{\boldsymbol{x} : \ \|P^s_n x_n\| \leq \frac{\alpha\mu_s}{1-\alpha\lambda_s}\|w\|, \ \|P^u_n x_n\| \leq \|w\|, \quad n \leq k\right\}$$

is invariant under $\mathscr{L}_{[\alpha,k,w]}$.

Corollary 4.11. *For any $w \in P^u\mathbb{R}^d$ the operator $\mathscr{L}_{[\alpha,k,w]}$ has a unique fixed point $\boldsymbol{y}^* = \boldsymbol{y}_{[\alpha,k,w]} = \{y^*_n\}$ for which $P^u_k y^*_k = w$ and*

$$\|P^s_n y^*_n\| \leq \frac{\alpha\mu_s}{1-\alpha\lambda_s}\|w\|, \quad \|P^u_n y^*_n\| \leq \|w\|, \qquad n \leq k.$$

Lemma 4.12. *The only bounded solution of the difference equation*

$$y_{n+1} = \alpha A_n y_n, \qquad n \le k-1,$$

which satisfies the condition $P_k^u y_k = w$ *is the fixed point* $\boldsymbol{y}_{[\alpha,k,w]}$ *of the operator* $\mathscr{L}_{[\alpha,k,w]}$.

Corollary 4.13. *The fixed point* $\boldsymbol{y}^* = \boldsymbol{y}_{[\alpha,k,w]} = \{y_n^*\}$ *of the operator* $\mathscr{L}_{[1,k,w]}$ *satisfies*

$$\|y_n^*\| \le \frac{\varrho}{(1+\omega)^{|n-k|}}\|w\|, \qquad n \le k.$$

4.2.2 An Equivariant Splitting

Now let $\boldsymbol{x}_{[\alpha,k,v]} = \{\boldsymbol{x}_{[\alpha,k,v],n}\}$ denote the fixed point of an operator $\mathscr{R}_{[\alpha,k,v]}$ and define the operator $X_{[\alpha,k]} : P_k^s\mathbb{R}^d \mapsto \mathbb{R}^d$ by

$$X_{[\alpha,k]}v = \boldsymbol{x}_{[\alpha,k,v],k}.$$

Lemma 4.14. *The operator* $X_{[\alpha,k]}$ *is linear,* $P_k^s X_{[\alpha,k]} P_k^s = P_k^s$ *and*

$$\|P_k^u X_{[\alpha,k]}v\| \le \frac{\alpha\mu_u}{\alpha\lambda_u - 1}\|v\|, \qquad v \in P_k^s\mathbb{R}^d. \tag{4.20}$$

Proof. Let $\boldsymbol{x} = \{x_n\} = \boldsymbol{x}_{[\alpha,k,v]}$ be the fixed point of the operator $\mathscr{R}_{[\alpha,k,v]}$ and $\widetilde{\boldsymbol{x}} = \{\widetilde{x}_n\} = \boldsymbol{x}_{[\alpha,k,\widetilde{v}]}$ be the fixed point of the operator $\mathscr{R}_{[\alpha,k,\widetilde{v}]}$ for given v, $\widetilde{v} \in P_k^s\mathbb{R}^d$. Then by Lemma 4.8 $\boldsymbol{x} = \{x_n\}$ is the solution of equation (4.18) satisfying the condition $P_k^s x_k = v$, while $\widetilde{\boldsymbol{x}} = \{\widetilde{x}_n\}$ is the solution of equation (4.18) satisfying the condition $P_k^s \widetilde{x}_k = \widetilde{v}$. Hence for any real numbers ξ and ζ, the sequence $\boldsymbol{z} = \xi\boldsymbol{x} + \zeta\widetilde{\boldsymbol{x}} = \{\xi x_n + \zeta\widetilde{x}_n\} \in \ell^\infty([k,\infty),\mathbb{R}^d)$ is the solution of equation (4.18) satisfying the condition $P_k^s z_k = \xi v + \zeta\widetilde{v}$. Using Lemma 4.8 again, we find that $\boldsymbol{z}$ is the fixed point of the operator $\mathscr{R}_{[\alpha,k,\xi v+\zeta\widetilde{v}]}$, that is $\boldsymbol{z} = \boldsymbol{x}_{[\alpha,k,\xi v+\zeta\widetilde{v}]}$. Thus

$$\xi x_{[\alpha,k,v]} + \zeta\boldsymbol{x}_{[\alpha,k,\widetilde{v}]} = \boldsymbol{x}_{[\alpha,k,\xi v+\zeta\widetilde{v}]},$$

so the mapping $\boldsymbol{x}_{[\alpha,k,\cdot]}$ is linear. Hence the operator $X_{[\alpha,k]}(\cdot) = \boldsymbol{x}_{[\alpha,k,\cdot]}$ is linear too.

Now note that $P_k^s X_{[\alpha,k]}v = P_k^s\boldsymbol{x}_{[\alpha,k,v],k} = v$ for $v \in P_k^s\mathbb{R}^d$, which means that $P_k^s X_{[\alpha,k]}P_k^s = P_k^s$, as required.

Estimate (4.20) then follows immediately from the definition of the operator $X_{[\alpha,k]}$ and from Corollary 4.7. □

Similarly, denote the fixed point of the operator $\mathscr{L}_{[\alpha,k,w]}$ by $\boldsymbol{y}_{[\alpha,k,w]} = \{\boldsymbol{y}_{[\alpha,k,w],n}\}$ and define an operator $Y_{[\alpha,k]} : P_k^u\mathbb{R}^d \mapsto \mathbb{R}^d$ by

$$Y_{[\alpha,k]}w = y_{[\alpha,k,w],k}.$$

The proof of the next lemma is essentially the same as that of Lemma 4.14.

Lemma 4.15. *The operator* $Y_{[\alpha,k]}$ *is linear,* $P_k^u Y_{[\alpha,k]} P_k^u = P_k^u$ *and*

$$\|P_k^s Y_{[\alpha,k]} w\| \leq \frac{\alpha\mu_s}{1-\alpha\lambda_s}\|w\|, \qquad w \in P_k^u \mathbb{R}^d.$$

Now define $\widetilde{E}_k^s$ as the set of those $v \in \mathbb{R}^d$ for which the equation

$$v_{n+1} = A_n v_n, \qquad n \geq k, \tag{4.21}$$

has a bounded solution satisfying $v_k = v$, and define $\widetilde{E}_k^u$ as the set of those $w \in \mathbb{R}^d$ for which the equation

$$w_{n+1} = A_n w_n, \qquad n < k, \tag{4.22}$$

has a bounded solution satisfying $w_k = w$. In addition, define subspaces $\widehat{E}_k^s$ and $\widehat{E}_k^s$ of $\mathbb{R}^d$ by

$$\widehat{E}_k^s = X_{[1,k]} P_k^s \mathbb{R}^d, \quad \widehat{E}_k^u = Y_{[1,k]} P_k^u \mathbb{R}^d, \qquad k \in \mathbb{Z}. \tag{4.23}$$

Finally, recall that the *separation* $\delta(E_1, E_2)$ between two subspaces $E_1, E_2 \subseteq \mathbb{R}^d$ is defined as

$$\delta(E_1, E_2) = \inf\{\|x+y\| : \ x \in E_1, \ y \in E_2, \ \|x\| = \|y\| = 1\}.$$

Lemma 4.16. *Suppose that matrices* A_n, $n \in \mathbb{Z}$, *are invertible. Then for any integer* k *the following relations are valid:*

$$\widehat{E}_k^s = \widetilde{E}_k^s, \qquad \widehat{E}_k^u = \widetilde{E}_k^u, \tag{4.24}$$

$$\widehat{E}_k^s \cap \widehat{E}_k^u = 0, \qquad \widehat{E}_k^s \oplus \widehat{E}_k^u = \mathbb{R}^d, \tag{4.25}$$

$$A_k \widehat{E}_k^s = \widehat{E}_{k+1}^s, \qquad A_k \widehat{E}_k^u = \widehat{E}_{k+1}^u. \tag{4.26}$$

$$\delta(\widehat{E}_k^s, \widehat{E}_k^u) \geq \frac{(1-\lambda_s)(\lambda_u - 1) - \mu_s\mu_u}{h(\lambda_u - 1 + \mu_u)(1 - \lambda_s + \mu_s)} > 0. \tag{4.27}$$

Proof. We first prove the identity $\widehat{E}_k^s = \widetilde{E}_k^s$ from (4.24). Indeed, if $v \in \widehat{E}_k^s$ then, by definition of $\widehat{E}_k^s$, there exists $v^* \in P_k^s \mathbb{R}^d$ such that $v = X_{[1,k]} v^*$ and, by definition of the operator $X_{[1,k]}$, $v^* = P_k^s v_k$ where $\{v_n\}$, $n \geq k$, is the fixed point of the operator $\mathscr{R}_{[1,k,v^*]}$ satisfying $v_k = v$. Hence by Lemma 4.8, $\{v_n\}$ is a bounded solution of equation (4.21) and so $v \in \widetilde{E}_k^s$, from which it follows that $\widehat{E}_k^s \subseteq \widetilde{E}_k^s$. Similarly, if $v \in \widetilde{E}_k^s$ then there exist a bounded solution $\{v_n\}$ of equation (4.21) satisfying $v_k = v$. So by Lemma 4.8 $\{v_n\}$ is the fixed point of the operator $\mathscr{R}_{[1,k,P_k^s v]}$. Therefore, by the definition of the operator $X_{[1,k]}$, the vector v can be represented in the form $v = X_{[1,k]} P_k^s v$ and so $v \in \widehat{E}_k^s$, from which it follows that $\widetilde{E}_k^s \subseteq \widehat{E}_k^s$. Combining the two results gives the desired identity $\widetilde{E}_k^s = \widehat{E}_k^s$. The corresponding identity $\widetilde{E}_k^u = \widehat{E}_k^u$ is proved analogously.

To prove the first equality of (4.25) suppose that $\widehat{E}_k^s \cap \widehat{E}_k^u \neq 0$ for some k. Then there exist nonzero $v \in P_k^s \mathbb{R}^d$, $w \in P_k^u \mathbb{R}^d$ and $\{v_n\} \in \ell^\infty([k,\infty),\mathbb{R}^d)$, $\{w_n\} \in \ell^\infty((-\infty,k],\mathbb{R}^d)$ such that $\{v_n\}$ is the fixed point of the operator $\mathscr{R}_{[1,k,v]}$, $\{w_n\}$ is the fixed point of the operator $\mathscr{L}_{[1,k,w]}$ and

$$v_k = w_k \in \widehat{E}_k^s \cap \widehat{E}_k^u, \qquad v_k = w_k \neq 0. \tag{4.28}$$

By Lemmata 4.8 and 4.14 the sequence $\{v_n\}$ is a bounded solution of equation (4.21), and the sequence $\{w_n\}$ is a bounded solution of equation (4.22). In view of (4.28), the sequence $\{x_n\}$ defined for all integer k by

$$x_n = \begin{cases} v_n, & n \geq k, \\ w_n, & n < k \end{cases}$$

is thus a nonzero bounded solution of

$$x_{n+1} = A_n x_n, \qquad n \in \mathbb{Z}.$$

Set $\chi = \sup_{n\in\mathbb{Z}} \|x_n\|$. Then by Corollary 4.9

$$\|x_k\| \leq \frac{\varrho}{(1+\omega)^{|k-n|}} \|x_n\| \leq \frac{\varrho\chi}{(1+\omega)^{|k-n|}}$$

for any $n < k$. Letting n converge to $-\infty$, from the above relations we obtain $x_k = v_k = w_k = 0$, which contradicts (4.28). The first equality of (4.25) is thus proved.

To prove the second equality of (4.25) it suffices to note that by Lemmata 4.14 and 4.15

$$\dim \widehat{E}_k^s \geq \dim P_k^s \mathbb{R}^d, \quad \dim \widehat{E}_k^u \geq \dim P_k^u \mathbb{R}^d,$$

and so

$$\dim \widehat{E}_k^s + \dim \widehat{E}_k^u \geq \dim P_k^s \mathbb{R}^d + \dim P_k^u \mathbb{R}^d = \dim \mathbb{R}^d = d.$$

From this and from the first equality of (4.25) we obtain the second equality of (4.25).

Now from the definition of the subspaces $\widetilde{E}_k^s$, $\widetilde{E}_k^u$ and from invertibility of matrices A_n, $n \in \mathbb{Z}$, it follows immediately that

$$A_k \widetilde{E}_k^s = \widetilde{E}_{k+1}^s, \quad A_k \widetilde{E}_k^u = \widetilde{E}_{k+1}^u.$$

Then by (4.24)

$$\begin{aligned} A_k \widehat{E}_k^s &= A_k \widetilde{E}_k^s = \widetilde{E}_{k+1}^s = \widehat{E}_{k+1}^s, \\ A_k \widehat{E}_k^u &= A_k \widetilde{E}_k^u = \widetilde{E}_{k+1}^u = \widehat{E}_{k+1}^u. \end{aligned}$$

from which equalities (4.26) follow.

Finally, we prove the inequality (4.27). Set

$$\theta_s = \frac{\mu_s}{1-\lambda_s}, \quad \theta_u = \frac{\mu_u}{\lambda_u - 1}.$$

Fix an arbitrary vector $x \in \widehat{E}_k^s$, $\|x\| = 1$, and denote $v = P_k^s x$. Then, by the definition of the subspace $\widehat{E}_k^s$, $x = X_{[1,k]}v$ and from Lemma 4.14

$$\|P_k^u x\| \le \theta_u \|v\| = \theta_u \|P_k^s x\|. \tag{4.29}$$

Analogously, for an arbitrary vector $y \in \widehat{E}_k^u$, $\|y\| = 1$, from the definition of the subspace $\widehat{E}_k^u$ and from Lemma 4.15 the inequality

$$\|P_k^s y\| \le \theta_s \|P_k^u y\| \tag{4.30}$$

can be easily obtained.

In view of (4.12) and (4.30)

$$\begin{aligned}\|x+y\| &\ge \frac{1}{h}\|P_k^s(x+y)\| \\ &\ge \frac{1}{h}\left(\|P_k^s x\| - \|P_k^s y\|\right) \ge \frac{1}{h}\left(\|P_k^s x\| - \theta_s \|P_k^u y\|\right),\end{aligned}$$

and also in view of (4.12) and (4.29)

$$\begin{aligned}\|x+y\| &\ge \frac{1}{h}\|P_k^u(x+y)\| \\ &\ge \frac{1}{h}\left(\|P_k^u y\| - \|P_k^u x\|\right) \ge \frac{1}{h}\left(\|P_k^u y\| - \theta_u \|P_k^s x\|\right).\end{aligned}$$

From (4.29) and (4.30) it follows that in the above inequalities the terms $\|P_k^s x\|$ and $\|P_k^u y\|$ can be estimated as

$$\|P_k^s x\| \ge \frac{1}{1+\theta_u}\|x\| = \frac{1}{1+\theta_u}, \quad \|P_k^u y\| \ge \frac{1}{1+\theta_s}\|y\| = \frac{1}{1+\theta_s}.$$

Hence

$$\|x+y\| \ge \frac{1}{h} \inf_{s \ge (1+\theta_u)^{-1},\ t \ge (1+\theta_s)^{-1}} \max\{s - \theta_s t,\ t - \theta_u s\}.$$

We note now that the function $\max\{s-\theta_s t,\ t-\theta_u s\}$ can attain its *infimum* on the set $s \ge (1+\theta_u)^{-1}$, $t \ge (1+\theta_s)^{-1}$ only in the case when $s-\theta_s t = t-\theta_u s$, or that is the same, when

$$(1+\theta_u)s = (1+\theta_s)t.$$

Introduce the auxiliary variable $\tau = (1+\theta_u)s = (1+\theta_s)t$, so

$$s = \frac{\tau}{1+\theta_u}, \quad t = \frac{\tau}{1+\theta_s}.$$

From the above reasoning we see that

$$\|x+y\| \geq \frac{1}{h} \inf_{\tau \geq 1} \left\{ \frac{\tau}{1+\theta_u} - \theta_s \frac{\tau}{1+\theta_s} \right\} \geq \frac{1-\theta_u\theta_s}{h(1+\theta_u)(1+\theta_s)}.$$

From the last inequality and from definitions of $\delta(\widehat{E}_k^s, \widehat{E}_k^u)$, θ_s and θ_u we obtain the required estimate (4.27). □

Remark 4.17. If the invertibility of the matrices A_n, $n \in \mathbb{Z}$, in Lemma 4.16 is not assumed, then instead of the equalities (4.26) we can only conclude that

$$A_k \widehat{E}_k^s \subseteq \widehat{E}_{k+1}^s, \quad A_k \widehat{E}_k^u = \widehat{E}_{k+1}^u.$$

That is, the spaces $\widehat{E}_k^s$ and $\widehat{E}_k^u$ do not have symmetric properties in this case.

We now summarize the results of this Section in the following theorem.

Theorem 4.18 (Equivariant Splitting Theorem). *Let $\{A_n\}$, $n \in \mathbb{Z}$, be an $(\boldsymbol{s}, h)$-semi-hyperbolic sequence of invertible $d \times d$ matrices. Then the subspaces $\widehat{E}_n^s$, $\widehat{E}_n^u$ defined by (4.23) form a splitting $\mathbb{R}^d = \widehat{E}_n^s \oplus \widehat{E}_n^u$ with corresponding projections $\widehat{P}_n^s : \mathbb{R}^d \mapsto \widehat{E}_n^s$ and $\widehat{P}_n^u = I - \widehat{P}_n^s : \mathbb{R}^d \mapsto \widehat{E}_n^u$ satisfying the uniform matrix bounds*

$$\|\widehat{P}_n^s\| \leq \widehat{h}, \quad \|\widehat{P}_n^u\| \leq \widehat{h}. \tag{4.31}$$

This splitting is equivariant with respect to the matrix sequence $\{A_n\}$, i.e.

$$A_n \widehat{E}_n^s = \widehat{E}_{n+1}^s, \quad A_n \widehat{E}_n^u = \widehat{E}_{n+1}^u.$$

Moreover, there exist constants $\varrho > 0$ and $\lambda \in (0,1)$ depending only on the split $\boldsymbol{s}$ and the parameter h such that for any integer k and $v_k \in \widehat{E}_k^s$ the sequence $\{v_n\}$ defined by

$$v_{n+1} = A_n v_n, \qquad n \geq k,$$

satisfies

$$v_n \in \widehat{E}_n^s, \quad \|v_n\| \leq \varrho\lambda^{-|n-k|}\|v_k\|, \qquad n \geq k. \tag{4.32}$$

Analogously, for any integer k and $w_k \in \widehat{E}_k^u$ the sequence $\{w_n\}$ defined by

$$w_{n+1} = A_n w_n, \qquad n < k,$$

satisfies

$$w_n \in \widehat{E}_n^w, \quad \|w_n\| \leq \varrho\lambda^{-|n-k|}\|w_k\|, \qquad n \leq k. \tag{4.33}$$

Proof. Existence of equivariant splitting $\mathbb{R}^d = \widehat{E}_n^s \oplus \widehat{E}_n^u$ and inclusions in (4.32), (4.33) follow immediately from Lemma 4.16. The estimate for norms $\|v_n\|$ in (4.32) follows from Corollary 4.9, while the estimate for norms $\|w_n\|$

in (4.33) follows from Corollary 4.13. So it is remained only to prove estimates (4.31) for projections $\widehat{P}_n^s : \mathbb{R}^d \mapsto \widehat{E}_n^s$ and $\widehat{P}_n^u = I - \widehat{P}_n^s : \mathbb{R}^d \mapsto \widehat{E}_n^u$.

Denote

$$\widehat{h} = \frac{2(\lambda_u - 1 + \mu_u)(1 - \lambda_s + \mu_s)}{(1 - \lambda_s)(\lambda_u - 1) - \mu_s \mu_u} h$$

and prove estimates (4.31) with this $\widehat{h}$. Clearly

$$\|x\| = \|\widehat{P}_n^s x + \widehat{P}_n^u x\| \geq \big|\|\widehat{P}_n^s x\| - \|\widehat{P}_n^u x\|\big|. \tag{4.34}$$

If

$$\big|\|\widehat{P}_n^s x\| - \|\widehat{P}_n^u x\|\big| \geq \frac{1}{\widehat{h}} \max\{\|\widehat{P}_n^s x\|,\ \|\widehat{P}_n^u x\|\}$$

then from this and from (4.34) we obtain the estimation (4.31). So, suppose that

$$\big|\|\widehat{P}_n^s x\| - \|\widehat{P}_n^u x\|\big| < \frac{1}{\widehat{h}} \max\{\|\widehat{P}_n^s x\|,\ \|\widehat{P}_n^u x\|\}. \tag{4.35}$$

Without loss of generality we can also suppose that $\widehat{P}_n^s x \neq 0$, $\widehat{P}_n^u x \neq 0$ and $\|\widehat{P}_n^s x\| \geq \|\widehat{P}_n^u x\|$, which implies

$$\|\widehat{P}_n^s x\| = \max\{\|\widehat{P}_n^s x\|,\ \|\widehat{P}_n^u x\|\}. \tag{4.36}$$

Now let us estimate the norm $\|x\|$ from below in the following manner:

$$\begin{aligned}
\|x\| &= \|\widehat{P}_n^s x + \widehat{P}_n^u x\| \\
&= \left| \left(\frac{\widehat{P}_n^s x}{\|\widehat{P}_n^s x\|} + \frac{\widehat{P}_n^u x}{\|\widehat{P}_n^u x\|} \right) \|\widehat{P}_n^s x\| - \frac{\widehat{P}_n^u x}{\|\widehat{P}_n^u x\|} \left(\|\widehat{P}_n^s x\| - \|\widehat{P}_n^u x\| \right) \right| \\
&\geq \left| \frac{\widehat{P}_n^s x}{\|\widehat{P}_n^s x\|} + \frac{\widehat{P}_n^u x}{\|\widehat{P}_n^u x\|} \right| \|\widehat{P}_n^s x\| - \big|\|\widehat{P}_n^s x\| - \|\widehat{P}_n^u x\|\big|.
\end{aligned}$$

Here, by (4.27) and (4.36), the first term in the right-hand part can be bounded from below by the value $2\widehat{h}^{-1} \max\{\|\widehat{P}_n^s x\|,\ \|\widehat{P}_n^u x\|\}$ while for the second term we have estimate (4.35). Thus

$$\|x\| \geq \frac{1}{\widehat{h}} \max\{\|\widehat{P}_n^s x\|,\ \|\widehat{P}_n^u x\|\},$$

which completes the proof of estimates (4.31). The theorem is proved. □

Remark 4.19. By statements (4.32) and (4.33) of Theorem 4.18, the subspaces $\widehat{E}_n^s$ and $\widehat{E}_n^u$ in the splitting $\mathbb{R}^d = \widehat{E}_n^s \oplus \widehat{E}_n^u$ can be treated as 'asymptotically stable' and 'asymptotically unstable' ones. In such a situation it seems quite reasonable to draw the following conclusion from Theorem 4.18: *if $\{A_n\}$ is an $(\boldsymbol{s}, h)$-semi-hyperbolic sequence of invertible matrices, then it is $(\widehat{\boldsymbol{s}}, \widehat{h})$-semi-hyperbolic with a split $\widehat{\boldsymbol{s}} = (\widehat{\lambda}_s, \widehat{\lambda}_u, \widehat{\mu}_s, \widehat{\mu}_u)$ satisfying*

$$\widehat{\lambda}_s < 1, \quad \widehat{\lambda}_u > 1, \quad \widehat{\mu}_s = \widehat{\mu}_u = 0,$$

(i.e. the subspaces $\widehat{E}_n^s$ and $\widehat{E}_n^u$ in the equivariant splitting $\mathbb{R}^d = \widehat{E}_n^s \oplus \widehat{E}_n^u$ are stable and unstable, correspondingly, with respect to a unique fixed norm $\|\cdot\|$ in $\mathbb{R}^d$). Unfortunately, we do not know whether a similar generalization of Theorem 4.18 is valid or not. A possible way to generalize Theorem 4.18 in this direction will be discussed in the next Section.

4.2.3 Hyperbolicity of Sequences of Matrices

Here we generalize results of Sections 4.2 and 4.3.2 by defining a version of the semi-hyperbolic matrix sequence that uses a variable norm (cf. Definition 3.4).

Definition 4.20 (Semi-Hyperbolic Sequence of Matrices). *A bounded sequence of $d \times d$ matrices $\{A_n\}$ will be called $(\boldsymbol{s}, h)$-semi-hyperbolic if there exists a set of norms $\{\|\cdot\|_n\}$ on $\mathbb{R}^d$ satisfying the uniform boundedness condition*

$$q^{-1}\|x\| \le \|x\|_n \le q\|x\|, \qquad x \in \mathbb{R}^d \tag{4.37}$$

for some constant $q \ge 1$ such that for each matrix A_n, $n \in \mathbb{I}$, there exists a splitting $\mathbb{R}^d = E_n^s \oplus E_n^u$ with

$$\dim E_n^s = \dim E_{n+1}^s, \quad \dim E_n^u = \dim E_{n+1}^u \tag{4.38}$$

and projections $P_n^s : \mathbb{R}^d \mapsto E_n^s$, $P_n^u = I - P_n^s : \mathbb{R}^d \mapsto E_n^u$, with the uniform matrix norm bounds[1]

$$\|P_n^s\| \le h, \quad \|P_n^u\| \le h, \tag{4.39}$$

such that

$$\begin{aligned}
\|P_{n+1}^s A_n P_n^s x\|_{n+1} &\le \lambda_s \|P_n^s x\|_n, \\
\|P_{n+1}^s A_n P_n^u x\|_{n+1} &\le \mu_s \|P_n^u x\|_n, \\
\|P_{n+1}^u A_n P_n^s x\|_{n+1} &\le \mu_u \|P_n^s x\|_n, \\
\|P_{n+1}^u A_n P_n^u x\|_{n+1} &\ge \lambda_u \|P_n^u x\|_n.
\end{aligned} \tag{4.40}$$

The following trivial, though illustrative, example shows that this transition from ‘fixed norm version’ to ‘variable norm version’ broadens the class of semi-hyperbolic mappings.

Example 4.21. Consider the following sequence of linear operators

$$A_{2n}x = 2x, \quad A_{2n+1}x = \frac{1}{4}x, \qquad x \in \mathbb{R}^1, \ n \in \mathbb{Z},$$

acting in the one-dimensional space $\mathbb{R}^1$. Clearly, the matrix sequence $\{A_n\}$ is not semi-hyperbolic in the sense of Definition 4.5 for any possible choice of fixed norm in $\mathbb{R}^1$, but if we define the norms $\|\cdot\|_n$ on $\mathbb{R}^1$ by

[1] Point out that in inequalities (4.39) the original norm $\|\cdot\|$ on $\mathbb{R}^d$ can be used while in (4.40) the variable norms are involved.

$$\|x\|_{2n} = |x|, \quad \|x\|_{2n+1} = \frac{1}{3}|x|, \qquad n \in \mathbb{Z},$$

where $|x|$ denotes the absolute value of the number x, then

$$\|A_{2n}x\|_{2n+1} = \frac{1}{3}|2x| = \frac{2}{3}|x| = \frac{2}{3}\|x\|_{2n}$$

and

$$\|A_{2n-1}x\|_{2n} = |\frac{1}{4}x| = \frac{1}{4}|x| = \frac{3}{4}\|x\|_{2n-1}.$$

Hence the matrix sequence $\{A_n\}$ is semi-hyperbolic in the sense of Definition 4.20.

The special case of a semi-hyperbolic sequence of matrices corresponding to the situation in which the parameters μ_s and μ_u vanish is called hyperbolic sequence of matrices. In this case the spaces E_n^s and E_n^u are equivariant for the sequence $\{A_n\}$, so it is convenient to present the formal definition in slightly different terms to Definition 4.20.

Definition 4.22 (Hyperbolic Sequence of Matrices). *A bounded sequence $\{A_n\}$ of $d \times d$ matrices will be called* hyperbolic *if for each $n \in \mathbb{I}$ there exists a norm $\|\cdot\|_n$ in $\mathbb{R}^d$ satisfying the uniform boundedness condition* (4.37) *and a splitting $\mathbb{R}^d = E_n^s \oplus E_n^u$ with corresponding projections $P_n^s : \mathbb{R}^d \mapsto E_n^s$, $P_n^u = I - P_n^s : \mathbb{R}^d \mapsto E_n^u$ satisfying* (4.38) *and* (4.39) *such that*

$$A_n E_n^s = E_{n+1}^s, \quad A_n E_n^u = E_{n+1}^u \tag{4.41}$$

and the inequalities

$$\|A_n P_n^s x\|_{n+1} \le \lambda_s \|P_n^s x\|_n, \quad \|A_n P_n^u x\|_{n+1} \ge \lambda_u \|P_n^u x\|_n \tag{4.42}$$

hold for some constants $0 \le \lambda_s < 1 < \lambda_m$.

Using Definitions 4.20 and 4.22, the formulation of Theorem 4.18 now becomes more compact and elegant.

Theorem 4.23. *Every semi-hyperbolic sequence of invertible matrices $\{A_n\}$, $n \in \mathbb{Z}$, is hyperbolic and vice versa.*

Proof. Clearly, every hyperbolic sequence of invertible matrices $\{A_n\}$, $n \in \mathbb{Z}$, is semi-hyperbolic, so we need only prove that the semi-hyperbolicity of a matrix sequence implies its hyperbolicity.

An adjustment of the proof of Theorem 4.18 to the 'variable norm'case can be done in the same manner as was explained in the proof of Theorem 4.33. In this way we obtain that for a given $(\boldsymbol{s}, h)$-semi-hyperbolic sequence of invertible matrices $\{A_n\}$, $n \in \mathbb{Z}$, in the sense of Definition 4.20 there exists a splitting $\mathbb{R}^d = \widehat{E}_n^s \oplus \widehat{E}_n^u$ with corresponding projections $\widehat{P}_n^s : \mathbb{R}^d \mapsto \widehat{E}_n^s$ and $\widehat{P}_n^u = I - \widehat{P}_n^s : \mathbb{R}^d \mapsto \widehat{E}_n^u$ with uniform matrix bounds (4.31). This splitting is equivariant with respect to the matrix sequence $\{A_n\}$, i.e. it satisfies (4.41). Moreover, there exist constants $\varrho > 0$ and $\lambda \in (0,1)$ such that for any integer $k \in \mathbb{Z}$ and $v_k \in \widehat{E}^s_k$ the sequence $\{v_n\}$ defined by

$$v_{n+1} = A_n v_n, \qquad n \geq k, \tag{4.43}$$

also satisfies

$$v_n \in \widehat{E}^s_n, \quad \|v_n\|_n \leq \varrho\lambda^{-|n-k|}\|v_k\|_k, \qquad n \geq k. \tag{4.44}$$

Analogously, for any integer $k \in \mathbb{Z}$ and $w_k \in \widehat{E}^u_k$ the sequence $\{w_n\}$ defined by

$$w_{n+1} = A_n w_n, \qquad n < k, \tag{4.45}$$

also satisfies

$$w_n \in \widehat{E}^w_n, \quad \|w_n\|_n \leq \varrho\lambda^{-|n-k|}\|w_k\|_k, \qquad n \leq k. \tag{4.46}$$

Thus, to complete the proof of theorem it suffices to establish the existence of norms $\|\cdot\|^*_n$ in $\mathbb{R}^d$ satisfying uniform boundedness condition

$$\frac{1}{q^*}\|x\| \leq \|x\|^*_n \leq q^*\|x\|, \qquad x \in \mathbb{R}^d, \tag{4.47}$$

with some constant $q^* \geq 1$ such that

$$\|A_n\widehat{P}^s_n x\|^*_{n+1} \leq \lambda\|\widehat{P}^s_n x\|^*_n, \quad \|A_n\widehat{P}^u_n x\|^*_{n+1} \geq \lambda^{-1}\|\widehat{P}^u_n x\|^*_n \tag{4.48}$$

To construct the required norms, fix an integer $k \in \mathbb{Z}$ and $x \in \mathbb{R}^d$ and choose $v_k = \widehat{P}^s_k x$, $w_k = \widehat{P}^u_k x$. Then define sequences $\{v_n\}$, $n \leq k$, and $\{w_n\}$, $n \geq k$, satisfying (4.43) and (4.45), respectively. Finally, define

$$\|x\|^*_k = \max\left\{\sup_{n\geq k}\lambda^{k-n}\|v_n\|_n,\ \sup_{n\leq k}\lambda^{n-k}\|w_n\|_n\right\}.$$

From (4.44) and (4.46) inequalities (4.48) and estimates (4.47) immediately follow with an appropriately chosen q^*. The theorem is proved. □

4.3 A More Abstract Approach

In this Section an abstract approach in terms of linear operators in Banach spaces to investigation of properties of sequences of semi-hyperbolic mappings will be developed.

4.3.1 Semi-Hyperbolic Linear Operators

An abstract approach is also able to clarify to a greater extent the essence of relationships between conditions of semi-hyperbolicity and hyperbolicity concepts for sequences of matrices. To implement such an approach we start by investigating how the property of semi-hyperbolicity of a linear operator on a Banach space is related to the hyperbolicity property.

Let $A : E \mapsto E$ be a bounded linear operator on a Banach space E and let $\boldsymbol{s} = (\lambda_s, \lambda_u, \mu_s, \mu_u)$ be a split.

Definition 4.24 (Semi-Hyperbolic Linear Operator). *A linear bounded operator A on a Banach space E with a norm $\|\cdot\|$ is called* semi-hyperbolic *(with respect to a split $\boldsymbol{s} = (\lambda_s, \lambda_u, \mu_s, \mu_u)$, a constant h and the norm $\|\cdot\|$) if there is a splitting $E = E^s \oplus E^u$ with corresponding bounded projectors $P^s : E \mapsto E^s$ and $P^u = I - P^s : E \mapsto E^u$ satisfying*

$$\|P^s\| \le h, \quad \|P^u\| \le h, \tag{4.49}$$

and

$$\begin{aligned} \|P^s A P^s x\| &\le \lambda_s \|P^s x\|, \quad \|P^s A P^u x\| \le \mu_s \|P^u x\|, \\ \|P^u A P^s x\| &\le \mu_u \|P^s x\|, \quad \|P^u A P^u x\| \ge \lambda_u \|P^u x\|. \end{aligned} \tag{4.50}$$

Remark 4.25. In Definition 4.24 it is supposed that the linear operator A is real, i.e. that it acts on a real Banach space E. Often, however in studying spectral properties of linear operators it is convenient to treat them as if they were acting on a complex Banach spaces. The natural way to pass from real to complex operators is the *complexification* of a linear operators. For the sake of completness of presentation we briefly review the necessary details.

For a real Banach space E with a norm $\|\cdot\|$ we define the *complexification* of E, the complex Banach space E_c, as the set of complex vectors $z = x + \mathrm{i}y$ with $x, y \in E$ endowed with the norm

$$\|z\|_c = \|x + \mathrm{i}y\|_c := \max_{|\alpha + \mathrm{i}\beta| = 1} \|\alpha x + \beta y\|, \qquad \alpha, \beta \in \mathbb{R}^1.$$

For a linear operator $A : E \mapsto E$ we then define its complexification $A_c : E_c \mapsto E_c$ by

$$A_c(x + \mathrm{i}y) := Ax + \mathrm{i}Ay, \qquad x, y \in E.$$

With such definition of complexification, spectral properties of a real operator A are exactly the same as those of its complexification A_c, i.e. the both these operators have the same spectrum, the same set of eigenvalues, etc.

For our purposes it is important that the passage from a real linear operator A to its complexification A_c preserves all the essential features of semi-hyperbolicity (see Definition 4.24). More precisely, if A is a semi-hyperbolic linear operator satisfying (4.49) and (4.50) then *its complexification $A_c : E_c \mapsto E_c$ must satisfy*

$$\|P_c^s A_c P_c^s z\|_c \le \lambda_s \|P_c^s z\|_c, \quad \|P_c^s A_c P_c^u z\|_c \le \mu_s \|P_c^u z\|_c,$$
$$\|P_c^u A_c P_c^s z\|_c \le \mu_u \|P_c^s z\|_c, \quad \|P_c^u A_c P_c^u z\|_c \ge \lambda_u \|P_c^u z\|_c,$$

for all $z \in E_c$ *with corresponding bounded projectors* $P_c^s : E_c \mapsto E_c^s$ *and* $P_c^u = I - P_c^s : E_c \mapsto E_c^u$ *satisfying*

$$\|P_c^s\|_c \le h, \quad \|P_c^u\|_c \le h.$$

It seems quite reasonable now to define the concept of a hyperbolic linear operator by introducing an additional demand of invariance of the splitting subspaces E^s and E^u with respect to the operator A. Such an invariance condition can be expressed by one of the following three equivalent conditions:

$$AE^s \subseteq E^s, \quad AE^s \subseteq E^s,$$

or

$$P^s A P^u = 0, \quad P^u A P^s = 0,$$

or

$$P^s A P^u x \equiv 0, \quad P^u A P^s x \equiv 0.$$

Definition 4.26 (Hyperbolic Linear Operator: Metric Definition). *A linear bounded operator* A *on a Banach space* E *with a norm* $\|\cdot\|$ *is called* hyperbolic *(in the norm* $\|\cdot\|$*) if there is an invariant splitting* $E = E^s \oplus E^u$ *with corresponding bounded projectors* $P^s : E \mapsto E^s$ *and* $P^u = I - P^s : E \mapsto E^u$ *satisfying*

$$\|P^s A P^s x\| \le \lambda_s \|P^s x\|, \quad \|P^u A P^u x\| \ge \lambda_u \|P^u x\|,$$

with appropriate constants $\lambda_s < 1$ *and* $\lambda_u > 1$.

The concept of hyperbolicity for a linear operator A on a Banach space can also be defined in a more clear and constructive way via a description of the spectral properties of the operator A.

Definition 4.27 (Hyperbolic Linear Operator: Spectral Definition). *A linear bounded operator* A *on a Banach space is called* hyperbolic *if its spectrum does not intersect the unit circle* $|\lambda| = 1$ *in the complex plane.*

The link between the concepts of semi-hyperbolicity and hyperbolicity (in the both, metric and spectral, versions) is established by the following theorem. For linear operator the concepts of semi-hyperbolicity and (metric or spectral) hyperbolicity are, in fact, equivalent.

Theorem 4.28. *For a linear bounded operator* A *on a Banach space* E *with a norm* $\|\cdot\|$ *the following three statements are equivalent:*

(i) The operator A *is semi-hyperbolic with respect to some split* $\boldsymbol{s}$*, constant* h *and a norm* $\|\cdot\|_*$ *equivalent to the norm* $\|\cdot\|$.

(ii) The operator A *is metrically hyperbolic in some norm* $\|\cdot\|_*$ *equivalent to the norm* $\|\cdot\|$.

(iii) The operator A *is spectrally hyperbolic.*

Proof. As was remarked earlier, throughout the proof of theorem we can treat E as the complex Banach space. To prove the theorem we establish the following implications

$$\text{(i)} \Rightarrow \text{(iii)} \Rightarrow \text{(ii)} \Rightarrow \text{(i)}.$$

We first prove the implication (i) $\Rightarrow$ (iii), i.e. that semi-hyperbolicity of the operator A implies its hyperbolicity in the sense of Definition 4.27. Suppose the contrary, then there is a complex number $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ which belongs to the spectrum $\sigma(A)$ of the operator A. As is known from the general spectral theory, for some sequence of elements $x_n \in E$, $\|x_n\| = 1$, the relation

$$Ax_n - \lambda x_n = y_n \to 0, \qquad n \to \infty,$$

is then valid. This relation can be decomposed as

$$\begin{aligned} P^s AP^s x_n + P^s AP^u x_n - P^s x_n &= P^s y_n \to 0, \\ P^u AP^s x_n + P^u AP^u x_n - P^u x_n &= P^u y_n \to 0, \end{aligned}$$

so,

$$\begin{aligned} \|P^s x_n\| &\le \|P^s AP^s x_n\| + \|P^s AP^u x_n\| + \varepsilon_n^s, \\ \|P^u AP^u x_n\| &\le \|P^u AP^s x_n\| + \|P^u x_n\| + \varepsilon_n^u, \end{aligned} \tag{4.51}$$

where

$$\varepsilon_n^s = \|P^s y_n\| \to 0, \quad \varepsilon_n^u = \|P^u y_n\| \to 0. \tag{4.52}$$

Now, without loss of generality, we can regard that the inequalities (4.50) in Definition 4.24 are valid in the norm $\|\cdot\|$, but not in some equivalent norm $\|\cdot\|_*$. Then from (4.51) and (4.50) we obtain

$$\begin{aligned} \|P^s x_n\| &\le \lambda_s \|P^s x_n\| + \mu_s \|P^u x_n\| + \varepsilon_n^s, \\ \lambda_u \|P^u x_n\| &\le \mu_u \|P^s x_n\| + \|P^u x_n\| + \varepsilon_n^u, \end{aligned}$$

and so

$$\begin{aligned} \|P^s x_n\| &\le \frac{(\lambda_u - 1)\varepsilon_n^s + \varepsilon_n^u}{(1-\lambda_s)(\lambda_u - 1) - \mu_s \mu_u}, \\ \|P^u x_n\| &\le \frac{\varepsilon_n^s + (1-\lambda_s)\varepsilon_n^u}{(1-\lambda_s)(\lambda_u - 1) - \mu_s \mu_u}. \end{aligned}$$

Hence by (4.52) we have

$$\|x_n\| \le \frac{\lambda_u \varepsilon_n^s + (2-\lambda_s)\varepsilon_n^u}{(1-\lambda_s)(\lambda_u - 1) - \mu_s \mu_u} \to 0, \qquad n \to \infty,$$

which contradicts to the supposition that $\|x_n\| = 1$. The contradiction obtained completes the proof of the implication (i) $\Rightarrow$ (iii).

We now prove the implication (iii) $\Rightarrow$ (ii), i.e. that hyperbolicity of the operator A in the sense of Definition 4.27 implies its hyperbolicity in the sense of Definition 4.26. By supposition, the spectrum σ of the operator A does not intersect the unit circle $|\lambda| = 1$. Then, since the spectrum of any operator is a closed set, the set σ can be represented as the union $\sigma = \sigma_s \cup \sigma_u$ of closed subsets σ_s and σ_u where σ_s lies entirely inside the unit circle and σ_u lies entirely outside the unit circle.

Denote by $E^s \subseteq E$ the invariant spectral subspace of the operator A corresponding to the part σ_s of its spectrum, and denote by $E^u \subseteq E$ the invariant spectral subspace of the operator A corresponding to the part σ_u of its spectrum. Then, as is known from the general spectral theory,

$$E^s \cap E^u = \emptyset, \quad E^s \oplus E^u = E$$

hold. Denote by $P^s : E \mapsto E^s$ and $P^u = I - P^s : E \mapsto E^u$ the projections corresponding to the splitting $E = E^s \oplus E^u$, which are bounded in our case.

Now, a linear operator A on a Banach space with the spectral radius $\sigma(A)$ satisfies

$$\sigma(A) = \limsup_{n\to\infty} \|A^n\|^{1/n},$$

takes place, from which it follows that for any $\varepsilon > 0$ there exists a number c_ε such that

$$\|A^n\| \leq c_\varepsilon(\sigma(A) + \varepsilon)^n, \qquad n \geq 0. \tag{4.53}$$

By definition, the closed set σ_s lies inside the circle $|\lambda| < 1$. Thus numbers $\lambda_s \in (0,1)$ and $\varepsilon > 0$ can be found such that σ_s, in fact, belongs to the circle $|\lambda| \leq \lambda_s - \varepsilon$. Then, applying the formula (4.53) to the restriction $A|_{E^s}$ of the linear operator A to its invariant subspace E^s, we obtain

$$\|(A|_{E^s})^n\| \leq c_{s,\varepsilon}\lambda_s^n, \qquad n \geq 0. \tag{4.54}$$

Analogously, the closed set σ_u lies outside the circle $|\lambda| \leq 1$. So numbers $\lambda_u \in (0,1)$ and $\varepsilon > 0$ can be found such that σ_u belongs to the set $|\lambda| \geq \lambda_u + \varepsilon$. The restriction $A|_{E^s}$ of linear operator A to its invariant subspace E^s is an invertible operator, and applying the formula (4.53) to $(A|_{E^s})^{-1}$, we obtain

$$\|(A|_{E^s})^{-n}\| \leq c_{u,\varepsilon}\lambda_u^{-n}, \qquad n \geq 0. \tag{4.55}$$

Define the norm $\|\cdot\|_*$ in E by

$$\|x\|_* = \max_{n\geq 0} \lambda_s^{-n}\|(A|_{E^s})^n P^s x\| + \max_{n\geq 0} \lambda_u^n \|(A|_{E^u})^{-n} P^u x\|.$$

Clearly

$$\|x\|_* \geq \|P^s x\| + \|P^u x\| \geq \|P^s x + P^u x\| = \|x\|,$$

and, at the same time, by (4.54), (4.55)

$$\|x\|_* \le c_{s,\varepsilon}\|P^s x\| + c_{u,\varepsilon}\|P^u x\| \le \max\{c_{s,\varepsilon}\|P^s\|,\ c_{u,\varepsilon}\|P^u\|\}\,\|x\|.$$

Hence the norms $\|\cdot\|_*$ and $\|\cdot\|$ are equivalent. To complete the proof of the implication (iii) $\Rightarrow$ (ii), it remains only to note from (4.54), (4.55) that the inequalities

$$\|P^s A P^s x\|_* \le \lambda_s \|P^s x\|_*, \quad \|P^u A P^u x\|_* \ge \lambda_u \|P^u x\|_*,$$

then follow, which, together with invariance of the subspaces E^s and E^u with respect to A, mean that A is a linear hyperbolic operator in the sense of Definition 4.26. The implication (iii) $\Rightarrow$ (ii) is then proved.

Finally, the implication (ii) $\Rightarrow$ (i) is evident from the definition of the split $\boldsymbol{s}$ and constant h, namely $\boldsymbol{s} = (\lambda_s, \lambda_u, 0, 0)$, $h = \max\{\|P^s\|,\ \|P^u\|\}$. The proof of the theorem is thus completed. □

4.3.2 Equivalent Operators in Sequence Spaces

Let $\{A_n\}$ be a sequence of $d \times d$ matrices defined for n belonging to some interval $\mathbb{I} \subseteq \mathbb{Z}$ of the form $[0, N)$ or $[0, \infty)$ or $(-\infty, 0]$ or $\mathbb{Z}$. We shall associate with the sequence $\{A_n\}$ some linear operators acting on a sequence space. Consider first the linear operator $\mathscr{A} : \mathscr{X} \mapsto \mathscr{Y}$ defined by

$$(\mathscr{A} x)_{n+1} = A_n x_n, \qquad n \in \mathbb{I}. \tag{4.56}$$

Here the definition of the spaces $\mathscr{X}$ and $\mathscr{Y}$ depend on $\mathbb{I}$. If $\mathbb{I} = [0, N]$, then it is convenient to set

$$\mathscr{X} = \ell^\infty([0, N+1], \mathbb{R}^d), \quad \mathscr{Y} = \ell^\infty([1, N+1], \mathbb{R}^d); \tag{4.57}$$

if $\mathbb{I} = (-\infty, 0]$, then

$$\mathscr{X} = \mathscr{Y} = \ell^\infty((-\infty, 1], \mathbb{R}^d); \tag{4.58}$$

if $\mathbb{I} = [0, \infty)$, then set

$$\mathscr{X} = \ell^\infty([0, \infty), \mathbb{R}^d), \quad \mathscr{Y} = \ell^\infty([1, \infty), \mathbb{R}^d); \tag{4.59}$$

and, finally, if $\mathbb{I} = \mathbb{Z}$ set

$$\mathscr{X} = \mathscr{Y} = \ell^\infty(\mathbb{Z}, \mathbb{R}^d). \tag{4.60}$$

In every case $\mathscr{Y} \subseteq \mathscr{X}$, but only in the cases $\mathbb{I} = (-\infty, 0]$ and $\mathbb{I} = \mathbb{Z}$ do we have $\mathscr{Y} = \mathscr{X}$.

Consider also the linear operator $\mathscr{D} : \mathscr{X} \mapsto \mathscr{Y}$ defined by

$$(\mathscr{D} x)_{n+1} = x_{n+1} - A_n x_n, \qquad n \in \mathbb{I}, \tag{4.61}$$

that is, $\mathscr{D} = I - \mathscr{A}$.

4.3.3 An Inversion Theorem

Consider the invertibility problem for the operator $\mathscr{D}$. It turns out that this problem depends heavily on which set of indices, $\mathbb{I} = [0, N]$ or $\mathbb{I} = [0, \infty)$ or $\mathbb{I} = (-\infty, 0]$ or $\mathbb{I} = \mathbb{Z}$, the sequence of matrices A_n is defined.

For example, to find the inverse operator of $\mathscr{D}$ in the case $\mathbb{I} = [0, N]$ we need to find for each $\boldsymbol{z} = \{z_n\} \in \ell^\infty([1, N+1], \mathbb{R}^d)$ an $\boldsymbol{x} = \{x_n\} \in \ell^\infty([0, N+1], \mathbb{R}^d)$ which satisfies

$$x_{n+1} = A_n x_n + z_{n+1}, \qquad n \in \mathbb{I}. \tag{4.62}$$

Clearly, the set of equations (4.62) is not sufficient to define $\boldsymbol{x}$ uniquely when $\mathbb{I} = [0, N]$, as x_0 can be chosen arbitrarily here. Less evident, but similar kind of problem also occurs when we try to find a bounded sequence $\boldsymbol{x}$ satisfying (4.62) in the case $\mathbb{I} = [0, \infty)$ or $\mathbb{I} = (-\infty, 0]$.

Let $\mathbb{R}^d = E_n^s \oplus E_n^u$ be the splitting associated with the matrix A_n, for which the projectors are $P_n^s : \mathbb{R}^d \mapsto E_n^s$ and $P_n^u = I - P_n^s : \mathbb{R}^d \mapsto E_n^u$. Rewrite (4.62) in the decomposed form

$$\begin{aligned} P_{n+1}^s x_{n+1} &= P_{n+1}^s A_n P_n^s x_n + P_{n+1}^s A_n P_n^u x_n + P_{n+1}^s z_{n+1}, \\ P_{n+1}^u x_{n+1} &= P_{n+1}^u A_n P_n^s x_n + P_{n+1}^u A_n P_n^u x_n + P_{n+1}^u z_{n+1}. \end{aligned}$$

Note that the linear operator $U_n = P_{n+1}^u A_n P_n^u : E_n^u \mapsto E_{n+1}^u$ is invertible by (4.11) and by the last condition (4.13) so we can rewrite these equations as

$$P_{n+1}^s x_{n+1} = P_{n+1}^s \left(A_n P_n^s x_n + A_n P_n^u x_n + z_{n+1} \right), \tag{4.63}$$

$$P_n^u x_n = U_n^{-1} P_{n+1}^u \left(x_{n+1} - A_n P_n^s x_n - z_{n+1} \right). \tag{4.64}$$

Now introduce an auxiliary linear operator $\mathscr{H}_{\boldsymbol{z}} : \mathscr{X} \mapsto \mathscr{X}$, which, for a fixed sequence $\boldsymbol{z} = \{z_n\} \in \mathscr{Y}$, transforms every sequence $\boldsymbol{x} = \{x_n\} \in \mathscr{X}$ into a sequence $\boldsymbol{w} = \{w_n\} \in \mathscr{X}$ satisfying

$$P_{n+1}^s w_{n+1} = P_{n+1}^s \left(A_n P_n^s x_n + A_n P_n^u x_n + z_{n+1} \right), \tag{4.65}$$

$$P_n^u w_n = U_n^{-1} P_{n+1}^u \left(x_{n+1} - A_n P_n^s x_n - z_{n+1} \right), \tag{4.66}$$

where both equations are considered for $n \in \mathbb{I}$.

Observe that equations (4.65) do not define $P_0^s w_0$ in the case when 0 is the left end of the interval $\mathbb{I}$, while equations (4.66) do not define $P_{N+1}^u w_{N+1}$ when N is the right end of the interval $\mathbb{I}$. Hence, to define the operator $\mathscr{H}_{\boldsymbol{z}}$ correctly equations (4.65) and (4.66)need to be supplemented by one of the following conditions:

$$P_0^s w_0 = 0, \quad P_{N+1}^u w_{N+1} = 0 \qquad \text{if} \quad \mathbb{I} = [0, N], \tag{4.67}$$

or

$$P_1^u w_1 = 0 \qquad \text{if} \quad \mathbb{I} = (-\infty, 0], \tag{4.68}$$

or

$$P_0^s w_0 = 0, \qquad \text{if} \quad \mathbb{I} = [0, \infty). \tag{4.69}$$

The system of equations (4.65)–(4.66) is sufficient and does not need supplementing condition in the case when $\mathbb{I} = \mathbb{Z}$.

Now introduce an auxiliary norm $\|\cdot\|_\infty^\circ$ on the space $\mathscr{X}$ by

$$\|\boldsymbol{x}\|_\infty^\circ = \sup_n \max\left\{\gamma\|P_n^s x_n\|, \|P_n^u x_n\|\right\}, \tag{4.70}$$

where the indices n are taken from $[0, N+1]$ if $\mathbb{I} = [0, N]$, from $(-\infty, 1]$ if $\mathbb{I} = (-\infty, 0]$, and from $\mathbb{I}$ if $J = [0, \infty)$ or $\mathbb{I} = \mathbb{Z}$, and $\gamma = \gamma(\boldsymbol{s})$ is the value defined by (4.5). As was mentioned in Section 4.1, the split $\boldsymbol{s}$ under consideration may be treated without loss of generality as positive, in which case the norm $\|\cdot\|_\infty^\circ$ in $\mathscr{X}$ is equivalent to the norm $\|\cdot\|_\infty$ as by (4.12) for any admissible n. To see this, note that for any $x_n \in \mathbb{R}^d$ we have

$$\frac{1}{2}\|x_n\| \le \max\left\{\|P_n^s x_n\|,\ \|P_n^u x_n\|\right\} \le h\|x_n\|,$$

so by Lemma 4.1

$$\frac{1}{2}\min\{\gamma, 1\}\|x_n\| \le \max\left\{\gamma\|P_n^s x_n\|, \|P_n^u x_n\|\right\} \le h\max\{\gamma, 1\}\|x_n\|,$$

from which it follows that

$$\frac{1}{2}\min\{\gamma, 1\}\|\boldsymbol{x}\|_\infty \le \|\boldsymbol{x}\|_\infty^\circ \le h\max\{\gamma, 1\}\|\boldsymbol{x}\|_\infty. \tag{4.71}$$

Lemma 4.29. *For every $\boldsymbol{z} \in \mathscr{Y}$ the operator $\mathscr{H}_{\boldsymbol{z}} : \mathscr{X} \mapsto \mathscr{X}$ has a unique fixed point $\boldsymbol{x} \in \mathscr{X}$ for which*

$$\|\boldsymbol{x}\|_\infty \le \frac{h}{\nu(\boldsymbol{s})}\|\boldsymbol{z}\|_\infty. \tag{4.72}$$

Proof. We show first that the operator $\mathscr{H}_{\boldsymbol{z}} : \mathscr{X} \mapsto \mathscr{X}$ is contracting in the norm $\|\cdot\|_\infty^\circ$ with contraction constant $\sigma = \sigma(\boldsymbol{s})$ defined by (4.3).

Given $\boldsymbol{x}, \widetilde{\boldsymbol{x}} \in \mathscr{X}$, write $\boldsymbol{y} = \boldsymbol{x} - \widetilde{\boldsymbol{x}}$ and $\boldsymbol{v} = \mathscr{H}_{\boldsymbol{z}}(\boldsymbol{x}) - \mathscr{H}_{\boldsymbol{z}}(\widetilde{\boldsymbol{x}})$. Then by (4.65), (4.66)

$$P_{n+1}^s v_{n+1} = P_{n+1}^s\left(A_n P_n^s y_n + A_n P_n^u y_n\right),$$
$$P_n^u v_n = U_n^{-1} P_{n+1}^u\left(y_{n+1} - A_n P_n^s y_n\right),$$

which may by supplemented, if needed, by one or the both following equalities

$$P_0^s v_0 = 0, \quad P_{N+1}^u v_{N+1} = 0.$$

Hence, by definition of the operator U_n and by (4.13) we obtain

$$\|P_{n+1}^s v_{n+1}\| \le \lambda_s\|P_n^s y_n\| + \mu_s\|P_n^u y_n\|, \tag{4.73}$$

$$\|P_n^u v_n\| \le \frac{\mu_u}{\lambda_u}\|P_n^s y_n\| + \frac{1}{\lambda_u}\|P_{n+1}^u y_{n+1}\|, \tag{4.74}$$

which may again be supplemented, if needed, by

$$\|P_0^s v_0\| = 0, \quad \|P_{N+1}^u v_{N+1}\| = 0. \tag{4.75}$$

Denote

$$V_1 = \sup_n \{\gamma \|P_n^s v_n\|\}, \quad V_2 = \sup_n \{\|P_n^u v_n\|\}$$

and

$$Y_1 = \sup_n \{\gamma \|P_n^s y_n\|\}, \quad Y_2 = \sup_n \{\|P_n^u y_n\|\}$$

Then

$$\|\boldsymbol{v}\|_\infty^\circ = \max\{V_1, V_2\}, \quad \|\boldsymbol{y}\|_\infty^\circ = \max\{Y_1, Y_2\}$$

and so, from (4.73), (4.74) and from (4.75), if needed, we obtain

$$V_1 \le \lambda_s Y_1 + \gamma\mu_s Y_2 \le (\lambda_s + \gamma\mu_s)\,\|\boldsymbol{y}\|_\infty^\circ, \tag{4.76}$$

$$V_2 \le \frac{\mu_u}{\gamma\lambda_u} Y_1 + \frac{1}{\lambda_u} V_2 \le \left(\frac{\mu_u}{\gamma\lambda_u} + \frac{1}{\lambda_u}\right) \|\boldsymbol{y}\|_\infty^\circ. \tag{4.77}$$

But from (4.6) we have

$$\lambda_s + \gamma\mu_s = \sigma, \quad \frac{\mu_u}{\gamma\lambda_u} + \frac{1}{\lambda_u} = \sigma,$$

so, by (4.76) and (4.77) we get

$$\|\boldsymbol{v}\|_\infty^\circ = \max\{V_1, V_2\} \le \sigma \|\boldsymbol{y}\|_\infty^\circ,$$

or equivalently

$$\|\mathscr{H}_{\boldsymbol{z}}(\boldsymbol{x}) - \mathscr{H}_{\boldsymbol{z}}(\widetilde{\boldsymbol{x}})\|_\infty^\circ \le \sigma \|\boldsymbol{x} - \widetilde{\boldsymbol{x}}\|_\infty^\circ.$$

By the Banach Contraction Mapping Theorem the operator $\mathscr{H}_{\boldsymbol{z}}$ for any $\boldsymbol{z} \in \mathscr{Y}$ has a unique fixed point $\boldsymbol{x} := \mathscr{B}\boldsymbol{z} \in \mathscr{X}$ for which the estimate

$$\|\mathscr{B}\boldsymbol{z}\|_\infty^\circ \le \frac{1}{1-\sigma} \|\mathscr{H}_{\boldsymbol{z}}(0)\|_\infty^\circ$$

holds. Unfortunately, the obvious bound on $\|\boldsymbol{x}\|_\infty$ provided by Lemma 4.1 is only a rough estimate, so another approach is required.

Since by (4.71) the norms $\|\cdot\|_\infty^\circ$ and $\|\cdot\|_\infty$ are equivalent then, as follows from Banach Contraction Mapping Theorem, the sequence of successive iterates

$$\boldsymbol{x}^{(m)} = \mathscr{H}_{\boldsymbol{z}}(\boldsymbol{x}^{(m-1)}), \qquad m = 1, 2, \dots,$$

with $\boldsymbol{x}^{(0)} = 0$, converges in the norm $\|\cdot\|_\infty$ to the fixed point $\boldsymbol{x}$ of $\mathscr{H}_{\boldsymbol{z}}$. In particular,

$$\|\boldsymbol{x}\|_\infty \le \limsup_{m\to\infty} \|\boldsymbol{x}^{(m)}\|_\infty. \tag{4.78}$$

Set

$$a_m = \sup_n \{\|P_n^s x_n^{(m)}\|\}, \quad b_m = \sup_n \{\|P_n^u x_n^{(m)}\|\} \tag{4.79}$$

and

$$h^s = \sup_n \{\|P_{n+1}^s z_{n+1}\|\}, \quad h^u = \sup_n \{\|U_n^{-1} P_{n+1}^u z_{n+1}\|\}. \tag{4.80}$$

Then, by taking the *supremum* of the norms of the left and right hand sides of equations (4.65), (4.66) and, if necessary, of one or the both equalities (4.67), we obtain the system of inequalities

$$\begin{aligned} a_m &\le \lambda_s a_{m-1} + \mu_s b_{m-1} + h^s, \\ b_m &\le \frac{\mu_u}{\lambda_u} a_{m-1} + \frac{1}{\lambda_u} b_{m-1} + h^u, \end{aligned}$$

from which, by Lemma 4.2, it follows that

$$\limsup_{m\to\infty} \|\boldsymbol{x}^{(m)}\|_\infty \le \limsup_{m\to\infty} (a_m + b_m) \le \frac{\max\{h^s,\, \lambda_u h^u\}}{\nu(\boldsymbol{s})}. \tag{4.81}$$

But clearly, by (4.12) and the definition of the operator U_n,

$$h^s \le h\|\boldsymbol{z}\|_\infty, \quad h^u \le \frac{h}{\lambda_u}\|\boldsymbol{z}\|_\infty. \tag{4.82}$$

The required estimate (4.72) follows from this and from (4.78) and (4.81). The lemma is proved. □

We can now establish the invertibility of the bounded linear operator $\mathscr{D}$ that was defined by (4.61) for a semi-hyperbolic sequence of matrices $\{A_n\}$.

Theorem 4.30. *If $\{A_n\}$ is a $(\boldsymbol{s}, h)$-semi-hyperbolic sequence of $d\times d$ matrices, then the linear operator $\mathscr{D} : \mathscr{X} \mapsto \mathscr{Y}$ defined by* (4.61) *(with spaces $\mathscr{X}, \mathscr{Y}$ defined by* (4.57) *or* (4.58) *or* (4.59) *or* (4.60)*) is bounded and has a bounded right inverse $\mathscr{D}^{-1} : \mathscr{Y} \mapsto \mathscr{X}$ satisfying*

$$\left\|\mathscr{D}^{-1}\right\|_\infty \le \frac{h}{\nu(\boldsymbol{s})}. \tag{4.83}$$

Proof. By Lemma 4.29 and the Banach Contraction Mapping Theorem, the operator $\mathscr{H}_{\boldsymbol{z}}$ for any $\boldsymbol{z} \in \mathscr{Y}$ has a unique fixed point $\boldsymbol{x} := \mathscr{B}\boldsymbol{z}$ for which the estimate

$$\|\mathscr{B}\boldsymbol{z}\|_\infty \le \frac{h}{\nu(\boldsymbol{s})}\|\boldsymbol{z}\|_\infty \tag{4.84}$$

is valid.

The operator $\mathscr{B}$ is obviously linear. By comparing equations (4.65) and (4.66) with (4.63) and (4.64), we conclude that $\boldsymbol{x}$ is the solution of (4.62). Thus, the operator $\mathscr{B}$ is the right inverse to $\mathscr{D}$, i.e. we can write $\mathscr{B} = \mathscr{D}^{-1}$, and the estimate (4.83) follows from (4.84). □

Remark 4.31. When the sequence of matrices A_n is defined for the set of indices $\mathbb{I} = \mathbb{Z}$ the bounded right inverse to $\mathscr{D}$ operator $\mathscr{D}^{-1}$ satisfying (4.83) is defined uniquely, while in the cases $\mathbb{I} = [0, N]$, $\mathbb{I} = [0, \infty)$ and $\mathbb{I} = (-\infty, 0]$ it is not.

Remark 4.32. In Definition 4.5, and hence in Theorem 4.30, the matrix sequence $\{A_n\}$ is supposed to be bounded. This requirement has been imposed only to simplify proofs and can be easily removed, if required, with the obvious changes in the formulation of assertions and proofs.

With Definition 4.20 of semi-hyperbolic sequence of matrices an analog of Theorem 4.30 is valid with some insignificant changes resulting from using the variable norm inequalities (4.40).

Theorem 4.33. *If $\{A_n\}$ is a $(\boldsymbol{s}, h)$-semi-hyperbolic sequence of $d \times d$ matrices (according to Definition 4.20), then the linear operator $\mathscr{D} : \mathscr{X} \mapsto \mathscr{Y}$ defined by* (4.61) *(with spaces $\mathscr{X}, \mathscr{Y}$ defined by* (4.57) *or* (4.58) *or* (4.59) *or* (4.60)*) is bounded and has a bounded right inverse $\mathscr{D}^{-1} : \mathscr{Y} \mapsto \mathscr{X}$ satisfying*

$$\left\|\mathscr{D}^{-1}\right\|_\infty \le \frac{q^2 h}{\nu(\boldsymbol{s})},$$

where q is as in estimates (4.37) *of Definition 4.20.*

Proof. In the proof of Theorem 4.30, we adjust definition (4.70) of the norm $\|\cdot\|_\infty^\circ$ and definitions (4.79), (4.80) of a_n, b_n, h^s, h^u by replacing the norms $\|P_n^s x_n\|$, $\|P_n^u x_n\|$, etc. by the norms $\|P_n^s x_n\|_n$, $\|P_n^u x_n\|_n$ in the following manner

$$\|\boldsymbol{x}\|_\infty^\circ = \sup_n \max\left\{\gamma \|P_n^s x_n\|_n,\ \|P_n^u x_n\|_n\right\}.$$

With such an adjustment the proof of Theorem 4.33 is essentially the same as that of Theorem 4.30. We need only to multiply by the factor q the corresponding sides of inequalities (4.81) and (4.82) involving estimation of the $\|\cdot\|_\infty$-norms of elements via a_n, b_n, h, and vice versa. □

4.3.4 Hyperbolicity of the Linear Operator $\mathscr{A}$

To complete Section 4.3 we examine the relationship between the hyperbolicity of a matrix sequence and hyperbolicity of the linear operator $\mathscr{A}$ defined by (4.56) in Section 4.3.2. Then, via Theorem 4.23, this will reveal the relationship between the semi-hyperbolicity of a matrix sequence and hyperbolicity of the linear operator $\mathscr{A}$ and, via Theorem 4.28, the relationship between the semi-hyperbolicity of a matrix sequence and semi-hyperbolicity of the linear operator $\mathscr{A}$.

Theorem 4.34. *The sequence of invertible matrices $\{A_n\}$, $n \in \mathbb{Z}$, is hyperbolic if and only if the linear operator $\mathscr{A} :\ \ell^\infty(\mathbb{Z}, \mathbb{R}^d) \mapsto \ell^\infty(\mathbb{Z}, \mathbb{R}^d)$ defined by* (4.56) *is hyperbolic.*

Proof. To simplify notation, denote $\mathscr{L} = \ell^\infty(\mathbb{Z}, \mathbb{R}^d)$. Suppose that the sequence of invertible matrices $\{A_n\}$, $n \in \mathbb{Z}$, is hyperbolic. For each integer n, let the projections P_n^s, P_n^u and norm $\|\cdot\|_n$ be as in Definition 4.22 and define linear operators $\mathscr{P}^s$ and $\mathscr{P}^u$ on $\mathscr{L}$ by

$$(\mathscr{P}^s \boldsymbol{x})_n = P_n^s x_n, \quad (\mathscr{P}^u \boldsymbol{x})_n = P_n^u x_n, \qquad n \in \mathbb{Z}.$$

Clearly, the operators $\mathscr{P}^s$ and $\mathscr{P}^u$ are bounded projections and, moreover, $\mathscr{P}^s + \mathscr{P}^u = I$. Hence the subspaces

$$\mathscr{E}^s = \mathscr{P}^s \mathscr{L} \subseteq \mathscr{L}, \quad \mathscr{E}^u = \mathscr{P}^u \mathscr{L} \subseteq \mathscr{L} \tag{4.85}$$

have trivial intersection and their direct sum coincides with $\mathscr{L}$, i.e.

$$\mathscr{E}^s \cap \mathscr{E}^u = 0, \quad \mathscr{E}^s \oplus \mathscr{E}^u = \mathscr{L}. \tag{4.86}$$

In addition, it follows from the equivariance relations (4.41) and the definition of projections $\mathscr{P}^s$ and $\mathscr{P}^u$ that the subspaces $\mathscr{E}^s$ and $\mathscr{E}^u$ are invariant under the linear operator $\mathscr{A}$, i.e.

$$\mathscr{A}\mathscr{E}^s \subseteq \mathscr{E}^s, \quad \mathscr{A}\mathscr{E}^u \subseteq \mathscr{E}^u.$$

Introduce the norm

$$\|\boldsymbol{x}\|_\infty^* = \sup_{n \in \mathbb{Z}} \{\|P_n^s x_n\|_n, \|P_n^u x_n\|_n\}$$

on the space $\mathscr{L}$. Using of (4.37) and (4.39) we can obtain the chain of inequalities

$$\begin{aligned} \frac{1}{2q}\|x_n\| \le \frac{1}{2}\|x_n\|_n \le \frac{1}{2}\left(\|P_n^s x_n\|_n + \|P_n^u x_n\|_n\right) \\ \le \max\{\|P_n^s x_n\|_n, \|P_n^u x_n\|_n\} \le h\|x_n\|_n \le hq\|x_n\| \end{aligned}$$

for any integer n, from which it follows that

$$\frac{1}{2q}\|\boldsymbol{x}\|_\infty \le \|\boldsymbol{x}\|_\infty^* \le hq\|\boldsymbol{x}\|_\infty,$$

i.e. the norms $\|\cdot\|_\infty$ and $\|\cdot\|_\infty^*$ in the space $\mathscr{L}$ are equivalent.

From inequalities (4.42), which are valid for the matrix sequence $\{A_n\}$ due to its hyperbolicity, we obtain immediately that

$$\|\mathscr{A}\mathscr{P}^s \boldsymbol{x}\|_\infty^* \le \lambda_s \|\mathscr{P}^s \boldsymbol{x}\|_\infty^*, \quad \|\mathscr{A}\mathscr{P}^u \boldsymbol{x}\|_\infty^* \ge \lambda_u \|\mathscr{P}^u \boldsymbol{x}\|_\infty^*.$$

Together with (4.85), these inequalities mean that the spectrum σ_s of the restriction $\mathscr{A}|_{\mathscr{E}^s}$ of the linear operator $\mathscr{A}$ to its invariant subspace $\mathscr{E}^s$ lies entirely in the circle $|\lambda| \le \lambda_s < 1$. Similarly, the spectrum σ_u of the restriction $\mathscr{A}|_{\mathscr{E}^u}$ of the linear operator $\mathscr{A}$ to its invariant subspace $\mathscr{E}^u$ lies entirely outside of the circle $|\lambda| \leq \lambda_u$. Now, from (4.85) and (4.86) it follows that the spectrum of operator $\mathscr{A}$ coincides with the union of the sets σ_s and σ_u, and thus does not intersect the circle $|\lambda| = 1$. Hence, the linear operator $\mathscr{A}$ is hyperbolic.

We now consider the opposite implication that hyperbolicity of the operator $\mathscr{A}$ implies hyperbolicity of the matrix sequence $\{A_n\}$. By supposition, the spectrum σ of the operator $\mathscr{A}$ does not intersect the unit circle $|\lambda| = 1$. Since the spectrum of any operator is a closed set, the set σ can then be represented as union $\sigma = \sigma_s \cup \sigma_u$ of closed subsets σ_s which lies entirely inside the unit circle and σ_u which lies entirely outside the unit circle.

Denote by $\mathscr{E}^s \subseteq \mathscr{L}$ the invariant spectral subspace of the operator $\mathscr{A}$ corresponding to the subset σ_s of its spectrum, and denote by $\mathscr{E}^u \subseteq \mathscr{L}$ the invariant spectral subspace of the operator $\mathscr{A}$ corresponding to σ_u. Then, as is known from the general spectral theory, the relations (4.86) hold. Denote the projections corresponding to the splitting (4.86) by $\mathscr{P}^s : \mathscr{L} \mapsto \mathscr{E}^s$ and $\mathscr{P}^u = I - \mathscr{P}^s : \mathscr{L} \mapsto \mathscr{E}^u$; these are bounded in our case.

The spectral radius $\sigma(A)$ of a linear operator A on a Banach space satisfies

$$\sigma(A) = \limsup_{n\to\infty} \|A^n\|^{1/n},$$

from which follows the existence for each $\varepsilon > 0$ of a number c_ε such that

$$\|A^n\| \leq c_\varepsilon(\sigma(A) + \varepsilon)^n, \qquad n \geq 0. \tag{4.87}$$

By definition, the closed set σ_s lies in the circle $|\lambda| < 1$, so numbers $\lambda_s \in (0,1)$ and $\varepsilon > 0$ can be found such that σ_s will, in fact, belong to the circle $|\lambda| \leq \lambda_s - \varepsilon$. Applying (4.87) to the restriction $\mathscr{A}|_{\mathscr{E}^s}$ of the linear operator $\mathscr{A}$ to its invariant subspace $\mathscr{E}^s$ (and taking $\|\cdot\|_\infty$ as the operator norm $\|\cdot\|$ in (4.87)), we obtain

$$\|(\mathscr{A}|_{\mathscr{E}^s})^n\|_\infty \leq c_\varepsilon \lambda_s^n, \qquad n \geq 0,$$

for an appropriate constant c_ε, from which it follows that

$$\|\mathscr{A}^n \mathscr{P}^s \boldsymbol{x}\|_\infty \leq c_\varepsilon \lambda_s^n \|\boldsymbol{x}\|_\infty, \qquad \boldsymbol{x} \in \mathscr{L},\ n \geq 0. \tag{4.88}$$

Analogously, numbers $\lambda_u > 1$ and $\varepsilon > 0$ can be found such that σ_u belongs to the exterior of the circle $|\lambda| \leq \lambda_u + \varepsilon$. Then the restriction $\mathscr{A}|_{\mathscr{E}^u}$ of the linear operator $\mathscr{A}$ to its invariant subspace $\mathscr{E}^u$ is invertible and, applying the formula (4.87) to $(\mathscr{A}|_{\mathscr{E}^u})^{-1}$, we obtain

$$\|(\mathscr{A}|_{\mathscr{E}^u})^{-n}\|_\infty \leq c_\varepsilon \lambda_u^{-n}, \qquad n \geq 0,$$

for an appropriate constant c_ε, from which it follows

$$\|\mathscr{A}^n \mathscr{P}^u \boldsymbol{x}\|_\infty \geq c_\varepsilon^{-1} \lambda_u^n \|\boldsymbol{x}\|_\infty, \qquad \boldsymbol{x} \in \mathscr{L},\ n \geq 0. \tag{4.89}$$

Define a new norm on $\mathscr{L}$ by

$$\|\boldsymbol{x}\|_* = \sup_{n\ge 0} \lambda_s^{-n} \|\mathscr{A}^n \mathscr{P}^s \boldsymbol{x}\|_\infty + \inf_{n\ge 0} \lambda_u^{-n} \|\mathscr{A}^n \mathscr{P}^u \boldsymbol{x}\|_\infty. \tag{4.90}$$

From (4.88) and (4.89) we conclude that the norms $\|\cdot\|_\infty$ and $\|\cdot\|_*$ are equivalent, i.e. there exists such a number Q that

$$Q^{-1}\|\boldsymbol{x}\|_\infty \le \|\boldsymbol{x}\|_* \le Q\|\boldsymbol{x}\|_\infty \tag{4.91}$$

for all $\boldsymbol{x} \in \mathscr{L}$. From (4.90) it thus follows that

$$\|\mathscr{A}\mathscr{P}^s\boldsymbol{x}\|_* \le \lambda_s\|\mathscr{P}^s\boldsymbol{x}\|_*, \quad \|\mathscr{A}\mathscr{P}^u\boldsymbol{x}\|_* \ge \lambda_u\|\mathscr{P}^u\boldsymbol{x}\|_*. \tag{4.92}$$

Now fix an integer $k \in \mathbb{Z}$ and define $E_k^s \subseteq \mathbb{R}^d$ to be the set of those $v \in \mathbb{R}^d$ for which the equation

$$v_{n+1} = A_n v_n, \qquad n \ge k, \tag{4.93}$$

has a bounded solution satisfying $v_k = v$. Similarly, define $E_k^u \subseteq \mathbb{R}^d$ to be the set of those $w \in \mathbb{R}^d$ for which the equation

$$w_{n+1} = A_n w_n, \qquad n < k, \tag{4.94}$$

has a bounded solution satisfying $w_k = w$. It is not hard to see that E_k^s and E_k^u are subspaces of $\mathbb{R}^d$. We need to prove that they form a splitting of $\mathbb{R}^d$, i.e. with

$$E_k^s \cap E_k^u = 0, \quad E_k^s \oplus E_k^u = \mathbb{R}^d. \tag{4.95}$$

Let $x \in E_k^s \cap E_k^u$. By definition of the subspace E_k^s there exists a bounded sequence $\{v_n\}$, $n \ge k$, satisfying (4.93) and $v_k = x$. Analogously, by definition of the subspace E_k^u there exists a bounded sequence $\{w_n\}$, $n \le k$, satisfying (4.94) and $w_k = x$. Then the sequence $\boldsymbol{x} = \{x_n\}$ defined by

$$x_n = \begin{cases} v_n, & \text{for} \quad n \ge k, \\ w_n, & \text{for} \quad n \le k \end{cases}$$

will satisfy the equation

$$x_{n+1} = A_n x_n, \qquad n \in \mathbb{Z},$$

and hence

$$\boldsymbol{x} = \mathscr{A}\boldsymbol{x}.$$

In view of hyperbolicity of the operator $\mathscr{A}$, this implies that $\boldsymbol{x} = 0$. The first assertion of (4.95) is thus proved.

To prove the second assertion in (4.95) fix an $x \in \mathbb{R}^d$ and define a sequence $\boldsymbol{x} = \{x_n\} \in \mathscr{L}$ by

$$x_k = x, \quad x_n = 0 \qquad \text{for} \quad n \ne k, \tag{4.96}$$

and

$$\boldsymbol{x}^s = \mathscr{P}^s \boldsymbol{x}, \quad \boldsymbol{x}^u = \mathscr{P}^u \boldsymbol{x}. \tag{4.97}$$

Then $\boldsymbol{x} = \boldsymbol{x}^s + \boldsymbol{x}^u$ and thus $x = x_k^s + x_k^u$. We want to show that

$$x_k^s \in E_k^s, \qquad x_k^u \in E_k^u. \tag{4.98}$$

Write $\boldsymbol{v}^{(k)} = \boldsymbol{x}^s$ and define elements $\boldsymbol{v}^{(n)} \in \mathscr{L}$ recursively by

$$\boldsymbol{v}^{(n+1)} = \mathscr{A}\boldsymbol{v}^{(n)}, \qquad n \geq k.$$

From (4.92) we then get

$$\|\boldsymbol{v}^{(k)}\|_* \geq \|\boldsymbol{v}^{(k+1)}\|_* \geq \ldots \geq \|\boldsymbol{v}^{(n)}\|_* \geq \ldots, \qquad n \geq k,$$

which mean that the sequence with components $\boldsymbol{v}^{(n)} \in \mathscr{L}$, $n \geq k$, is bounded in the norm $\|\cdot\|_*$ and thus, by (4.91), is bounded in the norm $\|\cdot\|_\infty$ too. Hence the sequence with elements $v_n \in \mathbb{R}^d$ defined by

$$v_n = v_n^{(n)}, \qquad n \geq k,$$

is also bounded (in the norm $\|\cdot\|$). Clearly, the sequence $\{v_n\}$ satisfies (4.93) and $v_k = x_k^s$. Hence, in view of the definition of the subspace E_k^s, we have proved the first inclusion of (4.98). The second inclusion of (4.98) is proved analogously. The second equality in (4.95) is now proved.

Define now operators P_k^s and P_k^u by

$$P_k^s x = x_k^s, \quad P_k^u x = x_k^u.$$

By definition of the points x_k^s and x_k^u, these operators are linear and satisfy

$$P_k^s \mathbb{R}^d \subseteq E_k^s, \quad P_k^u \mathbb{R}^d \subseteq E_k^u, \quad P_k^s + P_k^u = I.$$

From these and (4.95) we conclude that the operators P_k^s and P_k^u are projections. To estimate the norms $\|P_k^s\|$ and $\|P_k^u\|$ we use the chain of inequalities

$$\|P_k^s x\| = \|x_k^s\| \leq \|\boldsymbol{x}^s\|_\infty = \|\mathscr{P}^s \boldsymbol{x}\|_\infty \leq \|\mathscr{P}^s\|_\infty \|\boldsymbol{x}\|_\infty = \|\mathscr{P}^s\|_\infty \|x\|$$

from which it follows that

$$\|P_k^s\| \leq \|\mathscr{P}^s\|_\infty \leq \max\{\|\mathscr{P}^s\|_\infty,\ \|\mathscr{P}^u\|_\infty\}.$$

Analogously,

$$\|P_k^u\| \leq \|\mathscr{P}^u\|_\infty \leq \max\{\|\mathscr{P}^s\|_\infty,\ \|\mathscr{P}^u\|_\infty\},$$

so a uniform norm bound exists for the projections P_k^s and P_k^u with $k \in \mathbb{Z}$ (cf. (4.39)).

The equivariant identities (4.41) for the matrix sequence $\{A_n\}$ then follows immediately from the definition of subspaces E_n^s and E_n^u and from invertibility of matrices A_n. It remains only to construct norms $\|\cdot\|_n$ which ensure the validity of inequalities (4.42). To do it, we will need more detailed information about properties of the projections $\mathscr{P}^s$ and $\mathscr{P}^u$.

Given a point $x \in \mathbb{R}^d$ we construct points $\boldsymbol{x}, \boldsymbol{x}^s, \boldsymbol{x}^u \in \mathscr{L}$ by formulae (4.96) and (4.97). We want to show that

$$\boldsymbol{x} = \boldsymbol{x}^s \qquad \text{if} \quad x \in E_k^s, \tag{4.99}$$

$$\boldsymbol{x} = \boldsymbol{x}^u \qquad \text{if} \quad x \in E_k^u. \tag{4.100}$$

For definiteness, let $x \in E_k^s$. Then, in the same way as the inclusions (4.98) were obtained, we obtain the inclusions

$$x_n^s \in E_n^s, \quad x_n^u \in E_n^u, \qquad n \in \mathbb{Z}. \tag{4.101}$$

But, from the equality $\boldsymbol{x} = \boldsymbol{x}^s + \boldsymbol{x}^u$ and from the definition (4.96) of the point $\boldsymbol{x}$, it follows immediately that

$$x_n^s = -x_n^u, \qquad n \neq k,$$

which, in view of (4.101) and (4.95), can only be satisfied if

$$x_n^s = x_n^u = 0, \qquad n \neq k.$$

The required relations (4.99) and (4.100) follow from this fact.

Now define the norm $\|\cdot\|_k$ on $\mathbb{R}^d$ by

$$\|x\|_k = \|\boldsymbol{x}\|_*, \qquad x \in \mathbb{R}^d,$$

where $\boldsymbol{x} \in \mathscr{L}$ is the element defined by (4.96). From (4.91) and from the obvious equality $\|x\| = \|\boldsymbol{x}\|_\infty$ we obtain

$$Q^{-1}\|x\| = Q^{-1}\|\boldsymbol{x}\|_\infty \le \|x\|_k = \|\boldsymbol{x}\|_* \le Q\|\boldsymbol{x}\|_\infty = Q\|x\|,$$

which means that the uniform boundedness condition (cf. (4.37)) are satisfied by the norms $\|\cdot\|_k$, $k \in \mathbb{Z}$.

Now, given $x \in E_k^s$, denote $y = A_k x$. Then, as is already proved, $y \in E_{k+1}^s$. Define the element $\boldsymbol{x} \in \mathscr{L}$ by (4.96) and the define element $\boldsymbol{y} \in \mathscr{L}$ in the same manner by

$$y_{k+1} = y, \quad y_n = 0 \qquad \text{for} \quad n \neq k+1.$$

Then clearly $\boldsymbol{y} = \mathscr{A}\boldsymbol{x}$, and from the inclusions $x \in E_k^s$, $y \in E_{k+1}^s$ and (4.99) we obtain that

$$\boldsymbol{x} \in \mathscr{E}^s, \qquad \boldsymbol{y} = \mathscr{A}\boldsymbol{x} \in \mathscr{E}^s,$$

from which, in view of (4.92), we have

$$\|A_k x\|_{k+1} = \|\mathscr{A}\boldsymbol{x}\|_* \le \lambda_s \|\boldsymbol{x}\|_* = \lambda_s \|x\|_k, \qquad x \in E_k^s.$$

The inequality

$$\|A_k x\|_{k+1} = \|\mathscr{A}\boldsymbol{x}\|_* \ge \lambda_u \|\boldsymbol{x}\|_* = \lambda_u \|x\|_k, \qquad x \in E_k^u$$

can be proved analogously. The proof of the theorem is now completed. □

5

Semi-Hyperbolicity and Hyperbolicity

In Chapter 4 it was shown that a semi-hyperbolic sequence of matrices is hyperbolic, i.e. in the linear case semi-hyperbolicity implies hyperbolicity. This is generally not true for the nonlinear mappings, which will be shown by an example in Section 5.1 which demonstrates that a semi-hyperbolic mapping may not possess an invariant splitting and so cannot be a hyperbolic. Nevertheless, it will be shown in Section 5.2 that a semi-hyperbolic mapping which is smooth and invertible in a neighborhood of a compact invariant set is, in fact, hyperbolic on that set. The proofs depend substantially on background material and results of Chapter 4.

5.1 Perturbation of Anosov Endomorphisms

In this Section an example will be constructed which demonstrates that generally a semi-hyperbolic mapping may not possess an invariant splitting, and cannot be a hyperbolic.

We briefly recall some facts about Anosov mappings already considered in Section 3.2.2. Write the elements of $\mathbb{R}^d$ as vectors x with coordinates x_1, x_2, …, x_d and let $\mathbb{T}^d$ be the standard d-dimensional torus, that is the factorization of $\mathbb{R}^d$ by the integer lattice. This torus $\mathbb{T}^d$ is a compact differentiable manifold in $\mathbb{R}^d$ with respect to the locally Euclidean metric

$$\varrho(x,y) = \sqrt{|x_1 - y_1|^2_{\mathrm{mod}\,1} + |x_2 - y_2|^2_{\mathrm{mod}\,1} + \cdots + |x_d - y_d|^2_{\mathrm{mod}\,1}}$$

on $\mathbb{T}^d$ where

$$|t - s|_{\mathrm{mod}\,1} = \min\left\{|t - s + 2k| : \quad k = 0, \pm 1\right\}, \qquad 0 \le t, s < 1.$$

The tangent space $T_x\mathbb{T}^d$ can then be identified with $\mathbb{R}^d$ by an appropriate choice of natural coordinates generated by those of $\mathbb{R}^d$, so $T_x\mathbb{T}^d = \mathbb{R}^d$ for each $x \in \mathbb{T}^d$. Denote the natural projection from $\mathbb{R}^d$ onto $\mathbb{T}^d$ by

$$\Pi(x) = (x_1 \bmod 1,\, x_2 \bmod 1,\, \dots,\, x_d \bmod 1), \qquad x = (x_1, x_2, \dots, x_d)$$

and associate with a given $d \times d$ matrix A with integer components a_{ij} the mapping $f : \mathbb{T}^d \mapsto \mathbb{T}^d$ defined by

$$f(x) = \Pi(Ax).$$

Theorem 5.1. *Let $d > 3$. There exists a hyperbolic $d \times d$ matrix A with integer components such that for every $\varepsilon > 0$ there can be found a differentiable semi-hyperbolic mapping $f_\varepsilon : \mathbb{T}^d \mapsto \mathbb{T}^d$ which is ε-close in the C^1-norm to the mapping $f(x) = \Pi(Ax)$ but for which there is no continuous hyperbolic splitting.*

Proof. Let first $d = 3$. Consider the 3×3 matrix

$$A = \begin{bmatrix} 2 & 3 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

which has eigenvalues $\lambda_1 = 2 - \sqrt{3}$, $\lambda_2 = 2 + \sqrt{3}$ and $\lambda_3 = 2$ with corresponding eigenvectors $v_1 = (1, -1/\sqrt{3}, 0)$, $v_2 = (1, 1/\sqrt{3}, 0)$ and $v_3 = (0, 0, 1)$, respectively. Let B be a fixed 3×3 matrix

$$B = \begin{bmatrix} b_{11} & b_{12} & 0 \\ b_{21} & b_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

such that v_2 is not an eigenvector, but otherwise arbitrary.

Consider the sequence of points

$$x^{(0)} = (0, 0, 0), \qquad x^{(n)} = (0, 0, 2^{-n}) \quad \text{for} \quad n = -1, -2, \dots$$

and for $\varepsilon \geq 0$ denote by $f_\varepsilon : \mathbb{T}^3 \mapsto \mathbb{T}^3$ a continuously differentiable mapping satisfying

(i) f_ε is ε-close to f in the C^1-norm;
(ii) $f_\varepsilon(x^{(-1)}) = f(x^{(-1)}) = 0$ and $(Tf_\varepsilon)_{x^{(-1)}} = A + \varepsilon B$;
(iii) $f_\varepsilon(x) \equiv f(x)$ for all x such that $\|x - x^{(-1)}\| \geq \frac{1}{8}$.

Such a mapping f_ε exists for sufficiently small $\varepsilon > 0$. It satisfies

$$x^{(n)} = f_\varepsilon(x^{(n-1)}), \qquad n = 0, -1, -2, \dots \tag{5.1}$$

and is semi-hyperbolic by Lemma 3.18.

Since the mapping $f = f_0$ is an Anosov endomorphism, cf. Example 3.17, it has a hyperbolic splitting at every point x which does not depend on the point x and can thus be written as

$$T_x\mathbb{T}^3 = E^s \oplus E^u, \qquad x \in \mathbb{T}^3, \tag{5.2}$$

where E^s is the eigenspace of the matrix A corresponding to the eigenvalue λ_1 and E^u is the eigenspace of the matrix A corresponding to the eigenvalues λ_2 and λ_3.

A proof by contradiction will be used to show that the mapping f_ε for sufficiently small $\varepsilon > 0$ does not have a hyperbolic splitting.

Fix $\varepsilon > 0$ sufficiently small and suppose that

$$T_x\mathbb{T}^3 = E^s_{x,\varepsilon} \oplus E^u_{x,\varepsilon}, \qquad x \in \mathbb{T}^3, \tag{5.3}$$

is a hyperbolic splitting for f_ε. By Property (iii) $f_\varepsilon(x) \equiv f(x)$ in a neighborhood of the point $x = 0$, so $x = 0$ is a fixed point of f_ε and by the Hartman–Grobman Theorem [51] the splitting (5.3) at the point $x = 0$ coincides with the splitting (5.2), that is

$$E^s_{0,\varepsilon} \equiv E^s, \quad E^u_{0,\varepsilon} \equiv E^u.$$

In fact, these equivalences can be complemented by

$$E^u_{x^{(n)},\varepsilon} \equiv E^u, \qquad n = 1, 2, \dots\ .$$

Indeed, by Properties (ii) and (iii)

$$(Tf_\varepsilon)_{x^{(n)}} = A \qquad \text{for} \quad n = 2, 3, \dots\ , \tag{5.4}$$

and since by supposition the subspaces $E^u_{x,\varepsilon}$ are equivariant for Tf_ε, then from (5.4) and (5.1) it follows that

$$\begin{aligned} E^u_{x^{(n)},\varepsilon} &= (Tf^{k-n}_\varepsilon)_{x^{(k)}} E^u_{x^{(k)},\varepsilon} \\ &= (Tf_\varepsilon)_{x^{(n+1)}} (Tf_\varepsilon)_{x^{(n+2)}} \dots (Tf_\varepsilon)_{x^{(k)}} E^u_{x^{(k)},\varepsilon} \\ &= A^{k-n} E^u_{x^{(k)},\varepsilon} \end{aligned} \tag{5.5}$$

for all $k \geq n$. But, by the continuity of the splitting (5.3), the subspace $E^u_{x^{(n)},\varepsilon}$ is close to E^u for large enough n and thus does not contain vectors from E^s. Hence, taking the limit as $k \to \infty$ in the right hand side of (5.5), we obtain

$$E^u_{x^{(n)},\varepsilon} = \lim_{k\to\infty} A^{k-n} E^u_{x^{(k)},\varepsilon} = E^u.$$

It remains to observe that

$$\begin{aligned} (Tf_\varepsilon)_{x^{(1)}} E^u_{x^{(1)},\varepsilon} &= (Tf_\varepsilon)_{x^{(1)}} E^u = (A + \varepsilon B) E^u \\ &\neq E^u = E^u_{0,\varepsilon} = E^u_{x^{(0)},\varepsilon} = E^u_{f_\varepsilon(x^{(1)}),\varepsilon} \end{aligned}$$

since $(Tf_\varepsilon)_{x^{(1)}} = A + \varepsilon B$, and this contradicts the assumed equivariance of the splitting (5.3) with respect to Tf_ε. This contradiction completes the proof in the case when $d = 3$.

To prove the theorem in the case when $d > 3$ it suffices to consider the $d \times d$ block diagonal matrices

$$\widehat{A} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \qquad \widehat{B} = \begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix}$$

and repeat the previous considerations for the mapping $\widetilde{f}(x) = \Pi(\widehat{A}x)$. □

5.2 Converse Case

Any hyperbolic mapping is clearly semi-hyperbolic. Theorem 5.2 below shows that the converse is also valid for invertible smooth mappings, so establishing equivalence between semi-hyperbolicity and hyperbolicity for invertible mappings. The proof is valid for the various definitions of semi-hyperbolicity considered earlier.

Theorem 5.2. *Let $f : X \mapsto \mathbb{R}^d$ be a continuously differentiable mapping which is semi-hyperbolic (according to variable norm Definition 3.4 or to single norm Definition 3.5) on a bounded open set $X \subseteq \mathbb{R}^d$. In addition, suppose that f is invertible in a neighborhood of some compact set $K \subset X$ and that K is an invariant set for both f and f^{-1}, that is $f(K) = f^{-1}(K) = K$. Then f is hyperbolic on K.*

Proof. First we consider an equivariant splitting of $\mathbb{R}^d$ for the mapping f on K. Associate with a point $x \in K$ the sequence of linear operators

$$A_{n,x} = (Df^{n+1})_x = (Df)_{f^n(x)}. \tag{5.6}$$

Due to semi-hyperbolicity of the mapping f, for any $x \in K$ the sequence of linear operators $\{A_{n,x}\}$ will be also semi-hyperbolic[1]. Then by Theorem 4.18 we can construct for the point x and for any integer n the splitting of the space $\mathbb{R}^N$ into the direct sum of subspaces $\widehat{E}^s_{n,x}$ and $\widehat{E}^u_{n,x}$ satisfying

$$A_{n,x}\widehat{E}^s_{n,x} = \widehat{E}^s_{n+1,x}, \quad A_{n,x}\widehat{E}^u_{n,x} = \widehat{E}^u_{n+1,x},$$

or, what is the same,

$$(Df^{n+1})_x\widehat{E}^s_{n,x} = \widehat{E}^s_{n+1,x}, \quad (Df^{n+1})_x\widehat{E}^u_{n,x} = \widehat{E}^u_{n+1,x}. \tag{5.7}$$

By Lemma 4.16 and by (5.6) the subspace $\widehat{E}^s_{n,x}$ is equal to the set of $v \in \mathbb{R}^d$ for which the equation

[1] More precisely, if the mapping f is semi-hyperbolic in variable norm sense (see Definition 3.4), then the sequence $\{A_{n,x}\}$ will be also semi-hyperbolic in the variable norm sense (see Definition 4.5). If the mapping f is semi-hyperbolic for a single norm (see Definition 3.5), then the sequence $\{A_{n,x}\}$ will be semi-hyperbolic for a single norm (see Definition 4.20).

$$v_{k+1} = (Df^{k+1})_x v_k, \qquad k \geq n,$$

has a bounded solution satisfying $v_n = v$, and the subspace $\widehat{E}^u_{n,x}$ can be described as the set of those $w \in \mathbb{R}^d$ for which the equation

$$w_{k+1} = (Df^{k+1})_x w_k, \qquad k < n,$$

has a bounded solution satisfying $w_n = w$. From such an interpretation of subspaces $\widehat{E}^s_{n,x}$, $\widehat{E}^u_{n,x}$ and from invertibility of the linear operators Df_x at any point $x \in K$ it follows that

$$\widehat{E}^s_{n,x} = \widehat{E}^s_{0,f^n(x)}, \quad \widehat{E}^u_{n,x} = \widehat{E}^u_{0,f^n(x)}, \qquad n \in \mathbb{Z}.$$

So, by denoting $\mathscr{E}^s_x = \widehat{E}^s_{0,x}$, $\mathscr{E}^u_x = \widehat{E}^u_{0,x}$ we obtain a splitting $\mathbb{R}^d = \mathscr{E}^s_x \oplus \mathscr{E}^u_x$, $x \in K$, satisfying in view of (5.7) the equivariance property

$$Df_x \mathscr{E}^s_x = \mathscr{E}^s_{f(x)}, \quad Df_x \mathscr{E}^u_x = \mathscr{E}^u_{f(x)}, \qquad x \in K.$$

Thus, the mapping f satisfies Condition A1 of Definition 2.1.

Now, from Theorem 4.18 (see inequalities (4.32) and (4.33)) it follows that the mapping f satisfies Condition A2 of Definition 2.1, and from Lemma 4.16 (see inequality (4.27)) Condition A3 of Definition 2.1 also holds for mapping f. The mapping f thus satisfies all the conditions of Anosov system (Definition 2.1) except that f is defined not on a closed Riemann manifold, but on a neighborhood of a compact invariant set. This distinction makes formally impossible to use Theorem 2.2 for proving continuous dependence of the splitting $\mathbb{R}^d = \mathscr{E}^s_x \oplus \mathscr{E}^u_x$ on $x \in K$. But a closer inspection shows that the fact that f being defined on a closed Riemann manifold is not essential in the proof of Theorem 2.2. The statement of this theorem remains valid under the supposition that f is defined on a compact invariant set. Hence, by this reasoning, the splitting $\mathbb{R}^d = \mathscr{E}^s_x \oplus \mathscr{E}^u_x$ continuously depends on $x \in K$, and thus, by Definition 2.5, the mapping f is hyperbolic on K. □

5.2.1 Alternative Proof

The investigation of the relationship between semi-hyperbolicity and hyperbolicity in Chapter 4 and in Section 5.2 was intentionally done in a more traditional analytical manner, via an explicit description of splitting subspaces, in order that their properties could be determined more fully. Modern techniques based on differential geometry and smooth dynamics (see, e.g., [38]) is also applicable here and provide an alternative proof of Theorem 5.2, based on the Mather Projection Lemma 5.3[2], by adapting the proof of Hirsch and Pugh [38].

[2] We refer the reader to [38] and other textbooks for an explanation of the terms used here.

Let E be a vector bundle over a compact set $K \subset \mathbb{R}^d$ and let $f : K \to K$ be a C^1 diffeomorphism on a neighborhood of K. Denote by $C(E)$ the Banach space of bounded continuous sections of E over K with the sup-norm topology. Let $\widetilde{f} : C(E) \to C(E)$ be the continuous linear mapping

$$\widetilde{f}(\varphi) = Df \circ \varphi \circ f^{-1}. \tag{5.8}$$

Lemma 5.3 (Mather Projection Lemma). *If $\widetilde{f} - I$ is a hyperbolic isomorphism, then f is hyperbolic on K.*

Now let $C(T_K\mathbb{R}^d) = \mathscr{C}$ be the Banach space of bounded continuous sections of $T_K\mathbb{R}^d$ and let $\widetilde{f} : \mathscr{C} \mapsto \mathscr{C}$ be the map (5.8) induced by f. Write $\mathscr{C} = \mathscr{C}(E_1|_K) \times \mathscr{C}(E_2|_K)$. Then, making use of Theorem 4.28, it can be shown that the conditions of Theorem 5.2 are satisfied by $\widetilde{f}$, so $\widetilde{f}$ is hyperbolic linear operator. Hence, by the Mather Projection Lemma 5.3, f is hyperbolic on the set K.

6

Expansivity and Shadowing

We saw in Section 2.2 that hyperbolic systems possess some rather strong and useful properties such as expansivity (see Definition 2.7 and Theorem 2.8) and shadowing (see Theorem 2.9). In this Chapter we will show that these and other properties remain valid for semi-hyperbolic systems. Moreover, explicit values or sharp estimates of relevant parameters and intervals of validity will be obtained.

6.1 Expansivity

In this Section, semi-hyperbolic dynamical systems are shown to be exponentially expansive, locally at least, and explicit rates of expansion are determined. Differentiability is not required, so the proof will be given for the Lipschitz mappings only.

6.1.1 Definitions

Let (X, ϱ) be a metric space.

Definition 6.1. *A* trajectory *of a dynamical system generated by a mapping $f : X \mapsto X$ on a state space X is a sequence $\boldsymbol{x} = \{x_n\} \subset X$ satisfying*

$$x_{n+1} = f(x_n)$$

for all n belonging to some contiguous set $\mathbb{I}$ of integers. The qualifier finite *may be appended when this set $\mathbb{I}$ is of finite length and* infinite *otherwise.*

Recall that the dynamical system generated by a homeomorphism $f : X \mapsto X$ is said to be ξ*-expansive* on X (see Definition 2.7) if the inequalities

$$\varrho(f^n(x_0), f^n(y_0)) \leq \xi \qquad \text{for} \quad n = 0, \pm 1, \pm 2, \ldots$$

imply that $x_0 = y_0$, i.e. any two bi-infinite trajectories $\{x_n\} = \{f^n(x_0)\}$ and $\{y_n\} = \{f^n(y_0)\}$ which always remain within a threshold ξ of each other are identical. Usually the metric space (X, ϱ) is also assumed to be compact as it is this together with expansivity that gives rise to complicated dynamical behavior.

An important characteristic of ξ-expansivity is the rate of divergence of different trajectories. Let $\mathrm{Tr}_{\pm k}(f, X)$ denote the set of all trajectories $\boldsymbol{x} = \{x_{-k}, \dots, x_0, \dots, x_k\}$ of the mapping f that are contained entirely in a set X, and for any $\boldsymbol{x}, \widetilde{\boldsymbol{x}} \in \mathrm{Tr}_{\pm k}(f, K)$ define

$$\varrho_k(\boldsymbol{x}, \widetilde{\boldsymbol{x}}) = \max_{-k \le n \le k} \|x_n - \widetilde{x}_n\|.$$

Lemma 6.2. *Let K be a compact subset of $\mathbb{R}^d$ and let $f : K \mapsto K$ be a continuous ξ-expansive mapping. Then for every $0 < \varepsilon, \theta < \xi$ there exists a positive integer $\varkappa(\varepsilon, \theta)$ such that*

$$\varrho_k(\boldsymbol{x}, \widetilde{\boldsymbol{x}}) > \theta$$

holds for all $\boldsymbol{x}, \widetilde{\boldsymbol{x}} \in \mathrm{Tr}_{\pm k}(f, K)$ with $\|x_0 - \widetilde{x}_0\| \ge \varepsilon$ and $k \ge \varkappa(\varepsilon, \theta)$.

Proof. Suppose the contrary. Then for any positive integer k there exist trajectories $\boldsymbol{x}^{(k)}, \widetilde{\boldsymbol{x}}^{(k)} \in \mathrm{Tr}_{\pm k}(f, K)$ satisfying

$$\|x_0^{(k)} - \widetilde{x}_0^{(k)}\| \ge \varepsilon \qquad \text{and} \quad \varrho_k(\boldsymbol{x}^{(k)}, \widetilde{\boldsymbol{x}}^{(k)}) \le \theta < \xi.$$

By compactness of the set K, the sequences $\{\boldsymbol{x}^{(k)}\}$ and $\{\widetilde{\boldsymbol{x}}^{(k)}\}$ can be assumed to converge componentwise to trajectories $\boldsymbol{x}^*$ and $\widetilde{\boldsymbol{x}}^* \in \mathrm{Tr}_{\pm\infty}(f, K)$. Then

$$\|x_0^* - \widetilde{x}_0^*\| \ge \varepsilon,$$

but at the same time

$$\|x_n^* - \widetilde{x}_n^*\| \le \theta < \xi, \qquad n = 0, \pm 1, \pm 2, \dots ,$$

which contradicts the ξ-expansivity of f. □

In expansive systems distinct trajectories often separate exponentially fast, at least locally, as will be seen from the next Example.

Example 6.3. Let (Σ, ϱ) be the compact metric space of bi-infinite binary sequences $\boldsymbol{x} = \{x_i\}_{i=-\infty}^{+\infty}$, i.e. with $x_i \in \{0, 1\}$, endowed with the metric

$$\varrho(\boldsymbol{x}, \boldsymbol{y}) := \sum_{i=-\infty}^{+\infty} 2^{-|i|} |x_i - y_i| ,$$

and consider the *shift mapping* $\sigma : \Sigma \mapsto \Sigma$ on Σ defined by

$$(\sigma \boldsymbol{x})_i = x_{i+1}, \qquad i \in \mathbb{Z}.$$

The inequality

$$\varrho\left(\sigma^n \boldsymbol{x}, \sigma^n \boldsymbol{y}\right) \leq \xi < 1 \tag{6.1}$$

can be rewritten as $|x_n - y_n| \leq \xi < 1$, which for binary sequences $\boldsymbol{x}$ and $\boldsymbol{y}$ can only be valid if

$$x_n = y_n. \tag{6.2}$$

By (6.2), the validity of inequality (6.1) for any n implies the coincidence of the sequences $\boldsymbol{x}$ and $\boldsymbol{y}$. Hence the shift mapping σ is ξ-expansive for any $\xi < 1$.

It turns out that the shift mapping σ possesses a stronger property than ξ-expansivity, namely, the inequalities

$$\max\left\{\varrho\left(\sigma^n \boldsymbol{x}, \sigma^n \boldsymbol{y}\right),\ \varrho\left(\sigma^{-n} \boldsymbol{x}, \sigma^{-n} \boldsymbol{y}\right)\right\} \geq 2^{n-1} \varrho(\boldsymbol{x}, \boldsymbol{y}) \tag{6.3}$$

are valid for $n = -n_-, \ldots, 0, \ldots, n_+$ whenever

$$\varrho\left(\boldsymbol{x}, \boldsymbol{y}\right) \leq \xi < 1,$$

where n_- and n_+ are the largest possible integers such that

$$\varrho\left(\sigma^n \boldsymbol{x}, \sigma^n \boldsymbol{y}\right) \leq \xi < 1 \qquad \text{for} \quad n = -n_-, \ldots, 0, \ldots, n_+. \tag{6.4}$$

To prove this, suppose that inequalities (6.4) are valid with n_- and n_+ being maximal. Then by (6.1) and (6.2)

$$x_n = y_n, \qquad n = -n_-, \ldots, 0, \ldots, n_+,$$

so $\varrho\left(\boldsymbol{x}, \boldsymbol{y}\right)$ can be represented as

$$\varrho\left(\boldsymbol{x}, \boldsymbol{y}\right) = \sum_{i<-n_-} 2^i \left|x_i - y_i\right| + \sum_{i>n_+} 2^{-i} \left|x_i - y_i\right|,$$

where one of summands on the right hand part is greater than or equal to $\frac{1}{2}\varrho\left(\boldsymbol{x}, \boldsymbol{y}\right)$. For definiteness, let

$$\sum_{i>n_+} 2^{-i} \left|x_i - y_i\right| \geq \frac{1}{2} \varrho\left(\boldsymbol{x}, \boldsymbol{y}\right). \tag{6.5}$$

Similarly for any $n = -n_-, \ldots, 0, \ldots, n_+$, we obtain

$$\begin{aligned}
\varrho\left(\sigma^n \boldsymbol{x}, \sigma^n \boldsymbol{y}\right) &= \sum_{i\in\mathbb{Z}} 2^{-|i|} \left|x_{i+n} - y_{i+n}\right| = \sum_{i\in\mathbb{Z}} 2^{-|i-n|} \left|x_i - y_i\right| \\
&= \sum_{i<-n_-} 2^{-|i-n|} \left|x_i - y_i\right| + \sum_{i>n_+} 2^{-|i-n|} \left|x_i - y_i\right| \\
&= \sum_{i<-n_-} 2^{i-n} \left|x_i - y_i\right| + \sum_{i>n_+} 2^{n-i} \left|x_i - y_i\right| \\
&\geq 2^n \sum_{i>n_+} 2^{-i} \left|x_i - y_i\right|.
\end{aligned}$$

Then by (6.5)

$$\varrho\left(\sigma^n \boldsymbol{x}, \sigma^n \boldsymbol{y}\right) \geq \frac{1}{2} 2^n \varrho\left(\boldsymbol{x}, \boldsymbol{y}\right),$$

and the inequalities (6.3) for $n = -n_-, \ldots, 0, \ldots, n_+$ are proved.

Example 6.3 is of fundamental interest because the shift mapping σ is conjugate to the hyperbolic diffeomorphism of the Smale horseshoe system which is one of the most prominent examples of a hyperbolic system. It motivates the following definition of exponential expansivity.

Let $\mathscr{O}_\varepsilon(S)$ denote the open ε-neighborhood of a nonempty subset $S \subset \mathbb{R}^d$.

Definition 6.4. *Let $X \subseteq \mathbb{R}^d$ be an open bounded set and let K be a compact subset of X. A continuous mapping $f : X \mapsto \mathbb{R}^d$ is said to be* exponentially expansive on K with exponent $r > 1$ *if there exist constants ξ and $c > 0$ such that $\mathscr{O}_\xi(K) \subseteq X$ and for any (finite) trajectories*

$$\boldsymbol{x} = \left\{x_{-n_-}, \ldots, x_0, \ldots, x_{n_+}\right\}, \quad \boldsymbol{y} = \left\{y_{-n_-}, \ldots, y_0, \ldots, y_{n_+}\right\}$$

satisfying $\boldsymbol{x} \subseteq K$ and $\|y_i - x_i\| \leq \xi$ for $n = -n_-, \ldots, 0, \ldots, n_+$ at least one of the following groups of inequalities holds:

$$\|x_n - y_n\| \geq c r^n \|x_0 - y_0\|, \qquad n = 1, 2, \ldots, n_+, \tag{6.6}$$

or

$$\|x_n - y_n\| \geq c r^{-n} \|x_0 - y_0\|, \qquad n = -1, -2, \ldots, -n_-. \tag{6.7}$$

The exponential expansivity of diffeomorphisms and homeomorphisms on a compact hyperbolic set is well known. It also holds under the weaker assumptions of semi-hyperbolicity, which will be proved in the next subsection for Lipschitz mappings satisfying an even weaker version of the inequalities in Condition SH2(Lip). Explicit values of the exponential expansivity parameters can be determined in terms of the split coefficients and the other semi-hyperbolicity parameters.

6.1.2 Lipschitz Mappings

Conditions SH0(Lip), SH1(Lip) and SH2(Lip) of semi-hyperbolicity Definition 3.6 for a Lipschitz mapping are stronger than necessary for establishing expansivity. In particular, the inequalities of Condition SH2(Lip) can be weakened. As before, let $\boldsymbol{s} = (\lambda_s, \lambda_u, \mu_s, \mu_u)$ be a split which, without loss of generality, may be supposed to be positive. Consider a mapping $f : X \mapsto \mathbb{R}^d$, where X is an open subset of $\mathbb{R}^d$, and let K a nonempty compact subset of X such that $K \cap f(K) \neq \emptyset$.

Theorem 6.5. *Let f be a continuous mapping defined on $X \subseteq \mathbb{R}^d$ and for each $x \in K$ there exists a uniform splitting $\mathbb{R}^d = E_x^s \oplus E_x^s$ satisfying Condition SH1(Lip) and the following modification of Condition SH2(Lip) of Definition 3.6:*

SH2(Mod): *For all x satisfying $x, f(x) \in K$ and for all $z \in \mathbb{R}^d$ satisfying $\|P_x^s z\|, \|P_x^u z\| \le \delta$, the inclusion $x + z \in X$ and the inequalities*

$$\begin{aligned} \|P_{f(x)}^s \left(f(x+z) - f(x)\right)\| &\le \lambda_s \|P_x^s z\| + \mu_s \|P_x^u z\|, \\ \|P_{f(x)}^u \left(f(x+z) - f(x)\right)\| &\ge \lambda_u \|P_x^u z\| - \mu_u \|P_x^s z\| \end{aligned} \tag{6.8}$$

hold.

Then the mapping f is exponentially expansive in K with exponent

$$r = \sigma(\boldsymbol{s})^{-1} = 2\left(\frac{1}{\lambda_u} + \lambda_s + \sqrt{\left(\frac{1}{\lambda_u} - \lambda_s\right)^2 + \frac{4\mu_s\mu_u}{\lambda_u}}\right)^{-1}$$

(see (4.3)*) and constants*

$$c = \frac{1}{2}h^{-1}\min\{\gamma(\boldsymbol{s}), \gamma(\boldsymbol{s})^{-1}\} \quad \text{and} \quad \xi = h^{-1}\delta, \tag{6.9}$$

where h as in Condition SH1(Lip) of Definition 3.6.

Proof. Two lemmata will form the basis of the proof. Since the split $\boldsymbol{s}$ is fixed write $M = M(\boldsymbol{s}), \sigma = \sigma(\boldsymbol{s})$ and $\gamma = \gamma(\boldsymbol{s})$ (for definitions of these values see equations (4.2), (4.3) and (4.5) in Section 4.1). In addition, we shall use the norm $\|\cdot\|_*$ on $\mathbb{R}^2$ defined by

$$\|(y_1, y_2)\|_* = \max\{\gamma|y_1|, |y_2|\},$$

(the split $\boldsymbol{s}$ is assumed to be positive, so $\gamma > 0$ here).

Lemma 6.6. *The inequalities*

$$\eta\|z\| \le \|(\|P_x^s z\|, \|P_x^u z\|)\|_* \le 2\eta h \max\{\gamma, \gamma^{-1}\}\|z\|$$

with $\eta = \frac{1}{2}\min\{\gamma, 1\}$ are valid for all $z \in \mathbb{R}^d$ and all x from the semi-hyperbolicity set K of the mapping f.

Proof. By Lemma 4.1

$$\begin{aligned} \min\{\gamma, 1\}\, \max\{\|P_x^s z\|, \|P_x^u z\|\} &\le \|(\|P_x^s z\|, \|P_x^u z\|)\|_* \\ &\le \max\{\gamma, 1\}\, \max\{\|P_x^s z\|, \|P_x^u z\|\}, \end{aligned}$$

so it suffices to check that the inequalities

$$\frac{1}{2}\|z\| \le \max\{\|P_x^s z\|, \|P_x^u z\|\} \le h\|z\|$$

hold for all $z \in \mathbb{R}^d$ and all $x \in K$. The inequality on the right follows from the semi-hyperbolicity Condition SH1(Lip), while that on the left is just

$$\|z\| = \|P_x^s z + P_x^u z\| \le \|P_x^s z\| + \|P_x^u z\| \le 2\max\{\|P_x^s z\|, \|P_x^u z\|\}.$$

The statement of the lemma now follows with $\eta = \frac{1}{2}\min\{\gamma, 1\}$. □

For the remaining lemma note that semi-hyperbolicity does not require the sets K and X to be invariant under the mapping f.

Lemma 6.7. *Suppose that $x, f(x), f^2(x) \in K$ and $y, f(y), f^2(y) \in X$ with*

$$\|x-y\|,\ \|f(x)-f(y)\|,\ \|f^2(x)-f^2(y)\| < h^{-1}\delta. \tag{6.10}$$

Then at least one of the following pair of inequalities

$$\begin{aligned}\sigma\|(\|P_x^s r_0\|, \|P_x^u r_0\|)\|_* &\geq \|(\|P_{f(x)}^s r_1\|, \|P_{f(x)}^u r_1\|)\|_*,\\ \sigma\|(\|P_{f^2(x)}^s r_2\|, \|P_{f^2(x)}^u r_2\|)\|_* &\geq \|(\|P_{f(x)}^s r_1\|, \|P_{f(x)}^u r_1\|)\|_*,\end{aligned}$$

holds, where

$$r_0 := x-y, \quad r_1 := f(x)-f(y), \quad r_2 := f^2(x)-f^2(y).$$

Proof. By (6.10) and by Condition SH1(Lip) of Definition 3.6

$$\begin{aligned}\|P_x^s r_0\|,\ \|P_{f(x)}^s r_1\|,\ \|P_{f^2(x)}^s r_2\| &< \delta,\\ \|P_x^u r_0\|,\ \|P_{f(x)}^u r_1\|,\ \|P_{f^2(x)}^u r_2\| &< \delta,\end{aligned}$$

from which, by (6.8),

$$\begin{aligned}\|P_{f(x)}^s r_1\| &\leq \lambda_s\|P_x^s r_0\| + \mu_s\|P_x^u r_0\|,\\ \|P_{f^2(x)}^u r_2\| &\geq \lambda_u\|P_{f(x)}^u r_1\| - \mu_u\|P_{f(x)}^s r_1\|\end{aligned}$$

or, what is the same,

$$\begin{aligned}\|P_{f(x)}^s r_1\| &\leq \lambda_s\|P_x^s r_0\| + \mu_s\|P_x^u r_0\|\\ \|P_{f(x)}^u r_1\| &\leq \frac{\mu_u}{\lambda_u}\|P_{f(x)}^s r_1\| + \frac{1}{\lambda_u}\|P_{f^2(x)}^u r_2\|.\end{aligned}$$

Here, by definition of the norm $\|\cdot\|_*$,

$$\begin{aligned}\|P_x^s r_0\| &\leq \gamma^{-1}\|(\|P_x^s r_0\|, \|P_x^u r_0\|)\|_*,\\ \|P_x^u r_0\| &\leq \|(\|P_x^s r_0\|, \|P_x^u r_0\|)\|_*,\\ \|P_{f(x)}^s r_1\| &\leq \gamma^{-1}\|(\|P_{f(x)}^s r_1\|, \|P_{f^2(x)}^u r_2\|)\|_*,\\ \|P_{f^2(x)}^u r_2\| &\leq \|(\|P_{f(x)}^s r_1\|, \|P_{f^2(x)}^u r_2\|)\|_*,\end{aligned}$$

and therefore

$$\begin{aligned}\|P_{f(x)}^s r_1\| &\leq \gamma^{-1}(\lambda_s + \mu_s\gamma)\|(\|P_x^s r_0\|, \|P_x^u r_0\|)\|_*,\\ \|P_{f(x)}^u r_1\| &\leq \gamma^{-1}\left(\frac{\mu_u}{\lambda_u} + \frac{1}{\lambda_u}\gamma\right)\|(\|P_{f(x)}^s r_1\|, \|P_{f^2(x)}^u r_2\|)\|_*.\end{aligned}$$

In these inequalities from the definition of the values $\sigma = \sigma(\boldsymbol{s})$ and $\gamma = \gamma(\boldsymbol{s})$ (see (4.6)),

$$\lambda_s + \mu_s\gamma = \sigma, \quad \frac{\mu_u}{\lambda_u} + \frac{1}{\lambda_u}\gamma = \sigma\gamma,$$

so

$$\|P^s_{f(x)}r_1\| \le \sigma\gamma^{-1}\|(\|P^s_x r_0\|, \|P^u_x r_0\|)\|_*, \tag{6.11}$$

$$\|P^u_{f(x)}r_1\| \le \sigma\|(\|P^s_{f(x)}r_1\|, \|P^u_{f^2(x)}r_2\|)\|_*. \tag{6.12}$$

Suppose now that the lemma is not true, that is both the inequalities

$$\sigma\|(\|P^s_x r_0\|, \|P^u_x r_0\|)\|_* < \|(\|P^s_{f(x)}r_1\|, \|P^u_{f(x)}r_1\|)\|_*, \tag{6.13}$$

and

$$\sigma\|(\|P^s_{f^2(x)}r_2\|, \|P^u_{f^2(x)}r_2\|)\|_* < \|(\|P^s_{f(x)}r_1\|, \|P^u_{f(x)}r_1\|)\|_* \tag{6.14}$$

are true. Then inequalities (6.11) and (6.13) imply that the strict inequality

$$\gamma\|P^s_{f(x)}r_1\| < \|(\|P^s_{f(x)}r_1\|, \|P^u_{f(x)}r_1\|)\|_* \tag{6.15}$$

is valid[1] from which, in view of inequality $\sigma = \sigma(\boldsymbol{s}) < 1$, we get

$$\sigma\gamma\|P^s_{f(x)}r_1\| < \|(\|P^s_{f(x)}r_1\|, \|P^u_{f(x)}r_1\|)\|_*$$

The last inequality together with (6.14) shows that

$$\sigma\|(\|P^s_{f(x)}r_1\|, \|P^u_{f^2(x)}r_2\|)\|_* < \|(\|P^s_{f(x)}r_1\|, \|P^u_{f(x)}r_1\|)\|_*,$$

and by (6.12) the strict inequality

$$\|P^u_{f(x)}r_1\| < \|(\|P^s_{f(x)}r_1\|, \|P^u_{f(x)}r_1\|)\|_* \tag{6.16}$$

is valid[2].

Now, by definition of the norm $\|\cdot\|_*$, the inequalities (6.15) and (6.16) imply that

$$\|(\|P^s_{f(x)}r_1\|, \|P^u_{f(x)}r_1\|)\|_* < \|(\|P^s_{f(x)}r_1\|, \|P^u_{f(x)}r_1\|)\|_*,$$

which is a contradiction and so lemma is proved. □

To complete the proof of Theorem 6.5 consider two trajectories

$$\boldsymbol{x} = \left\{x_{-n_-}, \ldots, x_0, \ldots, x_{n_+}\right\}, \quad \boldsymbol{y} = \left\{y_{-n_-}, \ldots, y_0, \ldots, y_{n_+}\right\}$$

satisfying $\boldsymbol{x} \subseteq K, \boldsymbol{y} \subseteq X$ and $\|y_n - x_n\| \le \xi$ for $n = -n_-, \ldots, n_+$ and write

$$r_n := x_n - y_n, \quad \nu_n := \|(\|P^s_{f(x_n)}r_n\|, \|P^u_{f(x_n)}r_n\|)\|_* \tag{6.17}$$

[1] The analogous nonstrict inequality is obvious by the definition of the norm $\|\cdot\|_*$.
[2] Again, the analogous nonstrict inequality is obvious by the definition of the norm $\|\cdot\|_*$.

for $n = -n_-, \dots, n_+$. From Lemma 6.7 it then follows that at least one of the inequalities

$$\nu_{-1} \geq \sigma^{-1}\nu_0, \quad \nu_1 \geq \sigma^{-1}\nu_0$$

holds.

For definiteness, suppose that $\nu_{-1} \geq \sigma^{-1}\nu_0$. Then, by Lemma 6.7 again, at least one of the inequalities

$$\nu_{-2} \geq \sigma^{-1}\nu_{-1}, \quad \nu_0 \geq \sigma^{-1}\nu_{-1}$$

holds. But the second of these inequalities can not be valid at the same time as the inequality $\nu_{-1} \geq \sigma^{-1}\nu_0$ because of $\sigma < 1$. Hence, $\nu_{-2} \geq \sigma^{-1}\nu_{-1}$. In the same way it can be shown that

$$\nu_{n-1} \geq \sigma^{-1}\nu_n, \quad \text{for} \quad n = -n_- + 1, \dots, -1, 0,$$

and hence that

$$\nu_n \geq \sigma^{-n}\nu_0, \qquad \text{for} \quad n = -n_- + 1, \dots, -1.$$

Recalling definition (6.17) of the ν_n and using Lemma 6.6 to estimate the norms $\|x_n - y_n\|$ via ν_n, we obtain the inequalities (6.7) with constants c and ξ as in (6.9).

When $\nu_1 \geq \sigma^{-1}\nu_0$, the validity of inequalities (6.6) follows analogously. This completes the proof of Theorem 6.5. □

Remark 6.8. As mentioned above, condition (6.8) is a weakened version of Condition SH2(Lip) of Definition 3.6. We will show here that the inequalities (3.13)–(3.16) of Condition SH2(Lip), namely

$$\|P_y^s\left(f(x+u+v) - f(x+\widetilde{u}+v)\right\| \leq \lambda_s\|u-\widetilde{u}\|, \tag{3.13}$$
$$\|P_y^s\left(f(x+u+v) - f(x+u+\widetilde{v})\right)\| \leq \mu_s\|v-\widetilde{v}\|, \tag{3.14}$$
$$\|P_y^u\left(f(x+u+v) - f(x+\widetilde{u}+v)\right)\| \leq \mu_u\|u-\widetilde{u}\|, \tag{3.15}$$
$$\|P_y^u\left(f(x+u+v) - f(x+u+\widetilde{v})\right)\| \geq \lambda_u\|v-\widetilde{v}\|, \tag{3.16}$$

do indeed imply the inequalities (6.8).

Putting $y = f(x), u = P^s z, v = P^u z, \widetilde{u} = 0$ in (3.13) we obtain

$$\|P_{f(x)}^s\left(f(x+z) - f(x+P^u z)\right)\| \;\leq\; \lambda_s\|P^s z\|,$$

while putting $y = f(x), u = 0, v = P^u z, \widetilde{v} = 0$ in (3.14) we obtain

$$\|P_{f(x)}^s\left(f(x+P^u z) - f(x)\right)\| \;\leq\; \mu_s\|P^u z\|,$$

and the first of inequalities of (6.8) follows immediately.

Similarly, the second of inequalities of (6.8) can be obtained by appropriate choice of elements $u, v, \widetilde{u}$ and $\widetilde{v}$ in (3.15) and (3.16).

6.2 Shadowing

The relationship between the behavior of a given (unperturbed) dynamical system and its perturbed counterparts are also very important for nonsmooth and nonhomeomorphic systems. For example, a computer simulation of a specific dynamical system is really only approximation of the given system due to the finiteness of computer arithmetic and the subsequent round off error: in particular, computed trajectories are only pseudo-trajectories of the original system. This relationship is, however, difficult to express in terms of conjugacy, as it possible for diffeomorphic systems. A convenient and fruitful substitute for a conjugacy relationship in such situation is provided by bi-shadowing, in which trajectories and pseudo-trajectories are compared rather than the mappings themselves.

A typical statement of a problem here can be formulated in the following general, a somewhat vague form: given a dynamical system

$$x_{n+1} = f(x_n)$$

generated by a mapping f on a metric or topological space X, suppose that the function f can be perturbed at any instant n so that another system

$$y_{n+1} = \varphi(n, y_n)$$

is, in fact, observed, where φ is close to f and can possibly also vary with n. Then following Direct Shadowing Problem can be posed: *for every trajectory of every small perturbation φ of a given mapping f does there exists a close trajectory of f?* In admitting of non-autonomous perturbations φ of the mapping f we have a considerably more 'freedom'. The Direct Shadowing Problem is usually formulated in terms of relationships between the true trajectories and pseudo-trajectories of a system f, as in Section 6.2.2.

The Inverse Shadowing Problem is also of interest *an every trajectory of a mapping f be approximated (shadowed) by a trajectory of any perturbation φ of f from a predefined class $\mathscr{T}$?* Here the class of perturbations $\mathscr{T}$ is usually taken to be rather 'small' to obtain meaningful and strong results. For example, the class $\mathscr{T}$ may consist of all continuous autonomous mappings φ or of all (possibly discontinuous) mappings resulting from some approximation problem or numerical method.

The asymmetrical roles of the classes of perturbations $\mathscr{T}$ in each case should be emphasized, with the class $\mathscr{T}$ being as 'fat' as possible in Direct Shadowing and as 'thin' as possible in indirect or Inverse Shadowing. In the direct shadowing of the classical Shadowing Lemma (refer to Chapter 2, Theorem 2.9) this class $\mathscr{T}$ consists of all possible non-autonomous perturbations of the given system, while in inverse shadowing a natural and convenient class $\mathscr{T}$ consists of trajectories of all continuous mappings φ that are sufficiently close to f.

6.2.1 Definitions

Consider a mapping $f : X \mapsto X$ where X is an open bounded subset of $\mathbb{R}^d$ and let $\|\cdot\|$ denote a fixed but otherwise arbitrary norm on $\mathbb{R}^d$.

Definition 6.9. *A* γ-pseudo-trajectory *of a dynamical system generated by a mapping f on X is a sequence $\boldsymbol{y} = \{y_n\} \subset X$ with*

$$\|y_{n+1} - f(y_n)\| \leq \gamma, \qquad \gamma > 0, \tag{6.18}$$

for $n = -n_-, \ldots, 0, \ldots, n_+$ where $n_\pm \leq \infty$. The qualifier finite *may be appended when $n_\pm < \infty$ and* infinite *otherwise.*

Clearly, any trajectory (see Definition 6.1) of a system is its 0-pseudo-trajectory but not every pseudo-trajectory is a trajectory. Pseudo-trajectories arise naturally due to variety of causes such as to the presence of roundoff error in computer calculations of trajectories, though accumulated roundoff error can rapidly destroy any meaningful connection between a computed pseudo-trajectory and an original trajectory. The concept of shadowing provides a form of comparison of trajectories and pseudo-trajectories.

Definition 6.10. *A trajectory $\boldsymbol{x} = \{x_n\}$ is said to* ε-shadow *a γ-pseudo-trajectory $\boldsymbol{y} = \{y_n\}$ on some contiguous (finite or infinite) set $\mathbb{I} \subseteq \mathbb{Z}$ of indices*[3] *if*

$$\|x_n - y_n\| \leq \varepsilon, \qquad n \in \mathbb{I}.$$

The gist of a *Shadowing Theorem* (cf. Theorem 2.9) is that, under certain assumptions on f such as hyperbolicity, for every $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that each δ-pseudo-trajectory is ε-shadowed by a true trajectory. This is what is meant by specifying *direct shadowing.*

6.2.2 Direct Shadowing

The proof of the Shadowing Theorem for hyperbolic diffeomorphisms does not require the full structure of hyperbolicity. The theorem stated and proved below establishes the same result for differentiable semi-hyperbolic mappings. The proof is not only much shorter in the differentiable semi-hyperbolic case than for hyperbolic case, but is more general.

Theorem 6.11. *Let $f : X \mapsto X$ be differentiable and semi-hyperbolic (in the sense of Definition 3.5) on an open bounded set $X \subset \mathbb{R}^d$ with Df_x continuous on $\overline{X}$.*[4] *Then for every sufficiently small $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that every δ-pseudo-trajectory $\boldsymbol{y} = \{y_n\}$ of f is ε-shadowed by a true trajectory $\boldsymbol{x} = \{x_n\}$.*

[3] Without loss of generality that the set $\mathbb{I}$ is one of the follows intervals: $\mathbb{I} = [0, N]$ or $\mathbb{I} = (-\infty, 0]$ or $\mathbb{I} = [0, \infty)$ or $\mathbb{I} = (-\infty, \infty)$.

[4] The continuity of Df_x for $x \in \overline{X}$ is needed only to ensure uniform continuity of Df_x.

Proof. The proof adapts and slightly simplifies those in [15] and [17]. Let $\boldsymbol{y} = \{y_n\}$ be a δ-pseudo-trajectory of f defined over some integer interval $\mathbb{I}$ of indices. Define the nonlinear mapping $\mathscr{F} : \mathscr{X} \mapsto \mathscr{Y}$ by

$$(\mathscr{F}(\boldsymbol{x}))_n = x_{n+1} - f(x_n), \qquad n \in \mathbb{I}, \tag{6.19}$$

where $\boldsymbol{x} = \{x_n\} \in \mathscr{X}$ and spaces $\mathscr{X}$ and $\mathscr{Y}$ are as follows

$$\begin{aligned}
&\mathscr{X} = \ell^\infty([0, N+1], \mathbb{R}^d), && \mathscr{Y} = \ell^\infty([1, N+1], \mathbb{R}^d), && \text{if} \quad \mathbb{I} = [0, N],\\
&\mathscr{X} = \ell^\infty((-\infty, 1], \mathbb{R}^d), && \mathscr{Y} = \ell^\infty((-\infty, 1], \mathbb{R}^d), && \text{if} \quad \mathbb{I} = (-\infty, 0],\\
&\mathscr{X} = \ell^\infty([0, \infty), \mathbb{R}^d), && \mathscr{Y} = \ell^\infty([1, \infty), \mathbb{R}^d), && \text{if} \quad \mathbb{I} = [0, \infty),\\
&\mathscr{X} = \ell^\infty((-\infty, \infty), \mathbb{R}^d), && \mathscr{Y} = \ell^\infty((-\infty, \infty), \mathbb{R}^d), && \text{if} \quad \mathbb{I} = (-\infty, \infty).
\end{aligned}$$

Then $\mathscr{F}$ is C^1 with Fréchet derivative

$$(D\mathscr{F}(\boldsymbol{x})u)_n = u_{n+1} - Df_{x_n} u_n.$$

Let $\omega(t)$ where $t \geq 0$ be the modulus of continuity of Df over $\overline{X}$, that is

$$\omega(t) = \sup\left\{\|Df_y - Df_x\| : \ x, y \in \overline{X}, \ \|y - x\| \leq t\right\},$$

and note that $\omega(t) \to 0+$ as $t \to 0+$ by the continuity of Df_x on the set $\overline{X}$. Define ε_0 to be the largest positive number less than or equal to ε such that $\omega(\varepsilon_0) \leq 1/(2\alpha(\boldsymbol{s}, h))$, where

$$\alpha(\boldsymbol{s}, h) := \frac{h}{\nu(\boldsymbol{s})} = \frac{\lambda_u - \lambda_s + \mu_s + \mu_u}{(1 - \lambda_s)(\lambda_u - 1) - \mu_s \mu_u} h.$$

Here $\nu(\boldsymbol{s})$ is the constant (3.3). Note that the sequence of matrices $\{Df_{x_n}\}$ is semi-hyperbolic on account of the semi-hyperbolicity of the mapping f. The derivative $D\mathscr{F}(x)$ thus has a right inverse $D\mathscr{F}(x)^{-1}$ which satisfies

$$\left\|D\mathscr{F}(x)^{-1}\right\|_\infty \leq \frac{h}{\nu(\boldsymbol{s})} = \alpha(\boldsymbol{s}, h).$$

Choose $\delta = \delta(\varepsilon) = \varepsilon_0/(2\alpha(\boldsymbol{s}, h))$. Now, if $\boldsymbol{y}$ is the δ-pseudo-trajectory sequence and $\boldsymbol{x}$ is any $\ell^\infty(\mathbb{I}, \mathbb{R}^d)$ sequence satisfying $\|\boldsymbol{x} - \boldsymbol{y}\|_\infty \leq \varepsilon_0$, then

$$\begin{aligned}
\|D\mathscr{F}(\boldsymbol{x}) - D\mathscr{F}(\boldsymbol{y})\|_\infty &\leq \sup_n \|Df_{x_n} - Df_{y_n}\| \\
&\leq \omega(\varepsilon_0) \leq \frac{1}{2\alpha(\boldsymbol{s}, h)} \leq \frac{1}{2\left\|D\mathscr{F}(\boldsymbol{y})^{-1}\right\|_\infty}.
\end{aligned}$$

To complete the proof we need the following fixed point lemma due to Chow, Lin and Palmer [15].

Lemma 6.12. *Let $\mathscr{X}, \mathscr{Y}$ be Banach spaces*[5] *and $\mathscr{F} : \mathscr{X} \mapsto \mathscr{Y}$ be a C^1 map. Let $y \in \mathscr{X}$ be a point such that $D\mathscr{F}(y)^{-1}$ is a bounded linear right inverse of $D\mathscr{F}(y)$ and let $\varepsilon_0 > 0$ be chosen so that*

$$\|D\mathscr{F}(x) - D\mathscr{F}(y)\| \leq \frac{1}{2\|D\mathscr{F}(y)^{-1}\|} \tag{6.20}$$

for $\|x - y\| \leq \varepsilon_0$. If $0 < \varepsilon \leq \varepsilon_0$ and

$$\|\mathscr{F}(y)\| \leq \frac{\varepsilon}{2\|D\mathscr{F}(y)^{-1}\|}$$

then the equation $F(x) = 0$ has a unique solution x such that $\|x - y\| \leq \varepsilon$.

Proof. We write

$$\mathscr{F}(x) = \mathscr{F}(y) + D\mathscr{F}(y)(x - y) + \eta(x).$$

For $\|x_1 - y\|, \|x_1 - y\| \leq \varepsilon_0$ by (6.20) we have

$$\begin{aligned} \|\eta(x_1) - \eta(x_2)\| &= \|\mathscr{F}(x_1) - \mathscr{F}(x_2) - D\mathscr{F}(y)(x_1 - x_2)\| \\ &\leq \left\| \int_0^1 (D\mathscr{F}(x_2 + \theta(x_1 - x_2)) - D\mathscr{F}(y))\, \mathrm{d}\theta \right\| \cdot \|x_1 - x_2\| \\ &\leq \frac{\|x_1 - x_2\|}{2\|D\mathscr{F}(y)^{-1}\|}. \end{aligned} \tag{6.21}$$

We can rewrite the equation $\mathscr{F}(x) = 0$ as

$$x = y - D\mathscr{F}(y)^{-1}\{\mathscr{F}(y) + \eta(x)\} := T(x).$$

For $0 < \varepsilon \leq \varepsilon_0$, we define $B(\varepsilon, y) = \{x \in \mathscr{X} : \|x - y\| \leq \varepsilon\}$ and show that T is a contraction on $B(\varepsilon, y)$. The lemma will then follow immediately from the Banach Contraction Mapping Theorem.

Note first that if $x \in B(\varepsilon, y)$ then

$$\begin{aligned} \|T(x) - y\| &= \|D\mathscr{F}(y)^{-1}\{\mathscr{F}(y) + \eta(x)\}\| \\ &\leq \left\|D\mathscr{F}(y)^{-1}\right\| \left\{ \frac{\varepsilon}{2\|D\mathscr{F}(y)^{-1}\|} + \frac{\|x_1 - x_2\|}{2\|D\mathscr{F}(y)^{-1}\|} \right\} \\ &= \frac{\varepsilon}{2} + \frac{\|x_1 - x_2\|}{2} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

where we have used (6.20) and (6.21) with $x_1 = x, x_2 = y$. Hence T maps $B(\varepsilon, y)$ into itself. Moreover if $x_1, x_2 \in B(\varepsilon, y)$ then, using (6.21),

[5] The norms in $\mathscr{X}$ and $\mathscr{Y}$ for the simplicity will be denoted by the same symbol $\|\cdot\|$.

$$\|T(x_1) - T(x_2)\| = \|D\mathscr{F}(y)^{-1} \{\eta(x_1) - \eta(x_2)\}\| \\ \leq \|D\mathscr{F}(y)^{-1}\| \frac{\|x_1 - x_2\|}{2\|D\mathscr{F}(y)^{-1}\|} = \frac{\|x_1 - x_2\|}{2}.$$

Thus T is indeed a contraction on $B(\varepsilon, y)$ and the proof of Lemma 6.12 is completed. □

We now apply Lemma 6.12 to the mapping $\mathscr{F} : \mathscr{X} \mapsto \mathscr{Y}$ defined in (6.19), with the Banach spaces $\mathscr{X}$ and $\mathscr{Y}$ introduced above and endowed with the norm $\|\cdot\|_\infty$. Then the equation $\mathscr{F}(\boldsymbol{x}) = 0$ has a unique solution $\boldsymbol{x} \in \mathscr{X}$ satisfying $\|\boldsymbol{x} - \boldsymbol{y}\|_\infty \leq \varepsilon$. That is, the δ-pseudo-trajectory $\boldsymbol{y} = \{y_n\}$ is ε-shadowed by the true trajectory $\boldsymbol{x} = \{x_n\}$. This completes the proof of Theorem 6.11. □

6.2.3 Inverse Shadowing

As a distance between the mappings $f, \varphi : \mathbb{R}^d \mapsto \mathbb{R}^d$ consider the semi-norm[6]

$$\|f - \varphi\|_\infty = \sup_{x \in \mathbb{R}^d} \|f(x) - \varphi(x)\|.$$

Definition 6.13. *A finite trajectory $\boldsymbol{x} = \{x_0, x_1, \ldots, x_N\}$ of a mapping f is called α-*robust *for some positive number α if there exists an $\varepsilon_0 > 0$ such that any continuous mapping $\varphi : \mathbb{R}^d \mapsto \mathbb{R}^d$ satisfying*

$$\|f - \varphi\|_\infty \leq \varepsilon_0$$

has at least one trajectory $\boldsymbol{y} = \{y_0, y_1, \ldots, y_N\}$ such that

$$\|y_n - x_n\| \leq \alpha \|f - \varphi\|_\infty, \qquad n = 0, 1, \ldots, N. \tag{6.22}$$

This is a form of *inverse shadowing* where the reference class $\mathscr{T}$ is the space of all continuous mappings on X.

Remark 6.14. The key condition in Definition 6.13 is the existence of α independent of the length of the given trajectory. The estimates (6.22) can be obtained easily if α is allowed to depend on the length of the trajectory. For example, any trajectory $\boldsymbol{x}$ is $(1 + L + \cdots + L^N)$–robust if the mapping f is Lipschitz with Lipschitz constant L in a neighborhood of the trajectory $\boldsymbol{x}$. Indeed, in this case, for a given trajectory $\boldsymbol{x}$ of the mapping f and for a trajectory $\boldsymbol{y}$ of a mapping φ satisfying $y_0 = x_0$ we have

$$\|y_{n+1} - y_{n+1}\| = \|\varphi(y_n) - f(x_n)\| \\ \leq \|\varphi(y_n) - f(y_n)\| + \|f(y_n) - f(x_n)\| \\ \leq \|f - \varphi\|_\infty + L\|y_n - x_n\|$$

[6] It not a norm because it may take infinite values.

from which immediately follows the estimate

$$\|y_n - x_n\| \leq (1 + L + \cdots + L^n)\|f - \varphi\|_\infty, \qquad n = 1, 2, \ldots, N.$$

Thus, any trajectory $\boldsymbol{x} = \{x_0, x_1, \ldots, x_N\}$ of the mapping f is α_N-robust with $\alpha_N = (1 + L + \cdots + L^N)$.

As the next theorem shows, semi-hyperbolicity allows the robustness constant α to be chosen independently of the particular trajectory and its length N, uniformly throughout the domain of semi-hyperbolicity X.

Theorem 6.15. *Let $f : X \mapsto X$ be differentiable and $(\boldsymbol{s}, h)$-semi-hyperbolic on an open set $X \subseteq \mathbb{R}^d$. Then every finite trajectory $\boldsymbol{x} \subset X$ is α-robust for any*

$$\alpha > \alpha(\boldsymbol{s}, h) := \frac{\lambda_u - \lambda_s + \mu_s + \mu_u}{(1 - \lambda_s)(\lambda_u - 1) - \mu_s \mu_u} h, \tag{6.23}$$

where $\lambda_u, \lambda_s, \mu_u, \mu_s$ are the parameters of the split $\boldsymbol{s} = (\lambda_u, \lambda_s, \mu_u, \mu_s)$.

Proof. Denote by $\mathscr{B}$ the space of $(N+1)$-sequences

$$\boldsymbol{z} = \{z_0, z_1, \ldots, z_N\}, \qquad z_n \in \mathbb{R}^d, \quad n = 0, 1, \ldots, N,$$

satisfying boundary conditions

$$P^s_{x_0} z_0 = P^u_{x_N} z_N = 0. \tag{6.24}$$

The set $\mathscr{B}$ can be regarded as a subspace of the $(N+1)d$-dimensional vector space $\mathbb{R}^d \times \ldots \times \mathbb{R}^d$ $(N+1$ times) with the norm

$$\|\boldsymbol{z}\|_\infty = \max_{0 \leq n \leq N} \|z_n\|.$$

Let $\varphi : \mathbb{R}^d \mapsto \mathbb{R}^d$ be a given mapping and $\boldsymbol{x} = \{x_0, x_1, \ldots, x_N\}$ be a given trajectory of the mapping f. Introduce the linear operator $U_n : E^u_{x_{n-1}} \mapsto E^u_{x_n}$ defined by $U_n v = P^u_{x_n} Df_{x_{n-1}} v$. This operator is surjective on account of inequality (3.12) from Definition 3.5 and so is invertible, i.e. U_n^{-1} is well defined.

Consider now an operator $\mathscr{H} : \mathscr{B} \mapsto \mathscr{B}$, which transforms every sequence $\boldsymbol{z} = \{z_0, z_1, \ldots, z_N\}$ into a sequence $\boldsymbol{w} = \{w_0, w_1, \ldots, w_N\}$ defined by the boundary conditions

$$P^s_{x_0} w_0 = P^u_{x_N} w_N = 0$$

(see (6.24)) and by the relations

$$\begin{aligned} P^s_{x_n} w_n &= P^s_{x_n}(\varphi(x_{n-1} + z_{n-1}) - x_n), \\ P^u_{x_{n-1}} w_{n-1} &= U_n^{-1}(P^u_{x_n} z_n - P^u_{x_n} Df_{x_{n-1}} P^s_{x_{n-1}} z_{n-1} + \\ &\quad + P^u_{x_n}(-\varphi(x_{n-1} + z_{n-1}) + x_n + Df_{x_{n-1}} z_{n-1})), \end{aligned}$$

for $n = 1, 2, \ldots, N$. The necessity of introducing the operator $\mathscr{H}$ is explained by the next lemma, proof of which is obvious.

Lemma 6.16. *The operator $\mathscr{H}$ is continuous, and for any of its fixed point $\boldsymbol{z} = \{z_0, z_1, \dots, z_N\}$ the sequence*

$$\boldsymbol{y} = \{x_0 + z_0,\ x_1 + z_1,\ \dots,\ x_N + z_N\}$$

is a trajectory of the mapping φ.

We require some more notation and definitions. Given $\beta > 0$, denote by $\delta_\beta(\varepsilon)$ the largest positive value δ such that

$$\|x_n + Df_{x_{n-1}}z - f(x_{n-1} + z)\| \le \beta\varepsilon, \qquad n = 1, 2, \dots, N,$$

for any $\|z\| \le \delta$. For each $\boldsymbol{z} \in \mathscr{B}$ define the pair of real numbers

$$m^s(\boldsymbol{z}) = \max_{0\le n\le N} \|P^s_{x_n} z_n\|, \quad m^u(\boldsymbol{z}) = \max_{0\le n\le N} \|P^u_{x_n} z_n\|,$$

and denote by $\boldsymbol{m}(\boldsymbol{z})$ the 2-dimensional column vector with components $m^s(\boldsymbol{z})$ and $m^u(\boldsymbol{z})$. Let $M = M(\boldsymbol{s})$ be the split matrix (4.2) and define the column vector

$$\boldsymbol{h} = h(1, \lambda_u^{-1})^T.$$

Lemma 6.17. *Let $\beta > 0$. Then the coordinate-wise inequality*

$$\boldsymbol{m}(\mathscr{H}(\boldsymbol{z})) \le M\boldsymbol{m}(\boldsymbol{z}) + (1+\beta)\|f - \varphi\|_\infty \boldsymbol{h}$$

is valid for every continuous mapping φ and every $\boldsymbol{z}$ from the set

$$\mathscr{W} = \{\boldsymbol{z} \in \mathscr{B} : \|\boldsymbol{z}\|_\infty \le \delta_\beta(\|f - \varphi\|_\infty)\}.$$

Proof. We estimate first the value of $m^s(\mathscr{H}(\boldsymbol{z}))$. By definition

$$m^s(\mathscr{H}(\boldsymbol{z})) = \max_{0\le n\le N} \|v^s_n\|, \tag{6.25}$$

where

$$v^s_n = P^s_{x_n}\left(\varphi(x_{n-1} + z_{n-1}) - x_n\right).$$

Rewrite this as

$$v^s_n = I_1 + I_2 + I_3 + I_4, \tag{6.26}$$

where

$$\begin{aligned}
I_1 &= P^s_{x_n} Df_{x_{n-1}} P^s_{x_{n-1}} z_{n-1},\\
I_2 &= P^s_{x_n} Df_{x_{n-1}} P^u_{x_{n-1}} z_{n-1},\\
I_3 &= P^s_{x_n}(\varphi(x_{n-1} + z_{n-1}) - f(x_{n-1} + z_{n-1})),\\
I_4 &= P^s_{x_n}(f(x_{n-1} + z_{n-1}) - (f(x_{n-1}) + Df_{x_{n-1}} z_{n-1})).
\end{aligned}$$

From (3.9),

$$\|I_1\| \le \lambda_s \|P^s_{x_{n-1}} z_{n-1}\|, \tag{6.27}$$

while from (3.10),

$$\|I_2\| \leq \mu_s \|P^u_{x_{n-1}} z_{n-1}\|.$$

Condition SH1(Lip) of Definition 3.6 implies that

$$\|I_3\| \leq h\|f - \varphi\|_\infty.$$

Finally, Condition SH1(Lip) of Definition 3.6 and the definition of the function $\delta_\beta(\varepsilon)$ imply that

$$\|I_4\| \leq h\beta\|f - \varphi\|_\infty. \tag{6.28}$$

From (6.26) and (6.27)–(6.28) it then follows that

$$\|v^s_n\| \leq \lambda_s \|P^s_{x_{n-1}} z_{n-1}\| + \mu_s \|P^u_{x_{n-1}} z_{n-1}\| + (1+\beta)\|f - \varphi\|_\infty h,$$

and by (6.25)

$$m^s(\mathscr{H}(\boldsymbol{z})) \leq \lambda_s m^s(\boldsymbol{z}) + \mu_s m^s(\boldsymbol{z}) + (1+\beta)\|f - \varphi\|_\infty h. \tag{6.29}$$

We now estimate the value of $m^u(\mathscr{H}(\boldsymbol{z}))$. By definition,

$$m^u(\mathscr{H}(\boldsymbol{z})) = \max_{0 \leq n \leq N} \|v^u_n\|, \tag{6.30}$$

where

$$\begin{aligned} v^u_{n-1} = U_n^{-1}(&P^u_{x_n} z_n - P^u_{x_n} Df_{x_{n-1}} P^s_{x_{n-1}} z_{n-1} \\ &+ \ P^u_{x_n}(-\varphi(x_{n-1} + z_{n-1}) + f(x_{n-1}) + Df_{x_{n-1}} z_{n-1})). \end{aligned}$$

Rewrite this as

$$v^u_{n-1} = U_n^{-1}(J_1 + J_2 + J_3 + J_4), \tag{6.31}$$

where

$$J_1 = P^u_{x_n} z_n, \tag{6.32}$$

$$J_2 = -P^u_{x_n} Df_{x_{n-1}} P^s_{x_{n-1}} z_{n-1}, \tag{6.33}$$

$$J_3 = P^u_{x_n}(-\varphi(x_{n-1} + z_{n-1}) + f(x_{n-1} + z_{n-1}), \tag{6.34}$$

$$J_4 = -f(x_{n-1} + z_{n-1}) + (f(x_{n-1}) + Df_{x_{n-1}} z_{n-1}). \tag{6.35}$$

The relations (3.12) and (6.32) imply that

$$\|U_n^{-1} J_1\| \leq \lambda_u^{-1} \|P^u_{x_n} z_n\|, \tag{6.36}$$

while (3.11), (3.12) and (6.33) imply that

$$\|U_n^{-1} J_2\| \leq \lambda_u^{-1} \mu_u \|P^s_{x_{n-1}} z_{n-1}\|.$$

In addition, (3.12), (6.34) and Condition SH1(Lip) from Definition 3.6 give

$$\|U_n^{-1} J_3\| \leq \lambda_u^{-1} h\|f - \varphi\|_\infty.$$

Finally, (3.12), (6.35), Condition SH1(Lip) from Definition 3.6 and the definition of the function $\delta_\beta(\varepsilon)$ imply that

$$\|U_n^{-1} J_4\| \leq \lambda_u^{-1} h\beta \|f - \varphi\|_\infty. \tag{6.37}$$

From (6.31) and (6.36)–(6.37) it thus follows that

$$\|v_{n-1}^u\| \leq \lambda_u^{-1} \left(\|P_{x_n}^u z_n\| + \mu_u \|P_{x_{n-1}}^s z_{n-1}\| + (1+\beta)\|f - \varphi\|_\infty h \right),$$

so by (6.30)

$$m^u(\mathscr{H}(\boldsymbol{z})) \leq \lambda_u^{-1} \left(m^u(\boldsymbol{z}) + \mu_u m^s(\boldsymbol{z}) + (1+\beta)\|f - \varphi\|_\infty h \right). \tag{6.38}$$

Inequalities (6.29) and (6.38) are equivalent to the assertion of the lemma. □

Let us now return to the proof of Theorem 6.15. Without loss of generality we can assume that the split $\boldsymbol{s} = (\lambda_s, \lambda_u, \mu_s, \mu_u)$ is positive, i.e. all of the numbers $\lambda_s, \lambda_u, \mu_s, \mu_u$ are positive. We recall from Section 4.1 that $\sigma(M) < 1$ is the spectral radius (4.3) of the split matrix M and that $\|\cdot\|_*$ is the auxiliary norm on $\mathbb{R}^2$ defined for a given positive split $\boldsymbol{s}$ by

$$\|(y_1, y_2)\|_* := \max\{\gamma|y_1|, |y_2|\}.$$

with $\gamma = \gamma(\boldsymbol{s}) > 0$ given by (4.5).

Choose any fixed number $\alpha > \alpha(\boldsymbol{s}, h)$, with $\alpha(\boldsymbol{s}, h)$ defined by (6.23), write

$$\beta = \frac{\alpha}{\alpha(\boldsymbol{s}, h)} - 1,$$

and consider the convex set

$$\mathscr{V} = \left\{ \boldsymbol{z} : \ \|\boldsymbol{m}(\boldsymbol{z})\|_* \leq \frac{1+\beta}{1-\sigma(M)} \|\boldsymbol{h}\|_* \|f - \varphi\|_\infty \right\},$$

which is well defined since $\sigma(M) < 1$.

Lemma 6.18. *There exists an $\varepsilon_0 > 0$ such that for*

$$\|f - \varphi\|_\infty \leq \varepsilon_0 \tag{6.39}$$

the inclusion $\mathscr{V} \subseteq \mathscr{W}$ is valid.

Proof. We show first that

$$\mathscr{V} \subseteq \mathscr{V}^+, \tag{6.40}$$

where

$$\mathscr{V}^+ = \left\{ \boldsymbol{z} : \ \|\boldsymbol{z}\|_\infty \leq 2 \frac{1+\beta}{1-\sigma(M)} \max\{\gamma^{-1}, 1\} \|\boldsymbol{h}\|_* \|f - \varphi\|_\infty \right\}.$$

Indeed, by definition

$$\|\boldsymbol{m}(\boldsymbol{z})\|_* = \max\left\{\gamma \max_{0\le n\le N}\|P^s_{x_n}z_n\|,\ \max_{0\le n\le N}\|P^u_{x_n}z_n\|\right\}$$
$$= \max_{0\le n\le N}\ \max\left\{\gamma\|P^s_{x_n}z_n\|,\ \|P^u_{x_n}z_n\|\right\}$$
$$= \max_{0\le n\le N}\ \|(\|P^s_{x_n}z_n\|, \|P^u_{x_n}z_n\|)\|_*,$$

and then by Lemma 6.6

$$\|\boldsymbol{m}(\boldsymbol{z})\|_* \ge \frac{1}{2}\min\{\gamma,1\}\max_{0\le n\le N}\|z_n\| = \frac{1}{2}\min\{\gamma,1\}\|\boldsymbol{z}\|_\infty.$$

The inclusion (6.40) then follows from the last inequality.

Now, by definition of the differential of a mapping,

$$\liminf_{\varepsilon\to 0+}\frac{\delta_\beta(\varepsilon)}{\varepsilon} = \infty$$

and thus a number $\varepsilon_0 > 0$ can be chosen such that

$$2\frac{1+\beta}{1-\sigma(M)}\max\{\gamma^{-1},1\}\|\boldsymbol{h}\|_*\varepsilon \le \delta_\beta(\varepsilon) \qquad \text{for} \quad \varepsilon \le \varepsilon_0.$$

Hence, for f and φ satisfying (6.39) the inclusion $\mathscr{V}^+ \subseteq \mathscr{W}$ holds, and thus by (6.40), $\mathscr{V} \subseteq \mathscr{W}$. The lemma is proved. □

Let us now complete the proof of Theorem 6.15. Choose $\varepsilon_0 > 0$ as in Lemma 6.18. Then $\mathscr{V} \subseteq \mathscr{W}$ and for any vector $\boldsymbol{z} \in \mathscr{V}$ by Lemma 6.17 the coordinate-wise inequality

$$\boldsymbol{m}(\mathscr{H}(\boldsymbol{z})) \le M\boldsymbol{m}(\boldsymbol{z}) + (1+\beta)\|f-\varphi\|_\infty\boldsymbol{h}$$

is valid. Hence, in view of positivity of the components of the matrix M,

$$\|\boldsymbol{m}(\mathscr{H}(\boldsymbol{z}))\|_* \le \|M\boldsymbol{m}(\boldsymbol{z})\|_* + (1+\beta)\|f-\varphi\|_\infty\|\boldsymbol{h}\|_*.$$

Recall from Section 4.1 that the corresponding matrix norm $\|M\|_*$ of the matrix M coincides with the spectral radius $\sigma(M) < 1$ of M and that $\|My\|_* \le \sigma(M)\|y\|_*$ for all $y \in \mathbb{R}^2$. Hence

$$\|\boldsymbol{m}(\mathscr{H}(\boldsymbol{z}))\|_* \le \sigma(M)\|\boldsymbol{m}(\boldsymbol{z})\|_* + (1+\beta)\|f-\varphi\|_\infty\|\boldsymbol{h}\|_*$$

and by definition of the set $\mathscr{V}$ we obtain that

$$\mathscr{H}(\boldsymbol{z}) \in \mathscr{V} \qquad \text{for} \quad \boldsymbol{z} \in \mathscr{V},$$

that is, the set $\mathscr{V}$ is invariant under the operator $\mathscr{H}$. Then, from continuity of $\mathscr{H}$ (see Lemma 6.16) and Schauder Fixed Point Theorem, there exists a point $\boldsymbol{z}^*$ satisfying $\mathscr{H}(\boldsymbol{z}^*) = \boldsymbol{z}^*$ such that

$$\boldsymbol{z}^* \in \mathscr{V} \subseteq \mathscr{W}.$$

This inclusion and Lemma 6.17 imply that

$$\boldsymbol{m}(\boldsymbol{z}^*) = \boldsymbol{m}(\mathscr{H}(\boldsymbol{z}^*)) \leq M\boldsymbol{m}(\boldsymbol{z}^*) + (1+\beta)\|f - \varphi\|_\infty \boldsymbol{h}.$$

Hence, by positivity of the components of the matrix M, we have

$$\boldsymbol{m}(\boldsymbol{z}^*) \leq (1+\beta)\|f-\varphi\|_\infty (I-M)^{-1}\boldsymbol{h} = \frac{\alpha\|f-\varphi\|_\infty}{\alpha(\boldsymbol{s},h)}(I-M)^{-1}\boldsymbol{h},$$

from which

$$\max_{0\leq n\leq N} \|\boldsymbol{z}_n^*\| \leq m^s(\boldsymbol{z}^*) + m^u(\boldsymbol{z}^*) \leq \alpha\|f-\varphi\|_\infty$$

(see the proof of Lemma 4.2 for additional details). By Lemma 6.16 and by the last estimate, for any continuous mapping φ satisfying $\|f-\varphi\|_\infty \leq \varepsilon_0$ the sequence $\{x_0 + z_0^*, x_1 + z_0^*, \ldots, x_N + z_N^*\}$ is a finite trajectory of φ and satisfies (6.22). That is, the trajectory $\boldsymbol{x} = \{x_0, x_1, \ldots, x_N\}$ is α-robust and Theorem 6.15 is proved. □

6.3 Bi-Shadowing

The concept of ε-shadowing as in Definition 6.10 and α-robustness as in Definition 6.13 allow us to obtain useful information about perturbations of a given system via Theorems 6.11 and 6.15. Their practical utility is somewhat limited by their existential rather than constructive character. A unifying approach for investigating both direct and inverse shadowing is provided by the concept of *bi-shadowing* which has the added advantage of providing explicit values of the parameters involved.

Let $\mathrm{Tr}(f,K,\gamma)$ denote the totality of finite or infinite γ-pseudo-trajectories (6.18) of f belonging entirely to the subset $K \subseteq X$. Since a true trajectory can be regarded as a γ-pseudo-trajectory for any $\gamma \geq 0$, in particular with $\gamma = 0$ itself, it makes sense to denote the corresponding set of true trajectories by $\mathrm{Tr}(f,K,0)$ or simply by $\mathrm{Tr}(f,K)$. Obviously $\mathrm{Tr}(f,K) \subset \mathrm{Tr}(f,K,\gamma)$, with strict inclusion as there are γ-pseudo-trajectories which are not trajectories.

As a distance between mappings φ and f on X we consider the semi-norm

$$\|\varphi - f\|_\infty = \sup_{x\in X} \|\varphi(x) - f(x)\|.$$

Definition 6.19. *A dynamical system generated by a mapping $f : X \mapsto X$ is said to be* bi-shadowing *on a subset K of X with positive parameters α and β if for any given finite pseudo-trajectory $\boldsymbol{x} = \{x_n\} \in \mathrm{Tr}(f,K,\gamma)$ with $0 \leq \gamma \leq \beta$ and any mapping $\varphi : X \mapsto X$ satisfying*

$$\|\varphi - f\|_\infty \leq \beta - \gamma \tag{6.41}$$

there exists a trajectory $\boldsymbol{y} = \{y_n\} \in \mathrm{Tr}(\varphi, X)$ *such that*

$$\|x_n - y_n\| \leq \alpha(\gamma + \|\varphi - f\|_\infty) \tag{6.42}$$

for all n *for which* $\boldsymbol{y}$ *is defined.*

Remark 6.20. Bi-shadowing conceptualizes the robustness of observed dynamical behavior of a dynamical system and its perturbations such as from computer simulations. It can also be interpreted as a form of dynamical structural stability when restricted to specific classes of mappings, such as continuous mappings. Moreover, it implies both the direct shadowing and the inverse shadowing properties (in the form of α-robustness) discussed above: taking $\varphi \equiv f$ in (6.41) and (6.42) gives $\alpha\gamma$-shadowing of any γ-pseudo-trajectory $\boldsymbol{x} \in \mathrm{Tr}(f, K, \gamma)$ by a true trajectory $\boldsymbol{y} \in \mathrm{Tr}(f, K)$, while α-robustness of any trajectory $\boldsymbol{x}$ of f follows because a trajectory $\boldsymbol{y} \in \mathrm{Tr}(\varphi, X)$ can always be found which $\alpha\|\varphi - f\|_\infty$-shadows a given true trajectory $\boldsymbol{x} \in \mathrm{Tr}(f, K)$, considered here as the γ-pseudo-trajectory with $\gamma = 0$.

6.3.1 Bi-Shadowing of Finite Trajectories

The main result of this section is that semi-hyperbolicity is sufficient to ensure bi-shadowing of a dynamical system generated by a Lipschitz mapping with respect to a class of perturbed systems generated by continuous mappings. It not only generalizes existing variants of the Shadowing Lemma to a far broader class of systems, but includes inverse as well as direct shadowing and provides explicit values of the shadowing parameters.

Theorem 6.21. *Let* $f : X \mapsto X$ *be a Lipschitz mapping which is semi-hyperbolic on a compact subset* $K \subset X$ *with a split* $\boldsymbol{s}$ *and constants* h, δ *(see Definition 3.6). Then it is bi-shadowing on* K *with respect to continuous mappings* $\varphi : X \mapsto X$ *with parameters* $\alpha = \alpha(\boldsymbol{s}, h)$ *and* $\beta = \beta(\boldsymbol{s}, h, \delta)$ *defined by*

$$\alpha(\boldsymbol{s}, h) := h \frac{\lambda_u - \lambda_s + \mu_s + \mu_u}{(1 - \lambda_s)(\lambda_u - 1) - \mu_s \mu_u} \tag{6.43}$$

and

$$\beta(\boldsymbol{s}, h, \delta) := \delta h^{-1} \frac{(1 - \lambda_s)(\lambda_u - 1) - \mu_s \mu_u}{\max\{\lambda_u - 1 + \mu_s, 1 - \lambda_s + \mu_u\}}. \tag{6.44}$$

Proof. We begin with an auxiliary fact. Denote by $B_x^u(r)$ the closed ball of the radius r centered at 0 in the linear space E_x^u. For each $x, y \in K$ with $\|f(x) - y\| \leq \delta$ and each $z \in \mathbb{R}^d$ satisfying $\|P_x^s z\| \leq \delta$ introduce the mapping $F_{x,y,z} : B_x^u(\delta) \mapsto E_y^u$ by the equality

$$F_{x,y,z}(v) = P_y^u \left(f(x + P_x^s z + v) - f(x + P_x^s z) \right).$$

Lemma 6.22. *Let* $0 \leq r \leq \delta$*. Then* $B_y^u(\lambda_u r) \subseteq F_{x,y,z}\left(B_x^u(r)\right)$*.*

Proof. The assertion of lemma is evident for $r = 0$, so consider the case $r > 0$. Denote by $\partial B^u_x(r)$ and $\operatorname{Int} B^u_x(r)$ the boundary and the interior of the ball $B^u_x(r)$. Clearly,

$$F_{x,y,z}(0) = P^u_y(f(x + P^s_x z) - f(x + P^s_x z)) = 0 \in \operatorname{Int} B^u_y(\lambda_u r). \qquad (6.45)$$

On the other hand, by inequality (3.16)

$$\|F_{x,y,z}(v) - F_{x,y,z}(\widetilde{v})\| \geq \lambda_u \|v - \widetilde{v}\| \qquad \text{for} \quad v, \widetilde{v} \in E^u_x,\ \|v\|, \|\widetilde{v}\| \leq \delta. \quad (6.46)$$

In particular, from (6.46) we get

$$\|F_{x,y,z}(v)\| \geq \lambda_u r \qquad \text{for} \quad v \in E^u_x,\ \|v\| = r \leq \delta,$$

which shows that

$$F_{x,y,z}(\partial B^u_x(r)) \cap \operatorname{Int} B^u_y(\lambda_u r) = \emptyset. \qquad (6.47)$$

By Condition SH0(Lip) of Definition 3.6 dimensions of linear spaces E^u_x and E^u_y coincide for any $x, y \in K$ with $\|f(x) - y\| \leq \delta$. Therefore, by (6.46), the mapping $F_{x,y,z}(v)$ is homeomorphic on the ball $B^u_x(r)$ with $0 < r \leq \delta$. Hence by the Principle of Domain Invariance [1, p. 396] the image of the boundary $\partial B^u_x(r)$ of the ball $B^u_x(r)$ under the map $F_{x,y,z}(v)$ coincides with the boundary $\partial F_{x,y,z}(B^u_x(r))$ of the set $F_{x,y,z}(B^u_x(r))$, i.e.

$$F_{x,y,z}(\partial B^u_x(r)) = \partial F_{x,y,z}(B^u_x(r)).$$

By (6.47) the latter equality implies

$$\partial F_{x,y,z}(B^u_x(r)) \cap \operatorname{Int} B^u_y(\lambda_u r) = \emptyset.$$

This, together with (6.45), implies the required inclusion

$$B^u_y(\lambda_u r) \subseteq F_{x,y,z}(B^u_x(r))$$

and the lemma is proved. □

As an immediate consequence of Lemma 6.22 and inequality (3.16) we have

Lemma 6.23. *Let $x, y \in K$ and $\|f(x) - y\| \leq \delta$. Then the operator $Q_{x,y,z} = F^{-1}_{x,y,z}$ is well defined and continuous on $B^u_y(\lambda_u \delta)$ and satisfies the estimate $\|Q_{x,y,z}(v)\| \leq \lambda_u^{-1} \|v\|$.*

Denote, as in the proof of Theorem 6.11, by $\mathscr{B}$ the space of $(N+1)$-sequences

$$\boldsymbol{z} = \{z_0, z_1, \ldots, z_N\}, \qquad z_n \in \mathbb{R}^d, \quad n = 0, 1, \ldots, N.$$

The set $\mathscr{B}$ can be regarded as an $(N+1)d$-dimensional vector space $\mathbb{R}^d \times \cdots \times \mathbb{R}^d$ ($N+1$ times), with the norm

$$\|\boldsymbol{z}\|_\infty = \max_{0\le n\le N} \|z_n\|.$$

Let $\boldsymbol{x} = \{x_0, x_1, \ldots, x_N\}$ be a given γ-pseudo-trajectory of the system f and let φ be a given continuous mapping such that

$$\beta = \gamma + \|f-\varphi\|_\infty \tag{6.48}$$

satisfies the inequality

$$\beta \le \beta(\boldsymbol{s}, h, \delta) \tag{6.49}$$

with $\beta(\boldsymbol{s}, h, \delta)$ as in (6.44).

Introduce an operator $\mathscr{H} : \mathscr{B} \mapsto \mathscr{B}$ which transforms the sequence $\boldsymbol{z} = \{z_0, z_1, \ldots, z_N\} \in \mathscr{B}$ into the sequence $\boldsymbol{w} = \{w_0, w_1, \ldots, w_N\} \in \mathscr{B}$ defined by the relations

$$P^s_{x_0} w_0 = 0 \tag{6.50}$$

and

$$P^s_{x_n} w_n = P^s_{x_n}(\varphi(x_{n-1} + z_{n-1}) - x_n) \tag{6.51}$$

for $n = 1, 2, \ldots, N$, and by the relations

$$P^u_{x_N} w_N = 0 \tag{6.52}$$

and

$$\begin{aligned} P^u_{x_{n-1}} w_{n-1} = Q_{x_{n-1},x_n,z_{n-1}}(P^u_{x_n}(&-\varphi(x_{n-1}+z_{n-1}) \\ &+ f(x_{n-1}+z_{n-1}) + x_n \\ &- f(x_{n-1} + P^s_{x_{n-1}} z_{n-1}) + z_n)) \end{aligned} \tag{6.53}$$

for $n = 1, 2, \ldots, N$.

Define an auxiliary parameters a and b as the coordinates of the two-dimensional vector

$$(a, b)^T = (I - M(\boldsymbol{s}))^{-1}\boldsymbol{h}, \qquad \text{with} \quad \boldsymbol{h} = h(1, \lambda_u^{-1})^T, \tag{6.54}$$

where $M(\boldsymbol{s})$ is the 2×2 split matrix given by (4.2). Explicitly, we have

$$a = \frac{\lambda_u - 1 + \mu_s}{(1-\lambda_s)(\lambda_u - 1) - \mu_s\mu_u} h, \tag{6.55}$$

$$b = \frac{1 - \lambda_s + \mu_u}{(1-\lambda_s)(\lambda_u - 1) - \mu_s\mu_u} h, \tag{6.56}$$

and thus

$$\alpha(\boldsymbol{s}, h) = a + b, \quad \beta(\boldsymbol{s}, h, \delta) = \delta \min\left\{a^{-1}, b^{-1}\right\}. \tag{6.57}$$

Consider the set

$$\mathscr{S}(\beta) = \{\boldsymbol{z} \in \mathscr{B} : \ \|P^s_{x_n} z_n\| \le a\beta, \ \|P^u_{x_n} z_n\| \le b\beta, \ 0 \le n \le N\}. \tag{6.58}$$

By (6.49) and (6.57)

$$a\beta \le a\beta(\boldsymbol{s}, h, \delta) \le \delta, \quad b\beta \le b\beta(\boldsymbol{s}, h, \delta) \le \delta \tag{6.59}$$

and so, by inclusion $x + u + v \in X$ from Condition SH2(Lip) of Definition 3.6, trajectories from $\mathscr{S}(\beta)$ belong to X.

Lemma 6.24. *The operator $\mathscr{H}$ is well defined and continuous on $\mathscr{S}(\beta)$ and for any of its fixed points $\boldsymbol{z} = \{z_0, z_1, \ldots, z_N\} \in \mathscr{S}(\beta)$ the sequence*

$$\boldsymbol{y} = \{x_0 + z_0, \ x_1 + z_1, \ \ldots, \ x_N + z_N\} \tag{6.60}$$

is a trajectory of the system φ.

Proof. We prove first that the operator $\mathscr{H}$ is well defined and continuous at any point $\boldsymbol{z} \in \mathscr{S}(\beta)$. It is evident by (6.59) that the right hand side of (6.51) is meaningful and depends continuously on $\boldsymbol{z} \in \mathscr{S}(\beta)$. Hence, we need only to prove for any $n = 1, 2, \ldots, N$ that the right hand side of the equality (6.53) is meaningful and continuous for $\boldsymbol{z} \in \mathscr{S}(\beta)$. By Lemma 6.23 it is sufficient to establish the inequality

$$\begin{aligned} \|P^u_{x_n}(-\varphi(x_{n-1} + z_{n-1}) &+ f(x_{n-1} + z_{n-1}) \\ &+ \ x_n - f(x_{n-1} + P^s_{x_{n-1}} z_{n-1}) + z_n)\| \le \lambda_u \delta. \end{aligned}$$

Rewrite the last inequality in the form

$$\|J_1 + J_2 + J_3\| \le \lambda_u \delta,$$

where

$$\begin{aligned} J_1 &= P^u_{x_n}(-\varphi(x_{n-1} + z_{n-1}) + f(x_{n-1} + z_{n-1})) \\ &\quad + \ P^u_{x_n}(x_n - f(x_{n-1})), \\ J_2 &= P^u_{x_n}(f(x_{n-1}) - f(x_{n-1} + P^s_{x_{n-1}} z_{n-1})), \\ J_3 &= P^u_{x_n} z_n. \end{aligned} \tag{6.61}$$

To estimate $\|J_1\|$ note that by (6.48)

$$\|\varphi(x_{n-1} + z_{n-1}) - f(x_{n-1} + z_{n-1})\| \le \|\varphi - f\|_\infty = \beta - \gamma$$

and also that $\|x_n - f(x_{n-1})\| \le \gamma$ since $\boldsymbol{x}$ is a γ-pseudo-trajectory of the mapping f, so by Condition SH1(Lip)

$$\|J_1\| \le \beta h. \tag{6.62}$$

By inequality (3.15),

$$\|J_2\| \le \mu_u \|P^s_{x_{n-1}} z_{n-1}\|. \tag{6.63}$$

Clearly

$$\|J_3\| = \|P^u_{x_n} z_n\|. \tag{6.64}$$

On the other hand, by definition (6.58) of the set $\mathscr{S}(\beta)$, inclusion $\boldsymbol{z} \in \mathscr{S}(\beta)$ implies that

$$\|P^s_{x_{n-1}} z_{n-1}\| \le \beta a, \quad \|P^u_{x_n} z_n\| \le \beta b. \tag{6.65}$$

Hence by (6.62)–(6.65)

$$\|J_1 + J_2 + J_3\| \le \|J_1\| + \|J_2\| + \|J_3\| \le \beta(h + a\mu_u + b)$$

so we need to establish the inequality

$$\beta(h + a\mu_u + b) \le \lambda_u \delta. \tag{6.66}$$

From (6.55) and (6.56) we have $h + a\mu_u + b = \lambda_u b$, so (6.66) can be rewritten as

$$\beta\lambda_u b \le \lambda_u \delta,$$

but this follows from (6.57). Hence the operator $\mathscr{H}$ is well defined and continuous at any point $\boldsymbol{z} \in \mathscr{S}(\beta)$.

It remains to prove that the sequence (6.60) is a trajectory of the system φ provided that $\boldsymbol{z} = \{z_0, z_1, \dots, z_N\} \in \mathscr{S}(\beta)$ is a fixed point of the operator $\mathscr{H}$. By assumption $\boldsymbol{z}$ is a fixed point of the operator $\mathscr{H}$, so equation (6.51) can be rewritten as

$$P^s_{x_n} z_n = P^s_{x_n}(\varphi(x_{n-1} + z_{n-1}) - x_n)$$

which can be rearranged to give

$$P^s_{x_n}(x_n + z_n) = P^s_{x_n}\varphi(x_{n-1} + z_{n-1}). \tag{6.67}$$

Analogously, equation (6.53) can be rewritten as

$$\begin{aligned} P^u_{x_{n-1}} z_{n-1} = Q_{x_{n-1},x_n,z_{n-1}}(&P^u_{x_n}(-\varphi(x_{n-1} + z_{n-1}) \\ &+ f(x_{n-1} + z_{n-1}) + x_n - f(x_{n-1} + P^s_{x_{n-1}} z_{n-1}) + z_n)). \end{aligned}$$

Applying the (nonlinear) operator $F_{x_{n-1},x_n,z_{n-1}} = Q^{-1}_{x_{n-1},x_n,z_{n-1}}$ to both sides of the last equation we get

$$\begin{aligned} F_{x_{n-1},x_n,z_{n-1}}(P^u_{x_{n-1}} z_{n-1}) = P^u_{x_n}(&f(x_{n-1} + z_{n-1}) \\ &- \varphi(x_{n-1} + z_{n-1}) + x_n - f(x_{n-1} + P^s_{x_{n-1}} z_{n-1}) + z_n). \end{aligned}$$

By definition of the operator $F_{x,y,z}(v)$,

$$
\begin{aligned}
&F_{x_{n-1},x_n,z_{n-1}}(P^u_{x_{n-1}}z_{n-1})\\
&\quad = P^u_{x_n}(f(x_{n-1}+P^s_{x_{n-1}}z_{n-1}+P^u_{x_{n-1}}z_{n-1})-f(x_{n-1}+P^s_{x_{n-1}}z_{n-1})),
\end{aligned}
$$

so, by comparing the last two equalities, we obtain

$$0 = P^u_{x_n}(z_n - \varphi(x_{n-1}+z_{n-1})+x_n)$$

or on rearranging

$$P^u_{x_n}(x_n+z_n) = P^u_{x_n}\varphi(x_{n-1}+z_{n-1}). \tag{6.68}$$

From (6.67) and (6.68) it now follows that

$$x_n + z_n = \varphi(x_{n-1}+z_{n-1}), \qquad n = 1, 2, \ldots, N.$$

which means that the sequence (6.60) is a trajectory of the system φ. The lemma is proved. □

Lemma 6.25. *The set $\mathscr{S}(\beta)$ is invariant under the operator $\mathscr{H}$.*

Proof. Let $\boldsymbol{z} \in \mathscr{S}(\beta)$ and write $\boldsymbol{w} = \mathscr{H}(\boldsymbol{z})$, cf. see (6.51) and (6.53). We can represent (6.51) in the form

$$P^s_{x_n} w_n = I_1 + I_2$$

where

$$
\begin{aligned}
I_1 &= P^s_{x_n}\left((\varphi(x_{n-1}+z_{n-1}) - f(x_{n-1}+z_{n-1})) + (f(x_{n-1}) - x_n)\right),\\
I_2 &= P^s_{x_n}\left(f(x_{n-1}+z_{n-1}) - f(x_{n-1})\right),
\end{aligned}
$$

and rewrite the equality (6.53) in the form

$$P^u_{x_{n-1}} w_{n-1} = Q_{x_{n-1},x_n,z_{n-1}}(J_1+J_2+J_3)$$

where J_1, J_2, J_3 are defined in (6.61).

To estimate $\|I_1\|$ note that by (6.48) $\|\varphi - f\|_\infty = \beta - \gamma$ and also that $\|x_n - f(x_{n-1})\| \le \gamma$ since $\boldsymbol{x}$ is a γ-pseudo-trajectory of the mapping f. Hence

$$
\begin{aligned}
&\|\varphi(x_{n-1}+z_{n-1}) - f(x_{n-1}+z_{n-1})\| + \|f(x_{n-1}) - x_n\|\\
&\qquad\qquad \le (\beta-\gamma)+\gamma = \beta
\end{aligned}
$$

and by Condition SH1(Lip)

$$\|I_1\| \le h\beta. \tag{6.69}$$

By (3.13), (3.14) and from Condition SH2(Lip)

$$\|I_2\| \le \lambda_s \|P^s_{x_{n-1}} z_{n-1}\| + \mu_s \|P^u_{x_{n-1}} z_{n-1}\|. \tag{6.70}$$

Now for each $\boldsymbol{z} \in \mathscr{B}$, define the pair of real nonnegative numbers

$$m^s(\boldsymbol{z}) = \max_{0 \le n \le N} \|P^s_{x_n} z_n\|, \quad m^u(\boldsymbol{z}) = \max_{0 \le n \le N} \|P^u_{x_n} z_n\|,$$

and denote by $\boldsymbol{m}(\boldsymbol{z})$ the two-dimensional column vector with coordinates $m^s(\boldsymbol{z})$ and $m^u(\boldsymbol{z})$. From the estimates (6.69), (6.70) and definition (6.58) of the set $\mathscr{S}(\beta)$ it follows that

$$m^s(\mathscr{H}\boldsymbol{z}) = m^s(\boldsymbol{w}) \le \beta h + \beta\lambda_s a + \beta\mu_s b. \tag{6.71}$$

Analogously, from (6.62)–(6.64), the definition (6.58) of the set $\mathscr{S}(\beta)$ and Lemma 6.23 it follows that

$$m^u(\mathscr{H}\boldsymbol{z}) = m^u(\boldsymbol{w}) \le \lambda_u^{-1}(\beta\mu_u a + \beta b + \beta h). \tag{6.72}$$

Inequalities (6.71) and (6.72) are equivalent to the coordinate-wise inequality

$$\boldsymbol{m}(\mathscr{H}\boldsymbol{z}) = \boldsymbol{m}(\boldsymbol{w}) \le \beta M(\boldsymbol{s})(a,b)^T + \beta\boldsymbol{h}, \qquad \boldsymbol{z} \in \mathscr{S}(\beta), \tag{6.73}$$

where, in view of (6.54), we have

$$\beta M(\boldsymbol{s})(a,b)^T + \beta\boldsymbol{h} = \beta(M(\boldsymbol{s})(I - M(\boldsymbol{s}))^{-1} + I)\boldsymbol{h}$$
$$= \beta(I - M(\boldsymbol{s}))^{-1}\boldsymbol{h} = \beta(a,b)^T.$$

Hence, (6.73) is equivalent to the coordinate-wise estimate

$$\boldsymbol{m}(\mathscr{H}\boldsymbol{z}) = \boldsymbol{m}(\boldsymbol{w}) \le \beta(a,b)^T, \qquad \boldsymbol{z} \in \mathscr{S}(\beta),$$

which, by the definition (6.58) of $\mathscr{S}(\beta)$, means that the set $\mathscr{S}(\beta)$ is invariant under the operator $\mathscr{H}$. □

Let us now finish the proof of Theorem 6.21. By Lemma 6.24 the operator $\mathscr{H}$ is continuous on the convex set $\mathscr{S}(\beta)$ and by Lemma 6.25 $\mathscr{H}(\mathscr{S}(\beta)) \subseteq \mathscr{S}(\beta)$, so by the Brower Fixed Point Theorem $\mathscr{H}$ has a fixed point $\boldsymbol{z} = \{z_0, z_1, \ldots, z_N\} \in \mathscr{S}(\beta)$. Hence, by Lemma 6.24 again, the sequence

$$\boldsymbol{y} = \{x_0 + z_0,\ x_1 + z_1,\ \ldots,\ x_N + z_N\}$$

is a trajectory of the mapping φ.

By definition (6.58) of the set $\mathscr{S}(\beta)$, the estimates

$$\|z_n\| \le \|P^s_{x_n} z_n\| + \|P^u_{x_n} z_n\| \le (a+b)\beta, \qquad n = 0, 1, \ldots, N,$$

are valid for the fixed point $\boldsymbol{z} \in \mathscr{S}(\beta)$ of the operator $\mathscr{H}$. Thus by the definitions of β and a, b (see (6.48) and (6.57), respectively) we have

$$\|z_n\| \le \alpha(\boldsymbol{s}, h)(\gamma + \|\varphi - f\|_\infty), \qquad n = 0, 1, \ldots, N,$$

or

$$\|x_n - y_n\| = \|z_n\| \le \alpha(\boldsymbol{s}, h)(\gamma + \|\varphi - f\|_\infty), \qquad n = 0, 1, \ldots, N.$$

Theorem 6.21 is proved. □

Remark 6.26. According to Definition 6.19 of bi-shadowing, Theorem 6.21 ensures the existence of a trajectory $\boldsymbol{y} = \{y_n\} \in \mathrm{Tr}(\varphi, X)$, for a given pseudo-trajectory $\boldsymbol{x} = \{x_n\} \in \mathrm{Tr}(f, K, \gamma)$ with $0 \le \gamma \le \beta(\boldsymbol{s}, h, \delta)$ satisfying

$$\|x_n - y_n\| \le \alpha(\boldsymbol{s}, h)(\gamma + \|\varphi - f\|_\infty).$$

Sometimes it is of importance that, in fact, the following inequalities

$$\|P^s_{x_n}(x_n - y_n)\|,\ \|P^u_{x_n}(x_n - y_n)\| \le \delta,$$

also hold. This which follows here from the inclusion $y_n - x_n = z_n \in \mathscr{S}(\beta)$, definition (6.58) of the set $\mathscr{S}(\beta)$ and estimates (6.59) established in the proof of Theorem 6.21.

6.3.2 Bi-Shadowing of Infinite Trajectories

Assertions on bi-shadowing of infinite trajectories follow from the expansivity of a semi-hyperbolic Lipschitz mappings and from Theorem 6.21.

Theorem 6.27. *Let $f : X \mapsto X$ be a Lipschitz mapping which is semi-hyperbolic on a compact subset $K \subset X$ with a split $\boldsymbol{s}$ and constants h, δ (see Definition 3.6) and let $\alpha(\boldsymbol{s}, h)$ and $\beta(\boldsymbol{s}, h)$ be given by* (6.43) *and* (6.44). *Then the following statements are valid.*

(i) For any infinite pseudo-trajectory $\boldsymbol{x} = \{x_n\} \in \mathrm{Tr}(f, K, \gamma)$ and any continuous mapping $\varphi : X \mapsto X$ satisfying $\|\varphi - f\|_\infty \le \beta(\boldsymbol{s}, h, \delta) - \gamma$ there exists an infinite trajectory $\boldsymbol{y} = \{y_n\} \in \mathrm{Tr}(\varphi, X)$ such that

$$\|x_n - y_n\| \le \alpha(\boldsymbol{s}, h)(\gamma + \|\varphi - f\|_\infty), \qquad n \in \mathbb{Z}.$$

(ii) For any given infinite pseudo-trajectory $\boldsymbol{x} = \{x_n\} \in \mathrm{Tr}(f, K, \gamma)$ with

$$\gamma \le \min\{\delta/(2h\alpha(\boldsymbol{s}, h)), \beta(\boldsymbol{s}, h, \delta)\}$$

there exists a unique infinite trajectory $\boldsymbol{y} = \{y_n\} \in \mathrm{Tr}(f, X)$ such that

$$\|x_n - y_n\| \le \alpha(\boldsymbol{s}, h)\gamma, \qquad n \in \mathbb{Z}.$$

Proof. To prove Assertion (i) we associate with the given γ-pseudo-trajectory $\boldsymbol{x} = \{x_n\} \in \mathrm{Tr}(f, K, \gamma)$ a sequence of finite γ-pseudo-trajectories

$$\boldsymbol{x}^{(k)} = \{x^{(k)}_{-k}, \ldots, x^{(k)}_0, \ldots, x^{(k)}_k\} \in \mathrm{Tr}(f, K, \gamma), \qquad k = 1, 2, \ldots$$

with

$$x^{(k)}_n = x_n, \qquad n = 0, \pm1, \pm2, \ldots, \pm k.$$

Then, for a given mapping φ satisfying $\|\varphi - f\|_\infty \le \beta(\boldsymbol{s}, h, \delta) - \gamma$ by Theorem 6.21 there exists for any $k = 1, 2, \ldots$ a finite trajectory

$$\boldsymbol{y}^{(k)} = \{y_{-k}^{(k)}, \dots, y_0^{(k)}, \dots, y_k^{(k)}\} \subseteq \mathrm{Tr}(\varphi, X)$$

such that

$$\|y_n^{(k)} - x_n\| = \|y_n^{(k)} - x_n^{(k)}\| \le \alpha(\boldsymbol{s}, h)(\gamma + \|\varphi - f\|_\infty) \tag{6.74}$$

for $n = 0, \pm 1, \pm 2, \dots, \pm k$. Moreover, by Remark 6.26 the inequalities

$$\|P_{x_n}^s(y_n^{(k)} - x_n)\| \le \delta, \quad \|P_{x_n}^u(y_n^{(k)} - x_n)\| \le \delta \tag{6.75}$$

hold for $n = 0, \pm 1, \pm 2, \dots, \pm k$.

In view of (6.74), for any integer n the sequence $\{y_n^{(k)}\}$ is bounded and thus, without loss of generality, we can suppose that it converges to a limit point element y_n, that is $y_n^{(k)} \to y_n$ as $k \to \infty$.

Taking the limit in (6.75) we obtain

$$\|P_{x_n}^s(y_n - x_n)\| \le \delta, \quad \|P_{x_n}^u(y_n - x_n)\| \le \delta,$$

and so, by the inclusion $x_n \in K$ and Condition SH2(Lip) of Definition 3.6, we have

$$y_n \in X, \qquad n \in \mathbb{Z}. \tag{6.76}$$

Taking the limit $k \to \infty$ on both sides of

$$y_{n+1}^{(k)} = \varphi(y_n^{(k)})$$

we obtain

$$y_{n+1} = \varphi(y_n).$$

Together with (6.76) this means that

$$\boldsymbol{y} = \{y_n\}_{n=-\infty}^{\infty} \in \mathrm{Tr}(\varphi, X).$$

Finally, taking the limit $k \to \infty$ in (6.74) we conclude that the infinite trajectory $\boldsymbol{y} \in \mathrm{Tr}(\varphi, X)$ of the system φ satisfies

$$\|x_n - y_n\| \le \alpha(\boldsymbol{s}, h)(\gamma + \|\varphi - f\|_\infty), \qquad n \in \mathbb{Z}.$$

This completes the proof of Assertion (i).

To prove Assertion (ii) note first that the existence of an infinite trajectory $\boldsymbol{y} \in \mathrm{Tr}(f, X)$ satisfying

$$\|x_n - y_n\| \le \alpha(\boldsymbol{s}, h)\gamma, \qquad n \in \mathbb{Z}.$$

follows from Assertion (i). To prove the uniqueness of such an infinite trajectory $\boldsymbol{y} \in \mathrm{Tr}(f, X)$ we use the exponential expansivity of a semi-hyperbolic mapping established in Theorem 6.5.

Suppose that the statement of Assertion (ii) is not true. Then there exist trajectories $\boldsymbol{y}, \widetilde{\boldsymbol{y}} \in \mathrm{Tr}(f, X)$ such that

$$\|y_n - x_n\|, \ \|\widetilde{y}_n - x_n\| \le \alpha(\boldsymbol{s}, h)\gamma, \qquad n \in \mathbb{Z},$$

and hence such that

$$\|y_n - \widetilde{y}_n\| \le 2\alpha(\boldsymbol{s}, h)\gamma \le h^{-1}\delta, \qquad n \in \mathbb{Z}.$$

This contradicts Theorem 6.5 and so Assertion (ii) must be true. □

6.3.3 Cyclic Bi-Shadowing

Cyclic, or periodic, behavior is often of particular interest in dynamical systems. Here we show that bi-shadowing is preserved when we restrict attention to periodic (or cyclic) trajectories.

Definition 6.28. *A trajectory* $\boldsymbol{x} = \{x_n\}_{n=0}^N \in \mathrm{Tr}(f, K)$ *is called a* cycle (or periodic trajectory) of period N *if* $x_N = x_0$ *and a pseudo-trajectory* $\boldsymbol{y} = \{y_n\}_{n=0}^N \in \mathrm{Tr}(f, K, \gamma)$ *is called a* γ-pseudo-cycle (or periodic γ-pseudo-trajectory) of period N *if* $\|y_N - y_0\| \le \gamma$.

Let $\mathrm{Cyc}(f, K, \gamma) \subset \mathrm{Tr}(f, K, \gamma)$ denote the totality of γ-pseudo-cycles of any period belonging entirely to the subset K of X, with $\mathrm{Cyc}(f, K, 0) \subset \mathrm{Tr}(f, K)$ or $\mathrm{Cyc}(f, K) \subset \mathrm{Tr}(f, K)$ denoting the totality of proper cycles of any period which are contained entirely in K. Obviously $\mathrm{Cyc}(f, K) \subset \mathrm{Cyc}(f, K, \gamma)$ for every $\gamma > 0$.

A counterpart of bi-shadowing for cycles and pseudo-cycles is also useful.

Definition 6.29. *A dynamical system generated by a mapping* $f : X \mapsto X$ *is said to be* cyclically bi-shadowing *with positive parameters* α *and* β *on a subset* K *of* X *if for any given pseudo-cycle* $\boldsymbol{x} \in \mathrm{Cyc}(f, K, \gamma)$ *with* $0 \le \gamma \le \beta$ *and any mapping* $\varphi : X \mapsto X$ *satisfying* (6.41) *there exists a proper cycle* $\boldsymbol{y} \in \mathrm{Cyc}(\varphi, X)$ *of period* N *equal to that of* $\boldsymbol{x}$ *such that* (6.42) *holds for* $n = 0, 1, \ldots, N$.

Note that the cycle $\boldsymbol{y}$ in Definition 6.29 is required only to be in X rather than in the subset K. Cyclic bi-shadowing is also a consequence of semi-hyperbolicity.

Theorem 6.30. *Let* $f : X \mapsto X$ *be a Lipschitz mapping which is semi-hyperbolic on a compact subset* $K \subseteq X$ *with a split* $\boldsymbol{s}$ *and constants* h, δ. *Then it is cyclically bi-shadowing on* K *with respect to continuous mappings* $\varphi : X \mapsto X$ *with parameters* $\alpha(\boldsymbol{s}, h)$ *and* $\beta(\boldsymbol{s}, h, \delta)$ *given by* (6.43) *and* (6.44).

The proof repeats verbatim that of Theorem 6.21 with the following two minor modifications.

First, the boundary conditions

$$P^s_{x_0}(w_0) = P^s_{x_N}(w_N), \quad P^u_{x_N}(w_N) = P^u_{x_0}(w_0)$$

should be used instead of (6.50) and (6.52) in the definition of the operator $\mathscr{H}$.

Second, in the proof the following 'cyclic' analog of Lemma 6.24 should be used.

Lemma 6.31. *The operator* $\mathscr{H}$ *is well defined and continuous on* $\mathscr{S}(\beta)$*, and for any of its fixed points* $\boldsymbol{z} = \{z_0, z_1, \ldots, z_N\} \in \mathscr{S}(\beta)$ *the sequence*

$$\boldsymbol{y} = \{x_0 + z_0,\ x_1 + z_1,\ \ldots,\ x_N + z_N\}$$

is a cycle of the system φ.

7

Structural Stability

In this Chapter we consider properties of semi-hyperbolic mappings which can be conditionally qualified as structure-stability-properties. More precisely, it will be shown that topological entropy can only increase following to continuous perturbation of a Lipschitz semi-hyperbolic mapping. Then problems of conjugation and factorization of semi-hyperbolic mappings will be studied. And, at last, investigation of chaotic phenomena for semi-hyperbolic mappings will be discussed. Throughout the Chapter $\|\cdot\|$ will denote a fixed but otherwise arbitrary norm on $\mathbb{R}^d$.

7.1 Topological Entropy

Let throughout this Section $X \subseteq \mathbb{R}^d$ be an open and $K \subset X$ be a compact subset of X. By $h_\varepsilon(f, K)$ and $h(f, K)$ it will be denoted the *ε-entropy* and *entropy* of the mapping f as they are defined in Definition 2.12.

It was seen in Chapter 6 that semi-hyperbolic mappings are expansive. The following lemma on connection between ε-entropy and entropy of expansive mappings, essentially well known [53], and is quite useful sometimes as provides a method for calculation of the entropy.

Lemma 7.1. *Let $K \subset \mathbb{R}^d$ be a compact set and $f : K \mapsto K$ be a continuous ξ-expansive mapping with $f(K) = K$. Then*

$$h(f, K) = h_\theta(f, K)$$

holds for every $\theta < \xi$.

Proof. Fix θ with $0 < \theta < \xi$. As was remarked in footnote on the page 16, $h_\varepsilon(f, K)$ is non-increasing in ε, then

$$h(f, K) = \sup_{\varepsilon > 0} h_\varepsilon(f, K)$$

and by Definition 2.12 of topological entropy it suffices to prove that

$$h_\varepsilon(f, K) \le h_\theta(f, K) \tag{7.1}$$

for $\varepsilon > 0$ sufficiently small. From Lemma 6.2 it follows the existence of such a $\varkappa(\varepsilon, \theta)$ for which the inequality

$$\varrho_N(\boldsymbol{x}, \widetilde{\boldsymbol{x}}) \ge \varepsilon \qquad \text{for} \quad \boldsymbol{x}, \widetilde{\boldsymbol{x}} \in \operatorname*{Tr}_{\pm(N+\varkappa(\varepsilon,\theta))}(f, K)$$

implies that $\varrho_{N+\varkappa(\varepsilon,\theta)}(\boldsymbol{x}, \widetilde{\boldsymbol{x}}) \ge \theta$ for any positive integer N. At the same time, in view of the equality $f(K) = K$ each trajectory $\boldsymbol{x} \in \mathrm{Tr}_{\pm N}(f, K)$ can be extended to some trajectory in $\mathrm{Tr}_{\pm(N+\varkappa(\varepsilon,\theta))}(f, K)$. Hence[1]

$$C_\varepsilon(\operatorname*{Tr}_{\pm N}(f, K)) \le C_\theta(\operatorname*{Tr}_{\pm(N+\varkappa(\varepsilon,\theta))}(f, K))$$

from which by Definition 2.12 immediately follows the inequality (7.1). □

A rich theory of topological entropy has been developed for hyperbolic mappings (cf. [53] and the references therein) and many of the results remain valid for semi-hyperbolic mappings too. The following theorem is illustrative of such possible generalizations.

Theorem 7.2. *Let $X \subseteq \mathbb{R}^d$ be an open set and $f : X \mapsto \mathbb{R}^d$ be a Lipschitz mapping which is continuously $\boldsymbol{s}$-semi-hyperbolic with constants h, δ on a compact invariant set $f(K) = K \subset X$. Then*

$$h(g, \overline{\mathscr{O}_{\delta/2h}(K)}) \ge h(f, K)$$

for each continuous mapping $g : X \mapsto X$ satisfying

$$\|g - f\|_C < \frac{\delta}{2h\alpha(\boldsymbol{s}, h)},$$

where $\alpha(\boldsymbol{s}, h)$ is defined by (6.43).

Proof. Given mappings f and g, choose real θ satisfying

$$2\alpha(\boldsymbol{s}, h)\|g - f\|_C < \theta < h^{-1}\delta,$$

where $\alpha(\boldsymbol{s}, h)$ is defined by (6.43).

By Theorem 6.5 the mapping f is ξ-expansive in K with the expansivity constant $\xi = h^{-1}\delta$, and then, by Lemma 7.1, $h(f, K) = h_\theta(f, K)$. By Definition 2.12 of topological entropy $h_\sigma(g, \overline{\mathscr{O}_{\delta/2h}(K)}) \le h(g, \overline{\mathscr{O}_{\delta/2h}(K)})$ for any

[1] Remind, that the value $C_\varepsilon(\mathrm{Tr}_{\pm N}(f, K))$ is defined in Section 2.2.4 as the binary logarithm of maximal number of elements $x^{(1)}, \dots, x^{(p)}$ in $\mathrm{Tr}_{\pm N}(f, K)$ such that $\varrho_N(x^{(i)}, x^{(j)}) \ge \varepsilon$ for all $i \ne j$.

$\sigma > 0$. Hence, to complete the proof of the theorem it remains only to prove the middle inequality in the following relations

$$h(f, K) = h_\theta(f, K) \le h_\sigma(g, \overline{\mathscr{O}_{\delta/2h}(K)}) \le h(g, \overline{\mathscr{O}_{\delta/2h}(K)})$$

with $\sigma = \theta - 2\alpha(\boldsymbol{s}, h)\|g - f\|_C > 0$. This, in turn, by Definition 2.12 will follow from validity for any positive N of the inequality

$$C_\theta(\operatorname*{Tr}_{\pm N}(f, K)) \le C_\sigma(\operatorname*{Tr}_{\pm N}(g, \overline{\mathscr{O}_{\delta/2h}(K)}) \tag{7.2}$$

To prove (7.2) denote by $\{\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(p)}\}$ the maximal subset of elements from $\mathrm{Tr}_{\pm N}(f, K)$ satisfying

$$\varrho_N(\boldsymbol{x}^{(i)}, \boldsymbol{x}^{(j)}) \ge \theta, \qquad i \ne j.$$

By Theorem 6.30, for each such $\boldsymbol{x}^{(i)} \in \mathrm{Tr}_{\pm N}(f, K)$ there exists a trajectory $\boldsymbol{y}^{(i)} \in \mathrm{Tr}_{\pm N}(g, X)$ for which by condition of the theorem

$$\varrho_N(\boldsymbol{y}^{(i)}, \boldsymbol{x}^{(i)}) \le \alpha(\boldsymbol{s}, h)\|g - f\|_C \le \delta/2h.$$

Hence $\boldsymbol{y}^{(i)} \in \mathrm{Tr}_{\pm N}(g, \overline{\mathscr{O}_{\delta/2h}(K)})$ for $i = 1, \ldots, p$, and

$$\begin{aligned}\varrho_N(\boldsymbol{y}^{(i)}, \boldsymbol{y}^{(j)}) \ge \varrho_N(\boldsymbol{x}^{(i)}, \boldsymbol{x}^{(j)}) - \varrho_N(\boldsymbol{x}^{(i)}, \boldsymbol{y}^{(i)}) - \varrho_N(\boldsymbol{x}^{(j)}, \boldsymbol{y}^{(j)}) \\ \ge \theta - 2\alpha(\boldsymbol{s}, h)\|g - f\|_C = \sigma\end{aligned}$$

for any $j \ne i$, from which (7.2) follows. □

7.2 Structural Stability Properties

Let $X \subset \mathbb{R}^d$ be a bounded set. Denote by $\Sigma(X)$ the metric space of all bi-infinite sequences $\boldsymbol{x} = \{x_n\}_{n=-\infty}^{\infty}$ with $x_n \in X$ for $n = 0, \pm 1, \pm 2, \ldots$ with the norm

$$\varrho(\boldsymbol{x}, \widetilde{\boldsymbol{x}}) = \sum_{n=-\infty}^{\infty} 2^{-|n|}\|x_n - \widetilde{x}_n\|. \tag{7.3}$$

Let σ denote the *shift operator* on $\Sigma(X)$ defined as

$$(\sigma\boldsymbol{x})_n = x_{n+1}, \qquad \boldsymbol{x} \in \Sigma(X), \quad n \in \mathbb{Z}.$$

Finally, for a set $Y \subseteq X$ and a mapping $f \in C(X, \mathbb{R}^d)$ let $\mathrm{Tr}_{\pm\infty}(f, Y) \subset \Sigma(X)$ be the totality of bi-infinite trajectories $\boldsymbol{y} = \{y_n\} \subseteq Y$.

For the convenience of references, let us formulate the following evident statement.

Lemma 7.3. *If the set $Y \subseteq X \subseteq \mathbb{R}^d$ is compact and $f \in C(X, \mathbb{R}^d)$, then the set $\mathrm{Tr}_{\pm\infty}(f, Y) \subset \Sigma(X)$ is also compact in the metric space $(\Sigma(X), \varrho)$.*

Definition 7.4. *Let* $f_1,\ f_2 \in C(X, \mathbb{R}^d)$ *and let* Y_1 *and* Y_2 *be closed subsets of* X *such that* $f_1(Y_1) = Y_1$ *and* $f_2(Y_2) = Y_2$. *The restriction* $f_1|_{Y_1}$ *is said to be a* weak factorization *of the restriction* $f_2|_{Y_2}$ *if there exists a continuous (in the metric* ϱ*) surjection*

$$\Phi : \operatorname*{Tr}_{\pm\infty}(f_2, Y_2) \mapsto \operatorname*{Tr}_{\pm\infty}(f_2, Y_2)$$

which is shift invariant, i.e. with $\Phi \circ \sigma = \sigma \circ \Phi$.

Another closely connected concept is that of weak conjugacy.

Definition 7.5. *Let* $f_1,\ f_2 \in C(X, \mathbb{R}^d)$ *and let* Y_1 *and* Y_2 *be closed subsets of* X *such that* $f_1(Y_1) = Y_1$ *and* $f_2(Y_2) = Y_2$. *The restricted mappings* $f_1|_{Y_1}$ *and* $f_2|_{Y_2}$ *are said to be* weakly conjugate *if there exists a continuous one-to-one correspondence*

$$\Psi : \operatorname*{Tr}_{\pm\infty}(f_1, Y_1) \mapsto \operatorname*{Tr}_{\pm\infty}(f_2, Y_2)$$

which is shift invariant.

In the above definitions Y_1 and Y_2 are closed subsets of the bounded set X. Hence they are compact, and so are the metric spaces $(\mathrm{Tr}_{\pm\infty}(f_1, Y_1), \varrho)$ and $(\mathrm{Tr}_{\pm\infty}(f_2, Y_2), \varrho)$. Therefore the one-to-one mapping Ψ in Definition 7.5 is a homeomorphism. In other words, the restricted mappings $f_1|_{Y_1}$ and $f_2|_{Y_2}$ are weakly conjugate if the restrictions of the shift operator σ to the sets $\mathrm{Tr}_{\pm\infty}(f_1, Y_1)$ and $\mathrm{Tr}_{\pm\infty}(f_2, Y_2)$ are topologically conjugate, i.e. topologically conjugate the mappings $\sigma_1 = \sigma|_{\mathrm{Tr}_{\pm\infty}(f_1, Y_1)}$ and $\sigma_2 = \sigma|_{\mathrm{Tr}_{\pm\infty}(f_2, Y_2)}$.

The notion of weak factorization extends an analogue of semi-conjugacy (see Section 2.2.2 for definitions) to semi-hyperbolic mappings. Weak conjugacy is a generalization of topological conjugacy of mappings and reduces to it in the case of invertible mappings. The suitability of such generalizations in the analysis of noninvertible mappings is well known; see, for instance book of Ruelle [68, Section 15.6]. In particular, topological entropy is an invariant with respect to weak conjugacy and does not increase under weak factorization.

The sets Y_1 and Y_2 above are often sets of chain recurrent points. Remind, in the convenient for the following usage form, the necessary definitions from Section 2.2.3.

Definition 7.6. *A point* $x \in Y \subseteq X$ *is called* ε-chain recurrent (in Y) *for the mapping* f *defined on a set* X *if there exists an* ε*-pseudo-cycle* $\{x_n\} \subseteq Y$ *of* f *with* $x_0 = x$. *A point* $x \in Y \subseteq X$ *is called* chain recurrent (in Y) *for the mapping* f *defined on a set* X *if it is* ε*-chain recurrent in* Y *for any sufficiently small* $\varepsilon > 0$.

Denote the totality of ε-chain recurrent in Y points of f by $\mathrm{CR}(f, \varepsilon, Y)$. The totality $\mathrm{CR}(f, Y)$ of chain recurrent in Y points of f is then can be expressed as

$$\mathrm{CR}(f,Y) = \bigcap_{\varepsilon>0} \mathrm{CR}(f,\varepsilon,Y).$$

If $\overline{Y} \subseteq X$ where X is bounded, then $\mathrm{CR}(f,\overline{Y})$ is compact and f-invariant, i.e.,

$$f(\mathrm{CR}(f,\overline{Y})) = \mathrm{CR}(f,\overline{Y}).$$

Lemma 7.7. *Let X and Y be open bounded subsets of $\mathbb{R}^d$, $f \in C(X,\mathbb{R}^d)$ and $\mathrm{CR}(f,\overline{Y}) \subset Y \subset \overline{Y} \subset X$. Then there exists a nondecreasing function $q(\varepsilon,f)$ of $\varepsilon \geq 0$ with*

$$\lim_{\varepsilon\to 0} q(\varepsilon,f) = q(0,f) = 0, \qquad q(\varepsilon,f) > 0 \quad \text{for} \quad \varepsilon > 0,$$

such that

$$\mathrm{CR}(f,\varepsilon,\overline{Y}) \subseteq \mathscr{O}_{q(\varepsilon,f)}(\mathrm{CR}(f,\overline{Y})).$$

In particular, there exists an $\varepsilon = \varepsilon(f,Y) > 0$ for which $\mathrm{CR}(f,\varepsilon,\overline{Y}) \subset Y$.

Proof. Suppose the contrary. Then, for a certain $\varepsilon_0 > 0$, there exists a sequence $\boldsymbol{x}^{(k)} = \{x_n^{(k)}\}$ of $1/m$-pseudo-cycles of f with

$$\boldsymbol{x}^{(k)} \subseteq \overline{Y}, \qquad x_0^{(k)} \notin \mathscr{O}_{\varepsilon_0}(\mathrm{CR}(f,\overline{Y})). \tag{7.4}$$

In view of boundedness of the set Y, the set $\overline{Y}$ is compact and then the sequence $\{x_0^{(k)}\}$ has a limit point $y \in \overline{Y}$. By the first inclusion (7.4) and by the definition of chain recurrence, $y \in \mathrm{CR}(f,\overline{Y})$. But by the second relation (7.4), $y \notin \mathscr{O}_{\varepsilon_0}(\mathrm{CR}(f,\overline{Y}))$. The contradiction proves the lemma. □

7.2.1 Weak Factorization

Let X and Y be open bounded subsets of $\mathbb{R}^d$ with $\overline{Y} \subseteq X$ and let Lipschitz mapping $f \in \mathrm{Lip}(X,\mathbb{R}^d)$ be continuously semi-hyperbolic on $\mathrm{CR}(f,\overline{Y})$ with $\mathrm{CR}(f,\overline{Y}) \subset Y$. By Lemma 3.12 there exists an $\eta = \eta(f,Y) > 0$ such that f is semi-hyperbolic in $\overline{\mathscr{O}_\eta(\mathrm{CR}(f,\overline{Y}))}$ with a certain split $\boldsymbol{s}$ and constants h,δ. Denote by α,β the corresponding constants (6.43), (6.44), i.e.

$$\alpha = \alpha(\boldsymbol{s},h) := h\frac{\lambda_u - \lambda_s + \mu_s + \mu_u}{(1-\lambda_s)(\lambda_u - 1) - \mu_s\mu_u},$$

$$\beta = \beta(\boldsymbol{s},h,\delta) := \delta h^{-1}\frac{(1-\lambda_s)(\lambda_u - 1) - \mu_s\mu_u}{\max\{\lambda_u - 1 + \mu_s, 1 - \lambda_s + \mu_u\}}.$$

Theorem 7.8. *Let Y be an open set with $\overline{Y} \subseteq X$ and let the mapping $f \in \mathrm{Lip}(X,\mathbb{R}^d)$ be continuously $\boldsymbol{s}$-semi-hyperbolic in $\mathrm{CR}(f,\overline{Y}) \subset Y$ with constants h,δ. If $g \in C(X,\mathbb{R}^d)$ with $\|g-f\|_C < \varepsilon$ where ε is such that*

$$\varepsilon < \delta/(2h\alpha), \quad q(\varepsilon,f) < \eta, \quad \mathscr{O}_{q(\varepsilon,f)+\alpha\varepsilon}(\mathrm{CR}(f,\overline{Y})) \subset Y \tag{7.5}$$

then the restricted mapping $f|_{\mathrm{CR}(f,\overline{Y}))}$ is a weak factorization of $g|_{\mathrm{CR}(g,\overline{Y}))}$.

Proof. Consider a mapping $g \in C(X, \mathbb{R}^d)$ such that $\|f - g\|_C < \varepsilon$ where ε satisfies (7.5). Denote the ball in $\Sigma(X)$ of radius ε centered at $\boldsymbol{x}$ by

$$B(\boldsymbol{x}, \varepsilon) = \{\boldsymbol{y} \in \Sigma(X) : \ \|x_i - y_i\| \le \varepsilon, \ i \in \mathbb{Z}\}$$

and define the mapping Φ for $\boldsymbol{x} \in \mathrm{Tr}_{\pm\infty}(g, \mathrm{CR}(g, \overline{Y}))$ by

$$\Phi(\boldsymbol{x}) = B(\alpha\varepsilon, \boldsymbol{x}) \cap \mathop{\mathrm{Tr}}_{\pm\infty}(f, \mathrm{CR}(f, \overline{Y})). \tag{7.6}$$

To prove the theorem we need to show that Φ is well-defined, i.e. that

(i) the set
$$B(\alpha\varepsilon, \boldsymbol{x}) \cap \mathop{\mathrm{Tr}}_{\pm\infty}(f, \mathrm{CR}(f, \overline{Y})) \tag{7.7}$$
is non-empty;
(ii) the set (7.7) contains exactly one point;
(iii) the mapping Φ is a surjection of the set $\mathrm{Tr}_{\pm\infty}(g, \mathrm{CR}(g, \overline{Y}))$ onto the set $\mathrm{Tr}_{\pm\infty}(f, \mathrm{CR}(f, \overline{Y}))$;
(iv) the mapping Φ is continuous;
(v) the mapping Φ is shift-invariant.

(i) Prove that the set (7.7) is non-empty. First, fix

$$\boldsymbol{x} = \{x_n\} \in \mathop{\mathrm{Tr}}_{\pm\infty}(g, \mathrm{CR}(g, \overline{Y})) \tag{7.8}$$

and show that for each positive integer m there exists a $1/m$-pseudo-cycle

$$\boldsymbol{x}^{(m)} = \{x_n^{(m)}\} \subset \mathrm{CR}(g, 1/m, \overline{Y})$$

of the mapping g, satisfying

$$\|x_n^{(m)} - x_n\| < 1/m, \qquad n = -m, \dots, 0, 1, \dots, m. \tag{7.9}$$

Indeed, remark first that by the continuity of g for a given m there can be found $\bar{\delta} = \bar{\delta}_m > 0$ such that each $\bar{\delta}$-pseudo-trajectory

$$\boldsymbol{y} = y_{-m}, \dots, y_0, y_1, \dots, y_m$$

of g satisfying $y_0 = x_{-m}$, satisfies also the estimates

$$\sup_{-m \le n \le m} \|y_n - x_n\| < 1/m.$$

Then, in view of inclusion $x_{-m} \in \mathrm{CR}(g, \overline{Y})$ followed from (7.8), by chain recurrency Definition 7.6 such a $\bar{\delta}$-pseudo cycle $\boldsymbol{x}^{(m)} \subseteq \overline{Y}$ of the mapping g can be found that $\|x_{-m}^{(m)} - x_{-m}\| < \bar{\delta}$. Hence, for this $\bar{\delta}$-pseudo cycle $\boldsymbol{x}^{(m)}$ the estimates (7.9) will be also valid. So, the existence of $\boldsymbol{x}^{(m)}$ is proved.

Now remark, that each $\boldsymbol{x}^{(m)} \in \mathrm{Tr}_{\pm\infty}(g, \mathrm{CR}(g, 1/m, \overline{Y}))$, $m = 1, 2, \dots$, is simultaneously a $(1/m + \|f - g\|_C)$-pseudo-cycle of f, i.e.

$$\boldsymbol{x}^{(m)} \subseteq \mathrm{CR}(f, 1/m + \|f-g\|_C, \overline{Y}).$$

Consequently, by Lemma 7.7,

$$\boldsymbol{x}^{(m)} \subseteq \mathscr{O}_{q(1/m+\|f-g\|_C,f)}(\mathrm{CR}(f,\overline{Y}))$$

and so, by suppositions $q(\varepsilon, f) < \eta(f, Y)$ (see (7.5)) and by $\|f-g\|_C < \varepsilon$,

$$\boldsymbol{x}^{(m)} \subseteq \mathscr{O}_{q(\varepsilon,f)}(\mathrm{CR}(f,\overline{Y})) \subseteq \mathscr{O}_{\eta(f,Y)}(\mathrm{CR}(f,\overline{Y})) \tag{7.10}$$

for all sufficiently large m. Hence, in view of the definition of $\eta(f, Y)$, the pseudo-cycles $\boldsymbol{x}^{(m)}$ belong to the region of semi-hyperbolicity of f for all sufficiently large m. From (6.43), (6.44) it is seen that $\delta/\alpha \leq \beta$, and then due to the first inequality of (7.5) $\varepsilon \leq \beta/2h \leq \beta$ holds. Therefore, by the cyclic bi-shadowing Theorem 6.30, for each $\boldsymbol{x}^{(m)}$ satisfying (7.10) for sufficiently large m there exists a cycle $\boldsymbol{y}^{(m)}$ of f such that

$$\|x_n^{(m)} - y_n^{(m)}\| < \alpha\varepsilon \qquad \text{for all} \quad m. \tag{7.11}$$

Hence

$$\boldsymbol{y}^{(m)} \subseteq \mathscr{O}_{\eta(f,Y)+\alpha\varepsilon}(\mathrm{CR}(f,\overline{Y})),$$

and by the inclusion of (7.5) $\boldsymbol{y}^{(m)} \subseteq Y$. Moreover, since the trajectory $\boldsymbol{y}^{(m)}$ is periodic, then, in fact, $\boldsymbol{y}^{(m)} \in \mathrm{Tr}_{\pm\infty}(f, \mathrm{CR}(f,\overline{Y}))$. Therefore, by Lemma 7.3 the sequence $\boldsymbol{y}^{(m)}$ has a limit point $\boldsymbol{y} \in \mathrm{Tr}_{\pm\infty}(f, \mathrm{CR}(f,\overline{Y}))$. By (7.9) and (7.11)

$$\|x_n - y_n\| \leq \alpha\varepsilon \qquad \text{for all} \quad n$$

which means that $\boldsymbol{y} \in B(\alpha\varepsilon, \boldsymbol{x})$.

So, it is proved that the set (7.7) is non-empty for each

$$\boldsymbol{x} \in \operatorname*{Tr}_{\pm\infty}(g, \mathrm{CR}(g,\overline{Y})).$$

(ii) Prove now that the set (7.7) contains exactly one point. Remark that because of the first inequality of (7.5) for any two trajectories

$$\boldsymbol{y}, \widetilde{\boldsymbol{y}} \in B(\alpha\varepsilon, \boldsymbol{x}) \cap \operatorname*{Tr}_{\pm\infty}(f, \mathrm{CR}(f,\overline{Y})),$$

the estimate $\|y_i - \widetilde{y}_i\| < 2\alpha\varepsilon \leq \delta/h$ hold. By Theorem 6.5, the set (7.7) then contains no more than one element, and so the mapping Φ is well-defined.

(iii) Prove that Φ is a surjection of the set $\mathrm{Tr}_{\pm\infty}(g, \mathrm{CR}(g,\overline{Y}))$ onto the set $\mathrm{Tr}_{\pm\infty}(f, \mathrm{CR}(f,\overline{Y}))$. By definition of the mapping Φ we need only to construct for each $\boldsymbol{y} \in \mathrm{Tr}_{\pm\infty}(f, \mathrm{CR}(f,\overline{Y}))$ an element $\boldsymbol{x} \in \mathrm{Tr}_{\pm\infty}(g, \mathrm{CR}(g,\overline{Y}))$ with $\|x_i - y_i\| < \alpha\varepsilon$. As above, for each positive integer m there exist a $1/m$-pseudo-cycle $\boldsymbol{y}^{(m)} \subseteq \mathrm{CR}(g, 1/m, \overline{Y})$ of the mapping f satisfying

$$\|y_n^{(m)} - y_n\| < 1/m, \qquad n = -m, \ldots, 0, 1, \ldots, m.$$

By the definition of chain recurrence, $\boldsymbol{y}$ is a limit in metric (7.3) of a sequence $\boldsymbol{y}^{(m)}$, $m = 1, 2, \ldots$, of $1/m$-pseudo-cycles of the mapping f. As was mentioned above, due to the first inequality of (7.5) we have $\varepsilon \le \beta$. So, by cyclic bi-shadowing Theorem 6.30 for sufficiently large m there exist cycles $\boldsymbol{x}^{(m)}$ of g satisfying

$$\|x_n^{(m)} - y_n^{(m)}\| < \alpha\varepsilon \qquad \text{for all} \quad n.$$

It remains to define $\boldsymbol{x}$ as a limit point of the sequence $\boldsymbol{x}^{(m)}$ in the metric space $(\Sigma(X), \varrho)$ (such a limiting point exists by Lemma 7.3).

(iv) To prove the continuity of the mapping Φ, we shall suppose the contrary. Then there exist

$$\boldsymbol{x}, \boldsymbol{x}^{(m)} \in \operatorname*{Tr}_{\pm\infty}(g, \mathrm{CR}(g, \overline{Y})), \qquad m = 1, 2, \ldots\ ,$$

such that

$$\varrho(\boldsymbol{x}, \boldsymbol{x}^{(m)}) \to 0 \tag{7.12}$$

but $\varrho(\Phi(\boldsymbol{x}), \Phi(\boldsymbol{x}^{(m)})) \ge \bar\varepsilon$ for some $\bar\varepsilon > 0$. In this case, without loss of generality, we may assume that

$$\|(\Phi(\boldsymbol{x}))_0 - (\Phi(\boldsymbol{x}^{(m)}))_0\| \ge \bar\varepsilon, \qquad m = 1, 2, \ldots\ . \tag{7.13}$$

Now, for any positive integer N denote by $\varrho_N(\boldsymbol{z}, \widetilde{\boldsymbol{z}})$ the semi-norm in $\Sigma(X)$ defined by

$$\varrho_N(\boldsymbol{z}, \widetilde{\boldsymbol{z}}) = \sup_{-N \le n \le N} \|z_n - \widetilde{z}_n\|.$$

Choose a θ satisfying $2\alpha\varepsilon < \theta < \delta/h$; such θ exists by (7.5). Since by Theorem 6.5 the mapping g is ξ-expansive with $\xi = \delta/h$ on the set $\mathrm{CR}(f, \overline{Y})$, then by Lemma 6.2 there exists a positive integer $\varkappa(\bar\varepsilon, \theta)$ not depending on m such that

$$\varrho_{\varkappa(\bar\varepsilon,\theta)}(\Phi(\boldsymbol{x}), \Phi(\boldsymbol{x}^{(m)})) \ge \theta \tag{7.14}$$

holds whenever (7.13) is valid. At the same time, from the definition of the mapping Φ it follows that $\Phi(\boldsymbol{x}) \in B(\alpha\varepsilon, \boldsymbol{x})$, and so $\varrho_{\varkappa(\bar\varepsilon,\theta)}(\Phi(\boldsymbol{x}), \boldsymbol{x}) \le \alpha\varepsilon$. Hence

$$\begin{aligned} &\varrho_{\varkappa(\bar\varepsilon,\theta)}(\Phi(\boldsymbol{x}), \Phi(\boldsymbol{x}^{(m)})) \\ &\quad\le\ \varrho_{\varkappa(\bar\varepsilon,\theta)}(\Phi(\boldsymbol{x}), \boldsymbol{x}) + \varrho_{\varkappa(\bar\varepsilon,\theta)}(\boldsymbol{x}, \boldsymbol{x}^{(m)}) + \varrho_{\varkappa(\bar\varepsilon,\theta)}(\boldsymbol{x}^{(m)}, \Phi(\boldsymbol{x}^{(m)})) \\ &\qquad\qquad\le\ \alpha\varepsilon + \varrho_{\varkappa(\bar\varepsilon,\theta)}(\boldsymbol{x}, \boldsymbol{x}^{(m)}). \end{aligned}$$

Here $2\alpha\varepsilon < \theta$ by the definition of θ and $\varrho_{\varkappa(\bar\varepsilon,\theta)}(\boldsymbol{x}, \boldsymbol{x}^{(m)}) \to 0$ in view of (7.12). Therefore, $\varrho_{\varkappa(\bar\varepsilon,\theta)}(\Phi(\boldsymbol{x}), \Phi(\boldsymbol{x}^{(m)})) < \theta$ for sufficiently large m, which contradicts to (7.14) and so the mapping Φ is continuous.

(v) The shift invariance identity $\Phi \circ \sigma \equiv \sigma \circ \Phi$ follows directly from the definition of the mapping Φ.

Theorem 7.8 is thus completely proved. □

7.2.2 Weak Conjugacy

Theorem 7.8 above is a weak form of semi-conjugacy of semi-hyperbolic mappings, while Theorem 7.9 below is a version of structural stability for such mappings. The explicit estimates of the 'radii' of semi-conjugacy and structural stability in these theorems are particularly useful in applications. Note that semi-conjugacy in Theorem 7.8 is a C^0-robust property, while conjugacy in Theorem 7.9 below is Lipschitz-robust.

Let again X and Y be open bounded subsets of $\mathbb{R}^d$ with $\overline{Y} \subseteq X$ and let Lipschitz mapping $f \in \mathrm{Lip}(X, \mathbb{R}^d)$ be continuously semi-hyperbolic on $\mathrm{CR}(f, \overline{Y})$ with $\mathrm{CR}(f, \overline{Y}) \subset Y$. Let $\varepsilon(f, Y) > 0$ be a constant defined in Lemma 7.7 such that $\mathrm{CR}(f, \varepsilon, \overline{Y}) \subset Y$ for $\varepsilon \le \varepsilon(f, Y)$, then by Lemma 3.12 there exists an $\eta = \eta(f, Y) < \varepsilon(f, Y)$ such that every mapping $g \in \mathrm{Lip}(X, \mathbb{R}^d)$ with $\|f - g\|_{\mathrm{Lip}} \le \eta$ is semi-hyperbolic in $\overline{\mathscr{O}_\eta(\mathrm{CR}(f, \overline{Y}))}$.

Theorem 7.9. *Let Y be an open set with $\overline{Y} \subseteq X$ and let $f \in \mathrm{Lip}(X, \mathbb{R}^d)$ be a Lipschitz mapping continuously $\boldsymbol{s}$-semi-hyperbolic in $\mathrm{CR}(f, \overline{Y}) \subset Y$ with constants h, δ. If $g \in \mathrm{Lip}(X, \mathbb{R}^d)$ with $\|g - f\|_{\mathrm{Lip}} < \eta(f, Y)$ then the restricted mappings $f|_{\mathrm{CR}(f, \overline{Y})}$ and $g|_{\mathrm{CR}(g, \overline{Y})}$ are weakly conjugate.*

Proof. Consider $g \in \mathrm{Lip}(X, \mathbb{R}^d)$ satisfying $\|g - f\|_{\mathrm{Lip}} < \eta(f, Y)$. Introduce the family of mappings $g_\lambda(x) = \lambda g(x) + (1 - \lambda) f(x)$, $0 \le \lambda \le 1$, $x \in X$. Clearly,

$$\|g_\lambda - f\|_{\mathrm{Lip}} \le \|g - f\|_{\mathrm{Lip}} < \eta(f, Y). \tag{7.15}$$

To prove the theorem it suffices, by transitivity of the weak conjugacy relation, to find a finite set of parameters $\lambda_0 = 0, \lambda_1, \ldots, \lambda_k = 1$ such that the mappings g_{λ_i} and $g_{\lambda_{i+1}}$ are weak conjugate for each $i = 0, 1, \ldots, k - 1$. Since the family $\{g_\lambda, 0 \le \lambda \le 1\}$ is a compact subset of $\mathrm{Lip}(X, \mathbb{R}^d)$, the existence of the required set of parameters $\{\lambda_i\}$ is followed from the following result

Lemma 7.10. *There exists $\zeta > 0$ such that $g_\lambda|_{\mathrm{CR}(g_\lambda, \overline{Y})}$ is weakly conjugate to $g_{\bar{\lambda}}|_{\mathrm{CR}(g_{\bar{\lambda}}, \overline{Y})}$ for all $\lambda, \bar{\lambda} \in [0, 1]$ with $|\lambda - \bar{\lambda}| < \zeta$.*

Proof. By (7.15) and by the definition of $\eta(f, Y)$, for any $\lambda \in [0, 1]$ the mapping g_λ is continuously semi-hyperbolic in $\mathrm{CR}(f, \eta(f, Y), \overline{Y})$. Since $\eta(f, Y) < \varepsilon(f, Y)$, then by Lemma 7.7 $\mathrm{CR}(f, \eta(f, Y), \overline{Y}) \subset Y$. Again from (7.15) it follows that $\|g_\lambda - f\|_C \le \|g - f\|_C < \eta(f, Y)$, and so any chain recurrent point of g_λ belonging $\overline{Y}$ is $\eta(f, Y)$-chain recurrent point of f. Hence, by definition of the value $\eta(f, Y)$,

$$\mathrm{CR}(g_\lambda, \overline{Y}) \subseteq \mathrm{CR}(f, \eta(f, Y), \overline{Y}) \subset Y, \qquad 0 \le \lambda \le 1. \tag{7.16}$$

Then by Lemmata 3.12 and 7.7 for a given $\bar{\lambda}$ there exist $\eta > 0$ and $\zeta_1 > 0$ such that all mappings $g_\lambda(x)$ with $|\lambda - \bar{\lambda}| < \zeta_1$ are uniformly semi-hyperbolic in $\mathscr{O}_\eta(\mathrm{CR}(g_{\bar{\lambda}}, \overline{Y}))$ with the same split $\boldsymbol{s}$ and constants h, δ.

By (7.16) $\mathrm{CR}(g_{\bar\lambda},\overline{Y}) \subset Y$ and then by Theorem 7.8 there exists a positive ε satisfying

$$\varepsilon < \delta/(2h\alpha) \tag{7.17}$$

such that for each g_λ with $\|g_\lambda - g_{\bar\lambda}\|_C < \varepsilon$ the following property is true:

(i) $g_{\bar\lambda}|_{\mathrm{CR}(g_{\bar\lambda},\overline{Y})}$ is a weak factorization of $g_\lambda|_{\mathrm{CR}(g_\lambda,\overline{Y})}$ with the corresponding mapping

$$\Phi_\lambda(\boldsymbol{x}) = B(\alpha\varepsilon, \boldsymbol{x}) \cap \operatorname*{Tr}_{\pm\infty}(g_{\bar\lambda}, \mathrm{CR}(g_{\bar\lambda},\overline{Y})),$$

for $\boldsymbol{x} \in \mathrm{Tr}_{\pm\infty}(g, \mathrm{CR}(g,\overline{Y}))$.

On the other hand, if we choose a positive ε satisfying

$$q(\varepsilon, g_{\bar\lambda}) < \eta \tag{7.18}$$

where the function $q(\varepsilon, f)$ is defined in Lemma 7.7, then for $g_\lambda, g_{\bar\lambda}$ with $\|g_\lambda - g_{\bar\lambda}\|_C < \varepsilon$ by Lemma 7.7 the following chain of inclusions is true

$$\mathrm{CR}(g_\lambda,\overline{Y}) \subseteq \mathrm{CR}(g_\lambda,\delta,\overline{Y}) \subseteq \mathrm{CR}(g_{\bar\lambda},\varepsilon,\overline{Y}) \subseteq O_\eta(\mathrm{CR}(g_{\bar\lambda},\overline{Y}))$$

with any $\delta < \varepsilon - \|g_\lambda - g_{\bar\lambda}\|_C$. Thus, the following property can by written:

(ii) $\mathrm{CR}(g_\lambda,\overline{Y}) \subset O_\eta(\mathrm{CR}(g_{\bar\lambda},\overline{Y}))$.

Let us choose $\zeta > 0$ and $\lambda \in [0,1]$ such that $\zeta < \zeta_1$ and $|\lambda - \bar\lambda| < \zeta$ imply that $\|g_\lambda - g_{\bar\lambda}\| < \varepsilon$. On account of property (i), $g_{\bar\lambda}|_{\mathrm{CR}(g_{\bar\lambda},\overline{Y})}$ is a weak factorization of $g_\lambda|_{\mathrm{CR}(g_\lambda,\overline{Y})}$. It remains to prove that Φ_λ is an injection and that the inverse mapping Φ_λ^{-1} is continuous.

To prove that Φ_λ is injective choose arbitrary

$$\boldsymbol{x}, \widetilde{\boldsymbol{x}} \in \operatorname*{Tr}_{\pm\infty}(g_\lambda, \mathrm{CR}(g_\lambda,\overline{Y}))$$

such that $\boldsymbol{x} \neq \widetilde{\boldsymbol{x}}$. By property (ii) and Theorem 6.5 the mapping g_λ is δ/h-expansive and so

$$\sup_{i\in\mathbb{Z}} \|x_i - \widetilde{x}_i\| \geq \delta/h. \tag{7.19}$$

On the other hand, by property (i) and the inequality $|\lambda - \bar\lambda| < \zeta$, the mapping Φ_λ satisfies

$$\sup_i \|\Phi_\lambda(\boldsymbol{x})_i - x_i\|,\ \sup_i \|\Phi_\lambda(\widetilde{\boldsymbol{x}})_i - \widetilde{x}_i\| < \alpha\varepsilon$$

from which by (7.17) it follows

$$\sup_i \|\Phi_\lambda(\boldsymbol{x})_i - x_i\|,\ \sup_i \|\Phi_\lambda(\widetilde{\boldsymbol{x}})_i - \widetilde{x}_i\| < \delta/2h. \tag{7.20}$$

The latter inequality together with (7.19) implies

$$\sup_i \|\Phi_\lambda(\boldsymbol{x})_i - \Phi_\lambda(\widetilde{\boldsymbol{x}})_i\| \\ \geq \sup_i \|x_i - \widetilde{x}_i\| - \sup_i \|\Phi_\lambda(\boldsymbol{x})_i - x_i\| - \sup_i \|\Phi_\lambda(\widetilde{\boldsymbol{x}})_i - \widetilde{x}_i\| > 0,$$

i.e. the mapping Φ_λ is an injection.

Finally, prove continuity of the mapping $\Phi_\lambda^{-1}(\boldsymbol{x})$ in metric (7.3), for which we shall follow the proof of continuity of the mapping $\Phi(\boldsymbol{x})$ in Theorem 7.8. Suppose the contrary. Then there exists a sequence

$$\boldsymbol{y}^{(m)} \in \operatorname*{Tr}_{\pm\infty}(g_{\bar{\lambda}}, \mathrm{CR}(g_{\bar{\lambda}}, \overline{Y})),$$

converging in the metric (7.3) to some

$$\boldsymbol{y} \in \operatorname*{Tr}_{\pm\infty}(g_{\bar{\lambda}}, \mathrm{CR}(g_{\bar{\lambda}}, \overline{Y}))$$

such that $\boldsymbol{x}^{(m)} = \Phi_\lambda^{-1}(\boldsymbol{y}^{(m)})$ does not converge to $\boldsymbol{x} = \Phi_\lambda^{-1}(\boldsymbol{y})$. Then without loss of generality we can suppose that $\|x_0^{(m)} - x_0\| \geq \eta$ for some positive η. Choose $\theta > 0$ with $2\alpha\varepsilon < \theta < \delta/h$; such a θ exists by (7.17). Then by Theorem 6.5 and Lemma 6.2 there exists a positive integer $\varkappa(\eta, \theta)$ satisfying

$$\varrho_{\varkappa(\eta,\theta)}(\boldsymbol{x}^{(m)}, \boldsymbol{x}) = \max_{-\varkappa(\eta,\theta) \leq i \leq \varkappa(\eta,\theta)} \|x_i^{(m)} - x_i\| \geq \theta > 2\alpha\varepsilon.$$

From (7.19), (7.20) and the last inequality it then follows that

$$\varrho_{\varkappa(\eta,\theta)}(\boldsymbol{y}^{(m)}, \boldsymbol{y}) \\ \geq \varrho_{\varkappa(\eta,\theta)}(\boldsymbol{x}^{(m)}, \boldsymbol{x}) - \varrho_{\varkappa(\eta,\theta)}(\boldsymbol{y}^{(m)}, \boldsymbol{x}^{(m)}) - \varrho_{\varkappa(\eta,\theta)}(\boldsymbol{y}, \boldsymbol{x}) \\ \geq \theta - 2\alpha\varepsilon > 0.$$

The inequality obtained contradicts to the relation

$$\lim_{m\to\infty} \varrho(\boldsymbol{y}^{(m)}, \boldsymbol{y}) = 0,$$

and thus $\Phi_\lambda^{-1}(\boldsymbol{x})$ is continuous. The proofs of Lemma 7.10 and Theorem 7.9 are completed. □

7.3 Chaos

7.3.1 Definition of Chaotic Behavior

Let X be an open bounded subset of $\mathbb{R}^d$. Consider a mapping $f : X \mapsto X$ and the corresponding discrete-time dynamical system generated by f. A *trajectory* of this system is a sequence $\boldsymbol{x} = \{x_n\}_{n=-n_-}^{\infty}$ satisfying $x_{n+1} = f(x_n)$, for $n = -n_-, \ldots, 0, 1, 2, \ldots$, where $0 \leq n_- \leq \infty$ (note that $n_- = n_-(\boldsymbol{x})$

depends on the particular trajectory $\boldsymbol{x}$). Let $\mathrm{Tr}(f)$ denote the totality of trajectories of the dynamical system generated by f and let $\mathrm{Tr}_\infty(f)$ be the subset of $\boldsymbol{x} \in \mathrm{Tr}(f)$ with $n_-(\boldsymbol{x}) = \infty$.

Important attributes of chaotic behavior include sensitive dependence on initial conditions, an abundance of unstable periodic trajectories and an irregular mixing effect describable informally by the existence of a finite number of separated subsets $U_1, \dots, U_m$ of $\mathbb{R}^d$ which can be visited by trajectories of some fixed iterate f^k of f in any prescribed order. Symbolic dynamics allows a more exact formulation of this last characteristic. Let $\Sigma(m)$ denote the totality of all bi-infinite sequences $\boldsymbol{b} = \{b_n\}_{-\infty}^{\infty}$ with $b_n \in \{1, \dots, m\}$ for $n = 0, \pm 1, \pm 2, \dots$, and let $\mathscr{W} = \{w_1, \dots, w_m\}$ be an unordered subset of distinct points in X. Sequences in $\Sigma(m)$ will be used to prescribe the order in which some disjoint balls of the form

$$U_i = \{z \in X : \ \|z - w_i\| < \varepsilon\}, \qquad i = 1, \dots, m,$$

are to be visited.

Let $\varepsilon > 0$ and k be a positive integer and let Y be a compact subset of X for which $\max_{x,y \in Y} \|x - y\| \geq 2\varepsilon$.

Definition 7.11. *A continuous mapping f is* (ε, k)-chaotic in a neighborhood of Y *if for each finite subset $\mathscr{W} = \{w_1, \dots, w_m\}$ of Y with $\min_{i \neq j} \|w_i - w_j\| \geq 2\varepsilon$ there exists a mapping $Z_f : \Sigma(m) \mapsto \mathrm{Tr}_\infty(f)$ such that*

S1: *For each $\boldsymbol{b} \in \Sigma(m)$ the trajectory $\boldsymbol{z} = Z_f(\boldsymbol{b})$ of f satisfies $z_{kj} \in U_{b_j}$ for all integers j;*
S2: *The mapping $\boldsymbol{b} \mapsto Z_f(\boldsymbol{b})$ is shift invariant in the sense that a 1-shift σ of a sequence $\boldsymbol{b} \in \Sigma(m)$ induces a k-shift σ^k of the corresponding trajectory $Z_f(\boldsymbol{b})$;*
S3: *If a sequence $\boldsymbol{b} \in \Sigma(m)$ is periodic with minimal period p, then the corresponding trajectory $\boldsymbol{z} = Z_f(\boldsymbol{b})$ is periodic with minimal period kp;*
S4: *For each $\eta > 0$ there exists an uncountable subset $\Sigma_0(\eta)$ of $\Sigma(m)$ such that*

$$\limsup_{n\to\infty} \|Z_f(\boldsymbol{a})_n - Z_f(\boldsymbol{b})_n\| \geq \frac{1}{2}\varepsilon \qquad \text{for all} \quad \boldsymbol{a}, \boldsymbol{b} \in \Sigma_0(\eta), \boldsymbol{a} \neq \boldsymbol{b},$$

and

$$\liminf_{n\to\infty} \|Z_f(\boldsymbol{a})_n - Z_f(\boldsymbol{b})_n\| < \eta \quad \text{for all} \qquad \boldsymbol{a}, \boldsymbol{b} \in \Sigma_0(\eta).$$

Note that if f is (ε, k)-chaotic on Y then the topological entropy $\mathscr{E}^{top}$ of f in the ε-neighborhood of Y is positive and satisfies the inequality $\mathscr{E}^{top} \geq k(\varepsilon)^{-1} \ln(C_{\varepsilon/4}(Y))$, where $C_\varepsilon(Y)$ denotes ε-capacity of the compact set Y [27, 39].

The above defining properties of chaotic behavior are similar to those in the Smale transverse homoclinic trajectory theorem (see [67, Theorem 16.2]) with an important difference being that we do not require the existence of an invariant Cantor set. In addition we assume neither the uniqueness of the trajectory $Z_f(\boldsymbol{b})$ for $\boldsymbol{b} \in \Sigma(m)$ nor the continuity of Z_f, so Z_f need not be a semi-conjugacy. Instead, the definition includes Condition S3, which is usually a corollary of uniqueness, and Condition S4 which is a form of sensitivity and irregular mixing as in the Li–Yorke definition of chaos [50], with the subset of trajectories $Z_f(\Sigma_0(\eta))$ corresponding to the Li–Yorke scrambled subset S_0 and the whole set $Z_f(\Sigma(m))$ to their whole set S.

From the physical point of view our definition means that the trajectories of the system f appear to behave chaotically if the measurements are carried out with precision no better than ε at equal time intervals between subsequent measurements no shorter than $k(\varepsilon)$.

Definition 7.12. *The minimal $\varepsilon_0 \geq 0$ with the property that for each $\varepsilon > \varepsilon_0$ the system f is (ε, k)-chaotic for an appropriate k is called the* chaos threshold *of the system f.*

The chaos threshold characterizes accuracy of measurements for which the behavior of the system in the vicinity of the subset Y appears completely chaotic if the time lapse between subsequent measurements is sufficiently large. For instance, a chaotic diffeomorphism f which is topologically conjugate on Y to the shift operator σ on $\Sigma(2)$ has zero chaos threshold.

7.3.2 Perturbations of Bi-Shadowing Systems

A trajectory $\boldsymbol{x} = \{x_n\}_{-\infty}^{\infty}$ of a continuous bounded mapping $f : X \mapsto X \subset \mathbb{R}^d$ is called *homoclinic trajectory* if its elements are not all identical and there exists a point $x_* \in X$ such that $\lim_{n\to\infty} x_{-n} = \lim_{n\to\infty} x_n = x_*$.

Theorem 7.13. *Let $\boldsymbol{x}$ be a homoclinic trajectory of a continuous bounded mapping $f : X \mapsto X$, suppose that f is both bi-shadowing and cyclically bi-shadowing on the unordered set $\{\boldsymbol{x}\}$ with parameters α and β, and define $\gamma(\varepsilon) = \frac{1}{2}\min\{\beta, \varepsilon/\alpha\} > 0$. Then every mapping $g \in C$ satisfying $\|g - f\|_C < \gamma(\varepsilon)$ is (ε, k)-chaotic on a neighborhood of $\{\boldsymbol{x}\}$ for any positive integer $k \geq k(\varepsilon)$, where*

$$k(\varepsilon) = \max\{m : \ \exists \text{ an integer } j_0 : \\ \|x_j - x_*\| \geq \gamma(\varepsilon),\ j = j_0, \ldots, j_0 + m\}\,. \quad (7.21)$$

Proof. In the proof below there will be fixed $\varepsilon \leq \max_{x,y\in\boldsymbol{x}} \|x - y\|$, a positive integer $k > k(\varepsilon)$ and a finite set $\mathscr{W} = \{w_1, \ldots, w_m\}, m > 1$, satisfying $\min_{i\neq j} \|w_i - w_j\| \geq 2\varepsilon$. To prove the theorem we have to construct a mapping Z_f which satisfies Conditions S1–S4. Let $\mathscr{O}_\varrho(S)$ denote the open ϱ-neighborhood of a subset S of $\mathbb{R}^d$.

For each $w \in \mathscr{W}$ with $w \neq x_*$ there can be found a positive integer $m_-(w)$, a non-negative integer $m_+(w)$ and a finite sequence

$$\boldsymbol{u} = \boldsymbol{u}(w) = \left\{u(w)_{-m_-(w)}, u(w)_{-m_-(w)+1}, \ldots, u(w)_{m_+(w)-1}\right\},$$

which are uniquely defined by

$$u(w)_0 = w, \qquad u(w)_j = f(u(w)_{j-1}), \quad j = -m_-(w)+1, \ldots, m_+(w)-1$$

such that

$$u(w)_{-m_-(w)}, f(u(w)_{m_+(w)-1}) \in \mathscr{O}_{\gamma(\varepsilon)}(\{x_*\}), \quad u(w)_j \notin \mathscr{O}_{\gamma(\varepsilon)}(\{x_*\})$$

for $-m_-(w) < j < m_+(w)$.

Consider a given integer $k > k(\varepsilon)$ and a given sequence $\boldsymbol{b} \in \Sigma(m)$. Define a sequence $\boldsymbol{v} = \{v_j\}$ in X by $v_{j+kn} = u(w_n)_j$ for $-m_-(w_n) \le j < m_+(w_n)$ when $w_n \neq x_*$ and by $v_j = x_*$ for all other j. This sequence is a $\gamma(\varepsilon)$-pseudo-trajectory of f. Hence, by the assumed bi-shadowing property of f, for any mapping $g \in C$ with $\|g - f\|_C < \gamma(\varepsilon)$ the set $\mathscr{Z}_g(\boldsymbol{b})$ of all trajectories $\boldsymbol{z}$ satisfying $\|z_{kn} - w_{\boldsymbol{b}_n}\| < \varepsilon$ for all n is not empty. Furthermore, by the assumed cyclically bi-shadowing property of f, this set $\mathscr{Z}_g(\boldsymbol{b})$ contains a trajectory of minimal period pk if $\boldsymbol{b}$ is periodic with minimal period p.

Standard constructions using Zorn's lemma [40, pp. 31–36] allow a (single-valued) selector Z_g of the multi-valued mapping $\boldsymbol{b} \to \mathscr{Z}_g(\boldsymbol{b})$ to be chosen which satisfies Conditions S1–S3 in Definition 7.11. Indeed, let us denote by $\boldsymbol{Z}$ the totality of single-valued selectors Z_g which are defined on subsets of $\mathscr{D}(Z) \subset \Sigma(m)$ and satisfy Conditions S1–S3 and consider this set as being partially ordered by inclusion of the corresponding graphs

$$\mathrm{Gr}(Z_g) = \{(\boldsymbol{b}, Z_g(\boldsymbol{b})) : \ \boldsymbol{b} \in \mathscr{D}(Z)\}.$$

By the construction every chain $\widehat{\boldsymbol{Z}}$ (that is, linearly ordered subset) of $\boldsymbol{Z}$ has an upper bound, the graph of which is defined as the union $\bigcup_{Z_g \in \widehat{\boldsymbol{Z}}} \mathrm{Gr}(Z_g)$.

Hence by Zorn's lemma there exists a maximal element Z_* in the set $\boldsymbol{Z}$. Suppose that the strict inclusion $\mathscr{D}(Z_*) \subset \Sigma(m)$ holds. Then there exist an element $\boldsymbol{b}_* \in \Sigma(m) \setminus \mathscr{D}(Z_*)$. If for some positive integer i the sequence $\boldsymbol{b}_*$ is the ith-shift of a sequence $\boldsymbol{b}_0 \in \mathscr{D}(Z_*)$ then the mapping

$$Z_0(\boldsymbol{b}) = \begin{cases} \quad Z_*(\boldsymbol{b}) & \text{if } \boldsymbol{b} \in \mathscr{D}(Z_*), \\ \sigma^{-ik} Z_*(\boldsymbol{b}_0) & \text{if } \boldsymbol{b} = \boldsymbol{b}_* \end{cases}$$

satisfies Conditions S1–S3 and strictly dominates Z_*, which contradicts the definition of Z_*. On the other hand, if the sequence $\boldsymbol{b}_*$ cannot be represented as a shift of a sequence $\boldsymbol{b} \in \mathscr{D}(Z_*)$ then define $Z_0(\boldsymbol{b})$ as an arbitrary element from the nonempty set $\mathscr{Z}_g(\boldsymbol{b})$ of all trajectories $\boldsymbol{z}$ satisfying $\|z_{kn} - w_{\boldsymbol{b}_n}\| < \varepsilon$ for all n; again the mapping Z_0 satisfies Conditions S1–S3 and strictly dominates Z_*, and we arrive again at a contradiction.

It remains to prove that the selected mapping Z_g also satisfies Condition S4. This follows immediately from the next general result.

Lemma 7.14. *Let (Ω, d) be a compact metric space with the power of the continuum and let $\mathscr{S}$ be the set of sequences $\boldsymbol{s} = \{s_j\}_{j\geq 1}$ with $s_j \in \Omega$ for all $j \geq 1$. Then for each $\eta > 0$ there exists a subset $\mathscr{S}(\eta)$ of $\mathscr{S}$ with the power of the continuum such that $\liminf_{j\to\infty} d(s_j, t_j) < \eta$ for any $\boldsymbol{s}, \boldsymbol{t} \in \mathscr{S}(\eta)$.*

Proof. By compactness of (Ω, d) there exists a finite partition $\mathscr{P}$ of Ω such that $\mathrm{diam}(A) < \eta$ for each $A \in \mathscr{P}$. Consider the equivalence relation E_η on $\mathscr{S}$ defined by: $E_\eta(\boldsymbol{s}, \boldsymbol{t})$ is true if and only if s_j and t_j belong to the same subset from the partition for all sufficiently large j. Denote the set of equivalence classes by $\mathscr{T}$ and note that each equivalence class $\mathscr{E} \in \mathscr{T}$ contains no more than a denumerable set of elements, because each element from $\mathscr{E}$ differs from a chosen element $\{t_j\}_{j\geq 1}$ only for a finite number of indices j. On the other hand, the set $\mathscr{S}$ itself has by construction the power of continuum, so $\mathscr{T}$ has the power of the continuum.

Choose a single element from each equivalence class in $\mathscr{T}$, denote the set of these sequences by $\mathscr{S}_*$ and say that two sequences $\boldsymbol{s}$, $\boldsymbol{t} \in \mathscr{S}_*$ are connected if there exist arbitrarily large j for which s_j and t_j belong to the same subset from the partition. Since there are connected elements in every set which contains more than $\#(\mathscr{P})$ elements, there are connected elements in every denumerable set. The assertion of the Lemma will hold if a subset $\mathscr{S}(\eta) \subseteq \mathscr{S}$ of pairwise connected sequences which has the power of the continuum can be constructed. That this can be done follows by an application of a transfinite analogue of the Ramsey Complete Graph Theorem (cf. [32, p. 608, Theorem 5.23]; see also [33, pp. 427–428]): *If G is a graph of power m, where m is a transfinite cardinal, and if every denumerable subset of G contains two connected elements, then G contains a complete graph of power m.*

This completes the proof of Lemma 7.14 and hence the proof of Theorem 7.13 too. □

The positive integer $k(\varepsilon)$ defined by (7.21) represents the maximum number of iterations of an element of the unordered set $\{\boldsymbol{x}\}$ that can remain outside of the $\gamma(\varepsilon)$ neighborhood of the homoclinic point x_*.

Corollary 7.15. *The chaos threshold of each continuous mapping g satisfying $\|g - f\|_C < \gamma(\varepsilon)$ does not exceed ε.*

The next theorem provides a simple means of locating homoclinic trajectories. Recall that a mapping $f : X \mapsto X$ is said to be *ξ-expansive in X* if for any infinite trajectories $\boldsymbol{x}, \boldsymbol{y} \in \mathrm{Tr}(f)$ either $\boldsymbol{x} = \boldsymbol{y}$ or $\sup_{n\in\mathbb{Z}} \|x_n - y_n\| \geq \xi$.

Theorem 7.16. *Let $\boldsymbol{w} = \{w_0, \ldots, w_{p-1}\}$ be a γ-pseudo-cycle of a continuous mapping $f : \mathscr{X} \mapsto \mathscr{X}$ which is bi-shadowing and cyclically bi-shadowing with parameters α, β on $\{\boldsymbol{w}\}$ and ξ-expansive in $\mathscr{X}$. Suppose that $\gamma \leq \beta$, $\|f(w_0) - w_0\| \leq \beta$ and*

$$2\alpha\gamma < \max_{i,j} \|w_i - w_j\|, \quad \alpha(\gamma + \|f(w_0) - w_0\|) < \xi \tag{7.22}$$

are valid. Then f has a homoclinic trajectory $\boldsymbol{x}$ in an open $\alpha\gamma$ neighborhood of $\{\boldsymbol{w}\}$.

Proof. The point w_0 is clearly an $\|f(w_0)-w_0\|$-pseudo-equilibrium of f. By the assumption that $\|f(w_0) - w_0\| \leq \beta$ and the cyclically bi-shadowing property there exists a proper equilibrium x_* of f satisfying

$$\|x_* - w_0\| \leq \alpha \, \|f(w_0) - w_0\|. \tag{7.23}$$

Now consider the bi-infinite sequence $\{y_j\}$ defined by

$$y_j = \begin{cases} w_0 & \text{if} \quad j < 0 \quad \text{or} \quad j \geq p, \\ w_j & \text{otherwise,} \end{cases}$$

which is obviously a γ-pseudo-trajectory of f. In view of the inequality $\gamma \leq \beta$, there thus exists a proper trajectory $\boldsymbol{x} = \{x_j\}$ in the $\alpha\gamma$-neighborhood of the pseudo-trajectory $\{y_j\}$. The elements of this trajectory are not all identical because of the first inequality in (7.22). To complete the proof it remains to show that the trajectory $\boldsymbol{x}$ is homoclinic. To this end it suffices to establish the limit relationships

$$\lim_{n\to\infty} x_{-n} = \lim_{n\to\infty} x_n = x_*. \tag{7.24}$$

Suppose that

$$\lim_{n\to\infty} x_n = x_*. \tag{7.25}$$

does not hold. Then there exists a sequence of indices $j_m \to \infty$ and an $\varepsilon_1 > 0$ such that

$$\|x_{j_m} - x_*\| > \varepsilon_1, \qquad m = 1, 2, \ldots \;. \tag{7.26}$$

Consider a coordinate-wise limit point $\boldsymbol{x}^* = \{x_n^*\}_{n>-\infty}$ of the sequence of shifted trajectories

$$\boldsymbol{x}^{(m)} = \left\{ x^{(m)}_{-j_m}, x^{(m)}_{-j_m+1}, \ldots \right\}$$

defined by $x^{(m)}_{n-j_m} = x_n$ for $n = 0, 1, 2, \ldots$. Then (7.26) implies

$$\|x_0^* - x_*\| > \varepsilon_1. \tag{7.27}$$

Now every sequence $\boldsymbol{x}^{(m)}$ is a trajectory of f, so $\boldsymbol{x}^*$ is also a trajectory of f because f is continuous. Furthermore, $\boldsymbol{x}^*$ satisfies the inequalities

$$\|x_n^* - x_*\| < \xi \tag{7.28}$$

for all n because of (7.23) and the second inequality in (7.22). The inequalities (7.28) and (7.27) contradict the ξ-expansivity property, so the limit (7.25) must be exist. The other limit is handled similarly, which completes the proof. □

Note that only the direct shadowing part of bi-shadowing has been used in the above proof.

7.3.3 Semi-Hyperbolic Lipschitz Mappings

From Theorems 7.13 and 6.30 we have the following corollary.

Corollary 7.17. *Let f be $(\boldsymbol{s}, h, \delta)$-semi-hyperbolic on a compact subset Y of X and $\boldsymbol{x}$ be a homoclinic trajectory of f contained in Y and define $k(\varepsilon)$ by (7.21) and $\gamma(\varepsilon)$ by*

$$\gamma(\varepsilon) = \frac{1}{3}\min\{\beta(\boldsymbol{s}, h, \delta), \varepsilon/(\alpha(\boldsymbol{s}, h)). \tag{7.29}$$

Then every mapping $g \in C$ satisfying $\|g - f\|_C < \gamma(\varepsilon)$ is (ε, k)-chaotic on a neighborhood of $\{\boldsymbol{x}\}$ for any positive integer $k \geq k(\varepsilon)$.

From this corollary we can establish the chaotic behavior of nonsmooth perturbations of a diffeomorphism with a hyperbolic homoclinic trajectory, see [7, 39]. In fact Theorems 7.13 and 6.30 allow us to consider homoclinic trajectories for much wider classes of systems, such as those with a Marotto snap-back repellor or generalizations thereof [52, 71, 81].

Example 7.18. Let 0 be a hyperbolic fixed point of a smooth mapping f on $\mathbb{R}^d$ and let W^s and W^u denote the local stable and unstable manifolds of f at 0. Suppose that there is a point $x_0 \in W^u \setminus \{0\}$ and a positive integer m with $f^m(x_0) = 0$ and that the linear space $D_{x_0}T^u_{x_0}$ is transversal to T^s_0, where D_x is the derivative of $f^{(m)}$ at the point x and T^u_x and T^s_x are the tangent spaces to the stable and unstable manifolds. Since $x_0 \in W^u \setminus \{0\}$, there exist points x_{-n}, $n > 0$, with $\lim_{n\to\infty} x_{-n} = 0$ and $f(x_{-n}) = x_{-n+1}$. Let N be an arbitrary positive integer, write $r(N) = (\|x_{-N}\| + \|x_{-N+1}\|)/2$ and $U(N, \varepsilon) = \mathscr{O}_{r(N)}(0) \bigcup \mathscr{O}_\varepsilon(x_{-N})$ for arbitrary $\varepsilon > 0$. It can be verified [27] for suitable $\varepsilon > 0$ and integers $N, M > 0$ the mapping

$$F_{N,M,\varepsilon} = \begin{cases} f(x) & \text{if } x \in \mathscr{O}_{r(N)}(0), \\ f^{N+M}(x) & \text{if } x \in \mathscr{O}_\varepsilon(x_{-N}) \end{cases}$$

is semi-hyperbolic on any compact subset of $U(N, \varepsilon)$ and, hence, that the mapping f itself is bi-shadowing and cyclically bi-shadowing on its homoclinic trajectory

$$\boldsymbol{x} = \left\{\ldots, x_{-n}, \ldots, x_{-1}, x_0, f(x_0), \ldots, f^{m-1}(x_0), 0, \ldots, 0, \ldots\right\}.$$

Consequently, for every $\varepsilon > 0$ there exists a $\gamma > 0$ such that the chaos threshold of every small continuous perturbation g with $\|g - f\|_C < \gamma$ does not exceed ε.

7.3.4 Perturbations on Sets of Chain Recurrent Points

Another mechanism for generating robust chaotic behavior in nonsmooth dynamical systems involves its set of chain recurrent points.

Let X is an open bounded subset of $\mathbb{R}^d$, let Y be on open subset of X, and let $f : X \mapsto X$. A point $x \in Y$ is called (ε, Y)-*chain recurrent* for f if there exists a finite ε-pseudo-trajectory $\{x_0, x_1, \ldots, x_L\}$ of f in Y with $x_0 = x_L = x$, that is connecting x with itself. Let $\mathrm{CR}(f, \varepsilon, Y)$ denote the set of all (ε, Y)-chain recurrent points of f. The set of Y-*chain recurrent points* of f is then defined as $\mathrm{CR}(f, Y) = \cap_{\varepsilon>0} \mathrm{CR}(f, \varepsilon, Y)$. Note that if f is continuous and $\overline{Y} \subseteq X$, then $\mathrm{CR}(f, \overline{Y})$ is compact and $f(\mathrm{CR}(f, \overline{Y})) = \mathrm{CR}(f, \overline{Y})$.

The following lemma, which is analogous to Smale's Spectral Decomposition Theorem for Axiom–A diffeomorphisms [39], is an important technical step in formulating and proving our main result in this section.

Lemma 7.19. *Let Y be an open set with $\overline{Y} \subseteq X$ and let $f \in \mathrm{Lip}(X, \mathbb{R}^d)$ be bi-shadowing and expansive on $\mathrm{CR}(f, \overline{Y})$ with $\mathrm{CR}(f, \overline{Y}) \subset Y$. Then*

(1) *the set $\mathrm{CR}(f, \overline{Y})$ can be decomposed into a disjoint union of a finite number of closed sets $\Omega_1, \ldots, \Omega_K$ such that $f(\Omega_i) = \Omega_i$ for $i = 1, \ldots, K$. Moreover, for each $i = 1, \ldots, K$ there exists a bi-infinite trajectory $\boldsymbol{z} = \{z_n\}_{n=-\infty}^{n=\infty}$ of f belonging to Ω_i such that the two 'half-trajectories' $\boldsymbol{z}^+ = \{z_n\}_{n=0}^{\infty}$ and $\boldsymbol{z}^- = \{z_n\}_{n=-\infty}^{0}$ are dense in Ω_i;*
(2) *each set Ω_i can itself be decomposed into a disjoint union of a finite number of closed sets $\Omega_i^j, j = 1, \ldots, n_i$, which are cyclically permutated under f, such that $f^{n_i}|_{\Omega_i^1}$ is topologically mixing, i.e. for any U,V relatively open in Ω_i^1 there exists an $N_{i,U,V}$ such that $(f^{n_i})^n(U) \cap V \neq \emptyset$ for every $n \geq N_{i,U,V}$;*
(3) *the periodic points of $f|_{\mathrm{CR}(f,\overline{Y})}$ are dense in $\mathrm{CR}(f, \overline{Y})$.*

Proof of the first property: Points $x, y \in \mathrm{CR}(f, \overline{Y})$ are called *chain connected* in $\mathrm{CR}(f, \overline{Y})$ if for any $\varepsilon > 0$ there exist an ε-pseudo-trajectory $\boldsymbol{x}$ of f with $x, y \in \{\boldsymbol{x}\} \subseteq \mathrm{CR}(f, \overline{Y})$. The set $\mathrm{CR}(f, X)$ is partitioned into equivalence classes of points connected pairwise with each other and every such class $\mathscr{E}$ is a closed subset of $\mathrm{CR}(f, X)$ satisfying $f(\mathscr{E}) = \mathscr{E}$. These equivalence classes are called the *chain connected components* of the restriction $f|_{\overline{Y}}$. Write $\mathscr{E}(x)$ for the chain connected component containing an element $x \in \mathrm{CR}(f, X)$.

The proofs of the next lemma and its corollary are straightforward, so will be omitted.

Lemma 7.20 (cf. [27, Lemma 4]). *For every $x \in \mathrm{CR}(f, \overline{Y})$ and every $\varepsilon > 0$ there exists a $q = q(x, \varepsilon) > 0$ such that each q-pseudo-cycle $\boldsymbol{x}$ of f with $x \in \{\boldsymbol{x}\} \subseteq \overline{Y}$ satisfies $\{\boldsymbol{x}\} \subseteq \mathscr{O}_\varepsilon(\mathscr{E}(x))$*

Corollary 7.21. *For every $x \in \mathrm{CR}(f, \overline{Y})$ and $q > 0$ there exists a q-pseudo-cycle $\boldsymbol{x}$ of f with $x \in \{\boldsymbol{x}\}$ and $\{\boldsymbol{x}\} \subseteq \mathscr{E}(x)$.*

A proof of the next lemma will be given.

Lemma 7.22. *Let $\mathrm{CR}(f, \overline{Y}) \subset Y$ and $f|_Y$ be $(\boldsymbol{s}, h, \delta)$-semi-hyperbolic on $\mathrm{CR}(f, \overline{Y})$ with respect to continuous mappings $g : X \mapsto X$. Let α, β and ξ be the corresponding bi-shadowing and expansivity parameters in Theorem 6.30* *and suppose that x and $\widehat{x}$ are members of distinct chain connected components $\mathscr{E}$ and $\widehat{\mathscr{E}}$. Then $\|x - \widehat{x}\| \geq \min\{\beta, \xi/\alpha\}$.*

Proof. Suppose the contrary; in particular suppose that

$$\|x - \widehat{x}\| = q < r = \min\{\beta, \xi/\alpha\}.$$

A contradiction to x and $\widehat{x}$ belonging to distinct chain connected components will be obtained by showing that for each $\varepsilon > 0$ there exists an ε-pseudo-cycle $\boldsymbol{x}^* \subseteq \overline{Y}$ which simultaneously contains both x and $\widehat{x}$. For this it suffices to construct two trajectories $\boldsymbol{y}$ and $\boldsymbol{z}$ satisfying, respectively,

$$\boldsymbol{y} \subseteq \overline{Y}, \quad x \in \overline{\{y_n : \ n < 0\}}, \quad \widehat{x} \in \overline{\{y_n : \ n > 0\}}, \tag{7.30}$$

$$\boldsymbol{z} \subseteq \overline{Y}, \quad \widehat{x} \in \overline{\{z_n : \ n < 0\}}, \quad x \in \overline{\{z_n : \ n > 0\}}. \tag{7.31}$$

To construct the sequence $\boldsymbol{y}$ fix a positive number $\varepsilon < r - q$. By Corollary 7.21 there exist ε-pseudo-cycles $\boldsymbol{x} = \{x_0, \ldots, x_m\} \subseteq \mathscr{E}(x)$ and $\widehat{\boldsymbol{x}} = \{\widehat{x}_0, \ldots, \widehat{x}_{\widehat{m}}\} \subseteq \mathscr{E}(\widehat{x})$ satisfying $x_0 = x$ and $\widehat{x}_0 = \widehat{x}$. Denote by $\boldsymbol{u}$ and $\widehat{\boldsymbol{u}}$ the infinite periodic sequences defined, respectively, by $u_k = x_k$ for $k = 0, \ldots, m$ and $\widehat{u}_k = \widehat{x}_k$ for $k = 0, \ldots, \widehat{m}$. Then consider the infinite sequence $\boldsymbol{w}$ defined by

$$w_k = \begin{cases} u_k & \text{if } k \leq 0, \\ \widehat{u}_k & \text{if } k > 0. \end{cases} \tag{7.32}$$

Since $\boldsymbol{u}$ and $\widehat{\boldsymbol{u}}$ are both ε-pseudo trajectories and $\|u_0 - \widehat{u}_0\| = \|x - \widehat{x}\| = q$, the sequence $\boldsymbol{w}$ in (7.32) is a $(q+\varepsilon)$-pseudo-trajectory of f with $\boldsymbol{w} \subseteq \mathrm{CR}(f, \overline{Y})$. As f is bi-shadowing on $\mathrm{CR}(f, \overline{Y})$ with the constants α, β and $q + \varepsilon < r \leq \beta$, there exists a true trajectory $\boldsymbol{y} \in \overline{Y}$ of f which is $\alpha(\boldsymbol{s}, h)q$ close to $\boldsymbol{w}$. It remains to prove that this trajectory satisfies the second and the third inclusions of (7.30). Because $u(-km) = x, \widehat{u}_{k\widehat{m}} = \widehat{x}$ for the positive integer k, it suffices to establish the limit relations

$$\lim_{n \to -\infty} \|y_n - u_n\| = \lim_{n \to \infty} \|y_n - \widehat{u}_n\| = 0,$$

which can be done in the same way as for the limits in (7.24) above using the ξ-expansivity.

The trajectory $\boldsymbol{z}$ is obtained similarly by bi-shadowing with respect to the pseudo-trajectory $\widehat{\boldsymbol{w}}$ defined by $\widehat{w}_k = w_{-k}$ for all integers k. □

The chain connected components of $\mathrm{CR}(f, \overline{Y})$ are the required sets Ω_i. Lemma 7.22 shows that there are only a finite number of them, while their closedness and invariance follow from the definition.

It remains to show that each set Ω_i contains a dense trajectory. Fix i and consider a countable subset of points $\{x^0, \ldots, x^\nu, \ldots\}$ which is dense in Ω_i. Fix $\varepsilon > 0$ sufficiently small. By definition, for each ν there exists ε/ν-pseudo-cycles $\boldsymbol{x}^\nu = \{x_0^\nu, \ldots, x_{n(\nu)-1}^\nu\}$ with $x_0^\nu = x^\nu$. For each ν concatenate $\nu + 1$

times $\boldsymbol{x}^\nu$ to form a pseudo-cycle $\boldsymbol{y}$ of length $(\nu+1)n(\nu)$. Then consider the bi-infinite sequence $\boldsymbol{w}$ of the form

$$\boldsymbol{w} = \{\ldots, \boldsymbol{y}^\nu, \ldots, \boldsymbol{y}^1, \boldsymbol{y}^0, \boldsymbol{y}^1, \ldots, \boldsymbol{y}^\nu, \ldots\}$$

By definition this is a bi-infinite ε-pseudo-trajectory of f, so by the bi-shadowing property there exists a unique trajectory $\boldsymbol{z}$ in Ω_i which is $\alpha\varepsilon$ close to $\boldsymbol{w}$. It is straightforward then to check that $\boldsymbol{z}$ has the necessary properties.

Proof of the second property: The proof follows the usual pattern, see [39] and the references therein, so will only be sketched here. Consider the set Ω_1. The essential step is to construct the subset Ω_1^1.

For each $\varepsilon > 0$ let $P(\varepsilon)$ be the set of positive integers p with the property that there exists a ε-pseudo-cycle $\boldsymbol{x}$ belonging to Ω_1 of the period p. Define $P = \cap_{\varepsilon>0} P(\varepsilon)$, which is non-empty because of cyclic shadowing property, and denote by n_1 the minimal common denominator of the integers in P.

Let x be an arbitrary point in Ω_1. For each $\varepsilon > 0$ let $A(\varepsilon)$ denote the set of all ε-pseudo-cycles containing the point x as their first element. Then denote by $A^*(\varepsilon)$ be totality of the elements of pseudo-cycles in $A(\varepsilon)$ with indices of the form $n_1 k$ for $k = 1, 2, \ldots$. Finally, define $\Omega_1^1 = \cap A_\varepsilon^*$. This set has all of the required properties.

Proof of the third property: This follows straightforwardly from Theorem 6.30.

This completes the proof of Lemma 7.19. □

Using Lemma 7.19, Theorem 7.13 can be modified to establish the existence and robustness of chaotic behavior under continuous perturbation of a semi-hyperbolic mapping on components of its chain recurrent set.

Let Y be an open set with $\overline{Y} \subseteq X$ and let $f \in \mathrm{Lip}(X, \mathbb{R}^d)$ be bi-shadowing and expansive on $\mathrm{CR}(f, \overline{Y})$ with $\mathrm{CR}(f, \overline{Y}) \subset Y$ and with spectral decomposition Ω_i^j, $j = 1, \ldots, n_i$ and $i = 1, \ldots, K$. Denote $\gamma(\varepsilon) = \frac{1}{3}\min\{\beta, \varepsilon/\alpha\}$ where α, β are the bi-shadowing parameters. Property (2) above implies the existence of a *finite* integer $k(\varepsilon)$ defined by

$$k(\varepsilon) = \max\Big\{m:\ \exists v, w \in \Omega_1^1 \text{ with } (f^{n_1})^{(m)}\left(U^1_{\gamma(\varepsilon)}(v)\right) \cap U^1_{\gamma(\varepsilon)}(w) = \emptyset\Big\} \quad (7.33)$$

where $U_\gamma^1(x) = U_\gamma(x) \cap \Omega_1^1$. Indeed, given $\varepsilon > 0$, consider a finite $\gamma(\varepsilon)/3$-net $\boldsymbol{a} = \{a_1, \ldots, a_I\}$ in the compact set Ω_1^1. By the topological mixing property, there then exists a positive integer N such that

$$(f^{n_1})^n\left(U^1_{\gamma(\varepsilon)/3}(a_i)\right) \cap U^1_{\gamma(\varepsilon)/3}(a_j) \neq \emptyset, \qquad i, j = 1, \ldots, I,\ n > N.$$

On the other hand, each set of the form $U^1_{\gamma(\varepsilon)}(x)$ for $x \in \Omega_1^1$ contains at least one set of the form $U^1_{\gamma(\varepsilon)/3}(a_i)$, $i = 1, \ldots, I$, so this N is a finite upper bound for $k(\varepsilon)$ defined by (7.33).

Theorem 7.23. *Let f, Ω_1^1 be as above. Then every mapping $g \in C$ satisfying $\|g - f\|_C < \gamma(\varepsilon)$ is (ε, k)-chaotic on a neighborhood of Ω_1^1 for any positive integer $k \geq k(\varepsilon)$.*

Theorems 7.13 and 7.23 have the following corollary.

Corollary 7.24. *Let $\mathrm{CR}(f, \overline{Y}) \subset Y$ and let f be $(\boldsymbol{s}, h, \delta)$-semi-hyperbolic on $\mathrm{CR}(f, \overline{Y})$ with the spectral decomposition $\Omega_i^j, j = 1, \ldots, n_i$, and $i = 1, \ldots, K$, where Ω_1^1 contains more than one element. Define $\gamma(\varepsilon)$ by (7.29) and $k(\varepsilon)$ by (7.33). Then every mapping $g \in C$ satisfying $\|g - f\|_C < \gamma(\varepsilon)$ is (ε, k)-chaotic on a neighborhood of Ω_1^1 for any positive integer $k > k(\varepsilon)$. In particular, the chaos threshold of every continuous mapping g satisfying $\|g - f\|_C < \gamma(\varepsilon)$ does not exceed ε.*

Example 7.25. Let f be the canonical Smale horseshoe mapping on the square $Q = [-1, 1] \times [-1, 1]$ with compression factor 1/5 and expansion factor 5 (cf. [39]) and let h be a Lipschitz mapping on Q with Lipschitz constant less than 2/3 and with $\|h\|_C < 1/5$. Then the mapping $H = f + h$ is semi-hyperbolic on $\mathrm{CR}(H, Q) \subset \mathrm{Int}\, Q$ [27]; note that H here need not be invertible on $\mathrm{CR}(H, Q)$. Corollary 7.24 is applicable here, so for each $\varepsilon > 0$ the chaos threshold of every sufficiently small continuous perturbation of f does not exceed ε.

8

Applications

In this Chapter we consider several applications of semi-hyperbolicity and its consequences, in particular to delay differential equations, systems with hysteresis and the numerical approximation of chaotic attractors.

8.1 Semi-Hyperbolic Mappings in Banach Spaces

Here it will be shown that semi-hyperbolicity of a Lipschitz mapping on a given set implies bi-shadowing for a wide class of dynamical systems in infinite dimensional Banach spaces. Definitions of shadowing and bi-shadowing are given in the next section and that of semi-hyperbolicity for Lipschitz mappings in Section 8.1.1. The main results are stated in Section 8.1.2, an example of its application to delay equation is introduced in Section 8.2. Note that infinite dimensional perturbations of finite-dimensional semi-hyperbolic mappings were also considered in [22].

8.1.1 Bi-Shadowing with Respect to Completely Continuous Perturbations

Let E be a Banach space with the norm $\|\cdot\|$. Consider a mapping $f : X \mapsto X$ where X is a subset of E.

Definition 8.1. *A dynamical system generated by a mapping $f : X \mapsto X$ is said to be* bi-shadowing with respect to completely continuous perturbations *on a subset K of X with positive parameters α and β if for any given finite pseudo-trajectory $\boldsymbol{x} = \{x_n\} \in \mathrm{Tr}(f, K, \gamma)$ with $0 \leq \gamma \leq \beta$ and any completely continuous mapping $\varphi : X \mapsto X$ satisfying*

$$\|\varphi - f\|_\infty \leq \beta - \gamma \tag{8.1}$$

there exists a trajectory $\boldsymbol{y} = \{y_n\} \in \mathrm{Tr}(\varphi, X)$ such that

$$\|x_n - y_n\| \le \alpha(\gamma + \|\varphi - f\|_\infty) \tag{8.2}$$

for all n *for which* $\boldsymbol{y}$ *is defined.*

Definition 8.1 generalizes in a natural way the notion of bi-shadowing given in Definition 6.19 for mappings in finite-dimensional spaces. As is in finite-dimensional case bi-shadowing with respect to completely continuous perturbations conceptualizes the robust relationship between observed dynamical behavior of a dynamical system and its computer simulations, and can also be interpreted as a form of dynamical structural stability when restricted to specific classes of mappings, such as continuous mappings.

A counterpart of bi-shadowing with respect to completely continuous perturbations for cycles and pseudo-cycles is also useful.

Definition 8.2. *A dynamical system generated by a mapping* $f : X \mapsto X$ *is said to be* cyclically bi-shadowing with respect to completely continuous perturbations *on a subset* K *of* X *with positive parameters* α *and* β *if for any given pseudo-cycle* $\boldsymbol{x} \in \mathrm{Cyc}(f, K, \gamma)$ *with* $0 \le \gamma \le \beta$ *and any completely continuous mapping* $\varphi : X \mapsto X$ *satisfying* (8.1) *there exists a proper cycle* $\boldsymbol{y} \in \mathrm{Cyc}(\varphi, X)$ *of period* N *equal to that of* $\boldsymbol{x}$ *such that* (8.2) *holds for* $n = 0, 1, \ldots, N$.

Note that the cycle $\boldsymbol{y}$ in Definition 8.2 is required only to be in X rather than in the subset K.

At last, we will need a generalization of the notion of semi-hyperbolicity for Lipschitz mappings acting in a Banach space.

Definition 8.3. *Let* $\boldsymbol{s} = (\lambda_s, \lambda_u, \mu_s, \mu_u)$ *be a split and* K *a closed bounded subset of an open set* $X \subseteq E$. *A Lipschitz mapping* $f : X \mapsto X$ *is said to be* $\boldsymbol{s}$-semi-hyperbolic *on* K *if there exist positive real numbers* h, δ *such that for each* $x \in K$ *there exists a splitting* $E = E^s_x \oplus E^u_x$ *with corresponding projectors* P^s_x *and* P^u_x *satisfying Properties SH0(Lip)–SH2(Lip) from Definition 3.6:*

SH0(Lip): *The space* E^u_x *is finite dimensional for all* x *and* $\dim E^u_x = \dim E^u_y$ *if* $x, y \in K$ *with* $\|f(x) - y\| \le \delta$*;*

SH1(Lip): $\sup_{x \in K}\{\|P^s_x\|, \|P^u_x\|\} \le h$;

SH2(Lip): *The inclusion*

$$x + u + v \in X \tag{8.3}$$

and the inequalities

$$\|P^s_y\,(f(x+u+v) - f(x+\widetilde{u}+v))\,\| \le \lambda_s\|u - \widetilde{u}\|, \tag{8.4}$$

$$\|P^s_y\,(f(x+u+v) - f(x+u+\widetilde{v}))\,\| \le \mu_s\|v - \widetilde{v}\|, \tag{8.5}$$

$$\|P^u_y\,(f(x+u+v) - f(x+\widetilde{u}+v))\,\| \le \mu_u\|u - \widetilde{u}\|, \tag{8.6}$$

$$\|P^u_y\,(f(x+u+v) - f(x+u+\widetilde{v}))\,\| \ge \lambda_u\|v - \widetilde{v}\| \tag{8.7}$$

hold for all $x, y \in K$ *with* $\|f(x) - y\| \le \delta$ *and all* $u, \widetilde{u} \in E^s_x$ *and* $v, \widetilde{v} \in E^u_x$ *such that* $\|u\|, \|\widetilde{u}\|, \|v\|, \|\widetilde{v}\| \le \delta$.

Note that continuity in x of the splitting subspaces E_x^s, E_x^u or of the projectors P_x^s, P_x^u is not assumed here, nor is invariance of the splitting subspaces, as is the case in the definition of hyperbolicity of a diffeomorphism. Note also that contrary to the finite-dimensional case, here we require the set K to be closed and bounded but not compact as is in Definition 3.6 since requirement of compactness of the set K in infinite-dimensional case is too restrictive.

8.1.2 Main Results

The main result of this Section is that semi-hyperbolicity is sufficient to ensure bi-shadowing of a dynamical system generated by a Lipschitz mapping with respect to perturbed systems generated by completely continuous mappings.

Theorem 8.4. *Let K be a closed bounded subset of an open set $X \subseteq E$ and $f : X \mapsto X$ a Lipschitz mapping which is $\boldsymbol{s}$-semi-hyperbolic on K with constants h, δ. Then it is bi-shadowing on K, with respect to completely continuous mappings $\varphi : X \mapsto X$ with parameters*

$$\alpha(\boldsymbol{s}, h) = h \frac{\lambda_u - \lambda_s + \mu_s + \mu_u}{(1 - \lambda_s)(\lambda_u - 1) - \mu_s \mu_u} \tag{8.8}$$

and

$$\beta(\boldsymbol{s}, h, \delta) = \delta h^{-1} \frac{(1 - \lambda_s)(\lambda_u - 1) - \mu_s \mu_u}{\max\{\lambda_u - 1 + \mu_s, 1 - \lambda_s + \mu_u\}}. \tag{8.9}$$

The proof of the theorem repeats verbatim that of its finite-dimensional analogue, Theorem 6.21, with the following three minor modifications.

First, throughout the proof the space $\mathbb{R}^d$ should be replaced by the Banach space E. Second, instead of continuity of the operator $\mathscr{H}$ on $\mathscr{I}(\beta)$ we should claim its complete continuity which follows from the complete continuity of the mapping φ. Third, instead of use the Brower Fixed Point Theorem for establishing the existence of a fixed point of the operator $\mathscr{H}$ we should use the Showder Fixed Point Theorem.

Cyclic bi-shadowing is also a consequence of semi-hyperbolicity.

Theorem 8.5. *Let $f : X \mapsto X$ be a Lipschitz mapping which is $\boldsymbol{s}$-semi-hyperbolic on a subset K of X with constants h, δ. Then it is cyclically bi-shadowing on K with parameters $\alpha(\boldsymbol{s}, h)$ and $\beta(\boldsymbol{s}, h)$ given by* (8.8) *and* (8.9)*, with respect to completely continuous mappings $\varphi : X \mapsto X$.*

Remark 8.6. If we want to get bi-shadowing of infinite trajectories for mappings in a Banach space then, in addition to the requirements of Theorem 8.4, we should demand the compactness of the set K (cf. Theorem 6.27).

8.2 Delay Differential Equations

Consider the linear delay equation

$$x'(t) = Ax(t) + Bx(t-h). \tag{8.10}$$

Here $x(t) \in \mathbb{R}^d$, A and B are real d-matrices and h is a positive constant. We shall call this equation *hyperbolic* if its *characteristic equation*

$$\det(wI - A - B\mathrm{e}^{-wh}) = 0 \tag{8.11}$$

does not have a purely imaginary solution $w = \mathrm{i}p$.

To each solution w of this equation there corresponds a solution of the delay equation (8.10) of the form

$$\mathrm{e}^{wt}a, \qquad -\infty < t < \infty \tag{8.12}$$

where a is an eigenvector of the matrix $A + B\mathrm{e}^{-wh}$ with the eigenvalue w.

We will also consider nonlinear delay equations of the form

$$y'(t) = Ay(t) + By(t-h) + F(y(t), y(t-h)). \tag{8.13}$$

Here $F(y, v)$ is a continuous $\mathbb{R}^d$-valued function, which is locally Lipschitz in y and A,B are as before. Denote by $L(F)$ the set of all continuous function $y(t), t \geq -h$ satisfying the equation (8.13) for $t > 0$. In particular, $L(0)$ denotes the set of all continuous function $x(t)$, $t \geq -h$ satisfying the equation (8.10) for $t > 0$.

Theorem 8.7. *Let the equation* (8.10) *be hyperbolic. Then there exists a constant* $\gamma > 0$ *with the following properties.*

a) For each $x(t) \in L(0)$ *and for each uniformly bounded* $F(x, u)$ *there exists a continuous function* $y(t) \in L(F)$*, satisfying the inequality*

$$|y(t) - x(t)| < \gamma \sup_{y,v} |F(y, v)|, \qquad t \geq -h. \tag{8.14}$$

b) Let $F(y, v)$ *be a uniformly bounded and* $y(t) \in L(F)$*. Then there exists a function* $x(t) \in L(0)$ *satisfying* (8.14).

This demonstrates robustness of solutions of a hyperbolic delay equation with respect to arbitrary continuous perturbations of small amplitude. In particular, any nonlinear perturbation (8.13) of a linear equation (8.10) has bounded at $t \to \infty$ solutions which shadow a given bounded at $t \to \infty$ solution of the linear equation. Let us concisely describe the main steps in the proof of this theorem.

Step 1. For each continuous function $\xi(s)$, $s \in [-h, 0]$ the equation (8.13) has a unique solution $y(t; \xi, F)$, $t \geq -h$ which is continuous and satisfies

$$y(s;\xi,F)=\xi(s),\qquad s\in[-h,0],$$

because $F(y,v)$ is supposed continuous, uniformly bounded and satisfy local Lipschitz condition in y. Introduce the shift operator S_F for equation (8.13) by

$$(S_F\xi)(\tau)=y(h-\tau;\xi,F),\qquad -h\le\tau\le 0.$$

The operator S_F is completely continuous in the space $C=C([-h,0],\mathbb{R}^d)$.

In particular, denote by S the shift operator for the linear equation (8.10). The operator S is a linear completely continuous operator in C.

STEP 2. Let $\xi(\tau)$, $\tau\in[-h,0]$ be an eigenfunction of the complexification of the operator S with a complex eigenvalue w. Then by the definition $\xi(s)$ satisfies the equation

$$w\xi'(\tau)=wA\xi(\tau)+B\xi(\tau),\qquad -h\le\tau\le 0.$$

Thus the set of nonzero eigenvalues of the linear operator S coincides with the set of complex number $z=\mathrm{e}^{hw}$ where w is a solution of the equation (8.11) (The corresponding complex eigenfunction are restrictions of functions (8.12) on $[-h,0]$.)

Since S is completely continuous, the spectrum of S consists of zero and all complex numbers e^{wh} where w is a solution of the equation (8.11).

STEP 3. By the previous step and the hyperbolicity of the linear equation (8.10) the spectrum $\sigma(S)$ of the linear operator S consists of two disjoint parts $\sigma(S)=\sigma^s(S)\bigcup\sigma^u(S)$, such that σ^s is located strictly inside the unit disc of a the complex plane and σ^u is located strictly outside the unit disc. By the Decomposition Theorem [66, p. 421], it means that the space C can be decomposed into a direct sum

$$C=E^s\oplus E^u$$

so that both E^s and E^u are invariant for S, the spectrum of the restriction $\sigma(S|_{E^s})=\sigma^s$ of S onto E^s and the spectrum of the restriction $\sigma(S|_{E^u})=\sigma^u$ Further, since S is completely continuous, the subspace E^u is finite-dimensional. Note that the parallel projection P^s of C onto E^s in the direction of E^u can be written in an explicit form as

$$P^s=-\frac{1}{2\pi\mathrm{i}}\int_{|z|=1}(S-zI)^{-1}\,\mathrm{d}z. \tag{8.15}$$

STEP 4. Introduce an auxiliary norm $\|\cdot\|_s$ onto the subspace E^s by

$$\|x\|_s=\sum_{n=0}^{\infty}\|S^nx\|_C.$$

Clearly this norm is equivalent to the norm $\|\cdot\|$ and the restriction of the operator S onto E^s contracts in this norm with some constant $\lambda_s < 1$. Analogously, introduce an auxiliary norm $\|\cdot\|_u$ onto the subspace E^u by

$$\|x\|_u = \sum_{n=0}^{\infty} \|S^{-n}x\|_C.$$

This norm is also equivalent to the C-norm and the restriction of the operator S onto E^u expands in this norm with some constant $\lambda_u > 1$. Introduce in C an auxiliary norm $\|\cdot\|_*$ by $\|\xi\|_* = \max\{\|P^s\xi\|_s, \|P^u\xi\|_u\}$ where P^s is defined by (8.15) and $P^u = I - P^s$. Denote by $\boldsymbol{s}$ the split $(\lambda_s, \lambda_u, 0, 0)$. By construction, the linear operator S is $\boldsymbol{s}$-semi-hyperbolic with constants h, δ where $k = \max\{\|P^s\|_*, \|P^u\|_*\}$ and δ is an arbitrary positive number.

Step 5. In Step 1 the shift operator S_F of the nonlinear equation (8.13) is completely continuous. This operator also satisfies the estimate

$$\|S_F\xi - S\xi\|_* < \gamma_1 \sup_{y,v} |F(y,v)|, \qquad t \geq -h.$$

for some positive γ_1. Thus Theorem 8.4 is applicable and, taking into account the equivalence of norms $\|\cdot\|_C$ and $\|\cdot\|_*$, as a corollary to that theorem it follows that:

Corollary 8.8. *There exist a constant $\gamma > 0$ with the following properties.*

a) For each trajectory

$$\boldsymbol{\eta} = \eta_0, \eta_1, \ldots \tag{8.16}$$

of the shift operator S there exists a trajectory

$$\boldsymbol{\eta}^F = \eta^F_{(0)}, \eta^F_1, \ldots \tag{8.17}$$

of the operator S_F with

$$\|\eta_n - \eta^F_n\|_C \leq \gamma \sup_{y,v} |F(y,v)|, \tag{8.18}$$

b) For each trajectory (8.17) *of the shift operator S_F there exists a trajectory* (8.16) *of the operator S satisfying* (8.18).

Theorem 8.7 then follows. □

Note that the construction of the last Section can be carried out also for some systems described by parabolic equations. Also note that some hysteresis perturbations, like Prandtl, Besseling and Ishlinskii models in plasticity or Preisach, Giltay and Madelung models in magnetism ([45]) can be taken into account both in analysis of delay and parabolic equations.

8.3 Systems with Hysteresis

Consider a smooth mapping $f : \mathbb{R}^d \to \mathbb{R}^d$. The dynamical system generated by a difference equation of the form

$$x_n = f(x_{n-1}), \quad n = 1, 2, \dots , \tag{8.19}$$

is often used in technical, physical or mechanical applications, where f usually occurs via a Poincaré section. Throughout, this will be referred to as the system f. Realistically, a system (8.19) can describe the actual underlying system only approximately. Thus an important mathematical problem is the robustness of the system to perturbations. Classical results in this direction state that a C^r dynamical system preserves some of its structural properties under a small *smooth* perturbation [34, 60, 72]. However, there are some kinds of *nonsmooth* perturbations which are very important. In this Section we analyze a specific class of perturbations which arise in systems with weak hysteresis nonlinearities. An important feature of such models is that hysteresis nonlinearities are treated as continuous but nonsmooth dynamical systems W, often with an infinite dimensional set $\Omega = \Omega(W)$ of internal states ω. This includes such nonlinearities as play, stop, the Besseling–Ishlinskii and Preisach–Giltay models and so on. Further details may be found in [45, 54].

In such situations the natural description of state space of a perturbed system (8.19) is $\mathscr{Q} = \mathbb{R}^d \times \Omega$. So it is more realistic to describe the dynamics of the perturbed system W by relations of the form

$$(x_n, \omega_n) = W(x_{n-1}, \omega_{n-1}) = (\varphi(x_{n-1}, \omega_{n-1}), \psi(x_{n-1}, \omega_{n-1})).$$

Here $\varphi : \mathbb{R}^d \times \Omega \to \mathbb{R}^d$ and $\psi : \mathbb{R}^d \times \Omega \to \Omega$ are continuous mappings. Some concrete examples of systems which arise in the theory of hysteresis nonlinearities are given in Sections 8.3.1 and 8.3.2.

We are concerned with the relationship between the trajectories of a smooth system f and those of systems W which are close to f in some sense. An appropriate measure of the distance between the two types of system was introduced and investigated in Section 6.2.3. It is important to note that, without extra assumptions, the system (8.19) is not structurally stable in general.

A natural, additional assumption is hyperbolicity. In these circumstances, estimates of the distance between trajectories of f and its perturbation W should not depend explicitly upon the time interval over which the trajectories are considered. Instead, it is preferable that any estimate should be uniform so long as the trajectories remain in the region in question. This is the principal question that we address in this Section.

8.3.1 Transducer Stop

Recall that the nonlinearity *stop with threshold value h* or *transducer stop* ([45, p. 23–24]) is a system $\mathbf{U}_h$ with the state space $[-h, h]$, scalar inputs $u(t)$

and outputs $\omega(t)$. For a smooth input $u(t)$, $t \geq 0$, and initial state $\omega_0 \in [-h, h]$ the corresponding output $\omega(t) = (U_h[\omega_0]u)(t)$, $t \geq 0$, is defined as a unique absolutely continuous solution of the problem

$$\omega' = q(\omega, u'(t)), \qquad \omega(0) = \omega_0$$

where

$$q(\omega, u) = \begin{cases} \min\{u, 0\} & \text{if } \omega \geq h, \\ u & \text{if } |\omega| < h, \\ \max\{u, 0\} & \text{if } \omega \leq -h. \end{cases}$$

Consider the system described by the equations

$$x' = G(x, \omega), \quad \omega(t) = (U_h[\omega_0](c, x))(t). \tag{8.20}$$

Here $x \in \mathbb{R}^d$; $h > 0$ and $\omega \in [-h, h]$ are parameters, c is a fixed vector from $\mathbb{R}^d$ and U_h is the stop nonlinearity with threshold value h. Equations of such type arise as description of mechanical systems with elastic-plastic Prager elements, technical systems with plays or stops and many control systems.

Suppose that the function G satisfies a global Lipschitz condition. Then the equation (8.20) has a unique solution for any initial condition $x(0) = x_0$ and each initial state ω_0 of the hysteresis nonlinearity U_h. Let the shift operator $S_h(x_0, \omega_0)$ denote the image of the initial value (x_0, ω_0) after unit time along the trajectories of the system (8.20). Suppose that $F(x) = G(x, 0)$ is a smooth function, satisfying $F(0) = 0$ and the matrix DF_0 does not have eigenvalues with zero real part. Similarly, let $S_0(x_0)$ be the image of the initial value x_0 after unit time along the trajectories of equation

$$x' = F(x). \tag{8.21}$$

The mappings $W(x, \omega) = S_h(x, \omega)$ and $f(x) = S_0(x)$ generate dynamical systems W and f respectively, where the state space of the system W is the product $\mathbb{R}^d \times [-h, h]$

Clearly, the system f is semi-hyperbolic in some open ball centered at the origin. From Theorem 6.15 it follows immediately

Theorem 8.9. *There exist $\alpha > 0$ and $h_0 > 0$ with the following property: for any trajectory $x(t) \in \mathscr{B}$, $0 \leq t < t_* \leq \infty$, of the equation* (8.21) *and any $h \leq h_0$ there exists a trajectory $(x_h(t), \omega_h(t))$, $0 \leq t < t_*$, of* (8.20) *satisfying*

$$|x(t) - x_h(t)| \leq \alpha h, \qquad 0 \leq t < t_*.$$

Corollary 8.10. *There exist $\alpha > 0$ and $h_0 > 0$ with the following property: for any $x_0 \in \mathscr{B}$ belonging to the stable manifold of the equation* (8.21) *there exists a trajectory $(x_h(t), \omega_h(t))$, $t \geq 0$, of* (8.20) *satisfying*

$$|x_0(t) - x_h(t)| \leq \alpha h, \qquad t > 0.$$

This result can be treated as a kind of 'the stable manifold robustness theorem' with respect to hysteresis perturbations of a system.

Analogues of Theorem 8.9 are valid for equations with such nonlinearities as play or generalized play, with multi-dimensional plays and stops, with Mizes and Treska models [45], and so on.

8.3.2 Ishlinskii and Besseling Systems

Let U_h be the stop nonlinearity with threshold h, as in Section 8.3.1. Consider h as a parameter, $0 \le h \le \infty$, and let μ be a Borel measure on $[0, \infty]$ satisfying

$$\int_0^\infty h \, \mathrm{d}\mu(h) < \infty.$$

Denote by $\mathscr{H}$ the totality of continuous functions $z(h)$, $h \ge 0$, satisfying $|z(h)| \le h$. Now introduce a system W_μ, with scalar inputs and outputs and with state space $\mathscr{H}$, as follows. For a given smooth input $u(t)$, $t \ge 0$, and an initial state $z_0 \in \mathscr{H}$, the corresponding output $z(t) = (W_\mu[z_0]u)(t)$, $t \ge 0$, is defined as

$$z(t) = \int_0^\infty (U[z_0(h)]u)(t) \, \mathrm{d}\mu(h).$$

A model of this type includes fundamental mechanical models such as the Ishlinskii and Besseling systems ([45, p. 342–346]). It might be thought of as describing a continuum of linked transducers.

Suppose that the function G is globally Lipschitz, as in previous Section. Consider the system described by equations

$$x' = G(x, z), \quad z(t) = (W_\mu[z_0](c, x))(t). \tag{8.22}$$

This extends the system (8.20). Again, (8.22) has a unique solution $x(t)$, $t \ge 0$, for each initial condition $x(0) = x_0$. Define the corresponding shift operator $S_\mu(x_0)$. From Theorem 6.15 it follows that

Theorem 8.11. *There exist $\alpha > 0$ and $\varepsilon_0 > 0$ with the following property: for any trajectory $x(t)$, $0 \le t < t_* \le \infty$, of the equation* (8.21)*, for any measure μ satisfying*

$$r(\mu) = \int_0^\infty h \, \mathrm{d}\mu(h) \le \varepsilon_0$$

and any $z(h) \in \mathscr{H}$, there exists a trajectory $(x_\mu(t), z_\mu(t))$, $0 \le t < t_*$, of (8.22) *satisfying*

$$|x(t) - x_\mu(t)| \le \alpha r(\mu), \qquad 0 \le t < t_*.$$

Analogues of Theorem 8.11 are valid for models such as the multi-dimensional Ishlinskii system, the Preisach–Giltay model [45] and its multi-dimensional analogue [46].

8.4 Chaotic Attractors under Discretization

The solutions $\phi(t;x_0)$ with initial values $\phi(0;x_0) = x_0$ of an ordinary differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x) \tag{8.23}$$

on $\mathbb{R}^d$ generate a continuous-time dynamical system on $\mathbb{R}^d$, which is a flow $\{\phi(t:\cdot)\}$ of diffeomorphisms on $\mathbb{R}^d$ if f is smooth enough. It is well-known (cf. [42, 76]) that if this system has a compact attractor A_0, then the discrete-time dynamical system generated by a one-step numerical scheme

$$x_{n+1} = F_h(x_n) := x_n + hf(h;x_n) \tag{8.24}$$

with constant time-step $h > 0$, such as an Euler or Runge–Kutta scheme, also has a compact attractor A_h such that

$$H^*(A_h, A_0) \to 0 \qquad \text{as } h \to 0+, \tag{8.25}$$

where H^* is the *Hausdorff separation* of nonempty compact subsets of $\mathbb{R}^d$ defined by

$$H^*(A,B) = \max_{a\in A} \operatorname{dist}(a,B) = \max_{a\in A} \min_{b\in B} \|a-b\|.$$

Convergence (8.25) is often described as the upper semi-continuity of the numerical attractor A_h at $h = 0$.

In general, a numerical attractor A_h need not be lower semi-continuity under time-discretization at $h = 0$, that is with

$$H^*(A_0, A_h) \to 0 \qquad \text{as } h \to 0+, \tag{8.26}$$

unless additional assumptions are made about the dynamics inside the attractor A_0. Here it will be assumed that A_0 is a hyperbolic invariant set for the flow generated by the differential equation (8.23) containing a point $\bar{x}_0 \in A_0$ for which the ω-limit set $\omega^+(\bar{x}_0) = A_0$, which is typical of many chaotic attractors, where

$$\omega^+(\bar{x}_0) = \bigcap_{T\geq 0} \overline{\bigcup_{s\geq T} \{\phi(s;\bar{x}_0\}}. \tag{8.27}$$

The main technical tool to be used in the proof is the bi-shadowing property satisfied by the time–1 map $\phi(1:\cdot)$ of the continuous-time system on the hyperbolic set A_0.

Recall that a one-step numerical scheme (8.24) with time-step $h > 0$ is of pth order if for every bounded subset D of $\mathbb{R}^d$ there exists a constant C_D such that its one-step discretization error satisfies the inequality

$$\|\phi(h;x_0) - F_h(x_0)\| \leq C_D h^{p+1}$$

for all $x_0 \in D$ and $h > 0$. There then exists a constant $\gamma > 0$ such that the global discretization error satisfies the inequality

$$\|\phi(nh; x_0) - F_h^n(x_0)\| \le C_D \mathrm{e}^{\gamma nh}\, h^p \tag{8.28}$$

for all $x_0 \in D$, $h > 0$ and $n = 0, 1, \ldots, N_{D,h}$ such that $\phi(t; x_0) \in D$ for $0 \le t \le hN_{D,h}$.

Theorem 8.12. *Suppose that the mapping $f : \mathbb{R}^d \mapsto \mathbb{R}^d$ in the differential equation* (8.23) *is $p+1$ times continuously differentiable and that the one-step numerical scheme $F_h : \mathbb{R}^d \mapsto \mathbb{R}^d$* (8.24) *is of pth order. If the differential equation* (8.23) *has a hyperbolic attractor A_0 which contains a dense recurrent trajectory, then the numerical scheme* (8.24) *has an attractor A_h which is continuous in h at $h = 0$, that is*

$$H(A_h, A_0) = \max\{H^*(A_h, A_0), H^*(A_0, A_h)\} \to 0 \qquad \textit{as } h \to 0+,$$

Proof. By the assumed existence of the attractor A_0 there is an open bounded subset $\mathscr{X}$ of $\mathbb{R}^d$ which is positively invariant for the flow ϕ generated by the differential equation (8.23) and attention can be restricted to this set without loss of generality. The results of [37, 42, 76] then establish the existence of a numerical attractor A_h of the numerical scheme (8.24) for sufficiently small $h > 0$ which are upper semi-continuous in h at $h = 0$ in the sense of convergence (8.25). Hence it remains only to show the lower semi-continuity of the A_h at $h = 0$.

Write $\Phi(x) := \phi(1 : x)$ for the time–1 mapping of the differential equation (8.23), so $\Phi : \mathscr{X} \mapsto \mathscr{X}$. Note that Φ is a diffeomorphism on $\Phi(\mathscr{X})$ and that the original attractor $A_0 \subset \Phi(\mathscr{X})$ is also an attractor for the discrete-time dynamical system generated by Φ, and hence A_0 is invariant under Φ. Moreover, the diffeomorphism Φ inherits the hyperbolicity of the flow ϕ on A_0. Hence by Theorem 6.19 there exist positive constants α and β such that the diffeomorphism Φ is bi-shadowing with parameters α and β on A_0 with respect to the space of continuous functions $\mathscr{C}(\mathscr{X})$.

It can be assumed without loss of generality that mapping F_h in the numerical scheme (8.24) satisfies $F_h(\mathscr{X}) \subseteq \mathscr{X}$ for sufficiently small h, in particular for $h \le 1$. The $F_h^n \in \mathscr{C}(\mathscr{X})$ for $n = 1, 2, \ldots$ and by (8.28)

$$\sup_{x \in \mathscr{X}} \|\phi(nh; x) - F_h^n(x)\| \le C_{\mathscr{X}} \mathrm{e}^{\gamma nh}\, h^p$$

$n = 1, 2, \ldots$. Suppose that $C_{\mathscr{X}} \mathrm{e}^{\gamma}\, h^p \le \beta$ and that there is an integer N_h such that $h = 1/N_h$, that is $N_h h = 1$. Then

$$\left\|\Phi - F_h^{N_h}\right\|_\infty := \sup_{x \in \mathscr{X}} \left\|\Phi(x) - F_h^{N_h}(x)\right\| \le C_{\mathscr{X}}\, \mathrm{e}^{\gamma}\, h^p \le \beta. \tag{8.29}$$

Now let $\bar{x}_0 \in A_0$ be any point such that $\omega^+(\bar{x}_0) = A_0$ and define $\bar{x}_0(t) = \phi(t, \bar{x}_0)$ for each $t \in [0, 1]$. For each such t consider the trajectory $\{\Phi^n(\bar{x}_0(t))\}_{n \in \mathbf{N}}$ of the mapping Φ. By the Bi-shadowing Property of Φin A_0 with $\delta = 0$ there then exists a $\bar{y}_0(t) \in \mathscr{X}$ such that

$$\left\| \Phi^n(\bar{x}_0(t)) - F_h^{nN_h}(\bar{y}_0(t)) \right\| \leq \alpha \left\| \Phi - F_h^{N_h} \right\|_\infty \leq \alpha C_{\mathscr{X}} \mathrm{e}^\gamma \, h^p \tag{8.30}$$

for all $n = 0, 1, \ldots$. In particular, for $n = 0$ we have

$$\|\bar{x}_0(t) - \bar{y}_0(t)\| \leq \alpha C_{\mathscr{X}} \mathrm{e}^\gamma \, h^p,$$

that is $\bar{y}_0(t) \in \mathscr{B}_{\alpha C_{\mathscr{X}} \mathrm{e}^\gamma \, h^p} \left(\{\bar{x}_0(t)\}\right) \subset \mathscr{B}_{\alpha C_{\mathscr{X}} \mathrm{e}^\gamma \, h^p} \left(A_0\right)$ where $\mathscr{B}_\varrho(S)$ is the closed ball of radius ϱ about a subset S of $\mathbb{R}^d$.

Since the points $\bar{y}_0(t)$ for $0 \leq t \leq 1$ belong to a common bounded subset of $\mathbb{R}^d$, by the absorbing property of the (maximal) numerical attractor A_h for each $\varepsilon > 0$ there exists an integer $N(\varepsilon, h)$ such that

$$y_j(t) := F_h^j \left(\bar{y}_0(t)\right) \in \mathscr{B}_{\varepsilon/2} \left(A_h\right) \qquad \text{for all } j \geq N(\varepsilon, h), 0 \leq t \leq 1.$$

But from (8.30) we also have $\|\Phi^n(\bar{x}_0(t)) - y_{nN_h}(t)\| \leq \alpha C_{\mathscr{X}} \mathrm{e}^\gamma \, h^p$, that is

$$\Phi^n(\bar{x}_0(t)) \in \mathscr{B}_{\alpha C_{\mathscr{X}} \mathrm{e}^\gamma \, h^p} \left(\{y_{nN_h}(t)\}\right) \qquad \text{for all } n \geq 0, 0 \leq t \leq 1.$$

Hence $\Phi^n(\bar{x}_0(t)) \in \mathscr{B}_{\alpha C_{\mathscr{X}} \mathrm{e}^\gamma \, h^p + \varepsilon/2} \left(A_h\right)$ for all $n \geq N(\varepsilon, h)/N_h$ and $0 \leq t \leq 1$, or equivalently

$$\bigcup_{\substack{n \geq N(\varepsilon,h)/N_h, \\ 0 \leq t \leq 1}} \{\Phi^n(\bar{x}_0(t))\} \subseteq \mathscr{B}_{\alpha C_{\mathscr{X}} \mathrm{e}^\gamma \, h^p + \varepsilon/2} \left(A_h\right). \tag{8.31}$$

Now the group property of the flow ϕ gives

$$\Phi^n(\bar{x}_0(t)) = \phi(n, \bar{x}_0(t)) = \phi(n, \phi(t, \bar{x}_0(t))) = \phi(n + t, \bar{x}_0),$$

so

$$\bigcup_{\substack{n \geq N(\varepsilon,h)/N_h, \\ 0 \leq t \leq 1}} \{\Phi^n(\bar{x}_0(t))\} = \bigcup_{s \geq N(\varepsilon,h)/N_h} \{\phi(s, \bar{x}_0)\}$$

and hence from (8.31)

$$\bigcup_{s \geq N(\varepsilon,h)/N_h} \{\phi(s, \bar{x}_0)\} \subseteq \mathscr{B}_{\alpha C_{\mathscr{X}} \mathrm{e}^\gamma \, h^p + \varepsilon/2} \left(A_h\right).$$

But $\omega^+(\bar{x}_0) = A_0$, so by 8.27

$$A_0 \subseteq \overline{\bigcup_{s \geq N(\varepsilon,h)/N_h} \{\phi(s, \bar{x}_0)\}} \subseteq \mathscr{B}_{\alpha C_{\mathscr{X}} \mathrm{e}^\gamma \, h^p + \varepsilon/2} \left(A_h\right)$$

or equivalently

$$H^* \left(A_0, A_h\right) \leq \alpha C_{\mathscr{X}} \mathrm{e}^\gamma \, h^p + \varepsilon/2.$$

Restricting h further to $0 < h < h_0(\varepsilon)$ so that $\alpha C_{\mathscr{X}} \mathrm{e}^\gamma h^p \leq \varepsilon/2$ also holds and so

$$H^* \left(A_0, A_h\right) \leq \varepsilon \tag{8.32}$$

for all $0 < h < h_0(\varepsilon)$ with h^{-1} an integer.

Finally, the case that h^{-1} is not an integer needs to be considered. Let N_h be the integer such that $hN_h < 1 < h(N_h + 1)$. Then

$$\begin{aligned}
\|F_h^{N_h}(x) - \Phi(x)\| &= \|F_h^{N_h}(x) - \phi(1;x)\| \\
&\leq \|F_h^{N_h}(x) - \phi(hN_h;x)\| + \|\phi(hN_h;x) - \phi(1;x)\| \\
&\leq C_{\mathscr{X}} \mathrm{e}^{\gamma h N_h} h^p + \left\| \int_{hN_h}^{1} f\left(\phi(s;x)\right) \, \mathrm{d}s \right\| \\
&\leq C_{\mathscr{X}} \mathrm{e}^{\gamma} h^p + M(1 - hN_h) \leq C_{\mathscr{X}} \mathrm{e}^{\gamma} h^p + Mh \leq Kh
\end{aligned}$$

where $M = \max_{x \in \mathscr{X}} \|f(x)\|$ and $K = 2 \max\{C_{\mathscr{X}} \mathrm{e}^{\gamma}, M\}$. Repeating the above argument with

$$\|F_h^{N_h} - \Phi\|_\infty \leq Kh$$

instead of the global discretization estimate (8.29), that is with $C_{\mathscr{X}} \mathrm{e}^{\gamma} h^p$ replaced by Kh. With an appropriate modification to $h_0(\varepsilon)$, the inequality (8.32) then holds for all h sufficiently small. Convergence (8.26), that is the lower semi-continuity of the numerical attractor at $h = 0$, then holds.

This completes the proof of Theorem 8.12. □

References

1. P. Alexandroff and H. Hopf, *Topologie*, Springer–Verlag, Berlin, 1974. 4, 91
2. A. Al–Nayef, P. Diamond, P.E. Kloeden, V. Kozyakin and A. Pokrovskii, Bishadowing and delay equations, *Dynamics & Stability of Systems,* **10** (1996), 121–129. 4
3. A. Al–Nayef, P.E. Kloeden und A. Pokrovskii, Expansivity of nonsmooth functional differential equations. *J. Math. Anal. Applns.,* **208** (1997), 453–461. 4
4. A. Al–Nayef, P.E. Kloeden and A. Pokrovskii, Condensing operators, semihyperbolic mappings and neutral delay equations, *J. Differential Equations.* (submitted) 4
5. D. Anosov, Geodesic flows on closed Riemannian manifolds with negative curvature, *Proc. Steklov Inst. Math.* **90** (1967), 1–235. 2
6. D.V. Anosov and Ya.G. Sinai, Some Smooth Ergodic Systems, *Uspekhi Matem. Nauk*, **22** 5 (1967), 107–172. 2, 9, 10
7. D.V. Anosov D.V. (Ed.), *Dynamical Systems 9. Dynamical systems with hyperbolic behaviour,* Encyclopaedia of Mathematical Sciences, Volume 1, Springer–Verlag, Berlin, 1988. 2, 117
8. A.L Badoev and B.N. Sadovskii, An example of a densifying operator in the theory of differential equations with a deviating argument of neutral type, *Soviet Math. Dokl.,* **10** (1969), 724–728.
9. J.-M. Beyn, On the Numerical Approximation of Phase Portraits near Stationary Points, *SIAM J. Numer. Anal.*, **24** (1987), 1095–1113. 3
10. R. Bowen, Topological entropy and Axiom A, *Global Analysis, Amer. Math. Soc. Proc. Symposia Pure Math.*, **14** (1968), p. 23–41.
11. P.M. Binder, Machine iteration of a linear function: Local behaviour, *Computers Math. Applns.*, **21** (1991), 133–140.
12. M. Blank, Small perturbations of chaotic dynamical systems, *Russian Math. Surveys*, **44** (6) (1989), 1–33.
13. L.A. Bunimovich, Ya.B. Pesin, Ya.G. Sinai and M.V. Yakobson, Ergodic Theory of Smooth Dynamical Stystems, II. *Modern problems of mathematics. Fundamental directions. Vol. 2 (Itogi Nauki i Tekhniki. VINITI AN SSSR)*, Moscow, 1985, 113–231. 9, 10
14. S.-N. Chow, X.B. Lin and K.J. Palmer, A shadowing lemma for maps in infinite dimensions, *Proc. EQUADIFF'87.* (Editors: C.M. Dafermos, G. Ladas and G. Papanicolaou). Marcel–Dekker, New York, 1989, pp. 127–136. 3

15. S.-N. Chow, X.B. Lin and K.J. Palmer, A shadowing lemma with applications to semilinear parabolic equations, *SIAM J. Math. Anal.*, **20** (1989), 547–557. 3, 81
16. S.-N. Chow and K.J. Palmer, On the numerical computation of orbits of dynamical systems: the higher dimensional case, *J. Complexity*, **8** (1992), 398–423. 3
17. S.-N Chow and E.S. van Vleck, A shadowing lemma for diffeomorphisms, *Random & Computational Dynamics,* **1** (1993), 197–218. 3, 81
18. R. Corless and S.Yu. Pilyugin, Approximate and real trajectories for generic dynamical systems, *J. Math. Anal. Applns.,* **189** (1995), 409–423. 3
19. K. Deimling, *Nonlinear Functional Analysis*, Springer–Verlag, Heidelberg, 1985.
20. Li Desheng and P.E. Kloeden, Equi-attraction and the continuous dependence of attractors on parameters, *Glasgow Math. J.,* **46** (2004), 131-141.
21. P. Diamond and P.E. Kloeden, Spatial discretization of mappings, *Computers Math. Applns.*, **25** (1993), 85–94. 4
22. P. Diamond, P.E. Kloeden, V.S. Kozyakin, M.A. Krasnosel'skii and A.V. Pokrovskii, Robustness of trajectories of dynamical systems with respect to a class of hysteresis perturbations, *Doklady Akad. Nauk,* **343** 1 (1995), 1–3. 4, 123
23. P. Diamond, P.E. Kloeden, V.S. Kozyakin, M.A. Krasnosel'skii and A.V. Pokrovskii, Robustness of dynamical systems to a class of semi-hyperbolic mappings, *Nonlinear Analysis TMA*, **26** (1996), 351–361. 4
24. P. Diamond, P.E. Kloeden, V.S. Kozyakin, M.A. Krasnosel'skii and A.V. Pokrovskii, Periodic solutions of nonsmooth systems with chaotic disturbances, *Automatika i Telemekhanika,* no. 5 (1995), 34–41. 4
25. P. Diamond, P.E. Kloeden, V.S. Kozyakin and A.V. Pokrovskii, Computer robustness of semi-hyperbolic mappings, *Random & Computational Dynamics,* **3** (1995), 57–70. 4
26. P. Diamond, P.E. Kloeden, V.S. Kozyakin and A.V. Pokrovskii, Expansivity of semi-hyperbolic Lipschitz mappings. *Bulletin Austral. Math. Soc.,* **51** (1995), 301–308. 4
27. P. Diamond, P.E. Kloeden, V.S. Kozyakin and A.V. Pokrovskii, Robustness of observed behaviour of semi-hyperbolic dynamical systems, *Automat. Remote Control,* **56** (1995), no. 11, part 2, (1996) 1627–1636. 4, 112, 117, 118, 121
28. P. Diamond, P.E. Kloeden, V.S. Kozyakin and A.V. Pokrovskii, Semihyperbolic mappings. *J. Nonlinear Science,* **5** (1995), 419–431. 4
29. P. Diamond, P.E. Kloeden, M.A. Krasnosel'skii and A. Pokrovskii, Chaotic dynamics in nonsmooth perturbations of bishadowing systems. *Arab J. Math. Sci.,* **6** (2000), 41–74. 4
30. P. Diamond, P.E. Kloeden and A.V. Pokrovskii, Shadowing and approximation in dynamical systems, in *CMA 3rd Miniconference on Analysis*, (Editors: G. Martin and H.B. Thompson), Proc. Centre Math. Applns. **33** (1994), 47–60. 4
31. P. Diamond, P.E. Kloeden and A.V. Pokrovskii, Weakly chain recurrent points and spatial discretizations of dynamical systems, *Random & Computational Dynamics*, **2** (1994), 97–110. 4
32. B. Dushnik and E.W. Miller, Partially ordered sets, *Amer. J. Math.,* **63** (1941), 600–610. 115
33. P. Erdös and R. Rado, Combinatorical theorems on classifications of subsets of a given set, *Proc. London Math. Soc. (Third Series),* **2** (1952), 115

34. J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems and Bifurcation of Vector Fields*, Springer-Verlag, New York, 1983. 129
35. J.K. Hale, *Theory of Functional Differential Equations*, Springer–Verlag, Heidelberg. 1977.
36. J.K. Hale, Functional differential equations with infinite delays, *J. Math. Anal. Applns.*, **48** (1974), 276–283.
37. J.K. Hale, *Asymptotic Behavior of Dissipative Dynamical Systems*, Amer. Math. Soc., Providence, RI, 1988. 133
38. M.W. Hirsch and C.C. Pugh, Stable manifolds and hyperbolic sets, *Proc. Symp. Pure Math.,* **14** (1970), 133–163. 2, 69
39. A.E. Katok and B. Hasselblatt, *Introduction to the Modern Theory of Dynamical Systems*, Cambridge University Press, Cambridge, 1995. 2, 3, 112, 117, 118, 120, 121
40. J. Kelley, *General Topology*, Princeton, N.J.: D. Van Nostrand Company, Inc., 1955. 114
41. Yu. Kifer, *Random Perturbations of Dynamical Systems*, Birkhäuser, Boston, 1988.
42. P.E. Kloeden and J. Lorenz, Stable attracting sets in dynamical systems and in their one–step discretizations, *SIAM J. Numer. Anal.,* **23** (1986), 986–995. 4, 132, 133
43. P.E. Kloeden and J. Ombach, Hyperbolic homeomorphisms and bi-shadowing. *Annales Polonici Mathematici.* (to appear). 4
44. P.E. Kloeden and S. Piskarev, Discrete convergence and the equivalence of equi-attraction and the continuous convergence of attractors, *Inter. J. Dyn.Syt. Diff. Eqns.* (to appear)
45. M.A. Krasnosel'skii and A.V. Pokrovskii, *Systems with Hysteresis,* Nauka, Moscow 1983 (English translation: Springer–Verlag, Berlin, 1992). 128, 129, 131
46. Krasnosel'skij M.A., Mejergojz I.D., Pokrovskii A.V. and Rachinskii D.G., State Spaces for Vector Continual Systems of Relays, *Doklady Akademii Nauk*, **330**, No 4, 1993. (Russian). 131
47. M.A. Krasnosel'skii and P.P Zabreiko, *Geometrical Methods of Nonlinear Analysis*, Springer-Verlag, Heidelberg, 1984.
48. N. Kuznetsov und P.E. Kloeden, The problem of information stability in computer studies of continuous systems. *Mathematics & Computation in Simulation,* **43** (1997), 143–158. 4
49. B. Lani–Wayda, *Hyperbolic Sets, Shadowing and Persistence for Noninvertible Mappings in Banach Spaces*, Pitman Research Notes in Mathematics Series, No. 334, 1995. 3
50. T.Y. Li and J.A. Yorke, Period three implies chaos, *Amer. Math. Monthly,* **82** (1975), 985–992. 17, 113
51. R. Mane, *Ergodic Theory and Differentiable Dynamics*, Springer-Verlag, Berlin, 1987. 67
52. F.R. Marotto, Snack-back repellors imply chaos in $\mathbb{R}^n$, *J. Math. Anal. Applns.,* **63** (1978), 199–223. 17, 117
53. N. Martin and J. England, *Mathematical Theory of Entropy*, Addison–Wesley Publishing Company, London, 1981. 16, 101, 102
54. I.D. Mayergoyz, *Mathematical Models of Hysteresis*, Springer–Verlag, New York, 1991. 129

55. M. Mazur, K. Stolot and J. Tabor, Semi-hyperbolicity implies hyperbolicity in the linear case, *CADSEM Report* **96–018**, Deakin University, June 1996.
56. R. Nussbaum, Existence and uniqueness theorems for some functional differential equations of neutral type, *J. Differential Equations*, **11** (1972), 607-623.
57. J. Ombach, Consequences of the pseudo-orbits tracing property and expansiveness, *J. Austral. Math. Soc. (Series A).* **43** (1987), 301-313.
58. J. Ombach, Equivalent conditions for hyperbolic coordinates, *Topology and its Applications,* **23** (1986), 87 –90.
59. J. Palis and W. de Melo, *Geometric Theory of Dynamical Systems. An Introduction*, Springer-Verlag, Heidelberg, 1982.
60. K.J. Palmer, Exponential dichotomies, the shadowing lemma and transerval homoclinic points, *Dynamics Reported, Vol. I*, U. Kirchgraber and H.O. Walther (ed.), John Wiley & Sons and B.G. Teubner, New York, 1988. 129
61. S. Yu. Pilyugin, *The Space of Dynamical Systems with the C^0–Topology,* Lecture Notes in Mathematics Vol. 1571, Springer–Verlag, Heidelberg, 1994.
62. N.D. Pokrovskaya, Forced periodic regimes in control systems with control with respect to derivatives, *Automation and Remote Control*, **52** (1991), No 5, Part 1, 643–648.
63. C. C. Pugh, On a theorem of P. Hartman, *Amer. J. Math.,* **91** (1969), 363–367. 2
64. J. Quandt, On the Hartman–Grobman theorem for maps, *J. Differential Equations*, **64**, (1986), 154–164. 2
65. C. Robinson, Stability theorems and hyperbolicity in dynamical systems, *Rocky Mountain J. Math.*, **7** (1977), 425–437.
66. F. Riesz and B.Sz. Nagy, *Functional Analysis*, Frederick Ungar Publ. Co., New York, 1978. 127
67. D. Ruelle, *Thermodynamic Formalism*, Addison-Wesley, 1978. 112
68. D. Ruelle, *Elements of Differentiable Dynamics and Bifurcation Theory*, Academic Press, Boston, 1989. 15, 104
69. B.N. Sadovskii, Limit–compact and condensing operators. *Russian Math. Surveys*, **27** (1972), 85–155.
70. E.A. Sataev, Invariant measures for hyperbolic maps with singularities, *Uspekhi. Matem. Nauk*, **41** 1 (1992), 148– 202.
71. K. Shiraiwa and M. Kurata, A generalization of a theorem of Marotto, *Nagoya Math. J.,* **82** (1981), 83–97. 17, 18, 117
72. M. Shub, *Global Analysis of Dynamical Systems*, Springer–Verlag, New York, 1987. 3, 129
73. S. Smale, Differentiable dynamical systems, *Bull. Amer. Math. Soc.*, **73** (1967), 747– 817.
74. H. Steinlein and H.-O. Walther, Hyperbolic sets and shadowing for nonivertible mappings, in *Advances in the Theory of Dynamical Systems,* (Editors: G. Fusco, M. Ianelli & L. Salvadori), Academic Press, New York, 1989; pp. 219–234. 3
75. H.J. Stetter, *Analysis of Discretization Methods for Ordinary Differential Equations*, Springer–Verlag, Berlin, 1973.
76. A.M. Stuart, Numerical analysis of dynamical systems, *Acta Numerica,* **2** (1994), 467–572. 132, 133
77. W.R. Utz, Unstable homeomorphisms, *Proc. Amer. Math. Soc.*, **1** (1950), p. 357–368.
78. A. Visintin, *Differential Models of Hysteresis*, Springer–Verlag, Berlin, 1994.

79. P. Walters, On the pseudo orbit tracing property and its relationship to stability, *Springer Lectures Notes in Mathematics,* Vol. 668, (1978); pp. 231- 244.
80. P. Walters, *An Introduction to Ergodic Theory*, Springer–Verlag, New York, 1982. 16
81. B.G. Zaslavskii, An investigation of the quasihomoclinic structure of a semigroup generating operator on a Banach space, *Sibirs. Matem. Zh.,* **23** 6 (1983), 80–90. 117

Index

List of Symbols

A^T	— transpose of the matrix A
$\mathscr{A}, \mathscr{D}$	— linear operators associated with the sequence of matrices $\{A_n\}$, p. 53
A^T	— transpose of the matrix A
$\mathscr{A}, \mathscr{D}$	— linear operators associated with the sequence of matrices $\{A_n\}$, p. 53
$B(r)$	— the closed ball of the radius r centered at 0
$B(r, x)$	— the closed ball of the radius r centered at x
$C(X, \mathbb{R}^d)$	— the Banach space of continuous functions $f : X \mapsto \mathbb{R}^d$, p. 24
$C_\varepsilon(S)$	— ε-capacity of the compact set S, p. 16
$\mathrm{CR}(f, X)$	— the set of chain recurrent points of the mapping f, p. 15
$\mathrm{CR}(f, \varepsilon, X)$	— the set of ε-chain recurrent points of the mapping f, p. 104
$\mathrm{Cyc}(f, K, \gamma)$	— the set of γ-pseudo-cycles of the mapping f, p. 99
$\mathrm{Cyc}(f, K, 0)$,	
$\mathrm{Cyc}(f, K)$	— the set of cycles of the mapping f, p. 99
Df	— differential of the mapping f
$\mathscr{D}(K)$	— the space of diffeomorphisms $f : K \mapsto K$
$\dim E$	— dimension of the linear space E
∂X	— boundary of the set X
E	— linear space or Banach space
E_c	— complexification of the linear space E, p. 49
E_x^s, E_x^u	— subspaces forming a splitting $\mathbb{R}^d = E_x^s \oplus E_x^u$, p. 19
$f, \varphi, \widetilde{f}, \widetilde{\varphi}, \ldots$	— mappings
$f\vert_X$	— restriction of the mapping f to the set X
$f \circ g$	— composition of mappings: $(f \circ g)(x) := f(g(x))$
$\mathscr{H}(K)$	— the space of homeomorphisms $f : K \mapsto K$
$h(f, K)$	— topological entropy of the mapping f, p.16

$h_\varepsilon(f, K)$	—	topological ε-entropy of the mapping f, p.16
$\operatorname{Int} X$	—	interior of the set X
$\operatorname{Lip}(X, \mathbb{R}^d)$	—	the Banach space of Lipschitz functions $f : X \mapsto \mathbb{R}^d$, p. 24
$\ell^\infty(\mathbb{I}, \mathbb{R}^d)$	—	the space of bounded sequences of vectors $x_n \in \mathbb{R}^d$ with indices n taking values in an interval $\mathbb{I}$, p. 31
$M(\boldsymbol{s})$	—	split matrix, p. 31
$\mathscr{O}_\varepsilon(X)$	—	ε-neighborhood of the set X
$\operatorname{Per}(f, X)$	—	the set of periodic points of the mapping f.
P_x^s, P_x^u	—	projectors corresponding to a splitting $\mathbb{R}^d = E_x^s \oplus E_x^u$, p. 19
$\mathbb{R}^d$	—	real coordinate space of dimension d
$\mathbb{R}^d = E_x^s \oplus E_x^u$	—	splitting of $\mathbb{R}^d$, p. 19
$\boldsymbol{s}$, $\boldsymbol{s} = (\lambda_s, \lambda_u, \mu_s, \mu_u)$	—	split, p. 20
$x, y, z, \widetilde{x}, \widetilde{y}, \widetilde{z} \ldots$	—	vectors from the space $\mathbb{R}^d$
TM	—	tangent bundle
T_xM	—	tangent space
Tf_x	—	tangent mapping
$\mathbb{T}^d$	—	torus of dimension d, p. 27
$\operatorname{Tr}(f, K, \gamma)$	—	the set of γ-pseudo-trajectories of the mapping f, p. 89
$\operatorname{Tr}(f, K, 0)$, $\operatorname{Tr}(f, K)$	—	the set of trajectories of the mapping f, p. 89
$\operatorname{Tr}_{\pm k}(f, K)$	—	the set of trajectories of the mapping f defined over the interval $[-k, k]$, p. 72
$W_\varepsilon^s(x)$, $W_\varepsilon^u(x)$, $W^s(x)$, $W^u(x)$	—	stable and unstable sets of the mapping f, p. 11
$X, Y, Z \ldots$	—	subsets of $\mathbb{R}^d$
$\mathbb{Z}$	—	the set of integers
$\overline{X}$	—	closure of the set X
$x = (x_1, x_2, \ldots, x_d)$	—	coordinate representation of the vector $x \in \mathbb{R}^d$
$\{x_n\}$, $\boldsymbol{x} = \{x_n\}$, $\boldsymbol{x} = \{x_0, x_1, \ldots\}$, $\boldsymbol{x} = x_0, x_1, \ldots$	—	sequence of elements $x_0, x_1, \ldots$, trajectory, pseudo-trajectory, cycle, pseudo-cycle
x^T	—	transpose of the vector x
$\alpha(\boldsymbol{s}, h)$	—	bi-shadowing parameter, p. 90
$\beta(\boldsymbol{s}, h, \delta)$	—	bi-shadowing parameter, p. 90
$\gamma(\boldsymbol{s})$	—	split parameter, p. 32
$\delta(E_1, E_2)$	—	separation between linear subspaces E_1, E_2, p. 9
$\nu(\boldsymbol{s})$	—	split parameter, p. 20
$\Omega(f, X)$	—	the set of non-wandering points of the mapping f, p. 15

$\varrho_N(\boldsymbol{x}, \boldsymbol{y})$	— metric or semi-metric on the space of sequences, p. 16
Σ	— the set of bi-infinite binary sequences $\boldsymbol{x} = \{x_n\}$, p. 14
$\Sigma(X)$	— the set of bi-infinite sequences $\boldsymbol{x} = \{x_n\} \in X$, p. 103
$\sigma : \Sigma \mapsto \Sigma$	— the shift mapping, p. 72
$\sigma(A)$	— spectral radius of the matrix or linear operator A
$\sigma(\boldsymbol{s})$	— spectral radius of the split matrix $M(\boldsymbol{s})$, p. 31
$\#S$	— cardinality of the set S
$\lvert \cdot \rvert$	— absolute value of a number
$\lvert t - s\rvert_{\bmod k}$	— distance between real t and s modulo k, p. 27
$\lVert \cdot \rVert$	— a norm in $\mathbb{R}^d$, a norm of a matrix or of a linear operator
$\lVert \cdot \rVert_x$	— Riemannian norm on a manifold
$\lVert \cdot \rVert_*$	— weighted 'cubic' norm in $\mathbb{R}^2$, p. 35
$\lVert \cdot \rVert_\infty$	— supremum norm in $\ell^\infty(\mathbb{I}, \mathbb{R}^d)$, p. 31
$\lVert \cdot \rVert_\infty^*$	— 'cubic' supremum norm in $\ell^\infty(\mathbb{I}, \mathbb{R}^d)$, p. 36
$\lVert \cdot \rVert_\infty^\alpha$	— weighted 'cubic' supremum norm in $\ell^\infty(\mathbb{I}, \mathbb{R}^d)$, p. 35
$\lVert f - \varphi \rVert_\infty$	— supremum distance between mappings f and φ, p. 83
$\lVert \cdot \rVert_C$	— norm in the Banach space of continuous functions $C(X, \mathbb{R}^d)$, p. 24
$\lVert \cdot \rVert_{\mathrm{Lip}}$	— norm in the Banach space of Lipschitz functions $\mathrm{Lip}(X, \mathbb{R}^d)$, p. 24
(X, ϱ)	— metric space X with the norm ϱ