THE FROBENIUS MORPHISM ON FLAG VARIETIES, I

ALEXANDER SAMOKHIN

ABSTRACT. In this paper, we continue the study of Frobenius direct images $\mathsf{F}_*\mathcal{O}_{\mathbf{G}/\mathbf{B}}$ and sheaves of differential operators $\mathcal{D}_{\mathbf{G}/\mathbf{B}}$ on flag varieties in characteristic p. We first show how the decomposition of $\mathsf{F}_*\mathcal{O}_{\mathbf{G}/\mathbf{B}}$ into a direct sum of indecomposable homogeneous bundles follows from appropriate semiorthogonal decompositions of the derived category of coherent sheaves $\mathrm{D}^b(\mathbf{G}/\mathbf{B})$, and then produce such decompositions when $\mathbf{G}$ is of type $\mathbf{A}_2$ (cf. [9]), and $\mathbf{B}_2$. In the exceptional type $\mathbf{G}_2$ the method works for $p < 11$, but for larger primes the non–standard cohomology vanishing of line bundles starts to play an essential rôle, and it requires further work to obtain a decomposition for generic p in this case.

When $p > h$, the Coxeter number of the corresponding group, the set of indecomposable summands in these decompositions forms a full exceptional collection in the derived category, thus confirming general prediction about decompositions of $\mathsf{F}_*\mathcal{O}_{\mathbf{G}/\mathbf{B}}$ (cf. [10]).

1. Introduction

This paper continues the study of sheaves of differential operators on flag varieties in characteristic p that we begun in a series of papers [14] – [16].

Recall briefly the setting. Fix an algebraically closed field k of characteristic $p > 0$. Consider a simply connected semisimple algebraic group $\mathbf{G}$ over k, and let $\mathbf{G}_n$ be the n–the Frobenius kernel. Our main interest is in understanding the vector bundle $\mathsf{F}^n_*\mathcal{O}_{\mathbf{G}/\mathbf{B}}$ on the flag variety $\mathbf{G}/\mathbf{B}$, where $\mathsf{F}_n : \mathbf{G}/\mathbf{B} \to \mathbf{G}/\mathbf{B}$ is the (n–th power of) Frobenius morphism. The bundle in question is homogeneous, and one would like to understand its $\mathbf{G}_n\mathbf{B}$–structure. However, such a description seems to be missing so far in general.

On the other hand, there are sheaves of differential operators with divided powers $\mathcal{D}_{\mathbf{G}/\mathbf{B}}$ that are related to $\mathsf{F}^n_*\mathcal{O}_{\mathbf{G}/\mathbf{B}}$ via the formula $\mathcal{D}_{\mathbf{G}/\mathbf{B}} = \cup_n\mathcal{E}nd(\mathsf{F}^n_*\mathcal{O}_{\mathbf{G}/\mathbf{B}})$. Higher cohomology vanishing for sheaves of differential operators is essential for localization type theorems, of which the Beilinson–Bernstein localization [6] is the prototype. It is known that, unlike the characteristic zero case, localization of $\mathcal{D}$–modules in characteristic p does not hold in general [11]. However, one hopes for a weaker, but still sensible statement. Namely, consider the sheaf $\mathcal{D}^{(1)}_{\mathbf{G}/\mathbf{B}} = \mathcal{E}nd(\mathsf{F}_*\mathcal{O}_{\mathbf{G}/\mathbf{B}})$ (the first term of the p–filtration on $\mathcal{D}_{\mathbf{G}/\mathbf{B}}$ on $\mathbf{G}/\mathbf{B}$). The category of sheaves of modules over $\mathcal{D}^{(1)}_{\mathbf{G}/\mathbf{B}}$ is precisely that of $\mathcal{D}_{\mathbf{G}/\mathbf{B}}$–modules with vanishing p–curvature, which is equivalent to the category of coherent sheaves via Cartier's equivalence.

The question is then whether localization theorem holds for the category $\mathcal{D}^{(1)}_{\mathbf{G}/\mathbf{B}}$–mod (see [9]).

Representation–theoretic approach to $\mathbf{G}_1\mathbf{B}$–structure of $\mathsf{F}_*\mathcal{O}_{\mathbf{G}/\mathbf{B}}$ is worked out in [10] and is based on the study of $\mathbf{G}_1\mathbf{T}$ (or $\mathbf{G}_1\mathbf{B}$)–structure of the induced $\mathbf{G}_1\mathbf{B}$–module $\hat{\nabla}(0) := \mathrm{Ind}_{\mathbf{B}}^{\mathbf{G}_1\mathbf{B}}\chi_0$ (the so-called Humphreys–Verma module). It is generally believed (see *loc.cit.*) that indecomposable subfactors of $\mathsf{F}^n_*\mathcal{O}_{\mathbf{G}/\mathbf{B}}$, which are defined over $\mathbb{Z}$, should form a full exceptional collection in the derived category $\mathrm{D}^b(\mathrm{Coh}(\mathbf{G}/\mathbf{B}))$. When it holds, it can be seen as a refinement of the above equivalence for $\mathcal{D}^{(1)}_{\mathbf{G}/\mathbf{B}}$. Such a decomposition was obtained for projective spaces and the flag variety $\mathbf{SL}_3/\mathbf{B}$ in [9], and for smooth quadrics in [1] and [13].

In this paper, we suggest an approach to decomposing the bundle $\mathsf{F}^n_*\mathcal{O}_{\mathbf{G}/\mathbf{B}}$ that is based on the existence of an appropriate semiorthogonal decomposition of the derived category of coherent sheaves $\mathrm{D}^b(\mathbf{G}/\mathbf{B})$. In a nutshell, the idea is as follows. Given a smooth algebraic variety X of dimension n over k, consider its derived category $\mathrm{D}^b(X)$, and assume that it admits a semiorthogonal decomposition $\mathrm{D}^b(X) = \langle \mathcal{A}_{-n}, \mathcal{A}_{-n+1}, \dots, \mathcal{A}_0\rangle$ with the following property: each admissible category $\mathcal{A}_{-i}$ is generated by an exceptional collection of vector bundles $\mathsf{E}^i_l \in \mathrm{D}^b(X)$ that are pairwise orthogonal to each other, so that the category $\mathcal{A}_{-i}$ is equivalent to a direct sum of copies $\mathrm{D}^b(\mathsf{Vect} - k)$. Further, assume that $\mathrm{H}^j(X, \mathsf{F}^*\mathsf{E}^i_l) = 0$ for $j \neq -i$ and for each exceptional bundle $\mathsf{E}^i_l \in \mathcal{A}_{-i}$.

One then shows that under these assumptions the bundle $\mathsf{F}_*\mathcal{O}_X$ decomposes into a direct sum of vector bundles. Indecomposable summands of this decomposition form the so-called right dual semiorthogonal decomposition with respect to $\mathcal{A}$. The task is therefore broken up into two parts: one has first to find a semiorthogonal decomposition of $\mathrm{D}^b(X)$ with the above properties; once such a decomposition has been found, one then calculates right dual collection, and irreducible summands of $\mathsf{F}_*\mathcal{O}_X$ are precisely terms of the latter collection. The multiplicity spaces at each indecomposable summand are identified with cohomology groups $\mathrm{H}^j(X, \mathsf{F}^*\mathsf{E}^i_l)$ that are non–trivial at a unique cohomological degree.

In Section 4 we show how the above method works for classical groups of rank 2. The case of exceptional group $\mathbf{G}_2$ is harder because non–standard cohomology vanishing of vector bundles on the flag variety $\mathbf{G}_2/\mathbf{B}$ starts to manifest itself more prominently (see [5]). At present, for $\mathbf{G}_2$ the method produces a decomposition of $\mathsf{F}_*\mathcal{O}_{\mathbf{G}_2/\mathbf{B}}$ for $p < 11$ (in particular, for $p = 7$ one extracts from it a full exceptional collection on $\mathbf{G}_2/\mathbf{B}$). In a subsequent paper [17] we treat this case in a greater detail, as well as the case of adjoint varieties of type $\mathbf{A}_n$.

Acknowledgements. While working on this paper, the author greatly benefited from generous supports of the SFB/Transregio 45 at the University of Mainz, and of the SFB 878 at the University of Münster. It is a great pleasure to thank Manuel Blickle and Christopher Deninger for their invitations and their interest in this work. I am also very much indebted to Alexander Kuznetsov for many useful conversations over the years.

Notation. Throughout we fix an algebraically closed field k of characteristic $p > 0$. Let $\mathbf{G}$ be a semisimple algebraic group over k. Let $\mathbf{T}$ be a maximal torus of $\mathbf{G}$, and $\mathbf{T} \subset \mathbf{B}$ a Borel subgroup containing it. Denote $\mathrm{X}(\mathbf{T})$ the weight lattice, R – the root lattice and $\mathrm{S} \subset \mathrm{R}$ the set of simple roots. The Weyl group $\mathcal{W} = \mathrm{N}(\mathbf{T})/\mathbf{T}$ acts on $X(\mathbf{T})$ via the dot–action: if $w \in \mathcal{W}$, and $\lambda \in \mathrm{X}(\mathbf{T})$, then $w \cdot \lambda = w(\lambda + \rho) - \rho$. For a simple root $\alpha \in \mathrm{S}$ denote $\mathbf{P}_\alpha$ the minimal parabolic subgroup of $\mathbf{G}$. For a weight $\lambda \in \mathrm{X}(\mathbf{T})$ denote $\mathcal{L}_\lambda$ the corresponding line bundle on $\mathbf{G}/\mathbf{B}$. For a dominant weight $\lambda \in \mathrm{X}(\mathbf{T})$, the induced module $\mathrm{Ind}_{\mathbf{B}}^{\mathbf{G}}\lambda$ is denoted ∇_λ. For a vector space V over k its n-th Frobenius twist $\mathsf{F}^{n*}\mathsf{V}$ is denoted $\mathsf{V}^{[n]}$.

2. Cohomology of vector bundles on G/B

2.1. **Cohomology of line bundles.** We collect here several results on cohomology groups of line bundles on $\mathbf{G}/\mathbf{B}$ that are due to H.H.Andersen.

2.1.1. *First cohomology group of a line bundle.* Let α be a simple root, and denote s_α a corresponding reflection in $\mathcal{W}$. One has $s_\alpha \cdot \chi = s_\alpha(\chi) - \alpha$. There is a complete description [2, Theorem 3.6] of (non)–vanishing of the first cohomology group of a line bundle $\mathcal{L}_\lambda$.

Theorem 2.1. $\mathrm{H}^1(\mathbf{G}/\mathbf{B}, \mathcal{L}_\chi) \neq 0$ *if and only if there exist a simple root α such that one of the following conditions is satisfied:*

- $-p \leq \langle \chi, \alpha^\vee \rangle \leq -2$ *and* $s_\alpha \cdot \chi = s_\alpha(\chi) - \alpha$ *is dominant.*
- $\langle \chi, \alpha^\vee \rangle = -ap^n - 1$ *for some* $a, n \in \mathbf{N}$ *with* $a < p$ *and* $s_\alpha(\chi) - \alpha$ *is dominant.*
- $-(a+1)p^n \leq \langle \chi, \alpha^\vee \rangle \leq -ap^n - 2$ *for some* $a, n \in \mathbf{N}$ *with* $a < p$ *and* $\chi + ap^n\alpha$ *is dominant.*

Some bits of Demazure's proof of the Bott theorem in characteristic zero are still valid in positive characteristic [2, Corollary 3.2]:

Theorem 2.2. *Let χ be a weight. If either* $\langle \chi, \alpha^\vee \rangle \geq -p$ *or* $\langle \chi, \alpha^\vee \rangle = -ap^n - 1$ *for some* $a, n \in \mathbf{N}$ *and* $a < p$ *then*

$$\mathrm{H}^i(\mathbf{G}/\mathbf{B}, \mathcal{L}_\chi) = \mathrm{H}^{i-1}(\mathbf{G}/\mathbf{B}, \mathcal{L}_{s_\alpha \cdot \chi}). \tag{2.1}$$

Further, Theorem 2.3 of [3] states:

Theorem 2.3. *If χ is a weight such that for a simple root α one has* $0 \leq \langle \chi + \rho, \alpha^\vee \rangle \leq p$ *then*

$$\mathrm{H}^i(\mathbf{G}/\mathbf{B}, \mathcal{L}_\chi) = \mathrm{H}^{i+1}(\mathbf{G}/\mathbf{B}, \mathcal{L}_{s_\alpha \cdot \chi}). \tag{2.2}$$

Finally, the Borel–Weil-Bott theorem holds in characteristic p for weights lying in the interior of the bottom alcove in the dominant chamber ([3], Corollary 2.4).

3. Semiorthogonal decompositions, exceptional collections, and mutations

Fix an algebraically closed field k of arbitrary characteristic. Let $\mathcal{D}$ be a k–linear triangulated category.

Definition 3.1. ([7]) *A semiorthogonal decomposition of a triangulated category $\mathcal{D}$ is a sequence of full triangulated subcategories $\mathcal{A}_1, \dots, A_n$ of $\mathcal{D}$ such that the two following conditions hold. Firstly, one has $\mathrm{Hom}_{\mathsf{D}}(\mathcal{A}_i, \mathcal{A}_j) = 0$ for $i > j$, and, secondly, for every object $D \in \mathcal{D}$ there exists a chain of morphisms $0 \to D_n \to D_{n-1} \to \dots \to D_1 \to D_0 = D$ such that a cone of the morphism $D_k \to D_{k-1}$ is contained in $\mathcal{A}_k$ for each $k = 1, 2, \dots, n$.*

Definition 3.2. ([8]) *An object $\mathsf{E} \in \mathcal{D}$ is said to be exceptional if there is an isomorphism of graded $\mathbf{k}$-algebras*

$$\mathrm{H}om^{\bullet}_{\mathcal{D}}(\mathsf{E}, \mathsf{E}) = \mathbf{k}. \tag{3.1}$$

A collection of exceptional objects $(\mathsf{E}_0, \dots, \mathsf{E}_n)$ in $\mathcal{D}$ is called exceptional if for $1 \le i < j \le n$ one has

$$\mathrm{H}om^{\bullet}_{\mathcal{D}}(\mathsf{E}_j, \mathsf{E}_i) = 0. \tag{3.2}$$

Denote $\langle \mathsf{E}_0, \dots, \mathsf{E}_n \rangle \subset \mathcal{D}$ the full triangulated subcategory generated by the objects $\mathsf{E}_0, \dots, \mathsf{E}_n$. One proves ([8], Theorem 3.2) that such a category is admissible. The collection $(\mathsf{E}_0, \dots, \mathsf{E}_n)$ in $\mathcal{D}$ is said to be *full* if $\langle \mathsf{E}_0, \dots, \mathsf{E}_n \rangle^{\perp} = 0$, in other words $\mathcal{D} = \langle \mathsf{E}_0, \dots, \mathsf{E}_n \rangle$.

Definition 3.3. *Let $(\mathsf{E}_0, \mathsf{E}_1)$ be an exceptional pair in $\mathcal{D}$, i.e. an exceptional collection of two elements. The left mutation of $(\mathsf{E}_0, \mathsf{E}_1)$ is a pair $(\mathrm{L}_{\mathsf{E}_0}\mathsf{E}_1, \mathsf{E}_0)$, where the object $\mathrm{L}_{\mathsf{E}_0}E_1$ is defined to be a cone of the triangle*

$$\dots \longrightarrow \mathrm{L}_{\mathsf{E}_0}\mathsf{E}_1[-1] \longrightarrow \mathrm{H}om^{\bullet}_{\mathcal{D}}(\mathsf{E}_0, \mathsf{E}_1) \otimes \mathsf{E}_0 \longrightarrow \mathsf{E}_1 \longrightarrow \mathrm{L}_{\mathsf{E}_0}\mathsf{E}_1 \longrightarrow \dots \quad . \tag{3.3}$$

The right mutation is a pair $(\mathsf{E}_1, \mathsf{R}_{\mathsf{E}_1}\mathsf{E}_0)$, where $\mathrm{R}_{\mathsf{E}_1}\mathsf{E}_0$ is defined to be a shifted cone of the triangle

$$\dots \longrightarrow \mathrm{R}_{\mathsf{E}_1}\mathsf{E}_0 \longrightarrow \mathsf{E}_0 \longrightarrow \mathsf{Hom}^{\bullet}_{\mathcal{D}}(\mathsf{E}_0, \mathsf{E}_1)^* \otimes \mathsf{E}_1 \longrightarrow \mathrm{R}_{\mathsf{E}_1}\mathsf{E}_0[1] \longrightarrow \dots \quad . \tag{3.4}$$

More generally, if $(\mathsf{E}_0, \dots, \mathsf{E}_n)$ is an exceptional collection of arbitrary length in $\mathcal{D}$ then one can define left and right mutations of an object $\mathsf{E} \in \mathcal{D}$ through the category $\langle \mathsf{E}_0, \dots, \mathsf{E}_n \rangle$. Denote $\mathrm{L}_{\langle \mathsf{E}_0, \dots, \mathsf{E}_n \rangle}\mathsf{E}$ and $\mathrm{R}_{\langle \mathsf{E}_0, \dots, \mathsf{E}_n \rangle}\mathsf{E}$ left and right mutations of E through $\langle \mathsf{E}_0, \dots, \mathsf{E}_n \rangle$, respectively. One proves ([8], Proposition 2.1) that mutations of an exceptional collection are exceptional collections.

Let $(\mathsf{E}_0, \dots, \mathsf{E}_n)$ be an exceptional collection in $\mathcal{D}$. One can extend it to an infinite sequence of objects (E_i) of $\mathcal{D}$, where $i \in \mathbb{Z}$, defined inductively by putting

$$\mathsf{E}_{\mathsf{i+n+1}} := \mathrm{R}_{\langle \mathsf{E}_{\mathsf{i+1}},\dots,\mathsf{E}_{\mathsf{i+n}}\rangle}\mathsf{E}_{\mathsf{i}}, \qquad \mathsf{E}_{\mathsf{i-n-1}} := \mathrm{L}_{\langle \mathsf{E}_{\mathsf{i-n}},\dots,\mathsf{E}_{\mathsf{i-1}}\rangle}\mathsf{E}_{\mathsf{i}}. \tag{3.5}$$

Let X be a variety of dimension m, and ω_X the canonical invertible sheaf on X.

Definition 3.4. *A sequence of objects (E_i) of $\mathrm{D}^b(X)$, where $i \in \mathbb{Z}$, is called a helix of period n if $\mathsf{E}_i = \mathsf{E}_{i+n} \otimes \omega_X[m-n+1]$. An exceptional collection $(\mathsf{E}_0, \dots, \mathsf{E}_\mathsf{n})$ in $\mathrm{D}^b(X)$ is called a thread of the helix if the infinite sequence (E_i), obtained from the collection $(\mathsf{E}_0, \dots, \mathsf{E}_\mathsf{n})$ as in (3.5), is a helix of period $n+1$.*

For a Fano variety X there is a criterion to establish whether a given exceptional collection $(\mathsf{E}_0, \dots, \mathsf{E}_\mathsf{n})$ is full, i.e. generates $\mathrm{D}^b(X)$.

Theorem 3.1 ([8], Theorem 4.1). *Let X be a Fano variety, and $(\mathsf{E}_0, \dots, \mathsf{E}_\mathsf{n})$ be an exceptional collection in $\mathrm{D}^b(X)$. The following conditions are equivalent:*

1. *The collection $(\mathsf{E}_0, \dots, \mathsf{E}_\mathsf{n})$ generates $\mathrm{D}^b(X)$.*
2. *The collection $(\mathsf{E}_0, \dots, \mathsf{E}_\mathsf{n})$ is a thread of the helix.*

3.1. **Dual exceptional collections.**

Definition 3.5. *Let X be a smooth variety, and assume given an exceptional collection $(\mathsf{E}_0, \dots, \mathsf{E}_\mathsf{n})$ in $\mathrm{D}^b(X)$. Right dual collection to (E_i) is defined as follows:*

$$\mathsf{F}_i := \mathrm{R}_{\langle \mathsf{E}_{\mathsf{n-i+1}},\dots,\mathsf{E}_\mathsf{n}\rangle}\mathsf{E}_{\mathsf{n-i}}. \tag{3.6}$$

Let X be a smooth variety, and assume there exists a full exceptional collection $(\mathsf{E}_0, \dots, \mathsf{E}_\mathsf{n})$ in $\mathcal{D}^b(X)$. Denote $(\mathsf{F}_0, \dots, \mathsf{F}_n)$ right dual collection. Then the base change for semiorthogonal decompositions (see [12]) implies that there exists a set of objects $D_i \in \mathcal{D}^b(X)$ and a chain of morphisms $0 \to D_n \to D_{n-1} \to \dots \to D_1 \to D_0 = \mathcal{O}_\Delta$, such that a cone of each morphism $D_i \to D_{i-1}$ is quasiisomorphic to $\mathsf{E}_i \boxtimes \mathsf{F}_i^*$, where $\mathsf{F}_i^* = \mathcal{R}Hom(\mathsf{F}_i, \mathcal{O}_X)$ is the dual object.

In particular, for any object $\mathcal{G}$ of $\mathrm{D}^b(X)$ there is a spectral sequence

$$\mathbb{H}^{q+p}(X, \mathcal{G} \otimes \mathsf{E}_p) \otimes \mathsf{F}_p^* \Rightarrow \mathcal{G}. \tag{3.7}$$

Claim 3.1. *Let X be a smooth algebraic variety of dimension n over k, and assume the category $\mathrm{D}^b(X)$, admits a semiorthogonal decomposition $\mathcal{A} = \langle \mathcal{A}_{-n}, \mathcal{A}_{-n+1}, \dots, \mathcal{A}_0\rangle$ with the following property: each admissible category $\mathcal{A}_{-i}$ is generated by an exceptional collection of vector bundles $\mathsf{E}_k^i \in \mathrm{D}^b(X)$ that are pairwise orthogonal to each other. Furthermore, $\mathrm{H}^j(X, \mathsf{F}^*\mathsf{E}_k^i) = 0$ for $j \neq -i$ and for any $\mathsf{E}_k^i \in \mathcal{A}_{-i}$. Denote $\mathsf{F}_0, \dots, \mathsf{F}_n$ right dual collection. Then one has:*

$$\mathsf{F}^n_*\mathcal{O}_X = \bigoplus \mathbb{H}^p(X, \mathsf{F}^*\mathsf{E}^i_p) \otimes (\mathsf{F}^i_p)^*, \tag{3.8}$$

where F^i_p are vector bundles.

Proof. Proposition 9.2 of [8] ensures that under assumptions of the claim right dual collection with respect to $\mathcal{A}$ consists of pure objects, that is of coherent sheaves. Given that $\mathrm{H}^j(X, \mathsf{F}^*\mathsf{E}^i_k) = 0$ for $j \neq -i$, one sees that spectral sequence (3.7) for $\mathcal{G} = \mathsf{F}_*\mathcal{O}_X$ totally degenerates, its only non–trivial terms being situated at the diagonal. Finally, since X is smooth, the sheaves F^i_p are locally free, being direct summands of locally free sheaf $\mathsf{F}_*\mathcal{O}_X$. □

4. Decomposition of $\mathsf{F}^n_*\mathcal{O}_{\mathbf{G}/\mathbf{B}}$

In this section, we find semiorthogonal decompositions of the derived categories of flag varieties in type $\mathbf{A}_2$ and $\mathbf{B}_2$ that satisfy the assumptions of Claim 3.1. Further, we explicitly compute right dual collections for all p in both cases.

Decomposition of $\mathsf{F}_*\mathcal{O}_{\mathbf{G}/\mathbf{B}}$ in type $\mathbf{A}_2$ was previously obtained in [9] using results of [4] about the structure of $\mathbf{G}_1\mathbf{T}$–socle series of the Humphreys–Verma $\mathbf{G}_1\mathbf{B}$–module $\hat{\nabla}(0) = \mathrm{Ind}_{\mathbf{B}}^{\mathbf{G}_1\mathbf{B}}\chi_0$.

4.1. **Type $\mathbf{A}_2$.** Let $\mathbf{G} = \mathbf{SL_3}$, and ω_1, ω_2 the two fundamental weights. Denote π_1, π_2 the two projections of $\mathbf{SL}_3/\mathbf{B}$ onto $\mathbb{P}^2, (\mathbb{P}^2)^\vee$, respectively. Consider a sequence $\mathcal{A} = \langle \mathcal{A}_i \rangle_{i=-3}^{i=0}$ of full triangulated subcategories of $\mathrm{D}^b(\mathbf{SL}_3/\mathbf{B})$:

$$\begin{gathered} \mathcal{A}_{-3} = \langle \mathcal{L}_{-\rho} \rangle, \quad \mathcal{A}_{-2} = \langle \mathcal{L}_{-\omega_1}, \mathcal{L}_{-\omega_2} \rangle, \\ \mathcal{A}_{-1} = \langle \pi_1^*\Omega^1_{\mathbb{P}^2} \otimes \mathcal{L}_{\omega_1}, \pi_2^*\Omega^1_{(\mathbb{P}^2)^\vee} \otimes \mathcal{L}_{\omega_2} \rangle, \quad \mathcal{A}_0 = \langle \mathcal{O}_{\mathbf{SL}_3/\mathbf{B}} \rangle. \end{gathered} \tag{4.1}$$

Lemma 4.1. *The sequence $\mathcal{A}$ forms a semiorthogonal decomposition of $\mathrm{D}^b(\mathbf{SL}_3/\mathbf{B})$. The bundles in each block $\mathcal{A}_i$ are mutually orthogonal. If $\mathcal{E} \in \mathcal{A}_i$, then one has $\mathrm{H}^j(\mathbf{SL}_3/\mathbf{B}, (\mathsf{F}^n)^*\mathcal{E}) = 0$ for $j \neq -i$. Right dual decomposition with respect to (4.1) consists of the following subcategories:*

$$\begin{gathered} \mathcal{A}_0^\vee = \langle \mathcal{O}_{\mathbf{SL}_3/\mathbf{B}} \rangle, \quad \mathcal{A}_1^\vee = \langle \mathcal{L}_{\omega_1}, \mathcal{L}_{\omega_2} \rangle, \\ \mathcal{A}_2^\vee = \langle \pi_1^*\Omega^1_{\mathbb{P}^2} \otimes \mathcal{L}_{2\omega_1+\omega_2}, \pi_2^*\Omega^1_{(\mathbb{P}^2)^\vee} \otimes \mathcal{L}_{\omega_1+2\omega_2} \rangle, \quad \mathcal{A}_3^\vee = \langle \mathcal{L}_\rho \rangle. \end{gathered} \tag{4.2}$$

Proof. Semiorthogonality of $\mathcal{A}$ is verified immediately using the Euler sequences:

$$0 \to \pi_1^*\Omega^1_{\mathbb{P}^2} \otimes \mathcal{L}_{\omega_1} \to \mathsf{V}^* \otimes \mathcal{O}_{\mathbf{SL}_3/\mathbf{B}} \to \mathcal{L}_{\omega_1} \to 0, \tag{4.3}$$

(4.4) $$0 \to \pi_2^*\Omega^1_{(\mathbb{P}^2)^\vee} \otimes \mathcal{L}_{\omega_2} \to \mathsf{V} \otimes \mathcal{O}_{\mathbf{SL}_3/\mathbf{B}} \to \mathcal{L}_{\omega_2} \to 0.$$

From these sequences one obtains that $\mathrm{H}^i(\mathbf{SL}_3/\mathbf{B}, (\mathsf{F}^{\mathsf{n}})^*(\pi_1^*\Omega^1_{\mathbb{P}^2} \otimes \mathcal{L}_{\omega_1})) = \mathsf{S}^{p^n}\mathsf{V}^*/\mathsf{V}^{[\mathsf{n}]^*}$, and $\mathrm{H}^i(\mathbf{SL}_3/\mathbf{B}, , (\mathsf{F}^{\mathsf{n}})^*(\pi_2^*\Omega^1_{\mathbb{P}^2} \otimes \mathcal{L}_{\omega_2})) = \mathsf{S}^{p^n}\mathsf{V}/\mathsf{V}^{[\mathsf{n}]^*}$, if $i = 1$; otherwise both groups are trivial. From the Serre duality and the Kempf vanishing one finds that if $p > 2$, then $\mathrm{H}^i(\mathbf{SL}_3/\mathbf{B}, \mathcal{L}_{-p^n\omega_k}) = (\nabla_{(p^n-2)\omega_k})^*$, if $i = 2$ and $k = 1, 2$; otherwise the above groups are trivial. If $p = 2$ and $n = 1$, then the bundles $\mathcal{L}_{-2\omega_k}$ are acyclic. Finally, $\mathrm{H}^3(\mathbf{SL}_3/\mathbf{B}, \mathcal{L}_{-p^n\rho}) = (\nabla_{(p^n-2)\rho})^*$, and is trivial otherwise.

To show that $\mathcal{A}$ is a semiorthogonal decomposition, consider the relative Euler sequence for the projections π_1 and π_2. One obtains short exact sequences:

(4.5) $$0 \to \mathcal{L}_{\omega_1-\omega_2} \to \pi_1^*\mathcal{T}_{\mathbb{P}^2} \otimes \mathcal{L}_{-\omega_1} \to \mathcal{L}_{\omega_2} \to 0,$$

(4.6) $$0 \to \mathcal{L}_{\omega_2-\omega_1} \to \pi_2^*\mathcal{T}_{(\mathbb{P}^2)^\vee} \otimes \mathcal{L}_{-\omega_2} \to \mathcal{L}_{\omega_1} \to 0.$$

Tensoring these sequences with $\mathcal{L}_{-\omega_1-\omega_2}$, one obtains:

(4.7) $$0 \to \mathcal{L}_{-2\omega_2} \to \pi_1^*\mathcal{T}_{\mathbb{P}^2} \otimes \mathcal{L}_{-2\omega_1-\omega_2} \to \mathcal{L}_{-\omega_1} \to 0,$$

(4.8) $$0 \to \mathcal{L}_{-2\omega_1} \to \pi_2^*\mathcal{T}_{(\mathbb{P}^2)^\vee} \otimes \mathcal{L}_{-\omega_1-2\omega_2} \to \mathcal{L}_{-\omega_2} \to 0$$

(cf. Section 5 of [9]). Dualizing sequence (4.6), we see that the bundle $\mathcal{L}_{\omega_1-\omega_2}$ belongs to $\mathcal{A}$. Given that $\mathcal{L}_{-\omega_2}, \mathcal{L}_{-\rho}$ also belong to $\mathcal{A}$, it follows that $\mathcal{A}$ generates the whole $\mathrm{D}^b(\mathbf{SL}_3/\mathbf{B})$ by the relative Beilinson theorem for $\mathbb{P}^n$–bundles.

The terms of right dual decomposition can easily be found using the Euler sequences. From (4.3) one obtains that right dual to $\pi_1^*\Omega^1_{\mathbb{P}^2} \otimes \mathcal{L}_{\omega_1}$ is isomorphic to $\mathcal{L}_{\omega_1}[-1]$, while from (4.4) right dual to $\pi_2^*\Omega^1_{(\mathbb{P}^2)^\vee} \otimes \mathcal{L}_{\omega_2}$ is found to be isomorphic to $\mathcal{L}_{\omega_2}[-1]$. On the other hand, to find right duals to $\mathcal{L}_{-\omega_1}, \mathcal{L}_{-\omega_2}$ one can mutate these to the left through $\mathcal{L}_{-\rho}$, and then to the right through the whole collection. The effect of the last action is described by Theorem 3.1, that is tensoring with $\mathcal{L}_{2\rho}$. Left mutations through $\mathcal{L}_{-\rho}$ can be found tensoring (4.3) and (4.4) with $\mathcal{L}_{-\rho}$. One obtains that right duals are isomorphic to $\pi_1^*\Omega^1_{\mathbb{P}^2} \otimes \mathcal{L}_{2\omega_1+\omega_2}[-2]$, and $\pi_2^*\Omega^1_{(\mathbb{P}^2)^\vee} \otimes \mathcal{L}_{\omega_1+2\omega_2}[-2]$. Finally, right dual to $\mathcal{L}_{-\rho}$ is isomorphic to $\mathcal{L}_{\rho}[-3]$, once again by Theorem 3.1.

□

Theorem 4.1. *The bundle $\mathsf{F}_{\mathsf{n}*}\mathcal{O}_{\mathbf{SL}_3/\mathbf{B}}$ decomposes into the direct sum of vector bundles with indecomposable summands being isomorphic to:*

(4.9) $$\mathcal{O}_{\mathbf{SL}_3/\mathbf{B}}, \quad \mathcal{L}_{-\omega_1}, \quad \mathcal{L}_{-\omega_2}, \quad \pi_1^*\mathcal{T}_{\mathbb{P}^2} \otimes \mathcal{L}_{-2\omega_1-\omega_2}, \quad \pi_2^*\mathcal{T}_{(\mathbb{P}^2)^\vee} \otimes \mathcal{L}_{-\omega_1-2\omega_2}, \quad \mathcal{L}_{-\rho}.$$

The multiplicity spaces at each indecomposable summand are isomorphic, respectively, to:

$$\mathsf{k}, \quad , \mathsf{S}^{p^n}\mathsf{V}^*/\mathsf{V}^{[\mathsf{n}]^*}, \quad \mathsf{S}^{p^n}\mathsf{V}^*/\mathsf{V}^{[\mathsf{n}]^*}, \quad \mathsf{S}^{p^n-3}\mathsf{V}, \quad \mathsf{S}^{p^n-3}\mathsf{V}, \quad \nabla_{(p^n-2)\rho}. \tag{4.10}$$

Proof. Semiorthogonal decomposition $\mathcal{A}$ from Lemma 4.1 satisfies the assumptions of Claim 3.1. Hence, one has:

$$\mathsf{F}_{\mathsf{n}*}\mathcal{O}_{\mathbf{SL}_3/\mathbf{B}} = \bigoplus \mathbb{H}^p(X, \mathsf{F}^*\mathsf{E}_p) \otimes \mathsf{F}_p^*, \tag{4.11}$$

where E_p are the terms of $\mathcal{A}$. Recall that F_p^* are obtained by dualizing the terms of right dual decomposition that are precisely the set in (4.9). □

4.2. **Type B$_2$.** Recall (see [15]) the necessary facts about the flag variety $\mathbf{Sp}_4/\mathbf{B}$. The group $\mathbf{Sp}_4$ has two parabolic subgroups $\mathbf{P}_\alpha$ and $\mathbf{P}_\beta$ that correspond to the simple roots α and β, the root β being the long root. The homogeneous spaces $\mathbf{G}/\mathbf{P}_\alpha$ and $\mathbf{G}/\mathbf{P}_\beta$ are isomorphic to the 3-dimensional quadric Q_3 and $\mathbb{P}^3$, respectively. Denote q and π the two projections of $\mathbf{G}/\mathbf{B}$ onto Q_3 and $\mathbb{P}^3$. The projection π is the projective bundle over $\mathbb{P}^3$ associated to a rank two null–correlation vector bundle N over $\mathbb{P}^3 = \mathbb{P}(\mathsf{V})$, and the projection q is the projective bundle associated to the spinor bundle $\mathcal{U}_2$ on Q_3. There is a short exact sequence on $\mathbb{P}^3$:

$$0 \to \mathcal{L}_{-\omega_\alpha} \to \Omega^1_{\mathbb{P}^3} \otimes \mathcal{L}_{\omega_\alpha} \to \mathsf{N} \to 0, \tag{4.12}$$

while the spinor bundle $\mathcal{U}_2$ fits into a short exact sequence on Q_3:

$$0 \to \mathcal{U}_2 \to \mathsf{V} \otimes \mathcal{O}_{\mathsf{Q}_3} \to \mathcal{U}_2^* \to 0. \tag{4.13}$$

The following short exact sequences on $\mathbf{Sp}_4/\mathbf{B}$ will also be useful:

$$0 \to \mathcal{L}_{\omega_\alpha - \omega_\beta} \to \pi^*\mathsf{N} \to \mathcal{L}_{\omega_\beta - \omega_\alpha} \to 0, \tag{4.14}$$

and

$$0 \to \mathcal{L}_{-\omega_\alpha} \to \mathcal{U}_2 \to \mathcal{L}_{\omega_\alpha - \omega_\beta} \to 0. \tag{4.15}$$

Finally, recall that the bundle Ψ_1 on Q_3 fits into a short exact sequence:

$$0 \to \Psi_1 \to \nabla_{\omega_\beta} \otimes \mathcal{O}_{\mathsf{Q}_3} \to \mathcal{L}_{\omega_\beta} \to 0. \tag{4.16}$$

Consider a sequence $\mathcal{A} = \langle \mathcal{A}_i \rangle_{i=-4}^{i=0}$ of full triangulated subcategories of $\mathrm{D}^b(\mathbf{Sp}_4/\mathbf{B})$:

$$\mathcal{A}_{-4} = \langle \mathcal{L}_{-\rho} \rangle, \quad \mathcal{A}_{-3} = \langle \mathcal{L}_{-\omega_\alpha}, \mathcal{L}_{-\omega_\beta} \rangle, \tag{4.17}$$
$$\mathcal{A}_{-2} = \langle \pi^*\Omega^2_{\mathbb{P}^3} \otimes \mathcal{L}_{2\omega_\alpha}, \mathcal{U}_2 \rangle, \quad \mathcal{A}_{-1} = \langle \pi^*\Omega^1_{\mathbb{P}^3} \otimes \mathcal{L}_{\omega_\alpha}, \Psi_1 \rangle, \quad \mathcal{A}_0 = \langle \mathcal{O}_{\mathbf{Sp}_4/\mathbf{B}} \rangle.$$

Lemma 4.2. *The sequence $\mathcal{A}$ forms a semiorthogonal decomposition of $\mathrm{D}^b(\mathbf{Sp}_4/\mathbf{B})$. The bundles in each block $\mathcal{A}_i$ are mutually ortogonal. If $\mathcal{E} \in \mathcal{A}_i$, then one has $\mathrm{H}^j(\mathbf{Sp}_4/\mathbf{B}, (\mathsf{F}^{\mathsf{n}})^*\mathcal{E}) = 0$ for $j \neq -i$. Right dual decomposition with respect to (4.17) consists of the following subcategories:*

$$\mathcal{A}_0^{\vee} = \langle \mathcal{O}_{\mathbf{Sp_4/B}} \rangle, \quad \mathcal{A}_1^{\vee} = \langle \mathcal{L}_{\omega_\alpha}, \mathcal{L}_{\omega_\beta} \rangle, \quad \mathcal{A}_2^{\vee} = \langle \mathcal{U}_2 \otimes \mathcal{L}_\rho, \pi^*\mathcal{T}_{\mathbb{P}^3} \otimes \mathcal{L}_{\omega_\beta - \omega_\alpha} \rangle,$$
$$\mathcal{A}_3^{\vee} = \langle q^*\Psi_1 \otimes \mathcal{L}_\rho, \pi^*\Omega^1_{\mathbb{P}^3} \otimes \mathcal{L}_{\omega_\alpha + \rho} \rangle, \quad \mathcal{A}_4^{\vee} = \langle \mathcal{L}_\rho \rangle. \tag{4.18}$$

Proof. The collections $\mathcal{L}_{-\omega_\alpha}, \Omega^2_{\mathbb{P}^3} \otimes \mathcal{L}_{2\omega_\alpha}, \Omega^1_{\mathbb{P}^3} \otimes \mathcal{L}_{\omega_\alpha}, \mathcal{O}$ and $\mathcal{L}_{-\omega_\beta}, \mathcal{U}_2, \Psi_1, \mathcal{O}$ are exceptional on $\mathbb{P}^3$ and Q_3, respectively. This implies many orthogonalities in the collection of the lemma. That the bundles in each $\mathcal{A}_i$ are mutually orthogonal can be verified using short exact sequences:

$$0 \to \mathcal{L}_{-\omega_\alpha} \to \mathsf{V} \otimes \mathcal{O} \to \mathcal{T}_{\mathbb{P}^3} \otimes \mathcal{L}_{-\omega_\alpha} \to 0, \tag{4.19}$$

$$0 \to \mathcal{L}_{-\omega_\beta} \to (\nabla_{\omega_\beta})^* \otimes \mathcal{O} \to \Psi_1^* \to 0, \tag{4.20}$$

and (4.15). Using these sequences and the Borel–Weil–Bott theorem, one deduces as well the remaining set of orthogonalities.

The property $\mathrm{H}^j(\mathbf{Sp}_4/\mathbf{B}, \mathsf{F}^*\mathcal{E}) = 0$ for $j \neq -i$ for a bundle $\mathcal{E}$ belonging to $\mathcal{A}_i$ can be obtained, for example, from decompositions $\mathsf{F}_*\mathcal{O}_{\mathbb{P}^3}$ and $\mathsf{F}_*\mathcal{O}_{\mathsf{Q}_3}$ into direct sum of indecomposables:

$$\mathsf{F}_*\mathcal{O}_{\mathbb{P}^3} = \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{L}_{-\omega_\alpha}^{\oplus p_1} \oplus \mathcal{L}_{-2\omega_\alpha}^{\oplus p_2} \oplus \mathcal{L}_{-3\omega_\alpha}^{\oplus p_3} \tag{4.21}$$

$$\mathsf{F}_*\mathcal{O}_{\mathsf{Q}_3} = \mathcal{O}_{\mathsf{Q}_3} \oplus \mathcal{L}_{-\omega_\beta}^{\oplus q_1} \oplus (\mathcal{U}_2 \otimes \mathcal{L}_{\omega_{-\beta}})^{\oplus q_2} \oplus \mathcal{L}_{-2\omega_\beta}^{\oplus q_3}. \tag{4.22}$$

Finally, that $\mathcal{A}$ generates $\mathcal{D}^b(\mathbf{Sp}_4/\mathbf{B})$ easily follows from sequences (4.14), (4.15), and (4.16). Using these, one sees that semiorthgonal sequence (4.17) contains the subcategories $q^*\,\mathrm{D}^b(\mathsf{Q}_3)$ and $q^*\,\mathrm{D}^b(\mathsf{Q}_3) \otimes \mathcal{L}_{-\omega_\alpha}$, hence is full by the Beilinson theorem for $\mathbb{P}^n$–bundles.

Let us compute right dual decomposition. From sequences dual to (4.19) and (4.15) one immediately sees that $\mathrm{R}_{\langle \mathcal{O} \rangle}(\Omega^1_{\mathbb{P}^3} \otimes \mathcal{L}_{\omega_\alpha}) = \mathcal{L}_{\omega_\alpha}[-1]$ and $\mathrm{R}_{\langle \mathcal{O} \rangle}(\Psi_1) = \mathcal{L}_{\omega_\beta}[-1]$. On the other hand, tensoring each of the above sequences with $\mathcal{L}_{-\rho}$, one obtains $\mathrm{L}_{\mathcal{L}_{-\rho}}(\mathcal{L}_{-\omega_\alpha}) = \Psi_1 \otimes \mathcal{L}_{-\rho}[1]$ and $\mathrm{L}_{\mathcal{L}_{-\rho}}(\mathcal{L}_{-\omega_\beta}) = \Omega^1_{\mathbb{P}^3} \otimes \mathcal{L}_{-\omega_\beta}[1]$. Thus, by Theorem 3.1, one obtains:

$$\mathrm{R}_{\langle \mathsf{A}_{-2}, \mathsf{A}_{-1}, \mathsf{A}_0 \rangle}(\mathcal{L}_{-\omega_\alpha}) = \Psi_1 \otimes \mathcal{L}_\rho[-3], \tag{4.23}$$

and

$$\mathrm{R}_{\langle \mathsf{A}_{-2}, \mathsf{A}_{-1}, \mathsf{A}_0 \rangle}(\mathcal{L}_{-\omega_\beta}) = \Omega^1_{\mathbb{P}^3} \otimes \mathcal{L}_{-\omega_\beta} \otimes \mathcal{L}_{2\rho} = \Omega^1_{\mathbb{P}^3} \otimes \mathcal{L}_{\omega_\beta + 2\omega_\alpha}[-3]. \tag{4.24}$$

To compute left dual bundle to $\mathcal{U}_2$, we mutate it to the left through the subcategories A_{-3} and A_{-4}, and then mutate the result to the right through the whole collection. The effect of the last action is described by Theorem 3.1, and is equal to the right mutation of $\mathcal{U}_2$ through the subcategories $\mathsf{A}_{-2}, \mathsf{A}_{-1}, \mathsf{A}_0$, i.e. $\mathrm{R}_{\langle \mathsf{A}_{-2},\mathsf{A}_{-1},\mathsf{A}_0\rangle}\mathcal{U}_2$.

From (4.15) one finds that $\mathrm{H}om^{\bullet}(\mathcal{L}_{-\omega_\alpha},\mathcal{U}_2) = \mathsf{k}$ and $\mathrm{H}om^{\bullet}(\mathcal{L}_{-\omega_\beta},\mathcal{U}_2) = \mathsf{V}$. Thus, $\mathrm{L}_{\mathcal{L}_{-\omega_\beta}}\mathcal{U}_2 = \mathcal{U}_2 \otimes \mathcal{L}_{-\omega_\beta}[1]$. Further, $\mathrm{H}om^{\bullet}(\mathcal{L}_{-\omega_\alpha},\mathcal{U}_2 \otimes \mathcal{L}_{-\omega_\beta}) = \mathsf{k}[-1]$, and a unique non–trivial extension corresponds to the non–split short exact sequence

$$0 \to \mathcal{U}_2 \otimes \mathcal{L}_{\omega_\alpha} \to \Omega^1_{\mathbb{P}^3} \otimes \mathcal{L}_{2\omega_\alpha} \to \mathcal{L}_{\omega_\beta} \to 0, \tag{4.25}$$

tensored with $\mathcal{L}_{-\rho}$. Indeed, concatenating two short exact sequences (4.14) and (4.15), one obtains an element of the group $\mathrm{E}xt^2(\mathcal{L}_{\omega_\beta-\omega_\alpha}, \mathcal{L}_{-\omega_\alpha}) = \mathrm{H}^2(\mathcal{L}_{-\omega_\beta}) = 0$, and the above sequence is the one that trivializes that element. Thus, there is a distinguished triangle

$$\dots \to \Omega^1_{\mathbb{P}^3} \otimes \mathcal{L}_{\omega_\alpha-\omega_\beta} \to \mathcal{L}_{-\omega_\alpha} \to \mathcal{U}_2 \otimes \mathcal{L}_{-\omega_\beta}[1] \to \dots, \tag{4.26}$$

and $\mathrm{L}_{\mathcal{L}_{-\omega_\alpha}}(\mathcal{U}_2 \otimes \mathcal{L}_{-\omega_\beta}[1]) = \mathrm{L}_{\mathsf{A}_{-3}}\mathcal{U}_2 = \Omega^1_{\mathbb{P}^3} \otimes \mathcal{L}_{\omega_\alpha-\omega_\beta}[1]$. Finally, $\mathrm{H}om^{\bullet}(\mathcal{L}_{-\rho}, \Omega^1_{\mathbb{P}^3} \otimes \mathcal{L}_{\omega_\alpha-\omega_\beta}) = \wedge^2\mathsf{V}^*$, and one finds:

$$\dots \to \mathrm{L}_{\mathcal{L}_{-\rho}}(\Omega^1_{\mathbb{P}^3} \otimes \mathcal{L}_{\omega_\alpha-\omega_\beta}) \to \wedge^2\mathsf{V}^* \otimes \mathcal{L}_{-\rho}[1] \to \Omega^1_{\mathbb{P}^3} \otimes \mathcal{L}_{\omega_\alpha-\omega_\beta}[1] \to \dots, \tag{4.27}$$

thus, $\mathrm{L}_{\mathcal{L}_{-\rho}}(\Omega^1_{\mathbb{P}^3} \otimes \mathcal{L}_{\omega_\alpha-\omega_\beta}[1]) = \mathrm{L}_{\mathcal{L}_{-\rho}}\mathrm{L}_{\mathsf{A}_{-3}}\mathcal{U}_2 = \Omega^2_{\mathbb{P}^3} \otimes \mathcal{L}_{\omega_\alpha-\omega_\beta}[2]$. Tensoring it with $\mathcal{L}_{2\rho}$, using an isomorphism $\Omega^2_{\mathbb{P}^3} \otimes \mathcal{L}_{3\omega_\alpha} = \mathcal{T}_{\mathbb{P}^3} \otimes \mathcal{L}_{-\omega_\alpha}$, and applying Theorem 3.1, one obtains:

$$\mathrm{R}_{\langle \mathsf{A}_{-2},\mathsf{A}_{-1},\mathsf{A}_0\rangle}\mathcal{U}_2 = \mathcal{T}_{\mathbb{P}^3} \otimes \mathcal{L}_{\omega_\beta-\omega_\alpha}[-2]. \tag{4.28}$$

Finally, let us compute right dual bundle to $\Omega^2_{\mathbb{P}^3} \otimes \mathcal{L}_{2\omega_\alpha}$. Similarly, we compute the mutation of $\Omega^2_{\mathbb{P}^3} \otimes \mathcal{L}_{2\omega_\alpha}$ to the left through A_{-3} and A_{-4}, and then mutate the result to the right through the whole collection. One obtains isomorphisms $\mathrm{H}om^{\bullet}(\mathcal{L}_{-\omega_\alpha}, \Omega^2_{\mathbb{P}^3} \otimes \mathcal{L}_{2\omega_\alpha}) = \mathsf{V}$, and $\mathrm{H}om(\mathcal{L}_{-\omega_\beta}, \Omega^2_{\mathbb{P}^3} \otimes \mathcal{L}_{2\omega_\alpha}) = \mathsf{k}$. Thus, $\mathrm{L}_{\mathcal{L}_{-\omega_\alpha}}(\Omega^2_{\mathbb{P}^3} \otimes \mathcal{L}_{2\omega_\alpha}) = \mathcal{L}_{-2\omega_\alpha}[1]$. Further, $\mathrm{H}om^{\bullet}(\mathcal{L}_{-\omega_\beta}, \mathcal{L}_{-2\omega_\alpha}) = \mathsf{k}[-1]$, and a unique non–trivial extension corresponds to the non–split short exact sequence

$$0 \to \mathcal{L}_{\omega_\beta-\omega_\alpha} \to \mathcal{U}_2^* \to \mathcal{L}_{\omega_\alpha} \to 0, \tag{4.29}$$

tensored with $\mathcal{L}_{-\rho}$. Thus, there is a distinguished triangle

$$\dots \to \mathcal{U}_2^* \otimes \mathcal{L}_{-\rho} \to \mathcal{L}_{-\omega_\beta} \to \mathcal{L}_{-2\omega_\alpha}[1] \to \dots, \tag{4.30}$$

and $\mathrm{L}_{\mathcal{L}_{-\omega_\beta}}\mathcal{L}_{-2\omega_\alpha} = \mathrm{L}_{\mathsf{A}_{-3}}(\Omega^2_{\mathbb{P}^3} \otimes \mathcal{L}_{2\omega_\alpha}) = \mathcal{U}_2^* \otimes \mathcal{L}_{-\rho}[1]$. Finally, $\mathrm{H}om^{\bullet}(\mathcal{L}_{-\rho}, \mathcal{U}_2^* \otimes \mathcal{L}_{-\rho}) = \mathsf{V}$, and there is a short exact sequence:

$$0 \to \mathcal{U}_2 \otimes \mathcal{L}_{-\rho} \to \mathsf{V} \otimes \mathcal{L}_{-\rho} \to \mathcal{U}_2^* \otimes \mathcal{L}_{-\rho} \to 0. \tag{4.31}$$

Thus, $\mathrm{L}_{\mathcal{L}_{-\rho}}(\mathcal{U}_2^* \otimes \mathcal{L}_{-\rho}[1]) = \mathrm{L}_{\mathcal{L}_{-\rho}}\mathrm{L}_{\mathcal{A}_{-3}}(\mathcal{U}_2^* \otimes \mathcal{L}_{-\rho}[1]) = \mathcal{U}_2 \otimes \mathcal{L}_{-\rho}[2]$. Theorem 3.1 then gives

$$\mathrm{R}_{\langle \mathsf{A}_{-2}, \mathsf{A}_{-1}, \mathsf{A}_0 \rangle}(\Omega^2_{\mathbb{P}^3} \otimes \mathcal{L}_{2\omega_\alpha}) = \mathcal{U}_2 \otimes \mathcal{L}_\rho[-2]. \tag{4.32}$$

□

Theorem 4.2. *The bundle $\mathsf{F}_{\mathsf{n}*}\mathcal{O}_{\mathbf{Sp_4/B}}$ decomposes into the direct sum of vector bundles with indecomposable summands being isomorphic to:*

$$\begin{gathered} \mathcal{O}_{\mathbf{Sp_4/B}}, \quad \mathcal{L}_{-\omega_\alpha}, \quad \mathcal{L}_{-\omega_\beta}, \quad q^*\mathcal{U}_2^* \otimes \mathcal{L}_{-\rho}, \\ \pi^*\Omega^1_{\mathbb{P}^3} \otimes \mathcal{L}_{\omega_\alpha - \omega_\beta}, \quad q^*\Psi_1^* \otimes \mathcal{L}_{-\rho}, \quad \pi^*\mathcal{T}_{\mathbb{P}^3} \otimes \mathcal{L}_{-\omega_\alpha - \rho}, \quad \mathcal{L}_{-\rho}. \end{gathered} \tag{4.33}$$

The multiplicity spaces at each indecomposable summand are isomorphic, respectively, to:

$$\begin{gathered} \mathsf{k}, \quad \mathrm{H}^1(\mathsf{Q}_3, (\mathsf{F}^\mathsf{n})^*\Psi_1), \quad \mathrm{H}^1(\mathbb{P}^3, (\mathsf{F}^\mathsf{n})^*(\Omega^1_{\mathbb{P}^3} \otimes \mathcal{L}_{\omega_\alpha})), \quad \mathrm{H}^1(\mathsf{Q}_3, (\mathsf{F}^\mathsf{n})^*\mathcal{U}_2), \\ \mathrm{H}^2(\mathbb{P}^3, (\mathsf{F}^\mathsf{n})^*(\Omega^2_{\mathbb{P}^3} \otimes \mathcal{L}_{2\omega_\alpha})), \quad (\nabla_{(p^n-3)\omega_\beta})^*, \quad (\nabla_{(p^n-3)\omega_\alpha})^*, \quad (\nabla_{(p^n-2)\rho})^*. \end{gathered} \tag{4.34}$$

Proof. Similar to proof of Theorem 4.1. □

As a corollary to Theorem 4.2, one obtains another proof of the main result of [15]:

Corollary 4.1. *One has $\mathrm{H}^i(\mathbf{Sp}_4/\mathbf{B}, \mathcal{D}_{\mathbf{Sp_4/B}}) = 0$ for $i > 0$, and the flag variety $\mathbf{Sp}_4/\mathbf{B}$ is* D*–affine.*

Proof. The decomposition from Theorem 4.2 implies $\mathcal{E}xt^i(\mathsf{F}_{\mathsf{n}*}\mathcal{O}_{\mathbf{Sp_4/B}}, \mathsf{F}_{\mathsf{n}*}\mathcal{O}_{\mathbf{Sp_4/B}}) = 0$ for $i > 0$, since the indecomposable summands in the decomposition form an exceptional collection by the very construction. Thus, $\mathcal{E}xt^i(\mathsf{F}_{\mathsf{n}*}\mathcal{O}_{\mathbf{Sp_4/B}}, \mathsf{F}_{\mathsf{n}*}\mathcal{O}_{\mathbf{Sp_4/B}}) = \mathrm{H}^i(\mathbf{Sp}_4/\mathbf{B}, \mathcal{D}^{(n)}_{\mathbf{Sp_4/B}}) = 0$ for $i > 0$, and we conclude as in [15]. □

References

[1] P. Achinger, *Frobenius push-forwards on quadrics*, Comm. in Algebra, **40**, 8, (2012)

[2] H.H.Andersen, *The first cohomology group of a line bundle on* $\mathbf{G}/\mathbf{B}$, Invent. Math. **51** (1979), no. 3, 287–296

[3] H.H.Andersen, *Cohomology of line bundles on* $\mathbf{G}/\mathbf{B}$, Ann. scient. Ec. Norm. Sup., **12** (1979), pp. 85–100

[4] H.H.Andersen, M. Kaneda, *Loewy series of modules for the first Frobenius kernel in a reductive algebraic group*, Proc. London Math. Soc. (3) **59** (1989), no. 1, pp. 74–98

[5] H.H.Andersen, M. Kaneda, *Cohomology of line bundles on the flag variety for type* $\mathbf{G}_2$, J. Pure Appl. Alg., **216**, (2012), pp. 1566–1579

[6] A. Beilinson, J. Bernstein, *Localisation de* $\mathfrak{g}$*-modules*, C.R.Acad. Sci. Paris, Ser. I 292 (1981), 15–18

[7] A. Bondal, M. Kapranov, *Representable functors, Serre functors, and mutations*, (Russian) Izv. Akad. Nauk SSSR Ser. Mat. **53** (1989), no. 6, pp. 1183–1205

[8] A. Bondal, *Representations of associative algebras and coherent sheaves*, (Russian) Izv. Akad. Nauk SSSR Ser. Mat. 53 (1989), no. 1, pp. 25–44; translation in Math. USSR Izv. 34 (1990), no. 1, pp. 23–42

[9] Y. Hashimoto, M. Kaneda, D. Rumynin, *On localization of* $\overline{D}$*-modules*, in "Representations of algebraic groups, quantum groups, and Lie algebras", pp. 43–62, Contemp. Math., **413**, Amer. Math. Soc., Providence, RI, (2006)

[10] M. Kaneda, J.Ye, *Some observations on Karoubian complete strongly exceptional posets on the projective homogeneous varieties*, `arXiv:0911.2568`

[11] M. Kashiwara, N. Lauritzen, *Local cohomology and* $\mathcal{D}$*-affinity in positive characteristic*, C. R. Acad. Sci. Paris, Ser. I, 335 (2002), 993–996

[12] A. Kuznetsov, *Base change for semiorthogonal decompositions*, Compos. Math. 147 (2011), no. 3, 852–876.

[13] A. Langer, D-*affinity and Frobenius morphism on quadrics*, Int. Math. Res. Not., (2008), no. 1, Art. ID rnm 145, 26 pp.

[14] A. Samokhin, *Tilting bundles via the Frobenius morphism*, `arXiv:0904.1235v2`

[15] A. Samokhin, *On the* D*-affinity of flag varieties in positive characteristic*, J. Algebra, (6) **4** (2010), pp. 1435–1446

[16] A. Samokhin, *A vanishing theorem for differential operators in positive characteristic*, Transform. Groups **15** (2010), no. 1, pp. 227–242

[17] A. Samokhin, *The Frobenius morphism on flag varieties,* II, to appear

Institute for Information Transmission Problems, Moscow, Russia

and

Math. Institut, Heinrich-Heine-Universität, D-40204 Düsseldorf, Germany
E-mail address: `alexander.samokhin@gmail.com`