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Upper bounds on the smallest sizes of a complete arc in $PG(2, q)$ based on computer search

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Abstract. In the projective plane $PG(2, q)$, upper bounds on the smallest size $t_2(2, q)$ of a complete arc are considered. The results of computer search for a huge region of q, done in the recent works of the authors and in this work, are investigated. New upper bounds valid in this region are proposed. Our investigations and results allow us to conjecture that these bounds hold for all q.

1 Introduction

Let $PG(2, q)$ be the projective plane over the Galois field F_q. An n-arc is a set of n points no three of which are collinear. An n-arc is called complete if it is not contained in an $(n + 1)$-arc of $PG(2, q)$ [1–7].

One of the most important open problems in the study of projective planes is the determination of the *smallest size* $t_2(2, q)$ of a complete arc in $PG(2, q)$.

This work is devoted to *upper bounds* on $t_2(2, q)$.

Let $t(\mathcal{P}_q)$ be the size of the smallest complete arc in any (not necessarily Galois) projective plane $\mathcal{P}_q$ of order q. In [7], for *sufficiently large* q, the following result is proven by *probabilistic methods*:

$$t(\mathcal{P}_q) \le D\sqrt{q}\ln^C q,\ C \le 300,$$

where C and D are constants independent of q. The authors of [7] conjecture that the constant can be reduced to $C = 10$. We denote

$T = \{q : 173 \le q \le 49727,\ q \text{ power prime}\} \cup \{q : 173 \le q \le 125003,\ q \text{ prime}\}$
$\cup\{59 \text{ sporadic prime } q\text{'s in the interval } [125101 \ldots 360007]\}$; $\quad T^{\#} = \{q : 125017 \le q \le 150001,\ q \text{ prime}\} \cup \{290011, 370003, 380041, 390001, 400009\}$;
$L = \{q \le 67993,\ q \text{ prime}\} \cup \{43 \text{ sporadic prime } q\text{'s in } [69997 \ldots 190027]\}$;
$L^{\#} = \{152501, 157513, 162517, 167521, 172507, 195023, 200003, 205019, 210011, 215051, 220009, 225023, 230003, 235003, 240007\}$; $\quad R = \{q \le 46337,\ q \text{ prime}\}$.

Let $\overline{t}_2(2,q)$ be the *smallest **known** size* of a complete arc in $PG(2,q)$. For $q \in T$, the values of $\overline{t}_2(2,q)$ (up to November 2013) are collected in [2].

In [4,5] the algorithm FOP (Fixed Order of Points) used to construct small complete arcs in $PG(2,q)$ is described. Let $t_2^L(2,q)$ be the size of a complete arc in $PG(2,q)$ obtained by Algorithm FOP with Lexicographical order of points. For $q \in L$, the sizes $t_2^L(2,q)$ are collected in [4]. Let $t_2^R(2,q)$ be the size of a *random* complete arc in $PG(2,q)$. For $q \in R$, the sizes $t_2^R(2,q)$ are given in [6].

In this work, by computer search[1] using randomized greedy algorithms similar to those in [2,3], the values of $\overline{t}_2(2,q)$ are obtained for $q \in T^{\#}$. Also, using Algorithm FOP we obtained the arc sizes $t_2^L(2,q)$ for $q \in L^{\#}$.

Investigating the results of [1–5] for $q \in T$ and $q \in L$ and the results of this work for $q \in T^{\#}$ and $q \in L^{\#}$, we obtained Theorems 2 and 3.

Conjecture 1. *Bounds* (1) *and* (2) *of Theorems* 2 and 3 *hold for all* $q \ge 109$.

2 Bounds based on greedy algorithms' results

Theorem 2. *In $PG(2,q)$, for the smallest size $t_2(2,q)$ of a complete arc, the following upper bounds hold:*

$$t_2(2,q) < \min\{\sqrt{q}\ln^{0.7295} q,\ \sqrt{q}\ln^{f(q)} q,\ 0.6\sqrt{q}\ln^{\varphi(q)} q\}, \tag{1}$$
$$109 \le q \le 169 \text{ and } q \in T \cup T^{\#},$$
$$\text{where}\quad f(q) = \frac{0.27}{\ln q} + 0.7, \quad \varphi(q) = \frac{1.5}{\ln q} + 0.802,$$

[1]The calculations are done using computational resources of Multipurpose Computing Complex of National Research Centre "Kurchatov Institute" (http://computing.kiae.ru/).

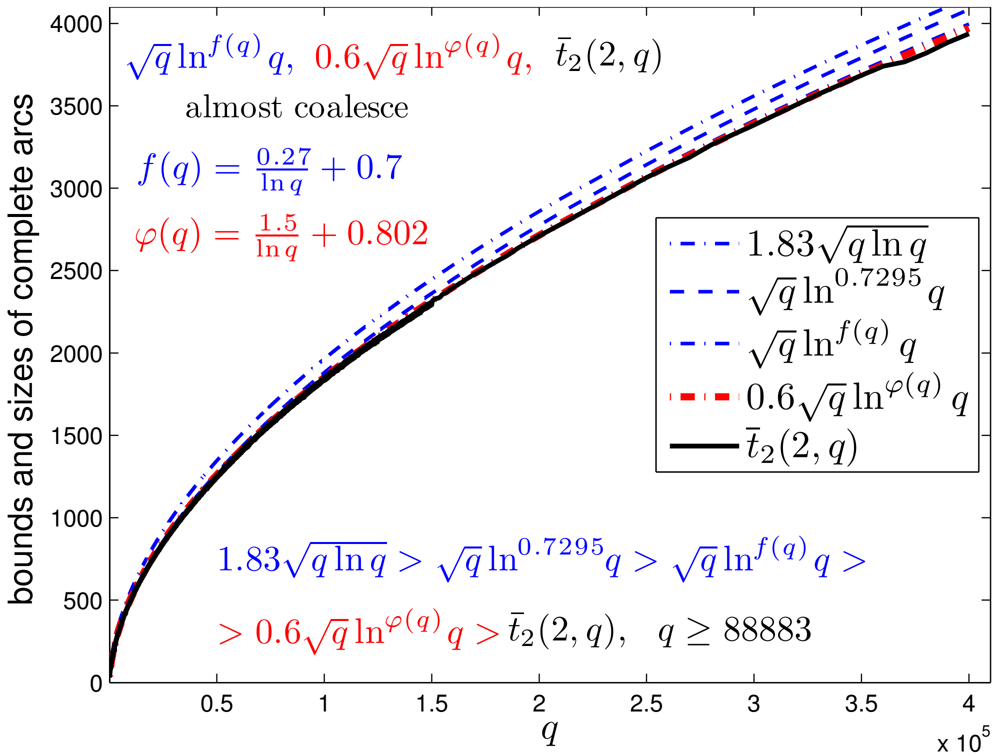

Figure 1: **Upper bounds on $t_2(2,q)$ vs greedy algorithms' results (I)** $1.83\sqrt{q\ln q}$ (the top dashed-dotted curve); $\sqrt{q}\ln^{0.7295} q$ (the 2-nd dashed curve); $\sqrt{q}\ln^{f(q)} q$ (the 3-rd dashed-dotted curve); $0.6\sqrt{q}\ln^{\varphi(q)} q$ (the 4-th dashed-dotted curve); sizes $\bar{t}_2(2,q)$ of complete arcs obtained by the greedy algorithms, $q \in T \cup T^{\#}$ (the bottom solid curve).

$$\min\{\sqrt{q}\ln^{0.7295} q,\ \sqrt{q}\ln^{f(q)} q,\ 0.6\sqrt{q}\ln^{\varphi(q)} q\} =$$
$$\begin{cases} \sqrt{q}\ln^{0.7295} q & \text{if} \quad 109 \leq q \leq 9437 \\ \sqrt{q}\ln^{f(q)} q & \text{if } 9437 \leq q \leq 88873,\ q \in T \\ 0.6\sqrt{q}\ln^{\varphi(q)} q & \text{if } 88883 \leq q \leq 400009,\ q \in T \cup T^{\#} \end{cases} .$$

Arcs satisfying these bounds can be obtained by the randomized greedy algorithms described in [2,3].

The statement of Theorem 2 is illustrated by Figures 1–4 where the values $\varphi^*(q)$, $f^*(q)$, $\overline{\varphi}(q)$, and $\overline{f}(q)$ are defined by relations $\varphi^*(q) = \varphi(q) - 0.5$, $f^*(q) = f(q) - 0.5$, $\bar{t}_2(2,q) = 0.6\sqrt{q}\ln^{\overline{\varphi}(q)} q$, $\bar{t}_2(2,q) = \sqrt{q}\ln^{\overline{f}(q)} q$.

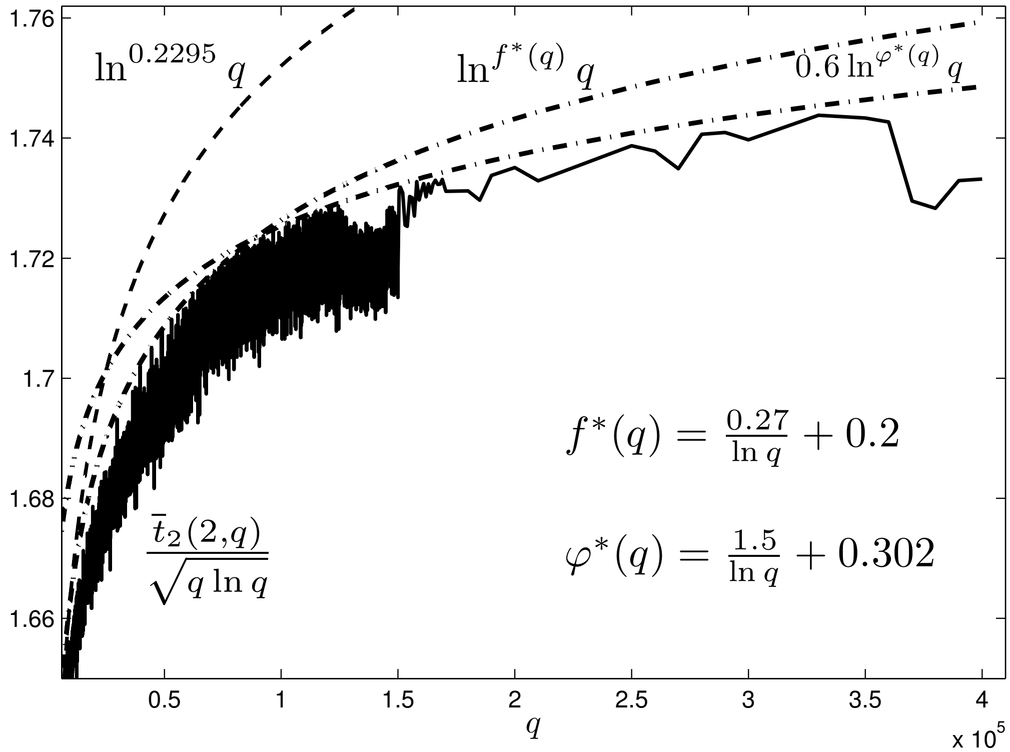

Figure 2: **Upper bounds on $t_2(2,q)$ vs greedy algorithms' results (II)** $\ln^{0.2295} q$ (the top dashed curve); $\ln^{f^*(q)} q$ (the 2-nd dashed-dotted curve); $0.6 \ln^{\varphi^*(q)} q$ (the 3-rd dashed-dotted curve); values $\frac{\bar{t}_2(2,q)}{\sqrt{q \ln q}}$ from complete arcs obtained by the greedy algorithms, $q \in T \cup T^{\#}$ (the bottom solid curve).

3 A bound based on results of Algorithm FOP

Algorithm FOP. Consider the projective plane $PG(2,q)$ and fix a particular order on its points. The algorithm builds a complete arc *iteratively*. Let $K^{(i-1)}$ be the arc obtained on the $(i-1)$-th step. On the next step, the first point in the fixed order not lying on the bisecants of $K^{(i-1)}$ is added to $K^{(i-1)}$. Suppose that the points of $PG(2,q)$ are ordered as $A_1, A_2, \ldots, A_{q^2+q+1}$. Consider the empty set as root of the search and let $K^{(j)}$ be the partial solution obtained in the j-th step, as extension of the root. We put $K^{(0)} = \emptyset$,

$$K^{(1)} = \{A_1\},\ K^{(2)} = \{A_1, A_2\},\ m(1) = 2,\ K^{(j+1)} = K^{(j)} \cup \{A_{m(j)}\},\ m(j) =$$
$$\min\{i \in [m(j-1)+1, q^2+q+1] \mid \nexists\, P, Q \in K^{(j)} :\ A_i, P, Q \text{ are collinear}\},$$

that is $m(j)$ is the minimum subscript i such that the corresponding point A_i is not saturated by $K^{(j)}$. The process ends when a complete arc is obtained.

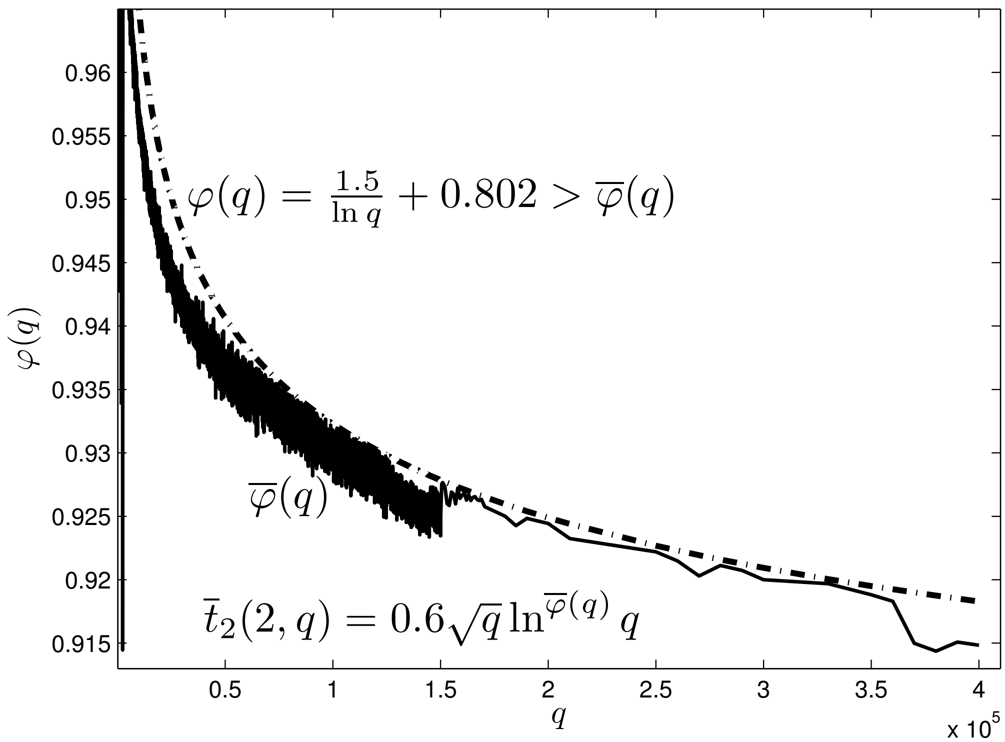

Figure 3: **The function $\varphi(q)$ vs $\overline{\varphi}(q)$ obtained by greedy algorithms.** $\varphi(q)$ (the top dashed-dotted curve); values $\overline{\varphi}(q)$ from complete arcs obtained by greedy algorithms, $q \in T \cup T^{\#}$ (the bottom solid curve).

Lexicographical order of points. Let q be prime and let the elements of the field $\mathbb{F}_q = \{0, 1, \ldots, q-1\}$ be treated as integers modulo q. Let the points A_i of $PG(2,q)$ be represented in homogenous coordinates so that $A_i = (x_0^{(i)}, x_1^{(i)}, x_2^{(i)})$, $x_j^{(i)} \in \mathbb{F}_q$, where the leftmost non-zero element is 1. For A_i, we put $i = x_0^{(i)} q^2 + x_1^{(i)} q + x_2^{(i)}$. So, the homogenous coordinates of a point A_i are treated as its number i written in the q-ary scale of notation.

Theorem 3. *In $PG(2,q)$, for the smallest size $t_2(2,q)$ of a complete arc, the following upper bound holds:*

$$t_2(2,q) < 1.83\sqrt{q \ln q}, \quad q \in L \cup L^{\#}. \tag{2}$$

Arcs satisfying these bounds can be obtained by Algorithm FOP with fixed Lexicographical order of points.

The statement of Theorem 3 is illustrated by Figure 5.

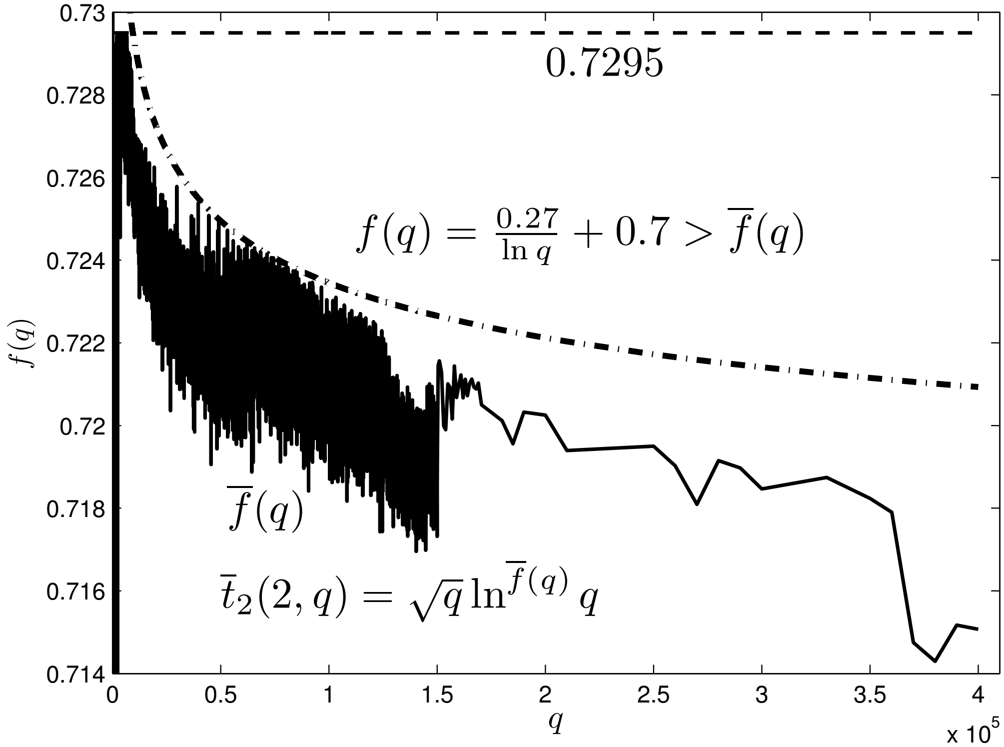

Figure 4: **The function $f(q)$ vs $\overline{f}(q)$ obtained by greedy algorithms.** $y = 0.7295$ (the top dashed line); $f(q)$ (the 2-nd dashed-dotted curve); values $\overline{f}(q)$ from complete arcs obtained by greedy algorithms (the bottom solid curve).

Lemma 4. *It holds that*

$$\min\{\sqrt{q}\ln^{0.7295} q,\ \sqrt{q}\ln^{f(q)} q,\ 0.6\sqrt{q}\ln^{\varphi(q)} q\} < 1.83\sqrt{q\ln q}.$$

Remark 5. In Coding Theory, *greedy codes* (or *lexicographical codes*, or *lexicodes*) are considered, see [8,9] and the references therein. In [5, Remark 2.1], it is noted that *formally* Algorithm FOP is an algorithm creating a parity check matrix of a lexicode with codimension 3 and minimum distance 4.

4 Random complete arcs in $PG(2,q)$

In [6] small complete arcs are obtained using an algorithm which *randomly* selects at each step the point to add. The corresponding values $t_2^R(2,q)/\sqrt{q\ln q}$, $q \in R$, are shown in Figure 6. One can see that Figures 5 and 6 have a very similar structure. This is expected, as Lexicographical order of points is a random order in the geometrical sense.

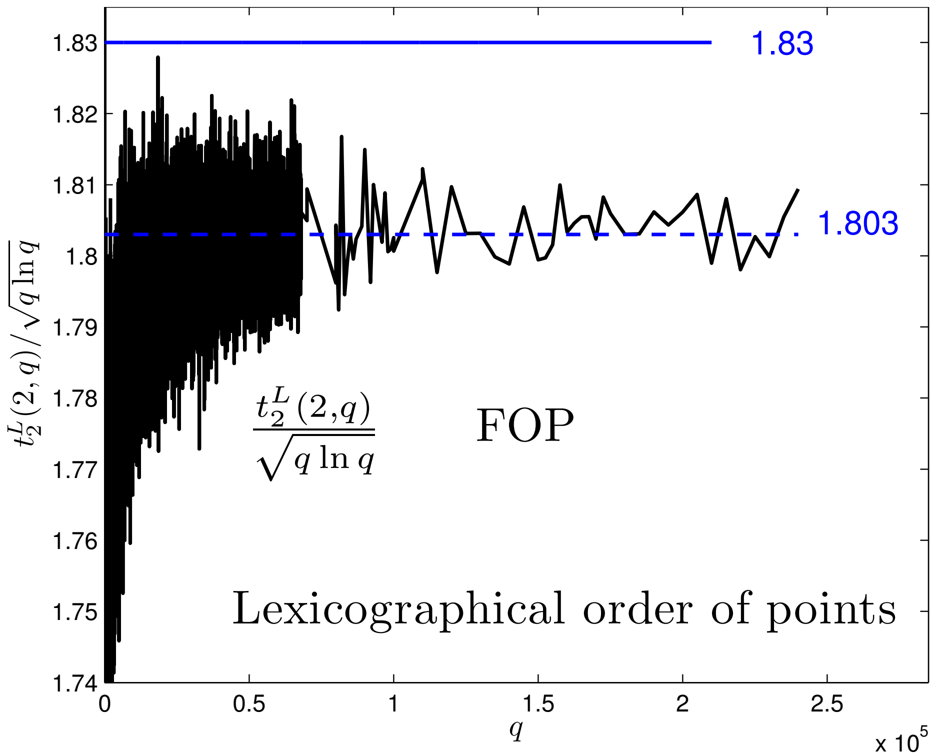

Figure 5: **Bound** $1.83\sqrt{q \ln q}$ **vs Algorithm's FOP results.** $y = 1.83$ (the top solid line); $y = 1.803$ (the 2-nd dashed line); $t_2^L(2,q)/\sqrt{q \ln q}$ obtained by Algorithm FOP with Lexicographical order of points, $q \in L \cup L^{\#}$ (the solid curve).

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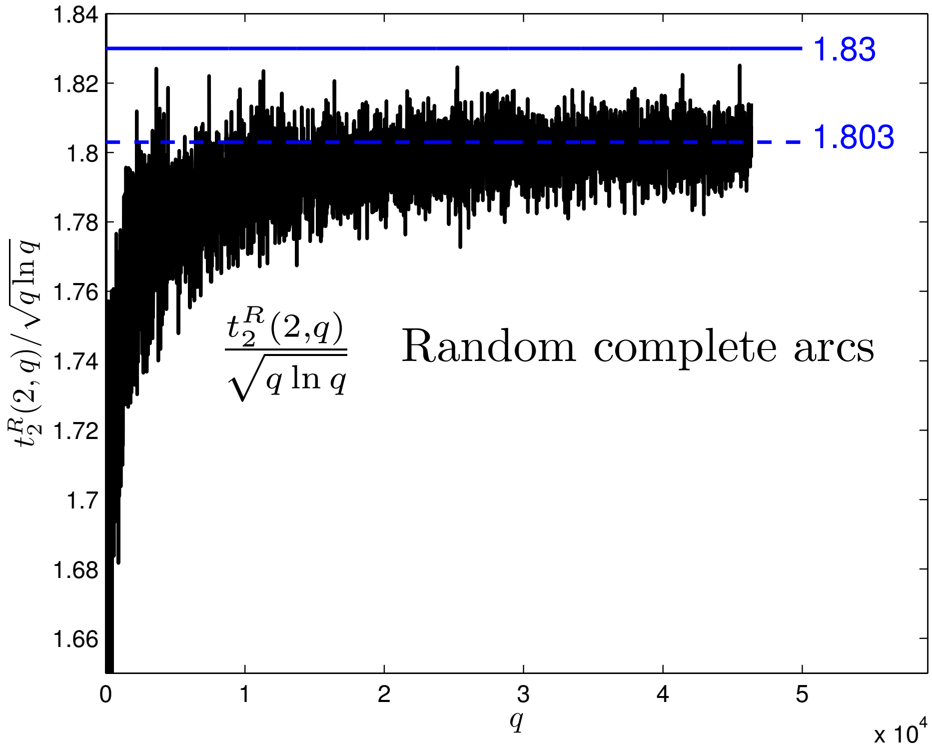

Figure 6: **Bound $1.83\sqrt{q\ln q}$ vs random complete arcs.** $y = 1.83$ (the top solid line); $y = 1.803$ (the 2-nd dashed line); values $t_2^R(2,q)/\sqrt{q\ln q}$ from random complete arcs, $q \in R$ (the solid curve).

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