

MULTIPLE COVERINGS OF THE FARTHEST-OFF POINTS WITH SMALL DENSITY FROM PROJECTIVE GEOMETRY

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ABSTRACT. In this paper we deal with the special class of covering codes consisting of multiple coverings of the farthest-off points (MCF). In order to measure the quality of an MCF code, we use a natural extension of the notion of density for ordinary covering codes, that is the μ -density for MCF codes; a generalization of the length function for linear covering codes is also introduced. Our main results consist in a number of upper bounds on such a length function, obtained through explicit constructions, especially for the case of covering radius $R = 2$. A key tool is the possibility of computing the μ -length function in terms of Projective Geometry over finite fields. In fact, linear (R, μ) -MCF codes with parameters $[n, n - r, d]_q R$ have a geometrical counterpart consisting of special subsets of n points in the projective space $PG(n - r - 1, q)$. We introduce such objects under the name of (ρ, μ) -saturating sets and we provide a number of example and existence results. Finally, Almost Perfect MCF (APMCF) codes, that is codes for which each word at distance R from the code belongs to exactly μ spheres centered in codewords, are considered and their connections with uniformly packed codes, two-weight codes, and subgroups of Singer groups are pointed out.

1. INTRODUCTION

For a code C with covering radius R , it is sometimes useful that for every word x at distance R from C there is more than one codeword in the Hamming sphere $S(x, R)$. The code C is said to be an (R, μ) -multiple covering of the farthest-off points (MCF for short) if for each $x \in \mathbb{F}_q^n$ with $d(x, C) = R$ the size of $S(x, R) \cap C$ is at least μ . Multiple coverings can be viewed as a particular variant of a natural generalization of 1-fold-coverings. One motivation for studying MCF codes arises

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from the generalized football pool problem; see e.g. [27, 28, 36] and the references therein. Another interesting connection is with the list decoding; see e.g. [43]. Results on MCF codes, mostly concerning the binary and the ternary cases, can be found in [11, 25, 26, 31, 32, 39, 40].

For a q -ary linear MCF code C , a natural parameter to consider is the average value $\gamma_\mu(C, R)$ of $\frac{1}{\mu} \#(S(x, R) \cap C)$, where x is a word at distance R from C . Clearly, $\gamma_\mu(C, R) \geq 1$ holds and equality is attained precisely when each $x \in \mathbb{F}_q^n$ with $d(x, C) = R$ belongs to exactly μ spheres centered in codewords; if this is the case, then C is called an APMCF code; here, AP stands for *almost-perfect*. The parameter $\gamma_\mu(C, R)$ will be referred to as the μ -density of C . If the minimum distance d of C is at least $2R - 1$, then the best μ -density among linear q -ary codes with same codimension r and covering radius R is achieved by the shortest ones. This motivates the introduction of the μ -length function $\ell_\mu(R, r, q)$ as the smallest length n of a linear (R, μ) -MCF code with parameters $[n, n - r, d]_q R$, $d \geq 3$. For $\mu = 1$, $\ell_1(R, r, q)$ is the usual length function $\ell(R, r, q)$ [11, 15] for 1-fold coverings.

The aim of this paper is to investigate the μ -length function, mainly for the case of covering radius $R = 2$. A key tool is the possibility of rephrasing the definition of the μ -length function in terms of Projective Geometry over finite fields. In fact, linear (R, μ) -MCF codes with parameters $[n, n - r, d]_q R$ have a geometrical counterpart consisting of special subsets of n points in the projective space $PG(n - r - 1, q)$. Such subsets will be called $(R - 1, \mu)$ -saturating sets. By using a few geometrical methods, we will provide a number of constructions of small (ρ, μ) -saturating sets which will produce significant upper bounds on the μ -length function. Our main results in this direction are contained in Section 5 for the case of codimension $r = 3$, and in Section 6 for higher codimensions. The main achievements of the paper are Proposition 5.2, together with Corollaries 5.5, 5.13, 5.15, 5.19, 6.4, and 6.5, and Theorem 7.8.

The paper is organized as follows. In Section 2 we recall the notion of μ -density for MCF codes. We also present some explicit formulae for the computation of the μ -density in the case of codes with minimum distance $d \geq 2R - 1$.

Section 3 presents the connection between MCF codes and (ρ, μ) -saturating sets in Projective Geometries over finite fields; the concept of an *optimal* (ρ, μ) -saturating set is introduced.

Results of Sections 2 and 3 are specialized to the case $R = 2$ in Section 4; the μ -length function is recalled, and some preliminary facts on $(1, \mu)$ -saturating sets are established.

Section 5 contains a number of constructions of small $(1, \mu)$ -saturating sets in the projective plane $PG(2, q)$. The cases where q is a square, a non-prime non-square, and a prime are distinguished. A number of significant upper bounds on the μ -length function $\ell_\mu(2, 3, q)$ are obtained.

In Section 6, we present an inductive method which allows to construct short MCF codes of arbitrarily high codimension from short MCF codes with codimension 3. Such a method is a generalization of the q^m -concatenating construction for ordinary covering codes, and for the sake of simplicity will be called *multiple concatenating construction* (MCC) in this paper. Combining the results of Section 5 with MCC provides significant upper bounds for the μ -length function for every codimension.

Finally, Section 7 presents some interesting connections between APMCF codes and uniformly packed codes, two-weight codes, and subgroups of Singer groups.

Some of the results of this paper were briefly presented in [4, 21, 38].

2. MULTIPLE COVERINGS

An $(n, M, d)_qR$ code C is a code of length n , cardinality M , minimum distance d , and covering radius R , over the finite field \mathbb{F}_q with q elements. If C is linear of dimension k over \mathbb{F}_q , then C is also said to be an $[n, k, d]_qR$ code. When either d or R are not relevant or unknown they can be omitted in the above notation. Let \mathbb{F}_q^n be the linear space of dimension n over \mathbb{F}_q , equipped with the Hamming distance. The Hamming sphere of radius j centered at $x \in \mathbb{F}_q^n$ is denoted by $S(x, j)$. The size $V_q(n, j)$ of such a sphere is

$$(2.1) \quad V_q(n, j) = \sum_{i=0}^j \binom{n}{i} (q-1)^i.$$

Let $\bar{S}(x, R)$ be the surface of the sphere $S(x, R)$. For an $(n, M)_qR$ code C , $A_w(C)$ denotes the number of codewords in C of weight w , and $f_\theta(e, C)$ denotes the number of codewords at distance θ from a vector e in \mathbb{F}_q^n ; equivalently, $f_\theta(e, C) = \#\{\bar{S}(e, \theta) \cap C\}$. Let $t = \lfloor \frac{d-1}{2} \rfloor$ be the number of errors that can be corrected by a code with minimum distance d .

Definition 2.1 ([11, 26, 27]). An $(n, M)_qR$ code C is said to be an (R, μ) multiple covering of the farthest-off points ((R, μ) -MCF code for short) if for all $x \in \mathbb{F}_q^n$ such that $d(x, C) = R$ the number of codewords c such that $d(x, c) = R$ is at least μ .

In the literature, MCF codes are also called multiple coverings of deep holes, see e.g. [11, Chapter 14].

One of the parameters that measure the quality of an $(n, M)_qR$ covering code C is its density, namely the average number of spheres of radius R centered in words of C containing a fixed element in \mathbb{F}_q^n , denoted as $\delta(C, R)$. Recall that

$$(2.2) \quad \delta(C, R) = \frac{\sum_{x \in \mathbb{F}_q^n} \#\{c \in C \mid d(c, x) \leq R\}}{q^n} = \frac{M \cdot V_q(n, R)}{q^n}$$

holds; also, $\delta(C, R) \geq 1$, and equality holds if and only if C is a perfect code. Now assume that C is an (R, μ) -MCF code in \mathbb{F}_q^n . In the terminology of [11, Sect. 13.1], C is a weighted covering or an \mathbf{m} -covering, where $\mathbf{m} = (m_0, \dots, m_n)$ with

$$m_0 = m_1 = \dots = m_{R-1} = 1, \quad m_R = 1/\mu, \quad m_i = 0 \text{ for } R < i \leq n.$$

According to [11, Sect. 13.1], the \mathbf{m} -density of C at a fixed vector $x \in \mathbb{F}_q^n$ is

$$\delta_{\mathbf{m}}(x) = f_0(x, C) + f_1(x, C) + \dots + f_{R-1}(x, C) + \frac{f_R(x, C)}{\mu}.$$

By definition $\delta_{\mathbf{m}}(x) \geq 1$ holds, and a natural parameter to estimate the quality of C as an (R, μ) -MCF code is the average \mathbf{m} -density $\delta_{\mathbf{m}}(x)$, denoted by $\delta_{\mu}(C, R)$. By definition, $\delta_{\mu}(C, R) \geq 1$; if equality holds then C will be called an (R, μ) perfect multiple covering of the farthest-off points ((R, μ) -PMCF code for short). Note that this terminology is consistent with [11, Def. 13.1.2].

Proposition 2.2. An $(n, M)_qR$ code C is an (R, μ) -PMCF code if and only if the following conditions hold:

- (i) each $x \in \mathbb{F}_q^n$ with $d(x, C) = R$ belongs to exactly μ spheres centered in codewords of C ;

(ii) $d(C) \geq 2R$.

Proof. Assume that both Condition (i) and Condition (ii) hold and fix a word $x \in \mathbb{F}_q^n$. If $d(x, C) = R$, then $f_0(x, C) + f_1(x, C) + \dots + f_{R-1}(x, C) = 0$; also, by Condition (i), $\frac{f_R(x, C)}{\mu} = 1$ holds. Therefore, $\delta_{\mathbf{m}}(x) = 1$. If $d(x, C) < R$, then Condition (ii) implies that there exists a unique codeword c with $d(x, c) \leq R$; as actually $d(x, c) \leq R - 1$ we have that $\delta_{\mathbf{m}}(x) = 1$. Whence, $\delta_{\mu}(C, R) = 1$ and C is an (R, μ) -PMCF code.

Conversely, assume that C is an (R, μ) -MCF code. If Condition (ii) does not hold, then there exist two codewords c_1, c_2 with $d(c_1, c_2) \leq 2R - 1$. Whence, for some word x we have

$$d(x, c_1) \leq R - 1, \quad d(x, c_2) \leq R.$$

Then clearly $\delta_{\mathbf{m}}(x) > 1$. The same holds if Condition (i) fails for x , as in this case $\delta_{\mathbf{m}}(x) \geq \frac{f_R(x, C)}{\mu} > 1$. \square

Note that beside the usual perfect codes with $\mu = 1$, Condition (ii) of Proposition 2.2 holds only for quasi-perfect codes with d even (that is, codes with $R = d/2$). This explains why perfect multiple coverings of the farthest-off points seem to be rare objects, even though a complete classification is not known. Some examples are provided in Section 7.

By Proposition 2.2, a necessary condition for C to be an (R, μ) -PMCF code is that each $x \in \mathbb{F}_q^n$ with $d(x, C) = R$ belongs to exactly μ spheres centered in codewords of C . As a matter of terminology, if this happens we say that C is an (R, μ) *almost-perfect* multiple covering of the farthest-off points ((R, μ) -APMCF code).

Let

$$\{x_1, \dots, x_{N_R(C)}\}$$

be the set of vectors in \mathbb{F}_q^n with distance R from C , and let

$$(2.3) \quad \gamma_{\mu}(C, R) = \frac{\sum_{i=1}^{N_R(C)} f_R(x_i, C)}{\mu N_R(C)}.$$

It is easily seen that $\gamma_{\mu}(C, R) \geq 1$, and that C is an (R, μ) -APMCF code precisely when equality holds. We will refer to the parameter $\gamma_{\mu}(C, R)$ as to the μ -density of the (R, μ) -MCF code C . Next we are going to discuss some formulas for $\gamma_{\mu}(C, R)$.

For a codeword $c \in C$, we denote by $F(c, R)$ the *farthest-off part of the sphere* $S(c, R)$, namely the set of points in $S(c, R)$ at distance R from C . Clearly,

$$(2.4) \quad \sum_{i=1}^{N_R(C)} f_R(x_i, C) = \sum_{c \in C} \#F(c, R)$$

holds. Also,

$$(2.5) \quad \#F(c, R) \leq \binom{n}{R} \cdot (q-1)^R;$$

$$(2.6) \quad \sum_{c \in C} \#F(c, R) \leq M \cdot \binom{n}{R} \cdot (q-1)^R.$$

It should be remarked that in both Inequalities (2.5),(2.6), equality holds precisely when the minimum distance $d(C)$ satisfies $d(C) > 2R - 1$. Also, it is easy to see that

$$(2.7) \quad N_R(C) = q^n - M \cdot V_q(n, R - 1) \text{ if } d(C) \geq 2R - 1;$$

$$(2.8) \quad N_R(C) > q^n - M \cdot V_q(n, R - 1) \text{ if } d(C) < 2R - 1.$$

Now, by Formulas (2.3)–(2.7), for an $(n, M, d)_q R$ code C which is an (R, μ) -MCF code we have

$$(2.9) \quad \gamma_\mu(C, R) = \frac{1}{\mu} \cdot \frac{\sum_{c \in C} \#F(c, R)}{N_R(C)} \leq \frac{1}{\mu} \cdot \frac{M \cdot \binom{n}{R} \cdot (q - 1)^R}{N_R(C)};$$

$$(2.10) \quad \gamma_\mu(C, R) = \frac{1}{\mu} \cdot \frac{M \cdot \binom{n}{R} \cdot (q - 1)^R}{q^n - M \cdot V_q(n, R - 1)} \text{ if } d(C) > 2R - 1.$$

If an (R, μ) -MCF code C is a linear $[n, k, d]_q R$ code then

$$(2.11) \quad \gamma_\mu(C, R) = \frac{\binom{n}{R} \cdot (q - 1)^R}{\mu \cdot (q^{n-k} - V_q(n, R - 1))} \text{ if } d(C) > 2R - 1.$$

The case $d(C) = 2R - 1$ is dealt with in the following lemma.

Proposition 2.3. *Let C be a linear $[n, k, d(C)]_q R$ code with $d(C) \geq 2R - 1$. If C is (R, μ) -MCF, then*

$$(2.12) \quad \gamma_\mu(C, R) = \frac{\binom{n}{R} \cdot (q - 1)^R - \binom{2R-1}{R-1} \cdot A_{2R-1}(C)}{\mu \cdot (q^{n-k} - V_q(n, R - 1))}.$$

Proof. If $d(C) > 2R - 1$ then $A_{2R-1}(C) = 0$ and the assertion follows from Equation (2.11). Assume then that $d(C) = 2R - 1$. Let $\mathbf{0}$ denote the zero vector in \mathbb{F}_q^n . As C is linear, $\#F(c, R)$ coincides with $\#F(\mathbf{0}, R)$ for every $c \in C$. By Equation (2.4), $\sum_{i=1}^{N_R(C)} f_R(x_i, C) = q^k \cdot \#F(\mathbf{0}, R)$, and hence

$$(2.13) \quad \gamma_\mu(C, R) = \frac{\#F(\mathbf{0}, R)}{\mu \cdot (q^{n-k} - V_q(n, R - 1))}.$$

Note that $d(C) = 2R - 1$ implies that for any two codewords $c_1, c_2 \in C$ with distance $2R - 1$, the spheres $S(c_1, R - 1)$ and $S(c_2, R - 1)$ are disjoint. Therefore, the complement of $F(\mathbf{0}, R)$ in $\bar{S}(\mathbf{0}, R)$ is the disjoint union of $\bar{S}(c, R - 1) \cap \bar{S}(\mathbf{0}, R)$, with c ranging over the codewords in C with weight $2R - 1$. For any word with weight $2R - 1$

$$\#(\bar{S}(c, R - 1) \cap \bar{S}(\mathbf{0}, R)) = \binom{2R - 1}{R - 1}$$

holds. Then the claim follows by Equation (2.13). □

In the rest of the paper we will assume that

$$(2.14) \quad d(C) \geq 2R - 1.$$

In this case, $N_R(C)$ only depends on the basic parameters of the code; see Equation (2.7). Note that under Condition (2.14), $R = t + 1$; equivalently, C is a quasi-perfect code in the classical sense.

3. (R, μ) -MCF LINEAR CODES AND $(R - 1, \mu)$ -SATURATING SETS IN PROJECTIVE GEOMETRY

In this section we assume that C is an $[n, k, d]_q$ code with $d \geq 3$ (or, equivalently, that C^\perp is a projective code). Let \mathbf{H} be a parity check matrix of C . For an element $x \in \mathbb{F}_q^n$, let $s(x) = \mathbf{H} \cdot x^{tr}$, denote the syndrome of x . Note that as $d \geq 3$, the columns of \mathbf{H} represent n pairwise linearly independent vectors of \mathbb{F}_q^{n-k} . Therefore, \mathbf{H} defines a set $S = \{P_1, \dots, P_n\}$ of n points in $PG(n - k - 1, q)$.

Lemma 3.1. *Let $x \in \mathbb{F}_q^n$. Then, the number of codewords $c \in C$ such that $d(x, c) = R$ is the number of distinct vectors v in \mathbb{F}_q^n of weight R such that*

$$(3.1) \quad s(x) = \mathbf{H} \cdot v^{tr}.$$

Proof. The assertion follows from the fact that $\mathbf{H} \cdot x^{tr} = \mathbf{H} \cdot v^{tr}$ holds if and only if $c = x - v$ is a codeword of C with $d(c, x)$ equal to the weight of v . \square

Note that $\mathbf{H} \cdot v^{tr}$ in Equation (3.1) is a linear combination of the columns of \mathbf{H} with exactly R non-zero coefficients.

We are going to translate the property of C being an (R, μ) -MCF code into some geometrical features of S , see Proposition 3.6 below. We recall that R being the covering radius of C corresponds to the following property of S : *every point in $PG(n - k - 1, q)$ is linearly dependent with R points from S , and there exists a point in $PG(n - k - 1, q)$ which is linearly independent with any set of $R - 1$ points from S .*

Remark 3.2. The condition $d(C) \geq 2R - 1$ reads as follows: *every $2R - 2$ points of S are linearly independent.* As $2R - 2 \geq R$, we have in particular that every R points of S are linearly independent.

Fix x in \mathbb{F}_q^n . A number of cases will be distinguished, according to the syndrome of x .

- $s(x)$ is a linear combination of $R - 1$ columns of \mathbf{H} . Here $d(x, C) < R$, and there is nothing to check. Geometrically, $s(x)$ is either the zero vector, or it represents a point belonging to some space of dimension less than or equal to $R - 2$ generated by some points of S .
- $s(x)$ is not a linear combination of $R - 1$ columns of \mathbf{H} . Let P be the point of $PG(n - k - 1, q)$ corresponding to $s(x)$. Then P does not belong to any space of dimension less than or equal to $R - 2$ generated by the points of S . As the covering radius of S is equal to R , the point P belongs to at least one subspace of dimension $R - 1$ generated by the points of S . Let $\{T_1, \dots, T_h\}$ be the set of distinct subspaces of dimension $R - 1$ generated by some points in S and containing P . Let $V_i := T_i \cap S$. Then V_i contains at least R independent points of S . Let u_i be the number of distinct sets of R independent points of S belonging to T_i . Then we have u_i distinct ways of expressing $s(x)$ as a linear combination of R columns of \mathbf{H} . In order for x to satisfy the condition of Definition 2.1 we need that $u_1 + u_2 + \dots + u_h$ is at least μ .

The definition of a (ρ, μ) -saturating set in $PG(N, q)$ can now be given.

Definition 3.3. Let $S = \{P_1, \dots, P_n\}$ be a subset of points of $PG(N, q)$. Then S is said to be (ρ, μ) -saturating if:

- (M1) S generates $PG(N, q)$;

- (M2) there exists a point Q in $PG(N, q)$ which does not belong to any subspace of dimension $\rho - 1$ generated by the points of S ;
- (M3) every point Q in $PG(N, q)$ not belonging to any subspace of dimension $\rho - 1$ generated by the points of S , is such that the number of subspaces of dimension ρ generated by the points of S and containing Q , counted with multiplicity, is at least μ . The multiplicity m_T of a subspace T is computed as the number of distinct sets of $\rho + 1$ independent points contained in $T \cap S$.

Note that if any $\rho + 1$ points of S are linearly independent (that is, the minimum distance of the corresponding code is at least $\rho + 2$), then

$$m_T = \binom{\#(T \cap S)}{\rho + 1}.$$

Definition 3.4. A (ρ, μ) -saturating n -set in $PG(N, q)$ is called *minimal* if it does not contain a (ρ, μ) -saturating $(n - 1)$ -set in $PG(N, q)$.

Definition 3.5. An $[n, k]_q R$ code C with $R = \rho + 1$ corresponds to a (ρ, μ) -saturating n -set S in $PG(n - k - 1, q)$ if C admits a parity-check matrix whose columns are homogeneous coordinates of the points in S .

As a consequence of the above discussion, the following result holds.

Proposition 3.6. A linear $[n, k]_q R$ code C corresponding to a (ρ, μ) -saturating n -set S in $PG(n - k - 1, q)$ is a $(\rho + 1, \mu)$ -MCF code.

Proposition 3.6 allows us to consider (ρ, μ) -saturating sets as linear $(\rho + 1, \mu)$ -MCF codes and vice versa.

Definition 3.7. Let S be a (ρ, μ) -saturating n -set in $PG(n - k - 1, q)$. The set S is called *optimal* (ρ, μ) -saturating set ((ρ, μ) -OS set for short) if every point Q in $PG(n - k - 1, q)$ not belonging to any subspace of dimension $\rho - 1$ generated by the points of S , is such that the number of subspaces of dimension ρ generated by the points of S and containing Q , counted with multiplicity, is exactly μ .

If S is a (ρ, μ) -OS set, then the corresponding linear $[n, k]_q R, \mu$ APMCF code with $\gamma_\mu(C, \rho + 1) = 1$. By Proposition 2.2, C is actually a $(\rho + 1, \mu)$ PMCF code with $\mu > 1$ if its minimum distance is precisely $2R$.

4. $(1, \mu)$ -SATURATING SETS

For $\rho = 1$ Conditions (M1)-(M3) read as follows:

- (M1) S generates $PG(N, q)$;
- (M2) S is not the whole $PG(N, q)$;
- (M3) every point Q in $PG(N, q)$ not belonging to S is such that the number of secants of S through Q is at least μ , counted with multiplicity. The multiplicity m_ℓ of a secant ℓ is computed as

$$m_\ell = \binom{\#(\ell \cap S)}{2}.$$

According to Definition 3.5, let C be the linear $[n, n - N - 1, d(C)]_q 2$ code corresponding to a $(1, \mu)$ -saturating n -set S . Then $\mu\gamma_\mu(C, 2)$ is equal to the average number of secants of S , counted with multiplicity, through a fixed point $Q \in PG(N, q) \setminus S$.

The following result then holds.

Proposition 4.1. *Let S be a $(1, \mu)$ -saturating set in $PG(N, q)$. Then S is a $(1, \mu)$ -OS set precisely when each point $Q \in PG(N, q) \setminus S$ belongs to exactly μ secants of S , counted with multiplicity.*

Note that as $R = 2$, the condition $d(C) > 2R - 1$ reads as $d(C) > 3$; also, $A_3(C) = 0$ holds. On the other hand, when $d(C) = 2R - 1 = 3$, we have $A_3(C) > 0$.

Let $B_3(S)$ denote the number of triples of collinear points in S .

Proposition 4.2. *Let S be a $(1, \mu)$ -saturating set in $PG(N, q)$. Then S is a $(1, \mu)$ -OS set if and only if*

$$(4.1) \quad \frac{n-1}{2}(q-1) - \frac{3}{n}B_3(S) = \mu \cdot \left(\frac{\#PG(N, q)}{n} - 1 \right).$$

Proof. Recall that $\#PG(N, q) = (q^{N+1} - 1)/(q - 1)$. Clearly,

$$B_3(S) = \frac{1}{q-1}A_3(C),$$

and hence, by Equations (2.1) and (2.12),

$$(4.2) \quad \gamma_\mu(C, 2) = \frac{\binom{n}{2}(q-1)^2 - 3A_3(C)}{\mu \cdot (q^{n-k} - V_q(n, 1))} = \frac{\frac{1}{2}(n-1)(q-1) - \frac{3}{n}B_3(S)}{\mu \cdot \left(\frac{\#PG(N, q)}{n} - 1 \right)}$$

holds. □

Taking into account Equation (2.7), it is clear that if q, N, μ are fixed then the best μ -density is achieved for small n . Therefore, the following parameter seems to be relevant in this context.

Definition 4.3. The μ -length function $\ell_\mu(2, r, q)$ is the smallest length n of a linear $(2, \mu)$ -MCF code with parameters $[n, n-r, d]_q$, $d \geq 3$, or equivalently the smallest cardinality of a $(1, \mu)$ -saturating set in $PG(r-1, q)$. For $\mu = 1$, $\ell_1(2, r, q)$ is the usual length function $\ell(2, r, q)$ [11, 15] for 1-fold coverings.

Remark 4.4. A number μ of disjoint copies of a 1-saturating set in $PG(N, q)$ give rise to a $(1, \mu)$ -saturating set in $PG(N, q)$. Therefore,

$$(4.3) \quad \ell_\mu(2, r, q) \leq \mu \ell(2, r, q).$$

Denote by $\gamma_\mu(2, r, q)$ the minimum μ -density of a linear $(2, \mu)$ -MCF code of codimension r over \mathbb{F}_q . Let $\delta(2, r, q)$ be the minimum density of a linear code with covering radius 2 and codimension r over \mathbb{F}_q . By Equation (4.2) and Inequality (4.3),

$$(4.4) \quad \gamma_\mu(2, r, q) \leq \frac{\frac{1}{2}(\mu \ell(2, r, q) - 1)(q-1)}{\mu \cdot \left(\frac{\#PG(r-1, q)}{\mu \ell(2, r, q)} - 1 \right)} - 1 \sim \mu \delta(2, r, q).$$

The same inequalities clearly hold for the *best known* lengths and densities, denoted, respectively, by $\bar{\ell}_\mu(2, r, q)$, $\bar{\ell}(2, r, q)$, $\bar{\gamma}_\mu(2, r, q)$, and $\bar{\delta}(2, r, q)$:

$$(4.5) \quad \bar{\ell}_\mu(2, r, q) \leq \mu \bar{\ell}(2, r, q).$$

$$(4.6) \quad \bar{\gamma}_\mu(2, r, q) \lesssim \mu \bar{\delta}(2, r, q).$$

From Equations (4.3)–(4.6), results for parameters $\ell_\mu(2, r, q)$, $\bar{\ell}_\mu(2, r, q)$, $\gamma_\mu(2, r, q)$, and $\bar{\gamma}_\mu(2, r, q)$, can be immediately obtained from the vast body of literature on 1-saturating sets in finite projective spaces; see e.g. [37], where $\ell(2, r, q)$ is established for some small q 's.

The aim of the present paper is to construct $(1, \mu)$ -saturating sets in $PG(N, q)$ giving rise to $(2, \mu)$ -MCF codes with size and density smaller than those in Inequalities (4.3)–(4.6).

5. CONSTRUCTIONS OF SMALL $(1, \mu)$ -SATURATING SETS IN $PG(2, q)$

We first point out a trivial upper bound on the largest size of a minimal $(1, \mu)$ -saturating set in $PG(2, q)$, $q > 2$.

Proposition 5.1. *Let A be a $(q + \mu + 1)$ -set in $PG(2, q)$, $q > 2$, $\mu < q^2$. Then A is a $(1, \mu)$ -saturating set.*

Proof. The inequality $\mu < q^2$ provides Condition (M2). Let P be a point of $PG(2, q) \setminus A$. On the $q + 1$ lines through P there are at least μ pairs of points of A and therefore A is a $(1, \mu)$ -saturating set, possibly not minimal. \square

Therefore, an upper bound for the largest size of a minimal $(1, \mu)$ -saturating set in $PG(2, q)$, $q > 2$, is $q + \mu + 1$.

Now we give some bounds on $\ell_\mu(2, 3, q)$ or, equivalently, bounds on the smallest size of a $(1, \mu)$ -saturating set in $PG(2, q)$.

Proposition 5.2. *For the length function $\ell_\mu(2, 3, q)$, the following relations hold.*

(i) *Trivial bound:*

$$(5.1) \quad \ell_\mu(2, 3, q) \geq \sqrt{2\mu q}.$$

(ii) *Probabilistic bound:*

$$(5.2) \quad \ell_\mu(2, 3, q) < 66\sqrt{\mu q \ln q}, \quad \text{if } \mu < 121q \log q.$$

(iii) *Baer bound for q a square:*

$$(5.3) \quad \ell_\mu(2, 3, q) \leq \mu(3\sqrt{q} - 1).$$

Proof. (i) Let S be a $(1, \mu)$ -saturating n -set in $PG(2, q)$. Then every point in $PG(2, q) \setminus S$ can be written in μ distinct ways as a linear combination of two distinct points in S . The total number of such combinations is $(q - 1)\binom{n}{2}$. Then $(q - 1)\binom{n}{2} \geq \mu(q^2 + q + 1 - n)$ which roughly gives $\#S \geq \sqrt{2\mu q}$.

(ii) The existence of 1-saturating sets of size $\lfloor 5\sqrt{q \ln q} \rfloor$ was shown by means of probabilistic methods, see [7, 33]. By adapting the proofs given in [7, 33] it is easily seen that Condition (5.2) holds.

(iii) By an explicit construction, in $PG(2, q)$, q a square, a 1-saturating set of size $3\sqrt{q} - 1$ is obtained in [12, Th. 5.2]. \square

The aim of this section is to provide some general constructions of $(1, \mu)$ -saturating sets in $PG(2, q)$ with size less than $\mu\bar{\ell}(2, 3, q)$. We remark that sometimes the sizes of the $(1, \mu)$ -saturating sets provided here exceed the probabilistic bound. The point of considering them here is to give explicit constructions, not only existence results.

5.1. q A SQUARE. When q is a square, the Baer bound in Proposition 5.2 can be improved for a large number of μ 's.

Theorem 5.3. *Let q be a square and let $3 \leq s \leq \sqrt{q} + 1$ be an even integer. Let $\mathcal{L}_1, \dots, \mathcal{L}_s$ be a set of s lines in $PG(2, \sqrt{q})$ no three of which concurrent. Then the union S of such lines is a $(1, \mu)$ -saturating set in $PG(2, q)$ of size $s(\sqrt{q} + 2 - s) + s(s - 1)/2$, with $\mu = \frac{1}{8}(s^2 - 2s)$.*

Proof. The size of S is clearly $s(\sqrt{q} + 2 - s) + s(s - 1)/2$. Through any point P in $PG(2, q)$ there passes a line of the subplane $PG(2, \sqrt{q})$. Such a line meets S in at least $s/2$ points, since no three lines in S are concurrent. Whence there are at least μ distinct pairs of points in S collinear with P . \square

Theorem 5.4. *Let q be a square and let $3 \leq s \leq \sqrt{q} + 1$ be an integer. Let $\mathcal{L}_1, \dots, \mathcal{L}_s$ be a set of s lines in $PG(2, \sqrt{q})$ through a common point P . For any other line \mathcal{L} through P choose $s - 1$ points $R_{\mathcal{L}}^{(1)}, \dots, R_{\mathcal{L}}^{(s-1)}$ in \mathcal{L} distinct from P . Then the union of the lines $\mathcal{L}_1, \dots, \mathcal{L}_s$ and the point set*

$$\bigcup_{\mathcal{L} \text{ through } P, \mathcal{L} \neq \mathcal{L}_i} \{R_{\mathcal{L}}^{(1)}, \dots, R_{\mathcal{L}}^{(s-1)}\}$$

is a $(1, \mu)$ -saturating set in $PG(2, q)$ of size $s\sqrt{q} + (s - 1)(\sqrt{q} + 1 - s) + 1$ with $\mu = \frac{1}{2}(s^2 - s)$.

Proof. The proof is analogous to that of Theorem 5.3. \square

Corollary 5.5. *Let q be a square.*

- *If $\mu = \frac{1}{8}(s^2 - 2s)$ for some positive even integer $3 < s \leq \sqrt{q} + 1$, then*

$$\ell_{\mu}(2, 3, q) \leq (1 + \sqrt{8\mu + 1})(\sqrt{q} + 1 - \sqrt{8\mu + 1}) + (1 + \sqrt{8\mu + 1})(\sqrt{8\mu + 1})/2.$$

- *If $\mu = \frac{1}{2}(s^2 - s)$ for some positive integer $3 \leq s \leq \sqrt{q} + 1$, then*

$$\ell_{\mu}(2, 3, q) \leq 1 + \sqrt{1 + 8\mu}\sqrt{q} - \left(\frac{1 + \sqrt{1 + 8\mu}}{2} - 1\right)^2.$$

5.2. $q = p^{\ell}$, p PRIME, $\ell \geq 3$. Let $q = p^{\ell}$ with p prime, and let H be an additive subgroup of \mathbb{F}_q of size p^s with $2s < \ell$. Also, let

$$(5.4) \quad L_H(X) = \prod_{h \in H} (X - h) \in \mathbb{F}_q[X].$$

Then L_H is a linearized polynomial, that is, there exist $\beta_0, \dots, \beta_s \in \mathbb{F}_q$ such that $L_H(X) = \sum_{i=0}^s \beta_i X^{p^i}$, see e.g. [34, Theorem 3.52].

For $m \in \mathbb{F}_q$, let

$$(5.5) \quad F_m(X, Y) = L_H(X) - mL_H(Y).$$

As the evaluation map $(x, y) \mapsto F_m(x, y)$ is an additive map from \mathbb{F}_q^2 to \mathbb{F}_q , the equation $F_m(X, Y) = 0$ has at least q solutions in \mathbb{F}_q^2 .

Let

$$(5.6) \quad \mathcal{M}_H := \left\{ \left(\frac{L_{H_1}(\beta_1)}{L_{H_2}(\beta_2)} \right)^p \mid H_1, H_2 \text{ subgroups of } H \text{ of index } p, \beta_i \in H \setminus H_i \right\}.$$

Lemma 5.6 ([19]).

$$-\mathcal{M}_H = \mathcal{M}_H.$$

Lemma 5.7 ([19]). *The size of \mathcal{M}_H is at most $(p^s - 1)^2 / (p - 1)$.*

Proposition 5.8 ([19]). *Let $F_m(X, Y)$ be as in Formula (5.5). Then the equation $F_m(X, Y) = 0$ has more than q solutions if and only if either $m \in \mathcal{M}_H$ or $m = 0$.*

For an element $\alpha \in \mathbb{F}_q$, define

$$(5.7) \quad D_{H,\alpha} = \{(L_H(a) : \alpha : 1) \mid a \in \mathbb{F}_q\} \subset PG(2, q).$$

As a corollary to Proposition 5.8, the following result is obtained.

Proposition 5.9. *Let α_1, α_2 be distinct elements in \mathbb{F}_q . Then a point $P = (u : v : 1)$ belongs to at least $p^{\ell-2s}$ lines joining two points of $D_{H,\alpha_1} \cup D_{H,\alpha_2}$ provided that $v \notin (\alpha_2 - \alpha_1)\mathcal{M}_H + \alpha_2$ and $v \neq \alpha_2$.*

Proof. Assume that $v \notin (\alpha_2 - \alpha_1)\mathcal{M}_H + \alpha_2$ and that $v \neq \alpha_2$. Then by Proposition 5.8, the equation

$$L_H(X) + \frac{v - \alpha_2}{\alpha_1 - \alpha_2} L_H(Y) = 0$$

has precisely q solutions, or, equivalently, the additive map

$$(x, y) \mapsto L_H(x) + \frac{v - \alpha_2}{\alpha_1 - \alpha_2} L_H(y)$$

is surjective and the counterimage of each element in \mathbb{F}_q consists of q distinct pairs (b, b') . Therefore, there exist q pairs $(b, b') \in \mathbb{F}_q$ such that

$$L_H(b) + \frac{v - \alpha_2}{\alpha_1 - \alpha_2} L_H(b') = u,$$

which is precisely the condition for the point $P = (u : v : 1)$ to belong to the line joining $(L_H(b' + b) : \alpha_1 : 1) \in D_{H,\alpha_1}$ and $(L_H(b) : \alpha_2 : 1) \in D_{H,\alpha_2}$.

We need to count the number of distinct pairs of points corresponding to the q pairs of elements (b, b') . The map $(x, y) \mapsto (L_H(x), L_H(y))$ is an \mathbb{F}_p -linear map whose kernel has dimension $2s$. This proves that the number of distinct pairs $(L_H(b), L_H(b'))$ is at least $q/p^{2s} = p^{\ell-2s}$. \square

Proposition 5.10. *Let α_1, α_2 be distinct elements in \mathbb{F}_q . Then a point $P = (u : \alpha_2 : 1)$ can be written in*

$$\binom{p^{\ell-s}}{2}$$

distinct ways as a linear combination of two points of $D_{H,\alpha_1} \cup D_{H,\alpha_2}$.

Proof. The assertion follows from the fact that $D_{H,\alpha_2} \cup \{P\}$ consists of $p^{\ell-s} + 1$ collinear points. \square

Theorem 5.11. *Let $q = p^\ell$, and let H be any additive subgroup of \mathbb{F}_q of size p^s , with $2s < \ell$. Let μ be any integer with $1 \leq \mu \leq p^{2\ell-s}$, and let $\tau_1, \tau_2, \dots, \tau_\mu$ be a set of distinct non-zero elements in \mathbb{F}_q . Let $L_H(X)$ be as in Equation (5.4), and \mathcal{M}_H be as in Formula (5.6). Then the set*

$$\begin{aligned} D = & \{(L_H(a) : 1 : 1), (L_H(a) : 0 : 1) \mid a \in \mathbb{F}_q\} \cup \\ & \{(\tau_i : m : 1) \mid m \in \mathcal{M}_H, i = 1, \dots, \mu\} \cup \\ & \{(1 : \tau_i : 0) \mid i = 1, \dots, \mu\} \cup \{(1 : 0 : 0)\} \end{aligned}$$

is a $(1, \mu)$ -saturating set of size at most

$$\frac{2q}{p^s} + \mu \frac{(p^s - 1)^2}{p - 1} + \mu.$$

Proof. Let $P = (u : v : 1)$ be a point in $PG(2, q)$. If $v \notin \mathcal{M}_H$, then P belongs to at least μ secants of D (counted with multiplicity) by Proposition 5.9, together with Lemma 5.6. If $v \in \mathcal{M}_H$, then P is collinear with $(\tau_i : v : 1) \in D$ and $(1 : 0 : 0) \in D$. Clearly the points $P = (u : v : 0)$ are covered by D at least μ times as they are collinear with $(1 : 0 : 0)$ and $(1 : \tau_i : 0)$. Then D is a $(1, \mu)$ -saturating set.

The set $\{L_H(a) \mid a \in \mathbb{F}_q\}$ is the image of an \mathbb{F}_p -linear map on $\mathbb{F}_q \cong \mathbb{F}_p^\ell$ whose kernel has dimension s , therefore its size is $p^{\ell-s}$. Note that the point $(0 : 1 : 1)$ belongs to both $\{(L_H(a) : 1 : 1) \mid a \in \mathbb{F}_q\}$ and $\{(0 : m : 1) \mid m \in \mathcal{M}_H\}$. Then the upper bound on the size of D follows from Lemma 5.7. \square

The order of magnitude of the size of D of Theorem 5.11 is $p^{\max\{\ell-s, \ln_p \mu \cdot (2s-1)\}}$. If s is chosen as $\lceil \ell/3 \rceil$, then the size of D satisfies

$$\#D \leq \begin{cases} 2q^{\frac{2}{3}} + \mu + \mu \frac{q^{\frac{2}{3}-2q^{\frac{1}{3}}+1}}{p-1} & \text{if } \ell \equiv 0 \pmod{3}; \\ 2 \left(\frac{q}{p}\right)^{\frac{2}{3}} + \mu + \mu \frac{p^2 \left(\frac{q}{p}\right)^{\frac{2}{3}} - 2p \left(\frac{q}{p}\right)^{\frac{1}{3}} + 1}{p-1} & \text{if } \ell \equiv 1 \pmod{3}; \\ 2 \frac{1}{p} (qp)^{\frac{2}{3}} + \mu + \mu \frac{(qp)^{\frac{2}{3}} - 2(qp)^{\frac{1}{3}} + 1}{p-1} & \text{if } \ell \equiv 2 \pmod{3}. \end{cases}$$

An existence result can be obtained by adapting another construction from [19].

Theorem 5.12. *Let $q = p^\ell$, with ℓ odd. Let $1 \leq \mu \leq p$, and let H be any additive subgroup of \mathbb{F}_q of size p^s , with $2s + 1 = \ell$. Let $L_H(X)$ be as in Formula (5.4), and \mathcal{M}_H be as in Equation (5.6). Then for any integer $v \geq 1$ there exists a $(1, \mu)$ -saturating set D in $PG(2, q)$ such that*

$$(5.8) \quad \#D \leq (v + 1)p^{s+1} + \mu \frac{\#\mathcal{M}_H^v}{(q - 1)^{v-1}} + 1 + \mu.$$

Proof. Let $A = \{\alpha_1, \dots, \alpha_{v+1}\}$ be any set of distinct $v + 1$ elements in \mathbb{F}_q . The idea is to consider the union of the $(v + 1)$ -sets D_{H, α_i} . Let

$$(5.9) \quad \begin{aligned} D(A) &= \bigcup_{i=1, \dots, v+1} D_{H, \alpha_i}, \\ \mathcal{M}(A) &= \bigcap_{i, j=1, \dots, v+1, i \neq j} (\alpha_j - \alpha_i) \mathcal{M}_H + \alpha_j. \end{aligned}$$

Also, let $\tau_1, \tau_2, \dots, \tau_\mu$ be a set of distinct non-zero elements in \mathbb{F}_q . Arguing as in the proof of Theorem 5.11 we obtain that the set

$$\begin{aligned} D(H, A) &= D(A) \cup \{(\tau_i : m : 1) \mid m \in \mathcal{M}(A), i = 1, \dots, \mu\} \cup \\ &\quad \{(1 : \tau_i : 0) \mid i = 1, \dots, \mu\} \cup \{(1 : 0 : 0)\} \end{aligned}$$

is a $(1, \mu)$ -saturating set in $PG(2, q)$. By Proposition 3.4 in [19], it is possible to choose A in such a way that

$$(5.10) \quad \#\mathcal{M}(A) \leq \frac{\#\mathcal{M}_H^v}{(q - 1)^{v-1}},$$

whence the claim follows. \square

Corollary 5.13. *Let $q = p^{2s+1}$, and let $1 \leq \mu \leq p$. Then*

$$\ell_\mu(2, 3, q) \leq \min_{v=1, \dots, 2s+1} \left\{ (v + 1)p^{s+1} + \mu \frac{(p^s - 1)^{2v}}{(p - 1)^v (p^{2s+1} - 1)^{(v-1)}} + 1 + \mu \right\}.$$

Proof. The claim follows from Theorem 5.12, together with Lemma 5.7. \square

For several values of s, p , and μ , Corollary 5.13 improves the probabilistic bound; namely, there exists some integer v such that

$$(5.11) \quad (v + 1)p^{s+1} + \mu \frac{(p^s - 1)^{2v}}{(p - 1)^v (p^{2s+1} - 1)^{(v-1)}} + 1 + \mu < \sqrt{\mu q \ln q}.$$

This happens for instance for the following 4-tuples (p, s, v, μ) with $\mu \leq 10$: $(3, 5, 5, 10), (5, 3, 3, 8), (5, 3, 3, 9), (5, 3, 3, 10), (5, 5, 4, 8), (5, 5, 4, 9), (5, 5, 4, 10), (5, 7, 5, 8), (5, 7, 5, 9), (5, 7, 5, 10), (5, 9, 6, 9), (5, 9, 6, 10), (5, 11, 7, 10), (7, 3, 3, 9), (7, 3, 3, 10), (7, 5, 4, 9), (7, 5, 4, 10), (7, 7, 5, 9), (7, 7, 5, 10), (7, 9, 6, 10)$. Also, as Inequality (5.10) is an upper bound, by computer search one can get even smaller saturating sets. The condition on the size of $\mathcal{M}(A)$ is easy to test, which allows us to consider also large q 's.

5.3. q A PRIME. For q a prime, the smallest known explicitly described saturating sets have about $Cq^{3/4}$ points; see e.g. [3, 18, 23, 41]. Here we show that a slight modification of a construction by Bartocci ([3]; see also [42, Example 4.3]) provides $(1, \mu)$ -saturating sets of about the same size with $\mu < \sqrt{q}$.

Theorem 5.14. *Let s be a divisor of $q - 1$ and let H_s be the subgroup of \mathbb{F}_q^* of index s . For an integer $\mu < \frac{q-1}{s}$, let V_1, \dots, V_μ be μ disjoint systems of representatives of the cosets of H_s different from H_s . Let*

$$S = \{(t, t^2) \mid x \in H_s\} \cup \{(0, -v) \mid v \in V_1 \cup \dots \cup V_\mu\} \cup B,$$

where B is any subset of the ideal line of size $\lceil (1 + \sqrt{1 + 8\mu})/2 \rceil$. Then S is a $(1, \mu)$ -saturating set in $\text{PG}(2, q)$ of size

$$\frac{q - 1}{s} + \mu(s - 1) + \lceil (1 + \sqrt{1 + 8\mu})/2 \rceil,$$

provided that

$$(5.12) \quad q > \left((s - 1)^2 + \sqrt{(s - 1)^4 + 2\mu s^2 + 4s} \right)^2.$$

Proof. Let $P = (a, b)$ be a point in $\text{AG}(2, q)$ off the parabola $\mathcal{P}(q)$ with equation $Y = X^2$. It is easily seen that P is covered by a secant of

$$S' = \{(t, t^2) \mid x \in H_s\}$$

passing through two distinct points $(x^s, x^{2s}), (y^s, y^{2s})$ if and only if (x, y) is an \mathbb{F}_q -rational point of the curve

$$\mathcal{C}_P : X^s Y^s - a(X^s + Y^s) + b = 0.$$

The curve \mathcal{C}_P is a generalized Fermat curve. Then \mathcal{C}_P is absolutely irreducible, and its genus is $(s - 1)^2$; see e.g. [17]. Then by the Hasse-Weil Theorem, together with Inequality (5.12), the curve \mathcal{C}_P has more than $2\mu s^2 + 4s$ \mathbb{F}_q -rational points. Then there are at least μ distinct secants of S passing through P . Now let $P = (a, a^2)$ be a point of $\mathcal{P}(q)$ not in S . For each $i = 1, \dots, \mu$, write $a = v_{i,j} d_i^s$ with $v_{i,j} \in V_i$. Then (a, a^2) is collinear with (d_i^{-s}, d_i^{-2s}) and $(0, -v_{i,j})$. Finally, if P is a point in $\text{PG}(2, q) \setminus \text{AG}(2, q)$, then it is covered at least μ times by the ideal line. \square

Corollary 5.15. *Assume s is a divisor of $q - 1$ such that $s < \sqrt[4]{(q/4)}$. Then for each $\mu \leq s - 3$*

$$\ell_\mu(2, 3, q) \leq \frac{q - 1}{s} + \mu(s - 1) + \lceil (1 + \sqrt{1 + 8\mu})/2 \rceil.$$

Proof. From Formula (5.12) we get that μ is at most

$$\frac{q - 2(s-1)^2\sqrt{q} - 4s}{2s^2}.$$

It is easy to check that any value $\mu \leq s - 3$ satisfies this bound. \square

Theorem 5.14 gives $(1, \mu)$ -saturating sets of size approximately $Cq^{3/4}$, provided that $q - 1$ has a divisor s of the same order of magnitude as $q^{1/4}$.

In the case where $q + 1$ admits such a divisor, a similar achievement can be obtained by using plane cubic curves. For an element β in $\mathbb{F}_{q^2} \setminus \mathbb{F}_q$ such that $\beta^2 \in \mathbb{F}_q$, let \mathcal{X}_β be the plane cubic with equation

$$Y(X^2 - \beta^2) = 1.$$

For $v \in \mathbb{K} \setminus \{0, 1\}$, let Q_v be the point on \mathcal{X}_β with affine coordinates $(\frac{v+1}{v-1}\beta, \frac{(v-1)^2}{4v\beta^2})$. Also, let $Q_0 = Y_\infty$ and $Q_1 = X_\infty$. For a divisor s of $q + 1$ with $(6, s) = 1$, let

$$K_\beta = \{Q_{(\frac{u+\beta}{u-\beta})^s} \mid u \in \mathbb{F}_q\} \cup \{X_\infty\}.$$

Let $T = Q_{\bar{t}}$ be a non-singular \mathbb{F}_q rational point in $\mathcal{X}_\beta \setminus K$, and let

$$(5.13) \quad K_{T,\beta} = \{Q_{\bar{t}(\frac{u+\beta}{u-\beta})^s} \mid u \in \mathbb{F}_q\} \cup \{Q_{\bar{t}}\}.$$

The following result was proved in [2].

Proposition 5.16. *Let $P = (a, b)$ be a point in $AG(2, q)$ off \mathcal{X}_β . Assume that*

$$(5.14) \quad (a, b) \notin \left\{ \left(0, -\frac{9}{8\beta^2}\right), (\beta\sqrt{-3}, 0), (-\beta\sqrt{-3}, 0) \right\}.$$

If

$$q + 1 - (6s^2 - 6s + 2)\sqrt{q} \geq 4s^2 + 8s + 1$$

then P is collinear with two distinct points of $K_{T,\beta}$.

Arguing as in the proof of Proposition 19 in [2], Proposition 5.16 can be extended as follows.

Proposition 5.17. *Let $P = (a, b)$ be a point in $AG(2, q)$ off \mathcal{X}_β . Assume that Condition (5.14) holds. If*

$$q + 1 - (6s^2 - 6s + 2)\sqrt{q} \geq 4s^2 + 8s + (\mu - 1)s^2 + 1,$$

then there are at least μ distinct pairs of points in $K_{T,\beta}$ collinear with P .

For distinct β, β' , the union $K_{T,\beta} \cup K_{T,\beta'}$, together with few extra points, provides a $(1, \mu)$ -saturating set.

Theorem 5.18. *Let s be a divisor of $q + 1$ such that $(6, s) = 1$ and*

$$q + 1 - (6s^2 - 6s + 2)\sqrt{q} \geq 4s^2 + 8s + (\mu - 1)s^2 + 1$$

holds. Let β, β' be distinct elements in $\mathbb{F}_{q^2} \setminus \mathbb{F}_q$ such that both β^2 and β'^2 belong to \mathbb{F}_q and $\beta^2 \neq \beta'^2$ holds. Let $K_{T,\beta}$ and $K_{T,\beta'}$ be as in Formula (5.13), and let B be any subset of the ideal line of size $\lceil (1 + \sqrt{1 + 8\mu})/2 \rceil$. Then

$$S = K_{T,\beta} \cup K_{T,\beta'} \cup B$$

is a $(1, \mu)$ -saturating set in $PG(2, q)$ of size less than or equal to

$$2\frac{q+1}{s} + \lceil (1 + \sqrt{1 + 8\mu})/2 \rceil.$$

Proof. By Proposition 5.17, the points in $AG(2, q)$ that are not covered μ times by $K_{T,\beta}$ are the points in $\mathcal{X}_\beta \setminus K_{T,\beta}$, together with those not satisfying Condition (5.14). As $\beta^2 \neq \beta'^2$, by Proposition 5.17 such points are covered μ times by $K_{T,\beta'}$. \square

Corollary 5.19. *Assume s is a divisor of $q + 1$ such that $(6, s) = 1$ and $s < \sqrt[4]{(q/36)}$. Then for each $\mu \leq 18(s - 1)$*

$$\ell_\mu(2, 3, q) \leq 2 \frac{q+1}{s} + \lceil (1 + \sqrt{1 + 8\mu})/2 \rceil.$$

It should be noted that the arguments used for cubics with an isolated double point would apply to nodal cubics as well; however, by using the results presented in [1], one would obtain a bound similar to that of Corollary 5.15.

6. RESULTS FOR CODIMENSION $r > 3$

In this section we show how the constructions of small $(1, \mu)$ -saturating sets in $PG(2, q)$ proposed in this paper actually provide short $(2, \mu)$ -MCF codes with higher codimension. The key tool will be a generalization of the q^m -concatenating construction for covering codes, see e.g. [15].

An $[n, k, d]_q(R, \mu)$ -code is a linear (R, μ) -MCF code of length n , dimension k , and minimum distance d over \mathbb{F}_q . Similarly, an $[n, k]_q(R, \mu)$ -code is a linear (R, μ) -MCF code of length n and dimension k . Using a *starting* $[n_0, n_0 - r_0]_q(R, \mu)$ code V_0 of length n_0 , we will provide an infinite family of $[n, n - (r_0 + 2m)]_q(2, \mu)$ codes with length

$$n \leq q^m n_0 + \max\{3, \mu\} \cdot \frac{q^m - 1}{q - 1},$$

where m ranges over an infinite set of integers. As a corollary, new significant upper bounds for the length function $\ell_\mu(2, r, q)$ will be obtained, see Corollaries 6.4 and 6.5 below.

Throughout this section we fix a basis of \mathbf{F}_{q^m} over \mathbb{F}_q . All matrices and columns are q -ary, and an element of \mathbb{F}_{q^m} written in a matrix denotes the m -dimensional column of its coordinates with respect to the fixed basis. Let

$$\theta_{m,q} = \frac{q^m - 1}{q - 1}.$$

Basic Multiple Concatenating Construction MCC. Let $\mathbf{H}_0 = [\mathbf{h}_1 \mathbf{h}_2 \dots \mathbf{h}_{n_0}]$, with $\mathbf{h}_j \in \mathbb{F}_q^{r_0}$, be a parity check matrix of an $[n_0, n_0 - r_0, 3]_q(2, \mu)$ *starting* code V_0 . Equivalently, the columns of \mathbf{H}_0 are projective coordinates of the points of a $(1, \mu)$ -saturating set in $PG(r_0 - 1, q)$.

Let $m \geq n_0$ be an integer. To each column \mathbf{h}_j we associate an element $\beta_j \in \mathbb{F}_{q^m} \cup \{*\}$ so that $\beta_i \neq \beta_j$ if $i \neq j$. The set $\mathcal{B} = \{\beta_1, \beta_2, \dots, \beta_{n_0}\}$ will be called an *indicator set*. For a given $(r_0 + 2m) \times N_m$ matrix \mathbf{C} , with $N_m \leq \max\{3, \mu\} \cdot \theta_{m,q}$, let V be the $[n, n - (r_0 + 2m)]_q$ code with $n = q^m n_0 + N_m$ and parity-check matrix

$$\mathbf{H}_V = [\mathbf{C} \mathbf{B}_1 \mathbf{B}_2 \dots \mathbf{B}_{n_0}],$$

where

$$(6.1) \quad \mathbf{B}_j = \begin{cases} \begin{bmatrix} \mathbf{h}_j & \mathbf{h}_j & \cdots & \mathbf{h}_j \\ \xi_1 & \xi_2 & \cdots & \xi_{q^m} \\ \beta_j \xi_1 & \beta_j \xi_2 & \cdots & \beta_j \xi_{q^m} \end{bmatrix} & \text{if } \beta_j \in \mathbb{F}_{q^m}, \\ \begin{bmatrix} \mathbf{h}_j & \mathbf{h}_j & \cdots & \mathbf{h}_j \\ 0 & 0 & \cdots & 0 \\ \xi_1 & \xi_2 & \cdots & \xi_{q^m} \end{bmatrix} & \text{if } \beta_j = *, \end{cases}$$

with $\{\xi_1, \xi_2, \dots, \xi_{q^m}\} = \mathbb{F}_{q^m}$, $\xi_1 = 0$, $\xi_2 = 1$.

If m , \mathbf{C} , and \mathcal{B} are carefully chosen, then the new code V has the same covering radius and the same covering multiplicity as the starting code V_0 .

Examples are shown in Constructions MCC_1 and MCC_2 below, where we use the following notation:

- $\mathbf{0}_k$ is a zero matrix with k rows (the number of columns will be clear by context);
- $\mathbf{W}_m = [\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_{\theta_{m,q}}]$ is a parity-check matrix of the $[\theta_{m,q}, \theta_{m,q} - m]_q$ Hamming code;
- for an element ξ in $\mathbb{F}_{q^m}^*$, let $k(\xi)$ be the element of \mathbb{F}_q^* and $i(\xi)$ the index such that the column $\mathbf{w}_{i(\xi)}$ of \mathbf{W}_m satisfies

$$(6.2) \quad \xi = k(\xi) \mathbf{w}_{i(\xi)}$$

(here the column $\mathbf{w}_{i(\xi_u)}$ is viewed as an element of $\mathbb{F}_{q^m}^*$).

Construction MCC_1 . Here we assume that

$$(6.3) \quad \mu = 2, \quad q^m + 1 \geq n_0, \quad N_m = 3\theta_{m,q},$$

and

$$\mathbf{C} = [\mathbf{C}_1 \mathbf{C}_2 \mathbf{C}_3]$$

with

$$\mathbf{C}_1 = \begin{bmatrix} \mathbf{0}_{r_0} \\ \mathbf{W}_m \\ \mathbf{0}_m \end{bmatrix}, \quad \mathbf{C}_2 = \begin{bmatrix} \mathbf{0}_{r_0} \\ \mathbf{W}_m \\ \mathbf{W}_m \end{bmatrix}, \quad \mathbf{C}_3 = \begin{bmatrix} \mathbf{0}_{r_0} \\ \mathbf{0}_m \\ \mathbf{W}_m \end{bmatrix}.$$

Theorem 6.1. *In Construction MCC_1 , the new code V is an $[n, n - (r_0 + 2m), 3]_q(2, 2)$ code with $n = q^m n_0 + 3\theta_{m,q}$.*

Proof. The covering radius of the new code V is $R_V = 2$ by [15, Sec. 2, Construction QM_1]. We need to prove that the covering multiplicity μ_V is equal to 2. To this end, we show that an arbitrary nonzero column (a, b, c) , such that $(a, b, c) \in \mathbb{F}_{q^{r_0}} \times \mathbb{F}_{q^m} \times \mathbb{F}_{q^m}$, (a, b, c) not a column of \mathbf{H}_V , can be represented at least in two ways by a linear combination of two columns of \mathbf{H}_V . We consider a number of cases.

1) $a \neq 0$, a not a column of \mathbf{H}_0 .

As the starting code has $R = 2$ and $\mu = 2$, we have that

$$a = d_1 \mathbf{h}_{i_1} + e_1 \mathbf{h}_{j_1} = d_2 \mathbf{h}_{i_2} + e_2 \mathbf{h}_{j_2}, \quad d_u, e_u \in \mathbb{F}_q^*$$

for some pairs of indices $\{i_1, j_1\} \neq \{i_2, j_2\}$. Let $\{\beta_{i_u}, \beta_{j_u}\} \subset \mathbb{F}_{q^m}$, $u \in \{1, 2\}$. Consider the linear system over \mathbb{F}_{q^m}

$$(6.4) \quad \begin{cases} d_u X + e_u Y = b \\ d_u \beta_{i_u} X + e_u \beta_{j_u} Y = c \end{cases}$$

the determinant of which $d_u e_u (\beta_{j_u} - \beta_{i_u}) \neq 0$ since $d_u, e_u \neq 0$ and $\beta_i \neq \beta_j$ if $i \neq j$. Let $X = \xi_x^{(u)}$, $Y = \xi_y^{(u)}$ be its unique solution. Then

$$(6.5) \quad (a, b, c) = d_u(\mathbf{h}_{i_u}, \xi_x^{(u)}, \beta_{i_u} \xi_x^{(u)}) + e_u(\mathbf{h}_{j_u}, \xi_y^{(u)}, \beta_{j_u} \xi_y^{(u)}).$$

Let $\beta_{i_u} \in \mathbb{F}_{q^m}$, $\beta_{j_u} = *$, $u \in \{1, 2\}$. Let $\xi_x^{(u)}$, $\xi_y^{(u)}$ be such that

$$(6.6) \quad \begin{cases} d_u \xi_x^{(u)} = b \\ d_u \beta_{i_u} \xi_x^{(u)} + e_u \xi_y^{(u)} = c \end{cases}.$$

We have

$$(6.7) \quad (a, b, c) = d_u(\mathbf{h}_{i_u}, \xi_x^{(u)}, \beta_{i_u} \xi_x^{(u)}) + e_u(\mathbf{h}_{j_u}, 0, \xi_y^{(u)}).$$

2) $a \neq 0$, $a = \mathbf{h}_j$ column of \mathbf{H}_0 .

We are going to show that there are at least two linear combinations of type

$$(a, b, c) = (\mathbf{h}_j, b, c) = \mathbf{b}_x^{(u)} + e_u \mathbf{t}_u, u = 1, 2,$$

where $\mathbf{b}_x^{(u)} = (\mathbf{h}_j, \xi_x^{(u)}, \beta_j \xi_x^{(u)})$, $u \in \{1, 2\}$, is a column of the submatrix \mathbf{B}_j of \mathbf{H}_V , and \mathbf{t}_u , $u \in \{1, 2\}$, is taken from the submatrix \mathbf{C} . Assume first that $\beta_j = 0$. On the one hand,

$$(a, b, c) = (\mathbf{h}_j, b, c) = (\mathbf{h}_j, b - c, 0) + k(c)(0, \mathbf{w}_{i(c)}, \mathbf{w}_{i(c)})$$

with $(0, \mathbf{w}_{i(c)}, \mathbf{w}_{i(c)})$ a column of \mathbf{C}_2 . On the other hand,

$$(a, b, c) = (\mathbf{h}_j, b, c) = (\mathbf{h}_j, b, 0) + k(c)(0, 0, \mathbf{w}_{i(c)})$$

with $(0, 0, \mathbf{w}_{i(c)})$ a column of \mathbf{C}_3 . If $\beta_j \neq 0$, then we can proceed analogously. Columns \mathbf{t}_1 and \mathbf{t}_2 need to be taken as follows:

- if $\beta_j = *$ then \mathbf{t}_1 is a column of \mathbf{C}_1 , \mathbf{t}_2 is a column of \mathbf{C}_2 ;
 - if $\beta_j = 1$ then \mathbf{t}_1 is a column of \mathbf{C}_1 , \mathbf{t}_2 is a column of \mathbf{C}_3 ;
 - if $\beta_j \notin \{0, 1, *\}$ then $\mathbf{t}_1, \mathbf{t}_2$ can be found in any two distinct submatrices of \mathbf{C} .
- 3) $a = 0$.

The proof is similar to that of case 2). Here both columns \mathbf{t}_1 and \mathbf{t}_2 are taken from the same submatrix \mathbf{C} . The choice of the submatrix depends on the values of b and c . □

Construction MCC₂. Here we assume that

$$(6.8) \quad \begin{aligned} \mu &\geq 3, \quad q^m + 1 - \mu \geq n_0, \\ \mathcal{B} &\subseteq \{\xi_\mu, \xi_{\mu+1}, \dots, \xi_{q^m}\}, \quad N_m = \mu\theta_{m,q}, \end{aligned}$$

and $\mathbf{C} = [\mathbf{C}_1 \mathbf{C}_2 \dots \mathbf{C}_\mu]$,

$$\mathbf{C}_i = \begin{bmatrix} \mathbf{0}_{r_0} \\ \mathbf{W}_m \\ \xi_i \mathbf{W}_m \end{bmatrix}, \quad i = 1, 2, \dots, \mu - 1, \quad \mathbf{C}_\mu = \begin{bmatrix} \mathbf{0}_{r_0} \\ \mathbf{0}_m \\ \mathbf{W}_m \end{bmatrix},$$

where $\{\xi_1, \xi_2, \dots, \xi_{q^m}\} = \mathbb{F}_{q^m}$, $\xi_i \mathbf{W}_m$ is an $m \times \theta_{m,q}$ matrix obtained from \mathbf{W}_m via multiplication of each column (treated as an element of \mathbb{F}_{q^m}) by ξ_i .

Theorem 6.2. *In Construction MCC₂, the new code V is an $[n, n - (r_0 + 2m), 3]_q(2, \mu)$ code with $n = q^m n_0 + \mu\theta_{m,q}$.*

Proof. The proof is similar to that of Theorem 6.1. □

Remark 6.3. If $\lceil (1 + \sqrt{1 + 8\mu})/2 \rceil \leq q$ then it is easy to see that $\lceil (1 + \sqrt{1 + 8\mu})/2 \rceil$ is the smallest size of an $(1, \mu)$ -saturating set in $\text{PG}(1, q)$.

Corollary 6.4. *Let $r > 2$ be an even integer with $\lceil(1 + \sqrt{1 + 8\mu})/2\rceil \leq \min\{q, q^{(r-2)/2} + 1 - \mu\}$. Then*

$$\ell_\mu(2, r, q) \leq q^{(r-2)/2} \lceil(1 + \sqrt{1 + 8\mu})/2\rceil + \max(3, \mu) \frac{q^{(r-2)/2} - 1}{q - 1}.$$

Proof. Let V_0 be the code corresponding to the smallest $(1, \mu)$ -saturating set in $\text{PG}(1, q)$. It means that $r_0 = 2$ and $n_0 = \lceil(1 + \sqrt{1 + 8\mu})/2\rceil$. Put $r = 2 + 2m$. Then applying construction MCC_1 for $\mu = 2$ and construction MCC_2 for $\mu > 2$ gives the assertion. \square

Corollary 6.5. *Let $r > 3$ be an odd integer with $q^{(r-3)/2} + 1 - \mu \geq \bar{\ell}_\mu(2, 3, q)$. Then*

$$(6.9) \quad \ell_\mu(2, r, q) \leq q^{(r-3)/2} \ell_\mu(2, 3, q) + \max(3, \mu) \frac{q^{(r-3)/2} - 1}{q - 1}$$

and

$$(6.10) \quad \bar{\ell}_\mu(2, r, q) \leq q^{(r-3)/2} \bar{\ell}_\mu(2, 3, q) + \max(3, \mu) \frac{q^{(r-3)/2} - 1}{q - 1}.$$

Proof. Let V_0 be the code with $r_0 = 3$ corresponding to the smallest $(1, \mu)$ -saturating set in $\text{PG}(2, q)$. Then by applying construction MCC_1 for $\mu = 2$ and construction MCC_2 for $\mu > 2$, Inequality (6.9) is obtained. If V_0 is chosen as the smallest known $(1, \mu)$ -saturating set in $\text{PG}(2, q)$, then Inequality (6.10) is obtained. \square

7. OPTIMAL (R, μ) -SATURATING SETS AND (R, μ) -APMCF CODES

In this section we deal with (R, μ) -APMCF codes, that is, codes C with $\gamma_\mu(C, R) = 1$.

7.1. (R, μ) -APMCF CODES AND UNIFORMLY PACKED CODES. In this section we discuss how uniformly packed codes (UP codes for short) can give rise to APMCF codes. For an introduction to UP codes, see e.g. [5, 24, 35] and the references therein. The UP codes C that are of interest in the context of MCF codes are those with $d(C) \in \{2R - 1, 2R\}$. There are different definitions of UP codes available in the literature; however, for codes with $d(C) \in \{2R - 1, 2R\}$ these definitions are consistent. Notation here is taken from [35]. Throughout the section, for a code C with minimum distance $d(C)$, let $t = \lfloor \frac{d(C)-1}{2} \rfloor$.

Definition 7.1. [35] A t -error-correcting $(n, M)_q R$ code C is called *uniformly packed with parameters α and β* if for any vector e of \mathbb{F}_q^n with distance t to C , there are exactly α codewords with distance $t + 1$ to e , and for any vector e of \mathbb{F}_q^n with distance $\geq t + 1$ to C , there are exactly β codewords with distance $t + 1$ to e .

Theorem 7.2. *An $(n, M, d)_q R$ UP code with parameters α and β , such that $d(C) \in \{2R - 1, 2R\}$, is an (R, μ) -APMCF code with $\mu = \beta$.*

Proof. As $d(C) \in \{2R - 1, 2R\}$, vectors with distance greater than $t + 1$ from the code do not exist. By Definition 7.1, for any vector e of \mathbb{F}_q^n with distance $t + 1 = R$ to C , there are exactly β codewords with distance R to e . \square

Lemma 7.3. (i) *Let C be an $(n, M, 2R)_q R$ UP code. Then C is an (R, μ) -PMCF code with*

$$(7.1) \quad \mu = \frac{M \binom{n}{R} (q-1)^R}{N_R(C)} = \frac{M \binom{n}{R} (q-1)^R}{q^n - M \cdot V_q(n, R-1)}.$$

(ii) Let C be a linear $[n, k, d]_q R$ UP code with $d(C) \in \{2R - 1, 2R\}$. Then C is an (R, μ) -APMCF code with

$$(7.2) \quad \mu = \frac{\binom{n}{R}(q-1)^R - \binom{d}{R-1} \cdot A_{2R-1}(C)}{q^{n-k} - V_q(n, R-1)}.$$

Proof. For $d = 2R$, assertions follow from Definition 7.1, the proof of Theorem 7.2, and Proposition 2.2. For $d = 2R - 1$ we use the proof of Proposition 2.3. \square

Example 7.4. By [5, 9, 24, 35], see also the references therein, the following linear nonbinary codes are UP with $d(C) = 2R$: the $[12, 6, 6]_3$ Golay code extended by parity check; the $[56, 50, 4]_3$ code corresponding to the largest cap in $PG(5, 3)$ (Hill’s cap); the $[q^2 + 1, q^2 - 3, 4]_q$ code corresponding to an elliptic quadric in $PG(3, q)$; the $[q + 2, q - 1, 4]_q$ code corresponding to a hyperoval in $PG(2, q)$, q even; the $[78, 72, 4]_4$ code [9] corresponding to a complete 78-cap in $PG(5, 4)$. By Theorem 7.2 and Equality (7.2), the codes above are $[12, 6, 6]_3(3, 4)$, $[56, 50, 4]_3(2, 10)$, $[q^2 + 1, q^2 - 3, 4]_q(2, \frac{1}{2}(q^2 - q))$, $[q + 2, q - 1, 4]_q(2, \frac{1}{2}(q + 2))$, and $[78, 72, 4]_4(2, 7)$ PMCF codes, respectively.

7.2. OPTIMAL $(1, \mu)$ -SATURATING SETS IN $PG(2, q)$ AND TWO-WEIGHT CODES. An n -set S in $PG(2, q)$ is a projective $(n, 3, h_1, h_2)$ set if every line meets S in either h_1 or h_2 points [10]. In [29], such sets are called sets of type (a, b) with $a = h_1, b = h_2$.

An $[n, 3]_q$ linear code C is called a two-weight projective code if the weight of any non-zero word of C is either w_1 or w_2 , and also no two columns of one of its generating matrix are linearly dependent.

By [10, Th. 3.1], two-weight projective $[n, 3]_q$ codes and projective $(n, 3, h_1, h_2)$ sets in $PG(2, q)$ are equivalent objects. In particular, the columns of a generating matrix of an $[n, 3]_q$ two-weight projective code with weights w_1 and w_2 , treated as points of $PG(2, q)$, form a projective $(n, 3, n - w_1, n - w_2)$ set and vice versa.

Interestingly, projective $(n, 3, h_1, h_2)$ sets give rise to $(1, \mu)$ -OS in $PG(2, q)$, and hence to APMCF codes.

Proposition 7.5. Let S be a projective $(n, 3, h_1, h_2)$ set in $PG(2, q)$.

(i) The numbers of h_i -secants of S through any point of $PG(2, q) \setminus S$ is

$$(7.3) \quad v_i = \frac{n - (q + 1)h_{3-i}}{h_i - h_{3-i}}, \quad i = 1, 2.$$

(ii) The set S is an optimal $(1, \mu)$ -saturating set in $PG(2, q)$ with

$$\mu = v_1 \binom{h_1}{2} + v_2 \binom{h_2}{2}.$$

Proof. Since there are $q + 1$ lines through every point of $PG(2, q)$, the values of v_i can be obtained from the linear system

$$(7.4) \quad \begin{cases} v_1 + v_2 = q + 1 \\ h_1 v_1 + h_2 v_2 = n \end{cases}.$$

Taking into account multiplicity, every point outside S is covered $v_1 \binom{h_1}{2} + v_2 \binom{h_2}{2}$ times. \square

Several examples of projective $(n, 3, h_1, h_2)$ sets in $PG(2, q)$ are given in [6, 10, 29]; see also the references therein. In [10] it is shown also that from a starting projective $(n, 3, h_1, h_2)$ set one can obtain such sets with new parameters using complementing and duality.

Note also that for $k > 3$, all two-weight projective $[n, k]_q$ codes and the corresponding projective (n, k, h_1, h_2) sets in $PG(k-1, q)$ give rise to $(2, \mu)$ -MCF codes and $(1, \mu)$ -saturating sets. However these sets in general can be non-optimal. Nevertheless there are cases when optimality occurs. For example, the affine space $AG(k-1, q)$ is a $(1, \mu)$ -OS in $PG(k-1, q)$. It corresponds to the two-weight code from [10, Example SU1]. Also, an elliptic quadric in $PG(3, q)$ is a $(1, \mu)$ -OS. It corresponds to the two-weight code of [10, Example TF3].

7.3. OPTIMAL $(1, \mu)$ -SATURATING SETS AND PARTITIONS OF $PG(2, q)$ IN SINGER POINT ORBITS. In [13, 14, 16], partitions of $PG(2, q)$ by Singer subgroups are considered.

Definition 7.6. [13] Let $v = dt$. A binary $v \times v$ matrix \mathbf{A} is said to be a *block double-circulant matrix* (BDC matrix for short) if

$$(7.5) \quad \mathbf{A} = \begin{bmatrix} \mathbf{C}_{0,0} & \mathbf{C}_{0,1} & \cdots & \mathbf{C}_{0,t-1} \\ \mathbf{C}_{1,0} & \mathbf{C}_{1,1} & \cdots & \mathbf{C}_{1,t-1} \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{C}_{t-1,0} & \mathbf{C}_{t-1,1} & \cdots & \mathbf{C}_{t-1,t-1} \end{bmatrix},$$

where $\mathbf{C}_{i,j}$ is a *circulant* $d \times d$ binary matrix for all i, j , and submatrices $\mathbf{C}_{i,j}$ and $\mathbf{C}_{l,m}$ with $j - i \equiv m - l \pmod{t}$ have the same weight. The matrix

$$(7.6) \quad \mathbf{W}(\mathbf{A}) = \begin{bmatrix} w_0 & w_1 & w_2 & w_3 & \cdots & w_{t-2} & w_{t-1} \\ w_{t-1} & w_0 & w_1 & w_2 & \cdots & w_{t-3} & w_{t-2} \\ w_{t-2} & w_{t-1} & w_0 & w_1 & \cdots & w_{t-4} & w_{t-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ w_1 & w_2 & w_3 & w_4 & \cdots & w_{t-1} & w_0 \end{bmatrix}$$

is a *circulant* $t \times t$ matrix whose entry in position i, j is the *weight* of $\mathbf{C}_{i,j}$. $\mathbf{W}(\mathbf{A})$ is called the *weight matrix* of \mathbf{A} . The vector $\overline{\mathbf{W}}(\mathbf{A}) = (w_0, w_1, \dots, w_{t-1})$ is called the *weight vector* of \mathbf{A} .

Let $q^2 + q + 1 = dt$. Then, by using the cyclic Singer group of $PG(2, q)$, the incidence matrix of the plane $PG(2, q)$ can be chosen as a BDC matrix \mathbf{A} as in (7.5). Let $P_1, P_2, \dots, P_{q^2+q+1}$ be the points of $PG(2, q)$, numbered so that P_i corresponds to the i -th column of \mathbf{A} . Let $\ell_1, \ell_2, \dots, \ell_{q^2+q+1}$ be the lines of $PG(2, q)$ numbered so that ℓ_i corresponds to the i -th row of \mathbf{A} . Denote by $\mathbf{P}_v = \{P_{dv+1}, P_{dv+2}, \dots, P_{dv+d}\}$, $0 \leq v \leq t-1$, the point set corresponding to the $(v+1)$ -th block column of \mathbf{A} . Let $\mathbf{L}_u = \{\ell_{du+1}, \ell_{du+2}, \dots, \ell_{du+d}\}$, $0 \leq u \leq t-1$, be the line set corresponding to the $(u+1)$ -th block row of \mathbf{A} . Here addition and subtraction of indices are calculated modulo t .

Lemma 7.7. Let $1 \leq m \leq t-1$. A set

$$\mathbf{P}^{(m)} = \mathbf{P}_0 \cup \mathbf{P}_1 \cup \dots \cup \mathbf{P}_{m-1}$$

corresponding to the first md columns of \mathbf{A} is a $(1, \mu)$ -saturating set in $PG(2, q)$ with $\mu = \min_v N_v^{(m)}$, where

$$(7.7) \quad N_v^{(m)} = \sum_{u=0}^{t-1} w_{t-u+v} \binom{w_u^{(m)}}{2} \geq 0,$$

$$w_u^{(m)} = \sum_{j=0}^{m-1} w_{t-u+j}, \quad 1 \leq m \leq v \leq t-1.$$

Proof. Every line of \mathbf{L}_u is a $w_u^{(m)}$ -secant of $\mathbf{P}^{(m)}$. Let $v \geq m$. Every point of \mathbf{P}_v is covered by as many as $w_{t-u+v} w_u^{(m)}$ -secants of $\mathbf{P}^{(m)}$ with multiplicity $\binom{w_u^{(m)}}{2}$ for $0 \leq u \leq t-1$. As a result, taking into account multiplicity, every point of \mathbf{P}_v , $m \leq v \leq t-1$, is covered by $N_v^{(m)}$ secants of $\mathbf{P}^{(m)}$. \square

Theorem 7.8. *Let \mathbf{A} be a BDC incidence matrix of the plane $PG(2, q)$, as in (7.5) with $v = q^2 + q + 1 = dt$. Let $\mathbf{P}^{(m)}$ and $N_v^{(m)}$ be as in Lemma 7.7. Then the following hold.*

- (i) *The set $\mathbf{P}^{(t-1)}$ is an optimal $(1, \mu)$ -saturating set of size $(t-1)d$ in $PG(2, q)$ with*

$$\mu = N_{t-1}^{(t-1)}.$$

- (ii) *If the weight vector of \mathbf{A} is of type*

$$\overline{\mathbf{W}}(\mathbf{A}) = (w_0, \underbrace{w, \dots, w}_{t-1 \text{ times}}),$$

then any set $\mathbf{P}^{(m)}$ with $1 \leq m \leq t-1$ is an optimal $(1, \mu)$ -saturating set in $PG(2, q)$ with

$$\mu = mw \binom{w_0 + (m-1)w}{2} + (w_0 + (t-m-1)w) \binom{mw}{2}.$$

- (iii) *If the weight vector $\overline{\mathbf{W}}(\mathbf{A})$ contains exactly two distinct weights, say \hat{w}_1 and \hat{w}_2 , then the set $\mathbf{P}^{(1)}$ is an optimal $(1, \mu)$ -saturating set in $PG(2, q)$ with parameters as in Proposition 7.5 for $n = d, h_1 = \hat{w}_1, h_2 = \hat{w}_2$.*

Proof. (i) The claim follows from Lemma 7.7.

- (ii) For $0 \leq u \leq m-1$, every line of \mathbf{L}_u is a $(w_0 + (m-1)w)$ -secant of $\mathbf{P}^{(m)}$. Every point of \mathbf{P}_v , $m \leq v \leq t-1$, is covered by mw such secants with multiplicity $\binom{w_0 + (m-1)w}{2}$. Also for $m \leq u \leq t-1$, every line of \mathbf{L}_u is an mw -secant of $\mathbf{P}^{(m)}$. Every point of \mathbf{P}_v , $m \leq v \leq t-1$, is covered by $w_0 + (t-m-1)w$ such secants with multiplicity $\binom{mw}{2}$.
- (iii) By construction, the set $\mathbf{P}^{(1)}$ is a projective $(d, 3, \hat{w}_1, \hat{w}_2)$ set in $PG(2, q)$. \square

Several examples of representations of the incidence matrix of $PG(2, q)$ in a BDC form (7.5) are considered in [16, Propositions 4,6,7, Table 1], where both theoretical and computer assisted results are listed. Both cases (ii) and (iii) of Theorem 7.8 occur for some q 's.

Note that codes corresponding to Theorem 7.8 are cyclic in the case (ii) if $m = 1$ and in case (iii); they are quasi-cyclic in the case (i) and in the case (ii) if $m \geq 2$.

Remark 7.9. In Theorem 7.8, the points of $\mathbf{P}^{(t-1)}$ as in case (i), viewed as the columns of a generator matrix, give rise to a projective code with $2 \leq s \leq t$ non-zero weights. On the other hand, the set $\mathbf{P}^{(m)}$ of case (ii) provides a projective two-weight code.

Remark 7.10. By Proposition 2.2 the APMCF codes corresponding to the optimal $(1, \mu)$ -saturating sets constructed in this section are actually PMFC precisely when no three points in the saturating set are collinear.

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